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John, P.; Suttorp, L.G.

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Equilibrium fluctuation formulae for the quantum one-component plasma in a magnetic field

P. John and L.G. Suttorp

Instituut voor Theoretische Fysica, Universiteit van Amsterdam, Valckenierstraat 65, 1018 XE Amsterdam, The Netherlands

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We derive a complete set of equilibrium fluctuation formulae for the charge density, the current density and the energy density of the quantum one-component plasma in a magnetic field. The derivation is based on the use of imaginary-time-dependent Green functions and their Kubo transforms. It is shown how the fluctuation formulae involving Kubo-transformed quantities can be established directly, even in the absence of a detailed knowledge of the full imaginary-time dependence of the Green functions. The resulting fluctuation formulae for the Kubo transforms are found to have a considerably simpler structure than those for the equal-imaginary-time observables.

1. Introduction

Fluctuation formulae are important tools in understanding the equilibrium behaviour of macroscopic systems. In general, they connect the large-scale correlations of fluctuating physical quantities to simple thermodynamic properties. For classical plasmas the fluctuation formulae for the charge density, the current density and the energy density are well known, both for the one-component plasma (OCP) [1,2] and for the general multi-component ionic mixture [3]. All these classical fluctuation formulae are equally valid if the plasma is situated in a uniform magnetic field. This is no longer true for quantum plasmas. For these plasmas only part of the fluctuation formulae are available in the literature. For the unmagnetized quantum plasma, fluctuation formulae involving the charge and the current densities can be found in refs. [4–6]. For the magnetized case not much is known. The fluctuation formula for the charge density in a magnetized OCP is given in ref. [6], while in ref. [4] some remarks are made about the current density fluctuations.

In this paper we will analyse the equilibrium fluctuation properties of the

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quantum OCP in a magnetic field in a systematic way. In particular, we shall derive the complete set of fluctuation formulae for the charge density, the current density and the energy density, considering both the auto-correlations and the cross-correlations of these quantities. In the standard type of fluctuation formulae these correlations refer to a pair of fluctuating local observables at the same time. However, it is a well-known feature of quantum statistics that the introduction of observables depending on an imaginary-time variable often helps in understanding the basic structure of the theory. For that reason we shall consider more generally imaginary-time-dependent fluctuation formulae, as has been done before [6].

In general it can be said that quantum fluctuation formulae are more complicated than their classical counterparts. This is true both for the equal-time and for the general imaginary-time-dependent formulae. The present case is no exception to that rule. Simpler fluctuation formulae are often encountered in linear-response theory. In this theory a central role is played by so-called Kubo-transformed fluctuation expressions which follow by integration over the imaginary time. The Kubo transformation has become a familiar tool in quantum statistical physics, for instance in the analysis of collective modes with the help of projection operators. In the following we shall derive these Kubo-transformed fluctuation formulae for the magnetized quantum OCP as well.

In contrast to what might be expected it is possible to derive the Kubo-transformed fluctuation formulae directly without making use of the corresponding imaginary-time dependent formulae. To stress this fact we shall start, in section 2, with a discussion of the former, while the latter will be discussed afterwards, in section 3. As a starting-point for the derivation of both sections we shall take the imaginary-time-dependent equations of motion for the charge and the current density, which follow by commutation with the Hamiltonian. In an appendix it will be shown that the results of section 2 (but not those of section 3) can be generalized to an ionic mixture in which several species of particles move in a neutralizing background.

2. Calculation of Kubo-transformed fluctuation formulae

The system we shall consider is the OCP in a uniform magnetic field in the fluid phase. The OCP consists of N quantum particles of charge e and mass m in a volume V. The particles move in an inert neutralizing background of charge density $-q_v = -eN/V$. The Hamiltonian reads

$$H = \frac{\hbar^2}{2mV} \sum_{k} \psi^{\dagger}(k) \left(k - \frac{ie}{2\hbar c} B \wedge \nabla_k \right)^2 \psi(k)$$
$$+ \frac{1}{2V^3} \sum_{kk'} \sum_{q \neq 0} \frac{e^2}{q^2} \psi^{\dagger}(k+q)\, \psi^{\dagger}(k'-q)\, \psi(k')\, \psi(k)\,, \tag{1}$$

where B denotes the magnetic field and where $\psi^{\dagger}(k)$ and $\psi(k)$ are the creation and annihilation operators satisfying the standard (anti)commutation relations. In the following we will forget about the spin dependence of the particles.

The charge density in Fourier language is given by $Q(k) = (e/V) \Sigma_{k'} \psi^{\dagger}(k'-k)\, \psi(k')$. The current density $J(k)$ is

$$J(k) = \frac{e\hbar}{2mV} \sum_{k'} \psi^{\dagger}(k'-k) \left(2k' - k - \frac{ie}{\hbar c} B \wedge \nabla_{k'} \right) \psi(k')\,. \tag{2}$$

The energy density is the sum of a kinetic part and a potential part. The former is defined as

$$E_{\mathrm{kin}}(k) = \frac{\hbar^2}{8mV} \sum_{k'} \psi'(k'-k) \left(2k' - k - \frac{ie}{\hbar c} B \wedge \nabla_{k'} \right)^2 \psi(k')\,. \tag{3}$$

For the potential part of the energy density one has a choice between several alternatives. If one adopts the expression $\frac{1}{2}E^2$, with E the electric field, and omits self-terms, one finds in Fourier language

$$E_{\mathrm{pot}}(k) = -\frac{1}{2V^3} \sum_{q\,(\neq 0, \neq k)} \frac{e^2 q \cdot (k-q)}{q^2 (k-q)^2}$$
$$\times \sum_{k'k''} \psi^{\dagger}(k'-k+q)\, \psi^{\dagger}(k''-q) \psi(k'')\, \psi(k')\,. \tag{4}$$

A somewhat different expression is found by localizing the energy at the particles and taking as the potential energy density the product of the local values of the charge density and the electric potential [2]. With this choice one gets in Fourier space

$$\tilde{E}_{\mathrm{pot}}(k) = \frac{1}{2V^3} \sum_{q\,(\neq 0, \neq k)} \frac{e^2}{q^2} \sum_{k'k''} \psi^{\dagger}(k'-k+q)\, \psi^{\dagger}(k''-q)\, \psi(k'')\, \psi(k')\,. \tag{5}$$

The total energy density corresponding to the alternatives (4) and (5) will be denoted as $E(k)$ and $\tilde{E}(k)$, respectively. For both choices one has $E(k=0) = \tilde{E}(k=0) = H$. We shall derive fluctuation formulas for each of these energy densities and compare the results.

In the following we will use the equations of motion for the charge density

and the current density. We will therefore need the commutation relations of these densities with the Hamiltonian. They read

$$[H, Q(\boldsymbol{k})] = -\hbar \boldsymbol{k} \cdot \boldsymbol{J}(\boldsymbol{k}) , \tag{6}$$

$$[H, \boldsymbol{J}(\boldsymbol{k})] = -\frac{\mathrm{i}e\hbar}{mc} \boldsymbol{J}(\boldsymbol{k}) \wedge \boldsymbol{B} - \frac{e\hbar}{m} \boldsymbol{k} \cdot \boldsymbol{T}_{\mathrm{kin}}(\boldsymbol{k}) - \frac{e\hbar}{mV^3} \sum_{\boldsymbol{q} \neq \boldsymbol{0}} \frac{e^2}{q^2} \boldsymbol{q} \sum_{\boldsymbol{k}'\boldsymbol{k}''} \psi^{\dagger}(\boldsymbol{k}' - \boldsymbol{k} + \boldsymbol{q}) \, \psi^{\dagger}(\boldsymbol{k}'' - \boldsymbol{q}) \, \psi(\boldsymbol{k}'') \, \psi(\boldsymbol{k}') , \tag{7}$$

where the kinetic pressure tensor $\boldsymbol{T}_{\mathrm{kin}}$ is defined by

$$\boldsymbol{T}_{\mathrm{kin}}(\boldsymbol{k}) = \frac{\hbar^2}{4mV} \sum_{\boldsymbol{k}'} \psi^{\dagger}(\boldsymbol{k}' - \boldsymbol{k}) \left(2\boldsymbol{k}' - \boldsymbol{k} - \frac{\mathrm{i}e}{\hbar c} \boldsymbol{B} \wedge \boldsymbol{\nabla}_{\boldsymbol{k}'}\right) \times \left(2\boldsymbol{k}' - \boldsymbol{k} - \frac{\mathrm{i}e}{\hbar c} \boldsymbol{B} \wedge \boldsymbol{\nabla}_{\boldsymbol{k}'}\right) \psi(\boldsymbol{k}') . \tag{8}$$

In the last expression the first gradient $\boldsymbol{\nabla}_{\boldsymbol{k}'}$ acts only on $\psi(\boldsymbol{k}')$ and not on the preceding factor.

The imaginary-time-dependent Green functions in a canonical ensemble are defined by

$$\langle A_{\tau} B \rangle = Z^{-1} \operatorname{tr}\{\exp[-(\beta - \tau)H] \, A \exp(-\tau H) \, B\} , \tag{9}$$

where Z is the partition function and β is the inverse temperature. In (9) we have taken the thermodynamic limit, as we will always do in the following. The Green functions satisfy the so-called KMS condition:

$$\langle A_{\tau} B \rangle = \langle B_{\beta - \tau} A \rangle , \qquad 0 \leqslant \tau \leqslant \beta , \tag{10}$$

which serves as a boundary condition for the imaginary-time-dependent equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \langle A_{\tau} B \rangle = \langle [H, A]_{\tau} B \rangle . \tag{11}$$

We are interested in the Green functions for operators A and B that depend on position. Introducing truncated Green functions as

$$\langle A(\boldsymbol{r}_1) \, B(\boldsymbol{r}_2) \rangle_{\mathrm{T}} = \langle [A(\boldsymbol{r}_1) - \langle A \rangle][B(\boldsymbol{r}_2) - \langle B \rangle] \rangle \tag{12}$$

we may take the Fourier transform and write

$$\frac{1}{V}\langle A(\boldsymbol{k})\, B(-\boldsymbol{k})\rangle_{\mathrm{T}} = \frac{1}{V}\int \mathrm{d}^3 r_1\, \mathrm{d}^3 r_2 \exp[-\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}_1-\boldsymbol{r}_2)]\,\langle A(\boldsymbol{r}_1)\, B(\boldsymbol{r}_2)\rangle_{\mathrm{T}}\,. \tag{13}$$

For $\boldsymbol{k}\neq\boldsymbol{0}$ there is no difference between the truncated and the ordinary Green functions, but for $\boldsymbol{k}=\boldsymbol{0}$ it is important to use the truncated version.

