Effective theories in cosmology
Mooij, S.

Citation for published version (APA):
Mooij, S. J. N. (2013). Effective theories in cosmology

General rights
It is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), other than for strictly personal, individual use, unless the work is under an open content license (like Creative Commons).

Disclaimer/Complaints regulations
If you believe that digital publication of certain material infringes any of your rights or (privacy) interests, please let the Library know, stating your reasons. In case of a legitimate complaint, the Library will make the material inaccessible and/or remove it from the website. Please Ask the Library: http://uba.uva.nl/en/contact, or a letter to: Library of the University of Amsterdam, Secretariat, Singel 425, 1012 WP Amsterdam, The Netherlands. You will be contacted as soon as possible.
Chapter 1

Cosmology and inflation

This first chapter is meant to give some background information on cosmology in general and inflation in particular. The emphasis is on the material needed to appreciate the work done in our articles [1, 2, 3, 4, 5], which will be presented in the chapters 3-7. Much more comprehensive reviews can be found in, for example, [6, 7].

1.1 Cosmology

In this first section we provide a short chronological overview of the history of the universe, highlighting the parts relevant for cosmological inflation. With time passing by, the universe expands. As a result it cools down, and matter and radiation dilute. An intuitive sketch of this expansion history, provided by the Particle Data Group, is in figure 1.1.

At $t = 0$ the Big Bang takes place. Much can be said, conjectured or dreamt about this beginning, but theories break down and nothing can be measured. Up to $t = 10^{-43}$ s (the Planck scale, $E = 10^{18}$ GeV), the universe should be described with a theory of quantum gravity. String theory is the most intensively pursued option. The only feature of string theory that we will deal with in this thesis is the fact that it predicts the existence of extra spacetime dimensions. At lower scales, these show up in quantum fields known as moduli fields. Below the Planck scale, gravity is much weaker than the three other fundamental forces (strong, weak and electromagnetic) and the universe can be described by a, possibly supersymmetric, quantum field theory in a curved background. It is widely conjectured that the three forces can still be described as one unified force in the framework of a Grand Unification Theory (GUT). However, so far no experimental evidence\footnote{Scenarios of grand unification predict a finite lifetime for protons, but no decaying proton has been observed yet.} has been found. After $t = 10^{-36}$ s ($E = 10^{16}$ GeV) we reach the Grand Unification scale. The three forces decouple from each other. (The energy scale of $10^{16}$ GeV is suggested by the supersymmetrical (MSSM) running of the coupling constants. In non-supersymmetric theories the grand unification scale is rather $10^{13}$ GeV, or there is no unification at all.)

Around this time, cosmological inflation takes place. This is a brief phase of accelerated expansion of the universe. The cosmological scale factor $a(t)$, defined in the metric (1.1), grows by a factor of at least $e^{60}$, but there is really no upper limit on this amount, see section 1.2. All pre-inflationary physics is therefore “washed out” (the universe is empty after inflation) and its observational features are extremely...
hard, if not impossible, to recover. On the other hand, the remnants of inflation itself leave a clear observational signal in the Cosmic Microwave Background (CMB) radiation, as we will discuss in section 1.3. Models of inflation in supersymmetric grand unification suggest that inflation took place at the Grand Unification scale as well, but we will see that so far only lower bounds on the inflationary energy scale have been found. After inflation and the subsequent process of reheating, during which the inflaton’s energy is transferred to other degrees of freedom, the universe is filled with radiation: relativistic elementary particles.

At $t = 10^{-10}$ s we reach the TeV-energy scale, so we enter the energy range observable by the Large Hadron Collider (LHC). From here on theories are comforted by experimental guidance. Given the current experimental results, the breaking of supersymmetry should already have taken place. Around 100 GeV we are at the scale of electroweak symmetry breaking: the $W$- and $Z$-bosons acquire a mass as the Higgs field settles down at its nonzero vacuum expectation value. The radiation that fills the universe now only consists of quarks, leptons, photons and gluons. After $10^{-4}$ s, we reach the QCD-energy scale of about 200 MeV. From here on individual quarks are confined inside hadrons (protons and neutrons) and mesons.
1.1. COSMOLOGY

After about three minutes ($E = 0.1$ MeV), Big Bang Nucleosynthesis (BBN) takes place: protons and neutrons combine into the light elements (H, He, Li,...). BBN manages to predict the abundances of these elements very precisely. Therefore, from this time on we have a precise quantitative description of the universe. Inflation should for sure take place before BBN (and as well before baryogenesis, which is meant to break the matter-antimatter symmetry in the post-inflationary phase).

With time passing further, the matter component of the energy density of the universe (non-relativistic particles) grows more quickly than its radiation component (relativistic particles). Both densities dilute in an expanding universe, but for radiation there is the extra effect from its wavelength that gets stretched out as well. After about $10^4$ years, the matter component becomes dominant. When the universe is about 380,000 years old, it becomes transparent as recombination takes place. Free electrons are caught by protons. Therefore, photons can begin to free-stream through the universe, as they do not scatter off free electrons anymore. There is light in the universe. After about $10^9$ years, stars, planets and galaxies begin to form. Also, the expansion of the universe begins to accelerate again. By now the universe is about 13.8 billion years old and the energy scale has dropped to about 1 meV. At least at one planet life has emerged. A subgroup of the population has begun to look up at the sky and to wonder where it all came from.