Fluctuation formulae are in general defined as relations for Fourier-transformed imaginary-time-dependent Green functions for small wavenumbers. In particular, one is interested in the so-called equal-time fluctuation formulae, in which one puts $\tau=0$. Alternatively, a Kubo transform of the imaginary-time-dependent fluctuation formulae may be considered. The Kubo transform of a quantity depending on the imaginary time τ is defined as

$$\mathscr{K} := \beta^{-1}\int_0^\beta \mathrm{d}\tau\,. \tag{14}$$

It appears naturally in linear-response theory. Often the Kubo transform of an imaginary-time-dependent fluctuation formula is simpler than the corresponding equal-time version. For that reason these transforms are preferred in calculations involving projection operator techniques, for instance in deriving the hydrodynamic modes. We will present in this section a method to calculate the Kubo-transformed fluctuation formulae of the OCP directly from the equations of motion, deriving a complete set of fluctuation formulae involving the charge density, the current density and the energy density. In section 3 we will treat the general imaginary-time-dependent fluctuation formulae.

We start with the simple observation that the Kubo transform of an imaginary-time derivative of a correlation function yields a commutator:

$$\begin{aligned}\mathscr{K}\,\frac{\partial}{\partial\tau}\langle A_\tau(\boldsymbol{k})\, B(-\boldsymbol{k})\rangle_{\mathrm{T}} &= \beta^{-1}\Big(\frac{1}{V}\langle A_\beta(\boldsymbol{k})\, B(-\boldsymbol{k})\rangle_{\mathrm{T}} - \frac{1}{V}\langle A(\boldsymbol{k})\, B(-\boldsymbol{k})\rangle_{\mathrm{T}}\Big)\\ &= \beta^{-1}\,\frac{1}{V}\,\langle[B(-\boldsymbol{k}),\, A(\boldsymbol{k})]\rangle\,,\end{aligned} \tag{15}$$

where we have used the KMS condition in the last equality. So the Kubo-transformed equations of motion have the form [5]

$$\mathscr{K}\,\frac{1}{V}\,\langle[H,\, A(\boldsymbol{k})]_\tau B(-\boldsymbol{k})\rangle_{\mathrm{T}} = \beta^{-1}\,\frac{1}{V}\,\langle[B(-\boldsymbol{k}),\, A(\boldsymbol{k})]\rangle\,. \tag{16}$$

In the following we drop the subscript τ in the Kubo-transformed fluctuation

formulae. We shall use (16) with the commutator relations (6) and (7) to derive a set of equations for $(1/V)\langle Q(\boldsymbol{k})\,\Omega(-\boldsymbol{k})\rangle_{\mathrm{T}}$ and $(1/V)\langle \boldsymbol{J}(\boldsymbol{k})\,\Omega(-\boldsymbol{k})\rangle_{\mathrm{T}}$ for arbitrary local operators Ω.

In its present form the commutator relation (7) is not very useful. It can be rewritten, however, by splitting the sum appearing at the right-hand side in two terms, with a different behaviour for small values of $\boldsymbol{k}$. The term with $\boldsymbol{q} = \boldsymbol{k}$ is singular for $\boldsymbol{k} \to \boldsymbol{0}$. It is proportional to the charge density. The remainder of the sum, with $\boldsymbol{q} \neq \boldsymbol{0}, \neq \boldsymbol{k}$, can be written in such a form that it obviously vanishes for $\boldsymbol{k} \to \boldsymbol{0}$:

$$\boldsymbol{k}\cdot\boldsymbol{T}_{\mathrm{pot}}(\boldsymbol{k}) = \frac{1}{2V^3}\sum_{\boldsymbol{q}\,(\neq\boldsymbol{0},\neq\boldsymbol{k})}\left(\frac{e^2}{q^2}\,\boldsymbol{q} + \frac{e^2}{(\boldsymbol{k}-\boldsymbol{q})^2}\,(\boldsymbol{k}-\boldsymbol{q})\right) \times \sum_{\boldsymbol{k}'\boldsymbol{k}''} \psi^{\dagger}(\boldsymbol{k}'-\boldsymbol{k}+\boldsymbol{q})\,\psi^{\dagger}(\boldsymbol{k}''-\boldsymbol{q})\,\psi(\boldsymbol{k}'')\,\psi(\boldsymbol{k}')\,, \tag{17}$$

with $\boldsymbol{T}_{\mathrm{pot}}$ the potential pressure tensor. As a result we have found from (7)

$$[H, \boldsymbol{J}(\boldsymbol{k})] = -\hbar\omega_{\mathrm{p}}^2\,\frac{\boldsymbol{k}}{k^2}\,Q(\boldsymbol{k}) - \mathrm{i}\hbar\omega_{\mathrm{c}}\boldsymbol{J}(\boldsymbol{k})\wedge\hat{\boldsymbol{B}} - \frac{e\hbar}{m}\,\boldsymbol{k}\cdot\boldsymbol{T}_{\mathrm{kin}}(\boldsymbol{k}) - \frac{e\hbar}{m}\,\boldsymbol{k}\cdot\boldsymbol{T}_{\mathrm{pot}}(\boldsymbol{k})\,, \tag{18}$$

where $\omega_{\mathrm{p}} = (ne^2/m)^{1/2}$ and $\omega_{\mathrm{c}} = (e/mc)|\boldsymbol{B}|$ are the plasmon and the cyclotron frequency, respectively. Furthermore, $\hat{\boldsymbol{B}}$ denotes the unit vector in the direction of the magnetic field. By expanding (17) in lowest order of $\boldsymbol{k}$, we can write the potential pressure tensor for $\boldsymbol{k} = \boldsymbol{0}$ as

$$\boldsymbol{T}_{\mathrm{pot}}(\boldsymbol{k}=\boldsymbol{0}) = \frac{1}{2V^3}\sum_{\boldsymbol{q}\neq\boldsymbol{0}}\frac{e^2}{q^2}\left(\boldsymbol{U} - \frac{2\boldsymbol{q}\boldsymbol{q}}{q^2}\right)\sum_{\boldsymbol{k}'\boldsymbol{k}''}\psi^{\dagger}(\boldsymbol{k}'+\boldsymbol{q})\,\psi^{\dagger}(\boldsymbol{k}''-\boldsymbol{q})\,\psi(\boldsymbol{k}'')\,\psi(\boldsymbol{k}')\,, \tag{19}$$

where $\boldsymbol{U}$ denotes the unit tensor.

Substituting (6) and (18) in the Kubo-transformed equation of motion (16) we get the following set of linear equations:

$$-\hbar\boldsymbol{k}\cdot\mathscr{K}\,\frac{1}{V}\,\langle \boldsymbol{J}(\boldsymbol{k})\,\Omega(-\boldsymbol{k})\rangle_{\mathrm{T}} = \beta^{-1}\,\frac{1}{V}\,\langle[\Omega(-\boldsymbol{k}), Q(\boldsymbol{k})]\rangle\,, \tag{20}$$

$$-\hbar\omega_{\mathrm{p}}^2\,\frac{\boldsymbol{k}}{k^2}\,\mathscr{K}\,\frac{1}{V}\,\langle Q(\boldsymbol{k})\,\Omega(-\boldsymbol{k})\rangle_{\mathrm{T}} - \mathrm{i}\hbar\omega_{\mathrm{c}}\mathscr{K}\,\frac{1}{V}\,\langle \boldsymbol{J}(\boldsymbol{k})\,\Omega(-\boldsymbol{k})\rangle_{\mathrm{T}}\wedge\hat{\boldsymbol{B}} - \frac{e\hbar}{m}\,\boldsymbol{k}\cdot\mathscr{K}\,\frac{1}{V}\,\langle \boldsymbol{T}(\boldsymbol{k})\,\Omega(-\boldsymbol{k})\rangle_{\mathrm{T}} = \beta^{-1}\,\frac{1}{V}\,\langle[\Omega(-\boldsymbol{k}), \boldsymbol{J}(\boldsymbol{k})]\rangle\,, \tag{21}$$

where we introduced the abbreviation $\boldsymbol{T} := \boldsymbol{T}_{\mathrm{kin}} + \boldsymbol{T}_{\mathrm{pot}}$ for the total pressure tensor.

We can solve this set of equations by taking the outer product with $\hat{\boldsymbol{k}}$ and the inner product with $\hat{\boldsymbol{B}}$ in (21). The resulting set of three equations can easily be solved to obtain

$$k^{-1}\mathcal{K}\,\frac{1}{V}\,\langle Q(\boldsymbol{k})\,\Omega(-\boldsymbol{k})\rangle_{\mathrm{T}} = -\,\frac{1}{\hbar\beta\omega_{\mathrm{p}}^2\cos\vartheta}\,\frac{1}{V}\,\langle[\Omega(-\boldsymbol{k}),\boldsymbol{J}(\boldsymbol{k})]\rangle\cdot\hat{\boldsymbol{B}} - \frac{e}{m\omega_{\mathrm{p}}^2\cos\vartheta}\,\boldsymbol{k}\cdot\mathcal{K}\,\frac{1}{V}\,\langle\boldsymbol{T}(\boldsymbol{k})\,\Omega(-\boldsymbol{k})\rangle_{\mathrm{T}}\cdot\hat{\boldsymbol{B}}\,, \tag{22}$$

$$\mathcal{K}\,\frac{1}{V}\,\langle\boldsymbol{J}(\boldsymbol{k})\,\Omega(-\boldsymbol{k})\rangle_{\mathrm{T}} = -\,\frac{1}{\hbar\beta k\cos\vartheta}\,\hat{\boldsymbol{B}}\,\frac{1}{V}\,\langle[\Omega(-\boldsymbol{k}),Q(\boldsymbol{k})]\rangle + \frac{\mathrm{i}}{\hbar\beta\omega_{\mathrm{c}}\cos\vartheta}\,\hat{\boldsymbol{k}}\wedge\frac{1}{V}\,\langle[\Omega(-\boldsymbol{k}),\boldsymbol{J}(\boldsymbol{k})]\rangle - \frac{\mathrm{i}e}{m\omega_{\mathrm{c}}\cos\vartheta}\left(\boldsymbol{k}\cdot\mathcal{K}\,\frac{1}{V}\,\langle\boldsymbol{T}(\boldsymbol{k})\,\Omega(-\boldsymbol{k})\rangle_{\mathrm{T}}\right)\wedge\hat{\boldsymbol{k}}\,, \tag{23}$$

in which we have defined $\cos\vartheta = \hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{B}}$.

These solutions are quite general, as they are valid for arbitrary wavevectors. They are of limited use, however. More useful results are obtained by expanding the Green functions in powers of k. Let us write

$$\frac{1}{V}\,\langle A(\boldsymbol{k})\,B(-\boldsymbol{k})\rangle_{\mathrm{T}} = \frac{1}{V}\,\langle A(\boldsymbol{k})\,B(-\boldsymbol{k})\rangle_{\mathrm{T}}^{(0)} + k\,\frac{1}{V}\,\langle A(\boldsymbol{k})\,B(-\boldsymbol{k})\rangle_{\mathrm{T}}^{(1)} + k^2\,\frac{1}{V}\,\langle A(\boldsymbol{k})\,B(-\boldsymbol{k})\rangle_{\mathrm{T}}^{(2)} + \cdots\,, \tag{24}$$

so that $(1/V)\langle A(\boldsymbol{k})\,B(-\boldsymbol{k})\rangle_{\mathrm{T}}^{(s)}$ is the sth coefficient in the wavenumber expansion of $(1/V)\langle A(\boldsymbol{k})\,B(-\boldsymbol{k})\rangle_{\mathrm{T}}$. Mark that $(1/V)\langle A(\boldsymbol{k})\,B(-\boldsymbol{k})\rangle_{\mathrm{T}}^{(s)}$ still depends on $\hat{\boldsymbol{k}}$. Using this expansion we will derive now explicit fluctuation formulae in leading orders in the wavenumber.

We start with the zeroth order in $\boldsymbol{k}$. One can derive an exact sum rule in this order, since (22) implies that the charge sum rule [5]

$$\mathcal{K}\,\frac{1}{V}\,\langle Q(\boldsymbol{k})\,\Omega(-\boldsymbol{k})\rangle_{\mathrm{T}}^{(0)} = 0 \tag{25}$$

holds for all operators Ω.

In the next order of the wavenumber a further general result can be

established by limiting ourselves to purely configurational operators Ω. For these type of operators one has, of course, $(1/V)\langle[\Omega(-\boldsymbol{k}), Q(\boldsymbol{k})]\rangle = 0$ in all orders of k, and moreover

$$\frac{1}{V}\langle[\Omega(-\boldsymbol{k}), \boldsymbol{J}(\boldsymbol{k})]\rangle^{(0)} = \boldsymbol{0}\,, \tag{26}$$

since $\boldsymbol{J}(\boldsymbol{k})$ for vanishing $\boldsymbol{k}$ is proportional to the generator for translations. Hence, one finds from (22) and (23)

$$\mathcal{K}\,\frac{1}{V}\langle Q(\boldsymbol{k})\,\Omega(-\boldsymbol{k})\rangle_{\mathrm{T}}^{(1)} = 0\,, \tag{27}$$

$$\mathcal{K}\,\frac{1}{V}\langle \boldsymbol{J}(-\boldsymbol{k})\,\Omega(-\boldsymbol{k})\rangle_{\mathrm{T}}^{(0)} = \boldsymbol{0}\,. \tag{28}$$

This is as far as we can go for general operators Ω. In addition one may derive symmetry properties of the Kubo-transformed Green functions by considering the transformation character of Ω with respect to rotations, spatial reflection and time reversal. In particular, from the latter one proves for any local gauge-invariant operator Ω, which is even (or odd) in the momenta, that the left-hand side of (22) is even (or odd) and that of (23) is odd (or even) under inversion of the direction of the magnetic field.

Further results follow by specifying the operator Ω. We will make three different choices for Ω, namely the charge density, the current density and the energy density.

Case $\Omega(\boldsymbol{k}) = Q(\boldsymbol{k})$

The charge density $Q(\boldsymbol{k})$ is purely configurational so that the zeroth- and first-order terms in the expansion of the charge–charge Kubo-transformed Green function vanish on account of (25) and (27). We therefore consider the second-order term. Using (25) once more we see that the terms with $\boldsymbol{T}$ drop out from (22). Making use of the commutator relation

$$\frac{1}{V}\langle[Q(-\boldsymbol{k}), \boldsymbol{J}(\boldsymbol{k})]\rangle = -\hbar\omega_{\mathrm{p}}^2\boldsymbol{k}\,, \tag{29}$$

we get

$$\mathcal{K}\,\frac{1}{V}\langle Q(\boldsymbol{k})\,Q(-\boldsymbol{k})\rangle_{\mathrm{T}}^{(2)} = \beta^{-1}\,. \tag{30}$$

This is the quantum-mechanical Kubo-transformed version of the famous Stillinger–Lovett condition [7], which has been derived here for a magnetized OCP. For the unmagnetized plasma it has been established in ref. [5]. It should

be noted that the dependence on the magnetic field has dropped out. We shall see in the next section that the general imaginary-time-dependent charge–charge fluctuation formula is a lot more complicated and that it does depend on the magnetic field.

The charge–current Kubo-transformed Green function in zeroth order vanishes on account of (28). The first-order term is calculated by again using the charge sum rule (25) and the commutator relation (29). It turns out to vanish as well:

$$\mathcal{K}\,\frac{1}{V}\,\langle \boldsymbol{J}(\boldsymbol{k})\,Q(-\boldsymbol{k})\rangle_{\mathrm{T}}^{(1)} = \boldsymbol{0}\,. \tag{31}$$

The charge–charge and charge–current Kubo-transformed fluctuation formulae can also be written in a form valid in all orders of k. This form reads

$$\mathcal{K}\,\frac{1}{V}\,\langle Q(\boldsymbol{k})\,Q(-\boldsymbol{k})\rangle_{\mathrm{T}} = \beta^{-1}k^2 - \frac{ek^2}{m\omega_{\mathrm{p}}^2\cos\vartheta}\,\hat{\boldsymbol{k}}\cdot\mathcal{K}\,\frac{1}{V}\,\langle \boldsymbol{T}(\boldsymbol{k})\,Q(-\boldsymbol{k})\rangle_{\mathrm{T}}\cdot\hat{\boldsymbol{B}}\,, \tag{32}$$

$$\mathcal{K}\,\frac{1}{V}\,\langle \boldsymbol{J}(\boldsymbol{k})\,Q(-\boldsymbol{k})\rangle_{\mathrm{T}} = -\frac{\mathrm{i}ek}{m\omega_{\mathrm{c}}\cos\vartheta}\left(\hat{\boldsymbol{k}}\cdot\mathcal{K}\,\frac{1}{V}\,\langle \boldsymbol{T}(\boldsymbol{k})\,Q(-\boldsymbol{k})\rangle_{\mathrm{T}}\right)\wedge\hat{\boldsymbol{k}}\,. \tag{33}$$

The Kubo-transformed Green function with the pressure tensor appearing in these formulae is at least of order k on account of the charge sum rule (25).

Case $\Omega(\boldsymbol{k}) = J(\boldsymbol{k})$

The charge–current fluctuation formula has already been obtained above. This leaves us with the Kubo transform of the current–current Green function. In order k^0 the terms with $\boldsymbol{T}$ drop out from the right-hand side of (23). On a par with (29) we need the commutator relation

$$\frac{1}{V}\,\langle[\boldsymbol{J}(-\boldsymbol{k}),\boldsymbol{J}(\boldsymbol{k})]\rangle = \mathrm{i}\hbar\omega_{\mathrm{p}}^2\omega_{\mathrm{c}}\boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{B}}\,, \tag{34}$$

with $\boldsymbol{\varepsilon}$ the Levi-Civita tensor. In writing (34) we have used that the average of the current density vanishes as can be seen by parity considerations. Employing (29) and (34) we get from (23)

$$\begin{aligned}\mathcal{K}\,\frac{1}{V}\,\langle \boldsymbol{J}(\boldsymbol{k})\,\boldsymbol{J}(-\boldsymbol{k})\rangle_{\mathrm{T}}^{(0)} &= -\frac{1}{\hbar\beta\cos\vartheta}\,\hat{\boldsymbol{B}}(-\hbar\omega_{\mathrm{p}}^2\hat{\boldsymbol{k}})\\ &\quad+\frac{\mathrm{i}}{\hbar\beta\omega_{\mathrm{c}}\cos\vartheta}\,[\mathrm{i}\hbar\omega_{\mathrm{p}}^2\omega_{\mathrm{c}}(-\boldsymbol{U}\cos\vartheta+\hat{\boldsymbol{B}}\hat{\boldsymbol{k}})]\\ &= \beta^{-1}\omega_{\mathrm{p}}^2\boldsymbol{U}\,,\end{aligned} \tag{35}$$

where $\boldsymbol{U}$ denotes the unit tensor. The dependence on the magnetic field has dropped out, as before. Because (34) is valid in all orders of the wavenumber, we can write the current–current Kubo-transformed fluctuation formula for any $\boldsymbol{k}$ as

$$\mathscr{K}\,\frac{1}{V}\,\langle \boldsymbol{J}(\boldsymbol{k})\,\boldsymbol{J}(-\boldsymbol{k})\rangle_{\mathrm{T}} = \beta^{-1}\omega_{\mathrm{p}}^{2}\boldsymbol{U} - \frac{\mathrm{i}ek}{m\omega_{\mathrm{c}}\cos\vartheta}\,\hat{\boldsymbol{k}}\cdot\mathscr{K}\,\frac{1}{V}\,\langle \boldsymbol{T}(\boldsymbol{k})\wedge\hat{\boldsymbol{k}}\,\boldsymbol{J}(-\boldsymbol{k})\rangle_{\mathrm{T}}\,. \tag{36}$$

Case $\Omega(\boldsymbol{k}) = E(\boldsymbol{k})$ *or* $\Omega(\boldsymbol{k}) = \tilde{E}(\boldsymbol{k})$

This case is somewhat more involved because the Kubo-transformed Green function with the pressure tensor in (22) and (23) does not vanish in zeroth order in k for this choice of Ω. We start by giving the relevant commutators. The energy density E is found to commute with the charge density Q for all values of the wavevector:

$$\frac{1}{V}\,\langle [E(-\boldsymbol{k}),\,Q(\boldsymbol{k})]\rangle = 0\,. \tag{37}$$

The same result is found for $\tilde{E}$, since the potential part of the energy density commutes with the charge density anyway.

Turning to the commutator of E with the current density $\boldsymbol{J}$ one gets a nontrivial result already in first order of k:

$$\frac{1}{V}\,\langle [E(-\boldsymbol{k}),\,\boldsymbol{J}(\boldsymbol{k})]\rangle^{(0)} = \boldsymbol{0}\,, \tag{38}$$

$$\frac{1}{V}\,\langle [E(-\boldsymbol{k}),\,\boldsymbol{J}(\boldsymbol{k})]\rangle^{(1)} = -\frac{e\hbar}{m}\,(\boldsymbol{k}\langle E\rangle + \boldsymbol{k}\cdot\langle \boldsymbol{T}\rangle)\,. \tag{39}$$

The averages of the energy density and the pressure appearing here are independent of position, since for any local observable Ω one may write $(1/V)\langle \Omega(\boldsymbol{k}=\boldsymbol{0})\rangle = \langle \Omega(\boldsymbol{r})\rangle = \langle \Omega(\boldsymbol{r}=\boldsymbol{0})\rangle \equiv \langle \Omega\rangle$ on account of translation invariance. The commutator of $\tilde{E}$ with $\boldsymbol{J}$ in zeroth and first order of k yields the same results as (38) and (39). Differences show up only in order k^2 or higher (see appendix A). Since we shall use the commutator relations up to first order in k only, the alternative definitions of the potential energy do not lead to different versions of the fluctuation formulae we are going to derive below. For that reason we shall write down the formulas for the energy density $E(\boldsymbol{k})$ only.