1.1.1 The metric of the universe

Already in the 16th century Copernicus has taught us that our position in the universe is not special at all. We are just one planet orbiting just one star in just some galaxy. In modern cosmology his principle has been translated in the notion that on large scales ($> 100$ Mpc $≈ 10^{24}$ m) the universe is homogeneous and isotropic. It has been proven right time and again by CMB and large scale structure experiments. In the $(+,−−)$ convention that we use in this thesis, the metric of the universe in spherical coordinates follows from the invariant interval

$$ds^2 = dt^2 - a^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right). \quad (1.1)$$

Here $a(t)$ is the cosmological scale factor, which takes the expansion of the universe into account. The physical distance between two objects with fixed $r$, $\theta$ and $\phi$ coordinates is given by the product of their fixed coordinate distance and this dynamical scale factor $a(t)$. The parameter $k$ specifies the global metric of the universe. The universe can be open ($k = 1$), flat ($k = 0$) or closed ($k = -1$).

In most of this thesis we will consider a flat universe, in line with the observations by Planck and earlier. (We will discuss the flatness of the universe further in the next section.) For a flat universe we can as well employ a Cartesian coordinate system, in which the metric follows from

$$ds^2 = dt^2 - a^2(t) (dx^2 + dy^2 + dz^2). \quad (1.2)$$

1.1.2 The dynamics of the universe

The mutual interaction between spacetime curvature and energy-momentum ("Matter tells space how to curve, and space tells matter how to move") is encoded in the Einstein equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}. \quad (1.3)$$

However, the apparent dimness of very far supernovas that is most often taken as an indication for the late-time accelerated expansion of the universe, can also be explained as the result of us living in the center of a local spherical underdensity ("void"), see [8] and references therein. Isotropy has also been questioned by "axis of evil"-scenarios [9].
CHAPTER 1. COSMOLOGY AND INFLATION

Here the Ricci tensor \( R_{\mu\nu} \) and the Ricci scalar \( R \) encode the curvature of spacetime. These follow directly from the metric \( g_{\mu\nu} \), see for example [10]. \( \Lambda \) is the cosmological constant, which is most probably responsible for the late-time acceleration of the universe. Treating the matter in the universe as a homogeneous and isotropic “cosmic fluid” the energy-momentum tensor \( T_{\mu\nu} \) can be written as

\[
T_{\mu\nu} = (\rho + p)u_{\mu}u_{\nu} - pg_{\mu\nu},
\]

where \( \rho \) denotes energy density and \( p \) stands for pressure. We will work in the rest frame where \( u_\mu = (1, 0, 0, 0) \).

In this whole thesis our units are such that \( \hbar = c = 1 \). Apart from this subsection, we set the reduced Planck mass \( \tilde{M}_p \equiv \sqrt{\frac{\hbar c}{8\pi G}} = 1 \) as well.

The (00) component of the Einstein equation gives, after inserting (1.4), the Friedmann equation

\[
H^2 + \frac{k}{a^2} - \frac{\Lambda}{3} = \frac{8\pi G \rho}{3}.
\]

Here \( H \) denotes the Hubble parameter, \( H \equiv \frac{\dot{a}}{a} \). From the Friedmann equation we can deduce the behaviour of the scale factor during the various epochs of expansion the universe has undergone. We get

\[
a(t) \sim \begin{cases} e^{\sqrt{(\Lambda/3)t}} = e^{Ht} & \text{(inflation, present day expansion)} \\ t^{1/2} & \text{(radiation domination)} \\ t^{2/3} & \text{(matter domination)} \end{cases}.
\]

Here we have used that during radiation domination we have \( \rho \sim a^{-4} \), while during matter domination \( \rho \sim a^{-3} \). The latter is easy to understand: when volumes grow as \( a^3 \), energy densities drop as \( a^{-3} \). The extra inverse power of the scale factor for radiation comes from the fact that its wavelength gets stretched out as well, as we already discussed above.

1.1.3 The CMB radiation

We can still observe the photons emitted at recombination in the Cosmic Microwave Background (CMB) radiation. The CMB provides a marvelous insight in early-universe cosmology, as it is literally a baby picture of the universe. Its study has led to the most precise determination of almost all cosmological parameters. Figure 1.2 shows a sky map of the measured CMB temperature, taken from the Planck results [11]. The background temperature is the same in all directions, but there are tiny fluctuations on top of that:

\[
T_{\text{CMB}} = 2.73 \pm 10^{-4} \text{K}.
\]

As we observe the CMB in a sphere around us, we are led to decompose the temperature fluctuations in spherical harmonics:

\[
\frac{\Delta T(\theta, \phi)}{T} = \sum_{l=1}^{\infty} \sum_{m=-l}^{m=l} a_{lm} Y_{lm}(\theta, \phi),
\]

with \( \theta \) and \( \phi \) angles at the sky. We assume that the fluctuations follow a Gaussian distribution and that each \( l \)-mode is unrelated to all others:

\[
\langle a_{lm} \rangle = 0, \quad \langle a_{lm} a_{l'm'}^* \rangle = \delta_{ll'} \delta_{mm'} C_l.
\]
The collection $C_l$, the angular power spectrum, contains all information about the Gaussian temperature fluctuations. Per value of $l$ we average over the $(2l + 1)$ contributions. We have

$$\langle \frac{\Delta T(\vec{n}) \Delta T(\vec{n}')}{T} \rangle = \frac{1}{4\pi} \sum_l (2l + 1) C_l P_l(\vec{n} \cdot \vec{n}'),$$

(1.10)

with $P_l$ the $l$th Legendre polynomial. We have converted $\theta$ and $\phi$ into the unit vector $\vec{n}$, indicating the direction from which a CMB photon enters the telescope.