In lowest order of k the $\boldsymbol{T}$ terms still drop out in (22) and (23) and we find the following fluctuation formulae:

$$\mathscr{K}\,\frac{1}{V}\,\langle Q(\boldsymbol{k})\,E(-\boldsymbol{k})\rangle_{\mathrm{T}}^{(1)} = 0 \tag{40}$$

and

$$\mathcal{K}\,\frac{1}{V}\,\langle \boldsymbol{J}(\boldsymbol{k})\,E(-\boldsymbol{k})\rangle_{\mathrm{T}}^{(0)} = \boldsymbol{0}\,. \tag{41}$$

To derive the fluctuation formulae in the next order of k we have to employ (39). The average of the pressure appearing in this commutator can be simplified somewhat by invoking the symmetry properties of the system. Since there is only one preferred direction, we can write the pressure tensor as

$$\langle \boldsymbol{T}\rangle = \lambda \boldsymbol{U} + \mu \hat{\boldsymbol{B}}\hat{\boldsymbol{B}}\,. \tag{42}$$

Here we used the fact that $\langle \boldsymbol{T}\rangle$ is a symmetric tensor. For the kinetic part of the pressure this property can be verified from (8). Furthermore, the Fourier transform of the potential pressure as given by (19) is symmetric as well. From (42) we derive

$$\hat{\boldsymbol{k}}\cdot\langle \boldsymbol{T}\rangle\cdot\hat{\boldsymbol{B}} = \cos\vartheta\;\hat{\boldsymbol{B}}\cdot\langle \boldsymbol{T}\rangle\cdot\hat{\boldsymbol{B}}\,, \tag{43}$$

$$(\hat{\boldsymbol{k}}\cdot\langle \boldsymbol{T}\rangle)\wedge\hat{\boldsymbol{k}} = -\tfrac{3}{2}\cos\vartheta\;\hat{\boldsymbol{k}}\wedge\hat{\boldsymbol{B}}\,(\hat{\boldsymbol{B}}\cdot\langle \boldsymbol{T}\rangle\cdot\hat{\boldsymbol{B}} - \tfrac{1}{3}\,\mathrm{tr}\langle \boldsymbol{T}\rangle)\,. \tag{44}$$

Insertion of (37) and (39) with (43) and (44) in (22) and (23) yields

$$\mathcal{K}\,\frac{1}{V}\,\langle Q(\boldsymbol{k})\,E(-\boldsymbol{k})\rangle_{\mathrm{T}}^{(2)} = \frac{e}{\beta m\omega_{\mathrm{p}}^2}\,(\hat{\boldsymbol{B}}\cdot\langle \boldsymbol{T}\rangle\cdot\hat{\boldsymbol{B}} + \langle E\rangle) - \frac{e}{m\omega_{\mathrm{p}}^2\cos\vartheta}\,\hat{\boldsymbol{k}}\cdot\mathcal{K}\,\frac{1}{V}\,\langle \boldsymbol{T}(\boldsymbol{k})\,E(-\boldsymbol{k})\rangle_{\mathrm{T}}^{(0)}\cdot\hat{\boldsymbol{B}} \tag{45}$$

and

$$\mathcal{K}\,\frac{1}{V}\,\langle \boldsymbol{J}(\boldsymbol{k})\,E(-\boldsymbol{k})\rangle_{\mathrm{T}}^{(1)} = -\frac{3\mathrm{i}e}{2\beta m\omega_{\mathrm{c}}}\,\hat{\boldsymbol{k}}\wedge\hat{\boldsymbol{B}}\,(\hat{\boldsymbol{B}}\cdot\langle \boldsymbol{T}\rangle\cdot\hat{\boldsymbol{B}} - \tfrac{1}{3}\,\mathrm{tr}\langle \boldsymbol{T}\rangle) - \frac{\mathrm{i}e}{m\omega_{\mathrm{c}}\cos\vartheta}\left(\hat{\boldsymbol{k}}\cdot\mathcal{K}\,\frac{1}{V}\,\langle \boldsymbol{T}(\boldsymbol{k})\,E(-\boldsymbol{k})\rangle_{\mathrm{T}}^{(0)}\right)\wedge\hat{\boldsymbol{k}}\,. \tag{46}$$

To evaluate the Kubo-transformed Green functions with the pressure tensor appearing at the right-hand sides of these equations we remark that in zeroth order of the wavenumber $(1/V)\langle \Omega_{\mathrm{T}}(\boldsymbol{k})\,E(-\boldsymbol{k})\rangle_{\mathrm{T}}^{(0)} = (1/V)\langle \Omega_{\mathrm{T}}(\boldsymbol{k})\,H\rangle_{\mathrm{T}}^{(0)}$ is independent of τ, so that the Kubo transformation becomes the identity operation for this case. Furthermore, we can use the thermodynamic relation $\langle \Omega H\rangle_{\mathrm{T}} = -\partial\langle \Omega\rangle/\partial\beta$. Hence, upon using once more (43) and (44) we can

rewrite the Kubo-transformed fluctuation formulae (45) and (46) as

$$\mathcal{K}\,\frac{1}{V}\,\langle Q(\boldsymbol{k})\,E(-\boldsymbol{k})\rangle_{\mathrm{T}}^{(2)} = \frac{e}{\beta m\omega_{\mathrm{p}}^2}\,\frac{\partial}{\partial\beta}\,(\beta\hat{\boldsymbol{B}}\cdot\langle\boldsymbol{T}\rangle\cdot\hat{\boldsymbol{B}}) + \frac{e}{\beta m\omega_{\mathrm{p}}^2}\,\langle E\rangle\,, \tag{47}$$

$$\mathcal{K}\,\frac{1}{V}\,\langle \boldsymbol{J}(\boldsymbol{k})\,E(-\boldsymbol{k})\rangle_{\mathrm{T}}^{(1)} = -\,\frac{3\mathrm{i}e}{2\beta m\omega_{\mathrm{c}}}\,\hat{\boldsymbol{k}}\wedge\hat{\boldsymbol{B}}\,\frac{\partial}{\partial\beta}\,[\beta(\hat{\boldsymbol{B}}\cdot\langle\boldsymbol{T}\rangle\cdot\hat{\boldsymbol{B}} - \tfrac{1}{3}\,\mathrm{tr}\langle\boldsymbol{T}\rangle)]\,. \tag{48}$$

These fluctuation formulae get a simpler form if the average pressure tensor is taken to be isotropic. It may be argued that in the fluid phase, i.e. below the Wigner crystallization point, the equilibrium quantum OCP cannot sustain shear forces. However, it is not clear whether this argument still holds if a magnetic field is present. If it is valid one may write $\hat{\boldsymbol{B}}\cdot\langle\boldsymbol{T}\rangle\cdot\hat{\boldsymbol{B}} = \frac{1}{3}\,\mathrm{tr}\langle\boldsymbol{T}\rangle = p$, with p the scalar pressure. The right-hand side of (48) vanishes then, while the right-hand side of (47) reduces to $(1/\beta)(\partial e_{\mathrm{v}}/\partial q_{\mathrm{v}})_{\beta}$, with $e_{\mathrm{v}} = \langle E\rangle$ the energy density.

By now we have nearly completed the task of deriving all Kubo-transformed fluctuation formulae for the charge, the current and the energy density. The only formula that is missing still is the energy–energy fluctuation formula. Its derivation is trivial, however, in view of the general properties of the Green function $(1/V)\langle\Omega_{\tau}(\boldsymbol{k})\,E(-\boldsymbol{k})\rangle_{\mathrm{T}}^{(0)}$ discussed above. We immediately find

$$\mathcal{K}\,\frac{1}{V}\,\langle E(\boldsymbol{k})\,E(-\boldsymbol{k})\rangle_{\mathrm{T}}^{(0)} = \frac{nc_V}{k_{\mathrm{B}}\beta^2}\,, \tag{49}$$

with c_V the specific heat at constant volume.

One should notice the relative simplicity of the Kubo-transformed versions of the fluctuation formulae. They should be compared to the more complicated expressions that occur in the imaginary-time-dependent fluctuation formulae, which will be derived in the next section. In fact, use of the Kubo transformation simplifies the calculations so greatly that we can even evaluate the Kubo-transformed fluctuation formulae for the multi-component ionic mixture in a magnetic field. This will be shown in appendix B.

3. Imaginary-time-dependent fluctuation formulae

Kubo-transformed fluctuation formulae are useful in describing the response of a quantum system to external disturbances. More detailed information on the equilibrium properties of the quantum OCP is obtained by studying the full imaginary-time dependence of the Green functions for the charge density, the

current density and the energy density. In the long-wavelength limit this dependence is given by the imaginary-time-dependent fluctuation formulae, the derivation of which will be considered now. The starting-point of our calculations will be once again the equation of motion (11) and the KMS condition (10). Employing the expressions for the commutators $[H, Q(\boldsymbol{k})]$ and $[H, \boldsymbol{J}(\boldsymbol{k})]$ as given by (6) and (18) we get the following set of linear differential equations:

$$\frac{\partial}{\partial\tau}\frac{1}{V}\langle Q_\tau(\boldsymbol{k})\,\Omega(-\boldsymbol{k})\rangle_{\mathrm{T}} = -\hbar\boldsymbol{k}\cdot\frac{1}{V}\langle \boldsymbol{J}_\tau(\boldsymbol{k})\,\Omega(-\boldsymbol{k})\rangle_{\mathrm{T}}\,, \tag{50}$$

$$\begin{aligned}\frac{\partial}{\partial\tau}\frac{1}{V}\langle \boldsymbol{J}_\tau(\boldsymbol{k})\,\Omega(-\boldsymbol{k})\rangle_{\mathrm{T}} = &-\hbar\omega_{\mathrm{p}}^2\frac{\boldsymbol{k}}{k^2}\frac{1}{V}\langle Q_\tau(\boldsymbol{k})\,\Omega(-\boldsymbol{k})\rangle_{\mathrm{T}}\\ &-\mathrm{i}\hbar\omega_{\mathrm{c}}\frac{1}{V}\langle \boldsymbol{J}_\tau(\boldsymbol{k})\,\Omega(-\boldsymbol{k})\rangle_{\mathrm{T}}\wedge\hat{\boldsymbol{B}}\\ &-\frac{e\hbar}{m}\boldsymbol{k}\cdot\frac{1}{V}\langle \boldsymbol{T}_\tau(\boldsymbol{k})\,\Omega(-\boldsymbol{k})\rangle_{\mathrm{T}}\,.\end{aligned} \tag{51}$$