The measured angular power spectrum of the CMB temperature fluctuations is in figure 1.3. For $l > 30$, the data can perfectly be fit by the picture of the universe we have described so far. At larger scales, there is some discrepancy, which might hint to some unknown large scale physics. However, at these larger scales the uncertainty in the measurements increases dramatically. This is mainly due to “cosmic variance”: we can observe the universe only from our position, and per value of $l$ we have less values of $m$ that we can average over. Therefore there is, at least at the time of writing, no statistical evidence for new physics on scales $l < 30$.

### 1.2 Inflation: why and how

#### 1.2.1 Naturalness problems

Inflation was originally proposed [12] to solve some naturalness problems. We will briefly review three of them in this section.

First there is the “horizon problem”. All over the CMB sky one measures one and the same background
temperature\textsuperscript{3}. Given the current age of the universe (and the finite speed of light), and the fact that the CMB came into existence after 380,000 years, one expects modes to be correlated only for $l \geq 30$ (that is, on scales $\leq 6^\circ$). The smaller $l$-modes correspond to scales that are too large to have been in causal contact already after 380,000 years. This is the horizon problem: how can it be that apparently all of the universe that we can see today has the same background temperature? Had the universe already been in causal contact in its infant days?

Next we have the “flatness” problem. The contributions to the total energy density $\rho_{\text{tot}}$ coming from matter ($\rho_m$), radiation ($\rho_r$) and dark energy ($\rho_\Lambda$) almost add up the critical density $\rho_{\text{crit}} = 3H^2/8\pi G$ needed to have a perfectly flat universe (a universe with Euclidean geometry). Indeed Planck \textsuperscript{[13]} finds, after switching to the normalized parameter $\Omega \equiv \rho/\rho_{\text{crit}}$,

$$1 - \Omega_{\text{tot}} \equiv 1 - (\Omega_m + \Omega_r + \Omega_\Lambda) = -0.0005^{+0.0065}_{-0.0066}, \quad (2\sigma). \quad (1.11)$$

In other words: the data are perfectly compatible with a completely flat universe.

That is already an unnatural result, but it gets much worse when we go back in time. The Friedmann equation (1.5) can be rewritten in such a way

$$\Omega_m + \Omega_r + \Omega_\Lambda = \frac{k}{H^2a^2} = 1, \quad (1.12)$$

that it lists the various contributions to the total energy density. Here we have introduced $\rho_\Lambda \equiv \Lambda/8\pi G$. We already discussed that in an expanding universe we have $\Omega_m \sim a^{-3}$ and $\Omega_r \sim a^{-4}$. As its name

\textsuperscript{3}Moreover, although this was not known when inflation was invented, we see in figure 1.3 that the temperature fluctuations are correlated on every scale.
1.2. INFLATION: WHY AND HOW

suggests, the cosmological constant’s contribution $\Omega_\Lambda$ is constant. Therefore we see that the curvature contribution is increasing when the universe expands. To have a pretty flat universe now, one would need an extremely flat universe in the past. In slightly other words: $\Omega_{\text{tot}} = 1$ is an unstable fixed point.

Furthermore it follows from the Friedmann equation that the immense flatness needed in the early universe to arrive at the current conditions is equivalent to having an enormous entropy in our Hubble volume:

$$\frac{k}{H^2a^2} \approx \frac{k}{T^4a^2} \approx \frac{k}{S^{2/3}T^2}. \quad (1.13)$$

Here $S$ is the total entropy per Hubble volume, and we have used that $H^2 \simeq \rho \simeq T^4$ and that $S \sim a^3S \sim a^3T^3$.

Inflation is by now regarded as a fundamental part of standard cosmology, as it solves the horizon problem, the flatness problem and the entropy problem in one go. One can compute that if inflation lasts at least $60-70$ e-folds (if the scale factor increases by at least a factor $e^{60-70}$, the precise number depends on the energy scale of inflation) the current homogeneous CMB sky and flatness follow from natural order one initial conditions before inflation. Moreover, the reheating of the universe after inflation, a process in which the inflation energy density is released into other (Standard Model) degrees of freedom, causes an enormous increase in the entropy ($T$ increases, $a$ remains approximately constant).

1.2.2 Expansion from negative equation of state

Now let us see how we can get such an enormous expansion of spacetime itself. All we need is a scalar field $\phi$ moving through an almost flat potential $V(\phi)$. For the energy density $\rho$ and the pressure $p$ we can then write, from evaluating the energy-momentum tensor in the cosmic rest frame,

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi), \quad p = \frac{1}{2}\dot{\phi}^2 - V(\phi). \quad (1.14)$$

Here we have neglected gradient terms, as we assume the field $\phi$ to be homogeneous and isotropic.