We will solve these equations using the KMS condition as a boundary condition. The latter implies

$$\frac{1}{V}\langle Q_\tau(\boldsymbol{k})\,\Omega(-\boldsymbol{k})\rangle_{\mathrm{T}}\Big|_{\tau=0}^{\tau=\beta} = \frac{1}{V}\langle[\Omega(-\boldsymbol{k}), Q(\boldsymbol{k})]\rangle\,, \tag{52}$$

$$\frac{1}{V}\langle \boldsymbol{J}_\tau(\boldsymbol{k})\,\Omega(-\boldsymbol{k})\rangle_{\mathrm{T}}\Big|_{\tau=0}^{\tau=\beta} = \frac{1}{V}\langle[\Omega(-\boldsymbol{k}), \boldsymbol{J}(\boldsymbol{k})]\rangle\,. \tag{53}$$

We start the calculation by giving a few definitions. First, we combine the imaginary-time Green functions involving the charge and the current density into a four-dimensional vector:

$$X_\tau^\mu := \begin{pmatrix}\omega_{\mathrm{p}}k^{-1}V^{-1}\langle Q_\tau(\boldsymbol{k})\,\Omega(-\boldsymbol{k})\rangle_{\mathrm{T}}\\ V^{-1}\langle \boldsymbol{J}_\tau(\boldsymbol{k})\,\Omega(-\boldsymbol{k})\rangle_{\mathrm{T}}\end{pmatrix}. \tag{54}$$

Furthermore, we introduce the Hermitian 4×4 matrix

$$L^{\mu\nu} := \hbar\begin{pmatrix}0 & -\omega_{\mathrm{p}}\hat{\boldsymbol{k}}\\ -\omega_{\mathrm{p}}\hat{\boldsymbol{k}} & -\mathrm{i}\omega_{\mathrm{c}}\boldsymbol{\varepsilon}\cdot\hat{\boldsymbol{B}}\end{pmatrix}, \tag{55}$$

with $\boldsymbol{\varepsilon}$ the Levi-Civita tensor, and the four-dimensional vector

$$M_\tau^\mu := \begin{pmatrix}0\\ -(e\hbar/m)\boldsymbol{k}\cdot V^{-1}\langle \boldsymbol{T}_\tau(\boldsymbol{k})\,\Omega(-\boldsymbol{k})\rangle_{\mathrm{T}}\end{pmatrix} \equiv \begin{pmatrix}0\\ \boldsymbol{M}_\tau\end{pmatrix}. \tag{56}$$

Making use of these definitions we can write (50) and (51) in the following way:

$$\frac{\partial}{\partial \tau} X^{\mu}_{\tau} = L^{\mu\nu} X^{\nu}_{\tau} + M^{\mu}_{\tau} \,. \tag{57}$$

The homogeneous equation related to (57) can easily be solved. The eigenvalues of $L^{\mu\nu}$ are $\hbar\rho\omega_\lambda$, with λ, $\rho = \pm 1$ and with the basic frequencies

$$\omega_\lambda = \tfrac{1}{2}\sqrt{\omega_p^2 + \omega_c^2 + 2\omega_p\omega_c \cos\vartheta} + \tfrac{1}{2}\lambda\sqrt{\omega_p^2 + \omega_c^2 - 2\omega_p\omega_c \cos\vartheta} \,. \tag{58}$$

The eigenvectors associated to these eigenvalues are

$$X^{\mu}_{\lambda\rho} = \begin{pmatrix} -1 \\ \boldsymbol{v}_{\lambda\rho} \end{pmatrix}, \tag{59}$$

with the vectors

$$\boldsymbol{v}_{\lambda\rho}(\hat{\boldsymbol{k}}) = \frac{\rho\omega_p\omega_\lambda}{\omega_\lambda^2 - \omega_c^2}\,\hat{\boldsymbol{k}}_\perp + \frac{\rho\omega_p}{\omega_\lambda}\,\hat{\boldsymbol{k}}_\parallel - \frac{i\omega_p\omega_c}{\omega_\lambda^2 - \omega_c^2}\,\hat{\boldsymbol{k}} \wedge \hat{\boldsymbol{B}} \,, \tag{60}$$

in which $\hat{\boldsymbol{k}}_\parallel = \cos\vartheta\,\hat{\boldsymbol{B}}$ and $\hat{\boldsymbol{k}}_\perp = \hat{\boldsymbol{k}} - \hat{\boldsymbol{k}}_\parallel$. The eigenfrequencies found here are the fundamental frequencies of the so-called "gyro-plasmon modes" that have been discussed before for the classical magnetized OCP [9]. The vectors $\boldsymbol{v}_{\lambda\rho}(\hat{\boldsymbol{k}})$ show up in the amplitudes of these modes. A few useful properties of these vectors have been collected in appendix C.

According to the method of variation of constants the general solution of the inhomogeneous equation (57) can be written as

$$X^{\mu}_{\tau} = \sum_{\lambda\rho} c_{\lambda\rho}(\tau)\, X^{\mu}_{\lambda\rho} \exp(\hbar\rho\omega_\lambda \tau) \,, \tag{61}$$

with time-dependent coefficients $c_{\lambda\rho}(\tau)$. These satisfy the equations

$$\sum_{\lambda\rho} \frac{\partial c_{\lambda\rho}(\tau)}{\partial \tau} \exp(\hbar\rho\omega_\lambda \tau) = 0 \,, \tag{62}$$

$$\sum_{\lambda\rho} \frac{\partial c_{\lambda\rho}(\tau)}{\partial \tau}\, \boldsymbol{v}_{\lambda\rho} \exp(\hbar\rho\omega_\lambda \tau) = \boldsymbol{M}_\tau \,. \tag{63}$$

Taking the inner product with $\boldsymbol{v}^*_{\lambda'\rho'}$ and using identity (C.2) of appendix C we can solve for $c_{\lambda\rho}(\tau)$:

$$c_{\lambda\rho}(\tau) = \frac{\omega_\lambda^2 - \omega_c^2}{2(\omega_\lambda^2 - \omega_{-\lambda}^2)} \int_0^\tau \mathrm{d}\tau' \, \boldsymbol{M}_{\tau'} \cdot \boldsymbol{v}_{\lambda\rho}^* \exp(-\hbar\rho\omega_\lambda \tau') + c_{\lambda\rho}(0) \,. \tag{64}$$

This leaves us with the problem of finding $c_{\lambda\rho}(0)$. Employing the boundary conditions (52) and (53) we get

$$\begin{aligned} c_{\lambda\rho}(0) = \frac{\omega_\lambda^2 - \omega_c^2}{2(\omega_\lambda^2 - \omega_{-\lambda}^2)} \frac{1}{\exp(\hbar\rho\omega_\lambda\beta) - 1} \Big(& - \frac{\omega_p}{k} \frac{1}{V} \langle [\Omega(-\boldsymbol{k}), Q(\boldsymbol{k})] \rangle \\ & + \boldsymbol{v}_{\lambda\rho}^* \cdot \frac{1}{V} \langle [\Omega(-\boldsymbol{k}), \boldsymbol{J}(\boldsymbol{k})] \rangle \\ & - \int_0^\beta \mathrm{d}\tau \, \boldsymbol{M}_\tau \cdot \boldsymbol{v}_{\lambda\rho}^* \exp[-h\rho\omega_\lambda(\tau - \beta)] \Big) \,. \end{aligned} \tag{65}$$

So our general solution for the imaginary-time-dependent Green functions is

$$\frac{1}{V} \langle Q_\tau(\boldsymbol{k}) \, \Omega(-\boldsymbol{k}) \rangle_T = - \frac{k}{\omega_p} \sum_{\lambda\rho} c_{\lambda\rho}(\tau) \exp(\hbar\rho\omega_\lambda \tau) \,, \tag{66}$$

$$\frac{1}{V} \langle \boldsymbol{J}_\tau(\boldsymbol{k}) \, \Omega(-\boldsymbol{k}) \rangle_T = \sum_{\lambda\rho} \boldsymbol{v}_{\lambda\rho} c_{\lambda\rho}(\tau) \exp(\hbar\rho\omega_\lambda \tau) \,, \tag{67}$$

with the coefficients

$$\begin{aligned} c_{\lambda\rho}(\tau) = \frac{\omega_\lambda^2 - \omega_c^2}{2(\omega_\lambda^2 - \omega_{-\lambda}^2)} \Big[& \frac{1}{\exp(\hbar\rho\omega_\lambda\beta) - 1} \Big(- \frac{\omega_p}{k} \frac{1}{V} \langle [\Omega(-\boldsymbol{k}), Q(\boldsymbol{k})] \rangle \\ & + \boldsymbol{v}_{\lambda\rho}^* \cdot \frac{1}{V} \langle [\Omega(-\boldsymbol{k}), \boldsymbol{J}(\boldsymbol{k})] \rangle \Big) \\ & - \frac{e\hbar}{m} \Big(\int_0^\tau \mathrm{d}\tau' \, \boldsymbol{k} \cdot \frac{1}{V} \langle \boldsymbol{T}_{\tau'}(\boldsymbol{k}) \, \Omega(-\boldsymbol{k}) \rangle_T \cdot \boldsymbol{v}_{\lambda\rho}^* \exp(-\hbar\rho\omega_\lambda \tau') \\ & - \frac{\exp(\hbar\rho\omega_\lambda\beta)}{\exp(\hbar\rho\omega_\lambda\beta) - 1} \int_0^\beta \mathrm{d}\tau' \, \boldsymbol{k} \cdot \frac{1}{V} \langle \boldsymbol{T}_{\tau'}(\boldsymbol{k}) \, \Omega(-\boldsymbol{k}) \rangle_T \cdot \boldsymbol{v}_{\lambda\rho}^* \\ & \times \exp(-\hbar\rho\omega_\lambda \tau') \Big) \Big] \,. \end{aligned} \tag{68}$$

Note that we recover eqs. (22) and (23) for the Kubo-transformed Green functions when we take the Kubo transform of (66) and (67). In proving this it is convenient to employ the properties (C.5) and (C.6) of the vectors $\boldsymbol{v}_{\lambda\rho}$.

The general solutions (66) and (67) can be used to derive explicit results in

successive order of the wavenumber. First, one immediately derives the charge sum rule

$$\frac{1}{V}\langle Q_\tau(\boldsymbol{k})\,\Omega(-\boldsymbol{k})\rangle_{\mathrm{T}}^{(0)} = 0 \tag{69}$$

for general operators Ω. For configurational operators Ω we can go one step further. Making use of the commutator relation (26) for a general configurational operator we get

$$\frac{1}{V}\langle Q_\tau(\boldsymbol{k})\,\Omega(-\boldsymbol{k})\rangle_{\mathrm{T}}^{(1)} = 0\,, \tag{70}$$

$$\frac{1}{V}\langle \boldsymbol{J}_\tau(\boldsymbol{k})\,\Omega(-\boldsymbol{k})\rangle_{\mathrm{T}}^{(0)} = \boldsymbol{0}\,, \tag{71}$$

because the integral terms in (68) vanish in this order. No further results for general operators Ω are available. We will derive specific fluctuation formulae for the charge density, the current density and the energy density by making use of the properties of these observables.