If the potential is sufficiently flat, $\dot{\phi}$ will be small, and the scalar field will have a negative equation of state: $p \approx -\rho$. From the conservation of energy-momentum it then follows that having such a negative equation of state leads to having a constant ($a$-independent) energy density:

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0 \quad \Rightarrow \quad \rho \sim a^0. \quad (1.15)$$

Therefore we see that the energy density $\rho$ of the scalar field $\phi$ behaves exactly like the energy density contribution from the cosmological constant $\rho_\Lambda$ we described before. From the analysis in section 1.1.2 it follows directly that indeed the scale factor will grow exponentially:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\rho}{3} \quad \Rightarrow \quad a(t) = e^{\sqrt{\rho/3}t}. \quad (1.16)$$

Note that if the potential is perfectly flat, the inflaton does not move at all. As a result, there is no end to inflation. (However, this may describe the accelerated expansion that we are currently observing.)
1.2.3 Slow-roll variables

To get inflation, one imposes that the rate of change of the Hubble constant (i.e. $\dot{H}/H$) be small over the Hubble time $1/H$:

$$\epsilon \equiv -\frac{\dot{H}}{H^2} \ll 1. \quad (1.17)$$

This is equivalent to demanding that the inflaton moves slowly through its potential, as we have $\dot{H} = -\frac{\dot{\phi}^2}{2}$.

The number of e-folds that inflation generates follows from

$$dN = -\int Hdt = -\int_{\phi_1}^{\phi_2} \frac{d\phi}{\sqrt{2\epsilon}}. \quad (1.18)$$

Since inflation is defined as an accelerated expansion of spacetime, we see from the relation $\ddot{a}/a = H^2 + \dot{H} = (1 - \epsilon)H^2$ that it stops when $\epsilon = 1$.

To keep inflating for a sufficient amount of time (or e-folds), $\epsilon$ has to remain small. We need a second slow-roll parameter $\eta$ that makes sure that $\epsilon$ does not change too fast over the Hubble volume. For inflation to happen we therefore demand

$$\eta \equiv -\frac{\dot{\epsilon}}{H\epsilon} \ll 1. \quad (1.19)$$

(This could also be written as $\eta \equiv -\frac{d\log \epsilon}{dN}$.)

To avoid having to solve the inflaton’s equation of motion, the slow-roll variables are often defined alternatively, in terms of the potential $V(\phi)$. We have

$$\tilde{\epsilon} \equiv \frac{1}{2} \left(\frac{V_{\phi}}{V}\right)^2, \quad (1.20)$$

$$\tilde{\eta} \equiv \frac{V_{\phi\phi}}{V}. \quad (1.21)$$

Here we used the notation $V_{\phi} \equiv \partial V/\partial \phi$. This “potential” $\tilde{\epsilon}$ can be expressed in terms of the kinematical slow-roll parameters defined before:

$$\tilde{\epsilon} = \epsilon \left(1 - \frac{\eta}{2(3 - \epsilon)}\right)^2 \approx \epsilon. \quad (1.22)$$

For $\tilde{\eta}$ it follows that

$$\tilde{\eta} = \frac{1}{3 - \epsilon} \left[6\epsilon + \frac{3}{2}\eta + \ldots\right] \approx 2\epsilon + \frac{\eta}{2}. \quad (1.23)$$

The last step assumes that kinematical $\epsilon$ and $\eta$ are small. We conclude that to have inflation, we need the kinematical slow-roll parameters $\epsilon$ and $\eta$ to be small (of order $10^{-2}$), which implies that the “potential” slow-roll parameters $\tilde{\epsilon}$ and $\tilde{\eta}$ are small as well. In other words: the inflaton needs to be light.

Strictly speaking, however, having small potential slow-roll variables does not necessarily imply being in an inflationary phase. The behaviour of the cosmological scale factor depends on the path the inflaton follows through field space. Only when the field is slowly rolling, this path precisely follows the direction of steepest descent through the potential. Then the “potential” slow-roll variables can be used to extract information about inflation, like the number of e-folds that inflation lasts. When there is no slow-roll, the
field can for example use its kinetic energy to climb up to a higher potential value. It does not follow the direction of steepest descent through the potential anymore, so $\tilde{\epsilon}$ and $\tilde{\eta}$ become useless.

In the rest of this thesis, we will be using the kinematical slow-roll variable $\epsilon$ and the potential slow-roll variable $\tilde{\eta}$, to which we will refer as $\eta$ from now on.

### 1.3 Inflating quantum perturbations

As we reviewed in the previous section, the mechanism of cosmological inflation was originally proposed to solve naturalness problems. However, it was soon realized [14] that inflation can also explain the tiny temperature fluctuations in the CMB. It had been known for a long time that basic Newtonian physics suffices to describe how these fluctuations grow out to form stars and planets. Before inflation was studied these tiny CMB temperature variations, the seeds for the formation of all structure in the universe, had to be put in by hand as initial conditions. Inflation gives an explanation for the correlated fluctuations in the CMB. In this section we will quickly review how the inflation of fluctuations of the quantum inflaton field grow out to the temperature fluctuations in the CMB. For simplicity we will focus on single field inflation here.