Case $\Omega(\boldsymbol{k}) = Q(\boldsymbol{k})$

In view of (69)–(71) we consider the charge–charge Green function in second order and the charge–current Green function in first order of the wavenumber. Hence, we need the coefficients (68) in first order. The $\boldsymbol{T}$ terms then drop out on account of the charge sum rule (69). Substituting the commutator relation (29) we arrive at the charge–charge and charge–current fluctuation formulae

$$\frac{1}{V}\langle Q_\tau(\boldsymbol{k})\,Q(-\boldsymbol{k})\rangle_{\mathrm{T}}^{(2)} = \tfrac{1}{2}\hbar\sum_{\lambda\rho}\rho\,\frac{\omega_\lambda^2-\omega_{\mathrm{c}}^2}{\omega_\lambda^2-\omega_{-\lambda}^2}\,\omega_\lambda\,\frac{\exp(\hbar\rho\omega_\lambda\tau)}{\exp(\hbar\rho\omega_\lambda\beta)-1}\,, \tag{72}$$

$$\frac{1}{V}\langle \boldsymbol{J}_\tau(\boldsymbol{k})\,Q(-\boldsymbol{k})\rangle_{\mathrm{T}}^{(1)} = -\tfrac{1}{2}\hbar\omega_{\mathrm{p}}\sum_{\lambda\rho}\rho\,\frac{\omega_\lambda^2-\omega_{\mathrm{c}}^2}{\omega_\lambda^2-\omega_{-\lambda}^2}\,\omega_\lambda\boldsymbol{v}_{\lambda\rho}\,\frac{\exp(\hbar\rho\omega_\lambda\tau)}{\exp(\hbar\rho\omega_\lambda\beta)-1}\,. \tag{73}$$

The first of these has already been given in ref. [6]. The fluctuation formulae found here depend explicitly on the magnetic field, in contrast to their Kubo-transformed counterparts (30) and (31). The Kubo transforms of the time-dependent fluctuation formulae given here can be evaluated conveniently with the help of (C.7). In particular, one checks in this way that the charge–current formula yields zero when Kubo transformed, in accordance with (31).

Case $\Omega(k) = J(k)$

In this case we confine ourselves to the evaluation of the current–current fluctuation formula in zeroth order of k. The relevant commutator expressions have been given in (29) and (34). The pressure terms again vanish in this order. We readily find

$$\frac{1}{V}\langle J_\tau(k)\, J(-k)\rangle_{\mathrm{T}}^{(0)} = \tfrac{1}{2}\hbar\omega_{\mathrm{p}}^2 \sum_{\lambda\rho} \rho\, \frac{\omega_\lambda^2 - \omega_{\mathrm{c}}^2}{\omega_\lambda^2 - \omega_{-\lambda}^2}\, \omega_\lambda \boldsymbol{v}_{\lambda\rho}\boldsymbol{v}_{\lambda\rho}^*\, \frac{\exp(\hbar\rho\omega_\lambda\tau)}{\exp(\hbar\rho\omega_\lambda\beta) - 1}\,. \tag{74}$$

Upon taking the Kubo transform and using (C.8) one recovers (35), from which any reference to the magnetic field has disappeared.

A special form of the current–current fluctuation formula has been given before [4]. It can be derived from (74) by choosing the wavevector to be parallel to the magnetic field. For that case (74) reduces to

$$\begin{aligned}\frac{1}{V}\langle J_\tau(k)\, J(-k)\rangle_{\mathrm{T}}^{(0)} &= \tfrac{1}{2}\hbar\omega_{\mathrm{p}}^3 \hat{k}\hat{k}\left(\frac{\exp(\hbar\omega_{\mathrm{p}}\tau)}{\exp(\hbar\omega_{\mathrm{p}}\beta) - 1} + \frac{\exp(-\hbar\omega_{\mathrm{p}}\tau)}{1 - \exp(-\hbar\omega_{\mathrm{p}}\beta)}\right)\\ &+ \tfrac{1}{2}\hbar\omega_{\mathrm{p}}^2\omega_{\mathrm{c}}(\boldsymbol{U} - \hat{k}\hat{k})\left(\frac{\exp(\hbar\omega_{\mathrm{c}}\tau)}{\exp(\hbar\omega_{\mathrm{c}}\beta) - 1} + \frac{\exp(-\hbar\omega_{\mathrm{c}}\tau)}{1 - \exp(-\hbar\omega_{\mathrm{c}}\beta)}\right)\\ &- \tfrac{1}{2}\mathrm{i}\hbar\omega_{\mathrm{p}}^2\omega_{\mathrm{c}}\boldsymbol{\varepsilon}\cdot\hat{k}\left(\frac{\exp(\hbar\omega_{\mathrm{c}}\tau)}{\exp(\hbar\omega_{\mathrm{c}}\beta) - 1} - \frac{\exp(-\hbar\omega_{\mathrm{c}}\tau)}{1 - \exp(-\hbar\omega_{\mathrm{c}}\beta)}\right).\end{aligned} \tag{75}$$

Clearly, the fluctuations of the longitudinal and the transverse components of the current density decouple for this particular configuration. Focussing on the transverse components we indeed recover the fluctuation formula given in ref. [4].

Case $\Omega(k) = E(k)$

For this choice of Ω the coefficients (68) vanish in zeroth order of the wavenumber, since the commutator expressions (37) and (39) do not contribute. Hence, one immediately finds

$$\frac{1}{V}\langle Q_\tau(k)\, E(-k)\rangle_{\mathrm{T}}^{(1)} = 0\,, \tag{76}$$

$$\frac{1}{V}\langle J_\tau(k)\, E(-k)\rangle_{\mathrm{T}}^{(0)} = \boldsymbol{0}\,. \tag{77}$$

In one order higher in k we have to insert (37) and (39). Moreover, we can

write the fluctuation expression with the pressure tensor as a derivative with respect to β, as before. Taking similar steps as in the previous section we obtain the fluctuation formulae for this case as

$$\frac{1}{V}\langle Q_\tau(\boldsymbol{k})\, E(-\boldsymbol{k})\rangle_{\mathrm{T}}^{(2)} = \frac{e\hbar}{2m\omega_{\mathrm{p}}} \sum_{\lambda\rho} \frac{\omega_\lambda^2 - \omega_{\mathrm{c}}^2}{\omega_\lambda^2 - \omega_{-\lambda}^2} (\hat{\boldsymbol{k}} \cdot \langle \boldsymbol{T}\rangle + \hat{\boldsymbol{k}}\langle E\rangle) \cdot \boldsymbol{v}_{\lambda\rho}^* \times \frac{\exp(\hbar\rho\omega_\lambda \tau)}{\exp(\hbar\rho\omega_\lambda \beta) - 1} + \frac{e}{m\omega_{\mathrm{p}}^2} \frac{\partial}{\partial\beta} (\hat{\boldsymbol{B}} \cdot \langle \boldsymbol{T}\rangle \cdot \hat{\boldsymbol{B}}), \tag{78}$$

$$\frac{1}{V}\langle \boldsymbol{J}_\tau(\boldsymbol{k})\, E(-\boldsymbol{k})\rangle_{\mathrm{T}}^{(1)} = -\frac{e\hbar}{2m} \sum_{\lambda\rho} \frac{\omega_\lambda^2 - \omega_{\mathrm{c}}^2}{\omega_\lambda^2 - \omega_{-\lambda}^2} \boldsymbol{v}_{\lambda\rho} (\hat{\boldsymbol{k}} \cdot \langle \boldsymbol{T}\rangle + \hat{\boldsymbol{k}}\langle E\rangle) \cdot \boldsymbol{v}_{\lambda\rho}^* \times \frac{\exp(\hbar\rho\omega_\lambda \tau)}{\exp(\hbar\rho\omega_\lambda \beta) - 1} - \frac{3\mathrm{i}e}{2m\omega_{\mathrm{c}}} \hat{\boldsymbol{k}} \wedge \hat{\boldsymbol{B}} \frac{\partial}{\partial\beta} (\hat{\boldsymbol{B}} \cdot \langle \boldsymbol{T}\rangle \cdot \hat{\boldsymbol{B}} - \tfrac{1}{3} \operatorname{tr}\langle \boldsymbol{T}\rangle). \tag{79}$$

Just as in the previous section the energy–energy fluctuation formula is the only one that we have not come across here. However, since it is independent of the imaginary time in zeroth order of the wavelength, we immediately get from (49)

$$\frac{1}{V}\langle E_\tau(\boldsymbol{k})\, E(-\boldsymbol{k})\rangle_{\mathrm{T}}^{(0)} = \frac{nc_V}{k_{\mathrm{B}}\beta^2}. \tag{80}$$

We have now obtained a complete set of imaginary-time-dependent fluctuation formulae for the charge density, the current density and the energy density. Comparing them with the Kubo-transformed fluctuation formulae of the previous section we see that the time-dependent fluctuation formulae are a lot more complicated. In contrast with the former the latter all depend explicitly on Planck's constant. Taking the classical limit is a trivial operation for the Kubo-transformed fluctuation formulae, whereas it leads to profound changes in the time-dependent formulae, with results that are identical to the simple Kubo-transformed expressions. Furthermore, the time-dependent formulae contain the basic eigenfrequencies of the magnetized OCP, which are sensitive to the orientation of the magnetic field with respect to the wavevector. The characteristic vectors $\boldsymbol{v}_{\lambda\rho}$ entering the time-dependent fluctuation formulae also depend on the magnetic field. The Kubo-transformed fluctuation

formulae, on the other hand, are mostly independent of the magnetic field, the only exception being those containing the energy density.

Having discussed the explicit magnetic-field dependence of the fluctuation formulae derived in this section we now consider the limit of vanishing magnetic field. In this limit ω_+ goes to ω_p and ω_- to 0. The associated eigenmodes of the unmagnetized plasma are the plasma oscillation modes and the viscous modes, respectively. The charge–charge and charge–current fluctuation formulae become

$$\lim_{|\boldsymbol{B}|\to 0} \frac{1}{V} \langle Q_\tau(\boldsymbol{k})\, Q(-\boldsymbol{k})\rangle_{\mathrm{T}}^{(2)} = \tfrac{1}{2}\hbar\omega_{\mathrm{p}}\left(\frac{\exp(\hbar\omega_{\mathrm{p}}\tau)}{\exp(\hbar\omega_{\mathrm{p}}\beta)-1} + \frac{\exp(-\hbar\omega_{\mathrm{p}}\tau)}{1-\exp(-\hbar\omega_{\mathrm{p}}\beta)}\right), \tag{81}$$

$$\lim_{|\boldsymbol{B}|\to 0} \frac{1}{V} \langle \boldsymbol{J}_\tau(\boldsymbol{k})\, Q(-\boldsymbol{k})\rangle_{\mathrm{T}}^{(1)} = -\tfrac{1}{2}\hbar\omega_{\mathrm{p}}^2\hat{\boldsymbol{k}}\left(\frac{\exp(\hbar\omega_{\mathrm{p}}\tau)}{\exp(\hbar\omega_{\mathrm{p}}\beta)-1} - \frac{\exp(-\hbar\omega_{\mathrm{p}}\tau)}{1-\exp(-\hbar\omega_{\mathrm{p}}\beta)}\right). \tag{82}$$

For general τ the first formula can be found in [6], while for $\tau = 0$ it has been known already for a long time [4,5,8]. The second one has been given in ref. [12].