We begin with the equation of motion for the quantum inflaton field $\phi$:

$$
\ddot{\phi} + 3H\dot{\phi} - \nabla^2 a^2 \phi + \frac{\partial V}{\partial \phi} = 0.
$$

(1.24)

Now expand the field in a classical background field and a quantum fluctuation $\phi(\vec{x}, t) = \phi(t) + \delta \phi(\vec{x}, t)$ and take the Fourier transform. In conformal time $\tau$ (defined by $dt = ad\tau$) we get, after rescaling the perturbations via $v_k \equiv a^2 \delta \phi_k$, up to first order

$$
\left(\frac{\rho^2}{2} \frac{\partial^2}{\partial \rho^2} + \frac{\rho^2}{2} - 2\right) v_k = 0.
$$

(1.25)

We will now first take a massless scalar field and work in exact de Sitter ($\epsilon = \eta = 0$). For the moment we work with a fixed metric: later we will take gravity dynamical and consider the metric’s perturbations as well. Now we have

$$
\tau \equiv \int \frac{dt}{a} = \int dt e^{-Ht} = \left[- \frac{1}{H} e^{-Ht}\right] = -\frac{1}{Ha}.
$$

(1.26)

This means $a'' = -\frac{2}{1+3H} = \frac{2}{\tau^2}$ and we get, after rewriting in terms of $\rho \equiv -k\tau$ (not to be confused with the energy density),

$$
\left(\frac{\rho^2}{2} \frac{\partial^2}{\partial \rho^2} + \rho^2 - 2\right) v_k = 0.
$$

(1.27)

This equation is solved by

$$
v_k = c \sqrt{\rho} H^{(1)}_{3/2}(\rho),
$$

(1.28)

where $H^{(1)}_{\nu}(\rho)$ denotes the Hankel function of the first kind. (Actually the solution contains a part proportional to a Hankel function of the second kind well, but that part should be set to zero if we want our solution to return the Bunch-Davies vacuum, defined below, in the infinite past.) For later reference we already notice that if we modify equation (1.27) a little bit into

$$
\left(\frac{\rho^2}{2} \frac{\partial^2}{\partial \rho^2} + \rho^2 - (2 + \alpha)\right) v_k = 0,
$$

(1.29)
the solution is still given by \( v_k = c \sqrt{\rho H^1(\rho)} \), but now we need
\[
\nu^2 - \frac{9}{4} - \alpha = 0 \quad \Rightarrow \quad \nu = \sqrt{\frac{9}{4} + \alpha} = \frac{3}{2} + \frac{\alpha}{3}.
\] (1.30)

We set the integration constant to \( c = \frac{i}{2} \sqrt{\frac{\pi}{k}} \). This makes \( v_k \) real in the limit \((-k\tau) \to 0\), as the Hankel function becomes purely imaginary there. Furthermore, for \((-k\tau) \to \infty\) we get, using the Hankel expansion for large argument \( H^1(\nu)(z) \to \sqrt{\frac{2}{\pi}} e^{iz} e^{-i\pi} \),
\[
v_k = \frac{i}{2} \sqrt{\frac{\pi}{k}} \sqrt{-k \tau} H^1_{3/2}(-k \tau) \quad \Rightarrow \quad \frac{i}{2} \sqrt{\frac{\pi}{k}} \sqrt{-k \tau} \frac{1}{\sqrt{-k \tau}} e^{-ik\tau} \times -1
\]
\[
= -i \times \frac{e^{-ik\tau}}{\sqrt{2k}},
\] (1.31)

which corresponds the Bunch-Davies vacuum.

To describe a more realistic situation, we need to consider a massive scalar field (but light compared to the Hubble scale). We should also acknowledge that inflation takes place in a “quasi-dS” spacetime: \( H \) does slowly change in time. In terms of kinematical \( \epsilon \equiv -\dot{H}/H^2 \) and “potential” \( \eta \equiv V''/V \) the equation of motion for the modes \( v_k \) becomes
\[
\left( \partial^2_{\tau} + k^2 - \frac{1}{\tau^2} [2 - 3\epsilon + 3\eta] \right) v_k = 0.
\] (1.32)

Here we have used the quasi-dS result \( \tau = -\frac{1}{\dot{H}} \frac{1}{\sqrt{-\epsilon - \eta}} \) which leads to \( \frac{a''}{a} = \frac{1}{\tau} (2 - \epsilon)(1 + 2\epsilon) = \frac{1}{\tau} (2 + 3\epsilon) \). After the discussion around (1.30) it is clear that the solution of (1.32) is given by
\[
v_k = \frac{i}{2} \sqrt{\frac{\pi}{k}} \sqrt{\rho H^1(\rho), \quad \nu = \frac{3}{2} + \epsilon - \eta.}
\] (1.33)

Plotting this solution, see figure 1.4, shows that for \( \rho > 1 \), which means inside the horizon\(^4\) \((kH > 1)\) the function is oscillating, with constant amplitude. This is not surprising because for large \( k \) the equation of motion (1.32) reduces to
\[
(\partial^2_{\tau} + k^2)v_k = 0 \quad \Rightarrow \quad v_k = A \sin(-k\tau) + B \cos(-k\tau).
\] (1.34)

Therefore: inside the horizon the modes \( v_k \) oscillate with constant amplitude, which means that the modes \( \delta \phi_k = \frac{v_k}{a} \) oscillate with an exponentially decreasing amplitude.

Outside the horizon, for \( \rho < 1 \), the solution grows exponentially. This can be seen from taking the small \( k \) limit of (1.32), which gives (again omitting slow-roll effects, reinserting \( \frac{a''}{a} = \frac{2}{\tau} \) and taking the growing solution)
\[
\left( \partial^2_{\tau} - \frac{a''}{a} \right) v_k = 0 \quad \Rightarrow \quad v_k = c_1 a.
\] (1.35)

So: outside the horizon the modes \( v_k \) grow exponentially, proportional to the scale factor, which means that the modes \( \delta \phi_k = \frac{v_k}{a} \) freeze. That is what we would expect from causality as well. As soon as the typical wavelength of a fluctuation \( a/k \) becomes larger then the physical horizon \( H^{-1} \), the wave can not evolve any further as its one end is not in causal contact anymore with its other end.

\(^4\)The Hubble horizon \( 1/H \) denotes the maximum distance over which physics can be in causal contact at a given point in time. When a mode crosses the horizon its physical wavelength \( a/k \) is equal to the horizon size \( 1/H \).
1.3. INFLATING QUANTUM PERTURBATIONS

Figure 1.4: On the left the numerical solution to equation (1.32) for the modes $v_k$, on the right the associated solution for the modes $\delta \phi_k$. Absolute values in red, real and imaginary values in blue and green. Inside the horizon ($\rho \equiv -k \tau > 1$) the modes $v_k$ oscillate, outside the horizon ($\rho < 1$) the modes $\delta \phi_k$ freeze.

Finally, we should take general relativity (GR) into account. So far we have only considered fluctuations of the inflaton field. However, the metric fluctuates as well. We can write the perturbed metric as

$$g_{\mu \nu} = a^2 \left( \frac{-1 + 2A}{\partial_i B} \frac{\partial_i B}{(1 - 2 \psi) \delta_{ij} - D_{ij} E} \right).$$

(1.36)

This suggests that there are four scalar degrees of freedom in the metric. However, two of these are redundant. We can work in the longitudinal gauge where $B = E = 0$. Then we can use one of the off-diagonal components of the perturbed Einstein equations to show that we also have $A = \psi$. Finally we can use a diagonal Einstein equation to relate the scalar metric perturbations to the field perturbation $\delta \phi$. At the end of the day we are thus left with one degree of freedom, that we can take to be a linear combination of $\psi$ and $\delta \phi$.

Now there are two frequently used diffeomorphism invariant variables to describe the fluctuations of inflaton and metric. We take the comoving curvature perturbation $R$ to be defined via

$$R \equiv \psi + \frac{H}{\dot{\phi}} \delta \phi.$$  

(1.37)

Alternatively, one often employs the curvature perturbation on uniform density hypersurfaces $\zeta$:

$$\zeta \equiv -\psi - \frac{H}{\dot{\rho}} \delta \rho.$$  

(1.38)

Now we can use energy conservation $\dot{\rho} + 3H(p + \rho)$, the expressions (1.14) and the Klein-Gordon equation (1.24) to show that during inflation we have

$$\zeta = -\psi - \frac{H}{\dot{\rho}} \delta \rho = -\psi - \frac{3H}{3H_0^2} \delta (V(\phi)) = -\psi - \frac{H}{3H_0^2} V'(\phi) \delta \phi = -\psi - \frac{H}{\phi} \delta \phi = -R.$$  

(1.39)

up to slow-roll corrections. Since after inflation $R$ and $\zeta$ are frozen, this equality remains. Finally we can use an off-diagonal perturbed Einstein equation to find that on superhorizon scales one can approximate

$$\psi \simeq \epsilon H \frac{\delta \phi}{\dot{\phi}} \rightarrow \zeta \simeq -(1 + \epsilon) \frac{\delta \phi}{\dot{\phi}} \simeq -\frac{H}{\phi} \delta \phi.$$  

(1.40)
We will use this last approximation to compute $\zeta$’s two-and three point function in the next section and especially in chapter 6.

To compute $\zeta$ itself, we still need to find out how the perturbation equation for $\delta \phi$ changes once we allow the metric to fluctuate as well. We get (in cosmic time)

$$\ddot{\delta \phi}_k + 3H \dot{\delta \phi}_k + \frac{k^2}{a^2} \delta \phi_k + \frac{\partial^2 V}{\partial \phi^2} \delta \phi_k = -2\psi_k \frac{\partial V}{\partial \phi} + 4\dot{\psi} \dot{\phi}. \quad (1.41)$$

On superhorizon scales we have $|4\dot{\psi} \dot{\phi}| \ll |2\psi \frac{\partial V}{\partial \phi}|$. The remaining term can be rewritten using $\frac{\partial V}{\partial \phi} = -3H \dot{\phi}$. (We are still working up to first order in the perturbations, and up to first order in slow-roll.) Finally we use the superhorizon result $\psi_k \approx \epsilon H \delta \phi_k \dot{\phi}$ again. We get

$$\ddot{\delta \phi}_k + 3H \dot{\delta \phi}_k + \frac{k^2}{a^2} \delta \phi_k + \left(\frac{\partial^2 V}{\partial \phi^2} - 6\epsilon H^2\right) \delta \phi_k = 0. \quad (1.42)$$

Now we use

$$\frac{\partial^2 V}{\partial \phi^2} - 6\epsilon H^2 = V \frac{V''}{V} - 6\epsilon H^2 = 3H^2 \eta - 6\epsilon H^2 = 3H^2 (\eta - 2\epsilon), \quad (1.43)$$

and it is clear that on superhorizon scales we will get

$$\zeta_k = -\frac{H v_k}{\dot{\phi} a}, \quad v_k = i \frac{\sqrt{\pi}}{2k} \sqrt{\rho H_\nu^{(1)}(\rho)}, \quad \nu = \frac{3}{2} + 3\epsilon - \eta. \quad (1.44)$$

### 1.4 Inflationary observables

During inflation the modes $\zeta_k$ freeze when they leave the horizon. After inflation, when time passes by, spacetime expands at a subluminal speed which makes that one by one the modes get back in causal contact. They begin to oscillate again. (Note however that during inflation the oscillations were of a quantum field, now they are classical oscillations of the pressure and the gravitational potential.) At the time the CMB radiation was emitted, the mode with $l \approx 180$ had just reached its first extremum, which causes the large first peak in figure 1.3. Modes with smaller wavelengths (larger $l$) had already done several oscillations, which generates the peak structure in the CMB in figure 1.3. For modes that were still frozen at that time ($l \lesssim 30$), the temperature correlation functions are directly proportional to the curvature correlation functions that can be computed from the superhorizon result (1.44).