The current–current fluctuation formula in the limit $|\boldsymbol{B}|\to 0$ takes the form

$$\lim_{|\boldsymbol{B}|\to 0} \frac{1}{V} \langle \boldsymbol{J}_\tau(\boldsymbol{k})\, \boldsymbol{J}(-\boldsymbol{k})\rangle_{\mathrm{T}}^{(0)} = \tfrac{1}{2}\hbar\omega_{\mathrm{p}}^3\hat{\boldsymbol{k}}\hat{\boldsymbol{k}}\left(\frac{\exp(\hbar\omega_{\mathrm{p}}\tau)}{\exp(\hbar\omega_{\mathrm{p}}\beta)-1} + \frac{\exp(-\hbar\omega_{\mathrm{p}}\tau)}{1-\exp(-\hbar\omega_{\mathrm{p}}\beta)}\right) + \beta^{-1}\omega_{\mathrm{p}}^2(\boldsymbol{U}-\hat{\boldsymbol{k}}\hat{\boldsymbol{k}}). \tag{83}$$

This formula is found to contain a term independent of the imaginary time. It originates from the $\lambda = -1$ term in (74) and is hence related to the presence of viscous modes in the unmagnetized OCP. Terms of this type are absent from the charge–charge and the charge–current fluctuation formulae.

A second remarkable feature of (83) is that it still depends on $\hat{\boldsymbol{k}}$ in the long-wave-length limit. Hence, the current–current fluctuation expression is not uniquely defined for $\boldsymbol{k}\to\boldsymbol{0}$; its value depends on the direction from which the origin is approached. Taking the inverse Fourier transform one finds that in position space the imaginary-time-dependent current–current Green function contains terms proportional to a dipole–dipole interaction tensor, which decays as r^{-3} for large separations r between the observation points. This slow non-integrable algebraic decay has been noticed before [4,6]. The slowly decaying tail disappears upon taking the classical limit. Recently, it has become clear that an algebraic decay of correlation functions is a general feature of

quantum plasmas [10–12]. It should be noted that the current–current fluctuation formula (74) for the magnetized OCP also depends on the orientation of the wavevector, via the frequencies ω_λ and the vectors $\boldsymbol{v}_{\lambda\rho}$. Hence, similar conclusions on the decay of the current–current Green function in position space can be drawn in that case as well.

Finally, we turn to the $|\boldsymbol{B}|\to 0$ limit of the fluctuation expressions involving the energy density. For the charge–energy fluctuations we find

$$\lim_{|\boldsymbol{B}|\to 0} \frac{1}{V} \langle Q_\tau(\boldsymbol{k})\, E(-\boldsymbol{k})\rangle_{\mathrm{T}}^{(2)} = \frac{e\hbar}{2m\omega_{\mathrm{p}}} (p + e_{\mathrm{v}}) \times \left(\frac{\exp(\hbar\omega_{\mathrm{p}}\tau)}{\exp(\hbar\omega_{\mathrm{p}}\beta) - 1} + \frac{\exp(-\hbar\omega_{\mathrm{p}}\tau)}{1 - \exp(-\hbar\omega_{\mathrm{p}}\beta)} \right) + \frac{e}{m\omega_{\mathrm{p}}^2} \frac{\partial p}{\partial \beta} . \tag{84}$$

Here we used the isotropy of the equilibrium pressure tensor of an unmagnetized OCP, so that we could write $\langle \boldsymbol{T} \rangle = p\boldsymbol{U}$. Furthermore, we introduced the notation $e_{\mathrm{v}} = \langle E \rangle$ for the energy density. For the current–energy fluctuation formula the limit of vanishing magnetic field is not as straightforward, because of the factor ω_{c} appearing in the denominator of the last term of (79). However, the equilibrium pressure tensor depends on the magnetic field only through the product $\omega_{\mathrm{c}}\hat{\boldsymbol{B}}$. Since the pressure tensor is symmetric, it is an even function of $\hat{\boldsymbol{B}}$ and hence also an even function of ω_{c}. As a consequence the off-diagonal terms of the pressure tensor are at least of order ω_{c}^2, so that the last term of (74) drops out in the limit of vanishing magnetic field. The other contributions yield

$$\lim_{|\boldsymbol{B}|\to 0} \frac{1}{V} \langle \boldsymbol{J}_\tau(\boldsymbol{k})\, E(-\boldsymbol{k})\rangle_{\mathrm{T}}^{(1)} = -\frac{e\hbar}{2m} \hat{\boldsymbol{k}} (p + e_{\mathrm{v}}) \times \left(\frac{\exp(\hbar\omega_{\mathrm{p}}\tau)}{\exp(\hbar\omega_{\mathrm{p}}\beta) - 1} - \frac{\exp(-\hbar\omega_{\mathrm{p}}\tau)}{1 - \exp(-\hbar\omega_{\mathrm{p}}\beta)} \right) . \tag{85}$$

This completes our discussion of the $|\boldsymbol{B}|\to 0$ limit of the imaginary-time-dependent fluctuation formulae.

4. Conclusion

The complete set of formulae determining the large-scale correlations in the fluctuations of the charge density, the current density and the energy density in

a quantum OCP has now been established. Starting from the equations of motion for the charge and the current density we have derived both Kubo-transformed and imaginary-time-dependent fluctuation formulae. The Kubo-transformed versions have a similar appearance as the classical formulae. In particular, they are independent of Planck's constant. On the other hand, the general imaginary-time-dependent fluctuation formulae have a more complicated structure. They depend on the basic mode frequencies of the OCP in a magnetic field and hence on both the strength of the magnetic field and on its orientation with respect to the wavevector. The frequency dependence also entails an explicit dependence on $\hbar$. In taking the classical limit all these interesting features disappear and one is left once more with the remarkable simple set of fluctuation formulae for the classical unmagnetized OCP, the equilibrium fluctuation properties of which are by now wellknown.

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Appendix A. Energy fluctuation formulae: $E(k)$ or $\tilde{E}(k)$?

In this appendix we will show that it makes no difference whether one takes the energy density $E(k)$ or $\tilde{E}(k)$ in the commutator (39). Taking account of (38) and of the remark following (37) we may conclude that in leading order of k the energy fluctuation formulae are the same for $E(k)$ and $\tilde{E}(k)$.

Since there is no difference in the kinetic parts, we will concentrate on the potential parts. For the commutator between $E_{\rm pot}$ and the current density J we find

$$\frac{1}{V}\langle [E_{\rm pot}(-k), \boldsymbol{J}(\boldsymbol{k})]\rangle = \frac{e\hbar}{mV^4} \sum_{\boldsymbol{q}(\neq \boldsymbol{0}, \neq \boldsymbol{k})} \frac{e^2 \boldsymbol{q}\cdot(\boldsymbol{k}-\boldsymbol{q})(\boldsymbol{k}-\boldsymbol{q})}{q^2(\boldsymbol{k}-\boldsymbol{q})^2} \times \sum_{\boldsymbol{k}'\boldsymbol{k}''} \langle \psi^+(\boldsymbol{k}'+\boldsymbol{q})\, \psi^+(\boldsymbol{k}''-\boldsymbol{q})\, \psi(\boldsymbol{k}'')\, \psi(\boldsymbol{k}')\rangle \,. \tag{A.1}$$

Expanding up to first order in k we get

$$\frac{\boldsymbol{q}\cdot(\boldsymbol{k}-\boldsymbol{q})(\boldsymbol{k}-\boldsymbol{q})}{q^2(\boldsymbol{k}-\boldsymbol{q})^2} \simeq \frac{\boldsymbol{q}}{q^2} - \frac{1}{2}\frac{\boldsymbol{k}}{q^2} - \frac{1}{2}\frac{\boldsymbol{k}}{q^2}\cdot\left(\boldsymbol{U} - \frac{2\boldsymbol{q}\boldsymbol{q}}{q^2}\right). \tag{A.2}$$

Upon substitution into (A.1) the first term vanishes on account of symmetry, while the last two terms yield the potential energy density and the potential pressure, respectively. Hence, we arrive at the result

$$\frac{1}{V}\langle[E_{\mathrm{pot}}(-\boldsymbol{k}), \boldsymbol{J}(\boldsymbol{k})]\rangle \simeq -\frac{e\hbar}{m}\left(\boldsymbol{k}\langle E_{\mathrm{pot}}\rangle + \boldsymbol{k}\cdot\langle \boldsymbol{T}_{\mathrm{pot}}\rangle\right), \tag{A.3}$$

which is equal to the potential part of (39). Calculating the commutator between $\tilde{E}_{\mathrm{pot}}$ and $\boldsymbol{J}$ we get

$$\frac{1}{V}\langle[\tilde{E}_{\mathrm{pot}}(-\boldsymbol{k}), \boldsymbol{J}(\boldsymbol{k})]\rangle = -\frac{e\hbar}{2mV^4}\sum_{\boldsymbol{q}\,(\neq\boldsymbol{0},\neq\boldsymbol{k})}(\boldsymbol{k}-\boldsymbol{q})\left(\frac{e^2}{q^2} + \frac{e^2}{(\boldsymbol{k}-\boldsymbol{q})^2}\right) \times \sum_{\boldsymbol{k}'\boldsymbol{k}''}\langle\psi^\dagger(\boldsymbol{k}'+\boldsymbol{q})\,\psi^\dagger(\boldsymbol{k}''-\boldsymbol{q})\,\psi(\boldsymbol{k}'')\,\psi(\boldsymbol{k}')\rangle, \tag{A.4}$$

and expanding up to first order in k as in (A.2) we find

$$\frac{1}{V}\langle[\tilde{E}_{\mathrm{pot}}(-\boldsymbol{k}), \boldsymbol{J}(\boldsymbol{k})]\rangle \simeq -\frac{e\hbar}{m}\left(\boldsymbol{k}\langle\tilde{E}_{\mathrm{pot}}\rangle + \boldsymbol{k}\cdot\langle \boldsymbol{T}_{\mathrm{pot}}\rangle\right). \tag{A.5}$$

Since $\langle\tilde{E}\rangle = \langle E\rangle$ we have again recovered the potential part of (39).