At the end of the day, CMB temperature measurements lead to precise values for the two- and three point correlation functions of the curvature perturbation $\zeta_k$. In this section we show how the solution (1.44) is connected to physical observables involving these correlation functions.

The power spectrum of $\zeta$ is defined as the power per mode in $\zeta$’s two point function in momentum space:

$$\langle 0 | \zeta(x) \zeta(x) | 0 \rangle = \int \frac{d^3 k}{(2\pi)^3} |\zeta_k|^2 \equiv \int \frac{dk}{k} \frac{1}{k} \Delta^2 \zeta(k). \quad (1.45)$$

We therefore have

$$\Delta^2 \zeta(k) = \frac{k^3}{2\pi^2} |\zeta_k|^2. \quad (1.46)$$

This quantity is directly connected to the temperature fluctuations in the CMB, so this is what we need to compute (in the superhorizon limit). In passing by we also give the momentum space two point function...
of $\zeta$:  
\[\langle \zeta(\vec{k})\zeta(\vec{l}) \rangle = (2\pi)^3 \delta^3 \left( \vec{k} + \vec{l} \right) P(k).\]  (1.47)

Now we can write  
\[\langle \zeta(x)^2 \rangle = \int \frac{d^3 k \cdot d^3 k'}{(2\pi)^3} \langle \zeta(\vec{k})\zeta(\vec{l}) \rangle = \int \frac{d^3 k}{(2\pi)^3} P(k)\]  (1.48)

to see that we have $P(k) = |\zeta|^2$.

Plugging our solution (1.44) in the definition (1.46) gives for the power spectrum of $\zeta$ from slow-roll inflation  
\[\Delta^2_\zeta(k) = \left( \frac{H^2}{2\pi\phi} \right)^2 \left( \frac{k}{aH} \right)^{2\eta - 6\epsilon}.\]  (1.49)

Therefore the picture is that every mode $k_\alpha$ oscillates as long as it is inside the horizon. At the moment of horizon crossing $t_\alpha$ we have by definition $k_\alpha = a(t_\alpha)H(t_\alpha)$ and the power in this mode freezes at the value $\left( \frac{H(t_\alpha)^2}{2\pi\phi(t_\alpha)} \right)^2$. If $H$ and $\dot{\phi}$ were exactly constant in time, all modes would freeze out at the same value and we would have a perfectly flat, scale invariant, spectrum. However, $H$ and $\dot{\phi}$ slowly decrease during inflation. Therefore we get a slightly red tilted spectrum: modes that leave the horizon later have smaller power. The most convenient way to describe this is to take a pivot scale $k_0$, compute the spectrum there, and compute the power on all other modes via  
\[\Delta^2_\zeta(k) = \Delta^2_\zeta(k_0) \left( \frac{k}{k_0} \right)^{n_s - 1}.\]  (1.50)

Unless stated otherwise, the pivot scale is taken at $k_0 = 0.002$ Mpc$^{-1}$, which corresponds to (more or less) the scale that leaves the horizon 60 e-folds before the end of inflation, about the largest observable scale today. The tilt in the power spectrum is defined via the spectral index $n_s$. (A scale invariant spectrum would correspond to $n_s = 1$.)

Planck finds [15]$^5$  
\[\Delta^2_\zeta(k_0 = 0.05 \text{ Mpc}^{-1}) = (2.20^{+0.05}_{-0.06}) \cdot 10^{-9}, \quad n_s = 0.9643 \pm 0.0059,\]  (1.51)

where the quoted errors show the 68% confidence levels (the 1$\sigma$ bounds) on $\Delta^2_\zeta$ and the 95% confidence levels (the 2$\sigma$ bounds) on $n_s$. Note that the spectral index is directly related to the slow-roll parameters: $n_s = 1 + 2\eta - 6\epsilon$.

In chapter 7 we will also look at the scale dependence of the spectral index itself.

Note that so far we have only looked at the scalar fluctuations observable in the CMB. However, there is also a tensor perturbation in the metric, denoted $D_{ij}$ in (1.36). Such a perturbation can be detected as a gravitational wave coming from inflation itself. At the moment of writing, these have never been observed, and Planck [15] can only give an upper bound for the so-called tensor-to-scalar ratio $r$:  
\[r = \frac{\Delta^2_\zeta(k_0)}{\Delta^2_\zeta(k_0)} = \frac{2\Delta^2_\zeta(k_0)}{\Delta^2_\zeta(k_0)} < 0.12 \quad (95\% \text{ CL}).\]  (1.52)

Here $\Delta^2_\zeta(k)$ denotes the total tensor power spectrum. It gets two equal contributions $\Delta^2_\zeta(k_0)$ from the two possible gravitational wave polarizations. Computing the tensor wave power spectrum $\Delta^2_\zeta(k)$ is easier than computing $\Delta^2_\zeta(k)$ since we only have to deal with metric fluctuations. The result is  
\[r = 16\epsilon.\]  (1.53)

$^5$Here we cite the result for a LCDM model with tensor waves and no running spectral index, acquired from both the Planck results and those from baryon acoustic oscillations.
The tensor to scalar ratio is connected to the energy scale of inflation $V$ as well. One has

$$V^{1/4} \sim \left(\frac{r}{0.01}\right)^{1/4} \cdot 10^{16} \text{ GeV}. \quad (1.54)$$

The detection of a tensor wave signal of order $10^{-2}$, which is the expected sensitivity for Planck, would point to inflation at the GUT scale. As no nonzero value for $r$ has been found at the time of writing, the inflation scale can still vary over many orders of magnitude.