Appendix B. Ionic mixture

In this appendix we will derive fluctuation formulae for a magnetized ionic mixture consisting of several species of particles with different charges and masses in an inert uniform background. The particles of species α are described by creation and annihilation operators $\psi^\dagger(\alpha, \boldsymbol{k})$ and $\psi(\alpha, \boldsymbol{k})$ satisfying the usual (anti)commutation relations. The Hamiltonian of the system is

$$H = \sum_\alpha \frac{\hbar^2}{2m_\alpha V}\sum_{\boldsymbol{k}}\psi^\dagger(\alpha,\boldsymbol{k})\left(\boldsymbol{k} - \frac{ie_\alpha}{2\hbar c}\boldsymbol{B}\wedge\boldsymbol{\nabla}_{\boldsymbol{k}}\right)^2\psi(\alpha,\boldsymbol{k}) + \frac{1}{2V^3}\sum_{\alpha\alpha'}\sum_{\boldsymbol{k}\boldsymbol{k}'}\sum_{\boldsymbol{q}\neq\boldsymbol{0}}\frac{e_\alpha e_{\alpha'}}{q^2}\,\psi^\dagger(\alpha,\boldsymbol{k}+\boldsymbol{q})\,\psi^\dagger(\alpha',\boldsymbol{k}'-\boldsymbol{q})\,\psi(\alpha',\boldsymbol{k}')\,\psi(\alpha,\boldsymbol{k}), \tag{B.1}$$

where e_α and m_α are the charge and the mass of the particles of species α. The total charge density is $Q(\boldsymbol{k}) = \Sigma_\alpha(e_\alpha/V)\,\Sigma_{\boldsymbol{k}'}\,\psi^\dagger(\alpha,\boldsymbol{k}'-\boldsymbol{k})\,\psi(\alpha,\boldsymbol{k}')$. The

current density $\boldsymbol{J}(\alpha, \boldsymbol{k})$ of species α is defined similarly as in (2). The total current density $\boldsymbol{J}(\boldsymbol{k})$ follows by summation over all species. For a general ionic mixture in which particles with different charge-to-mass ratio are present this current density is independent of the total momentum density $\boldsymbol{G}(\boldsymbol{k}) = \Sigma_\alpha (m_\alpha/e_\alpha)\boldsymbol{J}(\alpha, \boldsymbol{k})$.

The equations of motion are

$$[H, Q(\boldsymbol{k})] = -\hbar \boldsymbol{k} \cdot \boldsymbol{J}(\boldsymbol{k}) , \tag{B.2}$$

$$[H, \boldsymbol{G}(\boldsymbol{k})] = -\hbar q_v \frac{\boldsymbol{k}}{k^2} Q(\boldsymbol{k}) - i\hbar\gamma_c \boldsymbol{J}(\boldsymbol{k}) \wedge \hat{\boldsymbol{B}} - \hbar \boldsymbol{k} \cdot \boldsymbol{T}(\boldsymbol{k}) . \tag{B.3}$$

The pressure tensor $\boldsymbol{T}(\boldsymbol{k})$ consists of a kinetic and a potential part of which the definitions are similar to that of the OCP, with additional summations over all species. The background charge density is $-q_v = -\Sigma_\alpha n_\alpha e_\alpha$ with n_α the particle density of species α. The cyclotron ratio is given by $\gamma_c := |\boldsymbol{B}|/c$.

We can repeat the procedure given in section 2 of Kubo transforming and solving the equations of motion. As before we obtain in this way general expressions for the Green functions involving the charge and the current density:

$$\begin{aligned} k^{-1}\mathcal{K} \frac{1}{V} \langle Q(\boldsymbol{k})\, \Omega(-\boldsymbol{k})\rangle_T = &- \frac{1}{\hbar\beta q_v \cos\vartheta} \frac{1}{V} \langle [\Omega(-\boldsymbol{k}), \boldsymbol{G}(\boldsymbol{k})]\rangle \cdot \hat{\boldsymbol{B}} \\ &- \frac{1}{q_v \cos\vartheta} \boldsymbol{k} \cdot \mathcal{K} \frac{1}{V} \langle \boldsymbol{T}(\boldsymbol{k})\, \Omega(-\boldsymbol{k})\rangle_T \cdot \hat{\boldsymbol{B}} , \end{aligned} \tag{B.4}$$

$$\begin{aligned} \mathcal{K} \frac{1}{V} \langle \boldsymbol{J}(\boldsymbol{k})\, \Omega(-\boldsymbol{k})\rangle_T = &\frac{i}{\hbar\beta\gamma_c \cos\vartheta} \hat{\boldsymbol{k}} \wedge \frac{1}{V} \langle [\Omega(-\boldsymbol{k}), \boldsymbol{G}(\boldsymbol{k})]\rangle \\ &- \frac{1}{\hbar\beta k \cos\vartheta} \hat{\boldsymbol{B}} \frac{1}{V} \langle [\Omega(-\boldsymbol{k}), Q(\boldsymbol{k})]\rangle \\ &- \frac{i}{\gamma_c \cos\vartheta} \left[\boldsymbol{k} \cdot \mathcal{K} \frac{1}{V} \langle \boldsymbol{T}(k)\, \Omega(-\boldsymbol{k})\rangle_T\right] \wedge \hat{\boldsymbol{k}} . \end{aligned} \tag{B.5}$$

These Kubo-transformed fluctuation formulae are quite similar to those given in (22) and (23) for the OCP. The fact that the momentum density and the current density are independent quantities for a general ionic mixture has not prevented us from obtaining these results. One can easily convince oneself that the situation is not that simple for the imaginary-time-dependent fluctuation formulae: the treatment of section 3 cannot be followed for an ionic mixture,

precisely because of the lack of a general relation between the momentum density and the current density in a mixture.

As in section 2 one easily shows from (B.4) and (B.5) that the charge sum rule (25) and the higher-order rules (27) and (28) are valid for the ionic mixture as well. By making specific choices of the operator Ω a complete set of Kubo-transformed fluctuation formulae can once again be derived. The fluctuation formulae for the charge and the current density as given by (30), (31) and (35) are valid for the mixture as well. The squared plasma frequency appearing in the last of these formulae is given by $\Sigma_\alpha n_\alpha e_\alpha^2/m_\alpha$ in the present case. In addition, a few formulae containing the momentum density instead of the current density can be established as well. They read

$$\mathscr{K}\,\frac{1}{V}\,\langle Q(\boldsymbol{k})\,\boldsymbol{G}(-\boldsymbol{k})\rangle_{\mathrm{T}}^{(1)} = \boldsymbol{0}\,, \tag{B.6}$$

$$\mathscr{K}\,\frac{1}{V}\,\langle \boldsymbol{J}(\boldsymbol{k})\,\boldsymbol{G}(-\boldsymbol{k})\rangle_{\mathrm{T}}^{(0)} = \beta^{-1} q_{\mathrm{v}} \boldsymbol{U}\,. \tag{B.7}$$

As to the fluctuation formulae with the energy density, the formulae (40) and (41) remain valid as such, while (47) and (48) have to be adapted somewhat, by replacing the factors $e/(m\omega_{\mathrm{p}}^2)$ and $e/(m\omega_{\mathrm{c}})$ by $1/q_{\mathrm{v}}$ and $1/\gamma_{\mathrm{c}}$, respectively. The derivatives with respect to β in these formulae have to be taken at constant q_{v} and constant $\beta\tilde{\mu}_\alpha$, with $\tilde{\mu}_\alpha$ the chemical potentials as introduced in ref. [3]. Finally, the right-hand side of (49) gets the form $-\partial e_{\mathrm{v}}/\partial\beta$, with e_{v} the energy per volume.

Appendix C. Properties of the vector $\boldsymbol{v}_{\lambda\rho}$

In this appendix we will state a few properties of the vector $\boldsymbol{v}_{\lambda\rho}$ which is defined as

$$\boldsymbol{v}_{\lambda\rho}(\hat{\boldsymbol{k}}) = \frac{\rho\omega_{\mathrm{p}}\omega_\lambda}{\omega_\lambda^2 - \omega_{\mathrm{c}}^2}\,\hat{\boldsymbol{k}}_\perp + \frac{\rho\omega_{\mathrm{p}}}{\omega_\lambda}\,\hat{\boldsymbol{k}}_\parallel - \frac{\mathrm{i}\omega_{\mathrm{p}}\omega_{\mathrm{c}}}{\omega_\lambda^2 - \omega_{\mathrm{c}}^2}\,\hat{\boldsymbol{k}}\wedge\hat{\boldsymbol{B}}\,. \tag{C.1}$$

The inner product of two of these vectors is

$$\boldsymbol{v}_{\lambda\rho}\cdot\boldsymbol{v}^*_{\lambda'\rho'} = -1 + \frac{2(\omega_\lambda^2 - \omega_{-\lambda}^2)}{\omega_\lambda^2 - \omega_{\mathrm{c}}^2}\,\delta_{\lambda\lambda'}\delta_{\rho\rho'}\,. \tag{C.2}$$

To solve the eigenvalue equation for the matrix (55) one needs the relations

$$\hat{\boldsymbol{k}} \cdot \boldsymbol{v}_{\lambda\rho} = \rho \frac{\omega_\lambda}{\omega_p} , \tag{C.3}$$

$$\omega_p \hat{\boldsymbol{k}} - i\omega_c \boldsymbol{v}_{\lambda\rho} \wedge \hat{\boldsymbol{B}} = \rho\omega_\lambda \boldsymbol{v}_{\lambda\rho} . \tag{C.4}$$

The following equations are useful in Kubo transforming the general solutions (66) and (67) of the imaginary-time equations of motion:

$$\sum_{\lambda\rho} \frac{\omega_\lambda^2 - \omega_c^2}{2(\omega_\lambda^2 - \omega_{-\lambda}^2)} \frac{1}{\rho\omega_\lambda} \boldsymbol{v}_{\lambda\rho} = \frac{1}{\omega_p \cos\vartheta} \hat{\boldsymbol{B}} , \tag{C.5}$$

$$\sum_{\lambda\rho} \frac{\omega_\lambda^2 - \omega_c^2}{2(\omega_\lambda^2 - \omega_{-\lambda}^2)} \frac{1}{\rho\omega_\lambda} \boldsymbol{v}_{\lambda\rho} \boldsymbol{v}_{\lambda\rho}^* = -\frac{i}{\omega_c \cos\vartheta} \boldsymbol{\varepsilon} \cdot \hat{\boldsymbol{k}} . \tag{C.6}$$

Lastly, we state two properties which can be employed in taking the Kubo transform (or the classical limit) of the imaginary-time-dependent fluctuation formulae for the charge and the current density:

$$\sum_{\lambda\rho} \frac{\omega_\lambda^2 - \omega_c^2}{2(\omega_\lambda^2 - \omega_{-\lambda}^2)} \boldsymbol{v}_{\lambda\rho} = \boldsymbol{0} , \tag{C.7}$$

$$\sum_{\lambda\rho} \frac{\omega_\lambda^2 - \omega_c^2}{2(\omega_\lambda^2 - \omega_{-\lambda}^2)} \boldsymbol{v}_{\lambda\rho} \boldsymbol{v}_{\lambda\rho}^* = \boldsymbol{U} . \tag{C.8}$$

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