After analyzing the two point function, it is a logical step to turn to $\zeta$’s three point function. Analogously to (1.47) one can define a momentum space three point function $B(\vec{k}_1, \vec{k}_2, \vec{k}_3)$ (the bispectrum) via

$$\langle \zeta(\vec{k}_1)\zeta(\vec{k}_2)\zeta(\vec{k}_3) \rangle \equiv (2\pi)^3\delta^3(k_1 + k_2 + k_3)B(\vec{k}_1, \vec{k}_2, \vec{k}_3). \quad (1.55)$$

It is clear that while $P(\vec{k}_1, \vec{k}_2)$ depends only on the separation between its two arguments, the function $B(\vec{k}_1, \vec{k}_2, \vec{k}_3)$ can be studied for all possible shapes of the triangle formed by $\vec{k}_1$, $\vec{k}_2$ and $\vec{k}_3$.

If inflation is indeed the result of a single scalar field slowly rolling down a potential, all three-point functions will be very small. Maldacena already computed [16] that they will be of the order of the slow-roll parameters. That was to be expected: if the inflaton can be approximated as a free field, there are no cubic or quartic terms in its action. Therefore there is no correlation between the modes $\delta \phi_k$ at the linear level. The whole system can very adequately be described as a collection of uncoupled oscillators, that each follow a Gaussian distribution function.

However, if inflation is in fact the result of multiple fields conspiring together, the equation of motion (1.25) may not be a good approximation anymore. Then we really have to consider the full system of coupled oscillators. The system is not linear and perturbations are not completely Gaussian. One often defines the non-linearity parameter $f_{\text{NL}}$ via

$$\zeta(\vec{x}) = \zeta_0(x) + \frac{3}{5} f_{\text{NL}} \left[ \zeta_0(\vec{x})^2 - \langle \zeta_0(\vec{x})^3 \rangle \right], \quad (1.56)$$

where $\zeta_0$ denotes the Gaussian part of $\zeta$. This version of non-Gaussianity is called local, since its definition is local in real space. Now that the system is not linear anymore, three point functions do not have to be negligibly small anymore. In the local case we have

$$B(\vec{k}_1, \vec{k}_2, \vec{k}_3) = \frac{6}{5} f_{\text{NL}}^3 \left[ P(\vec{k}_1)P(\vec{k}_2) + P(\vec{k}_2)P(\vec{k}_3) + P(\vec{k}_3)P(\vec{k}_1) \right]. \quad (1.57)$$

The bispectrum for local non-Gaussianity is largest when the smallest of the three vectors $\vec{k}_1$, $\vec{k}_2$ and $\vec{k}_3$ is very small, such that the other two are almost equal (the squeezed limit).

Another shape for non-Gaussianity that we will compute in this thesis is the equilateral shape, which is largest when all three vectors are of equal size. In this case we can relate $B_{\text{eq}}(k)$ to the position space three point function via (compare with (1.48))

$$\langle \zeta(\vec{x})^3 \rangle = \int d^3k \frac{8\pi^2}{(2\pi)^6} k^6 B_{\text{eq}}(k) \approx \frac{8\pi^2}{(2\pi)^6} k^6 B_{\text{eq}}(k) \mathcal{O}(1). \quad (1.58)$$

Now it follows that we can extract a value for $f_{\text{NL}}^3$ via

$$\langle \zeta(\vec{k}_1)\zeta(\vec{k}_2)\zeta(\vec{k}_3) \rangle_{\text{eq}} = (2\pi)^3\delta^3 \left( \vec{k}_1 + \vec{k}_2 + \vec{k}_3 \right) \times (2\pi)^4 \frac{3}{10} f_{\text{NL}}^3 \Delta_\zeta^3(k) \sum_{i=1}^{3} k_i^3 \frac{\Delta k_i^3}{\Pi_i k_i^3}. \quad (1.59)$$

---

6The same happens in models with higher order derivative terms, such as DBI inflation [17].

7Here we follow the convention employed in [18], but we modified some factors of $(2\pi)$ to be consistent with our definition of the Fourier transform.
We therefore have

\[ f_{\text{NL}}^{eq} = B_{eq}(\vec{k}_1, \vec{k}_2, \vec{k}_3) \frac{10}{3} \frac{1}{(2\pi)^4} \frac{1}{\Delta^2(k)} \sum_i k_i^3 = \frac{10}{9} \frac{(2\pi)^2}{8\pi^2} \langle \zeta(x)^4 \rangle. \] (1.60)

The Planck results on these two shapes for non-Gaussianity are [19]:

\[ -192 < f_{\text{NL}}^{eq} < 108, \quad -8.9 < f_{\text{NL}}^{loc} < 14.3, \] (1.61)

where we have quoted the 2σ bounds. Therefore, especially local primordial non-Gaussianity will be difficult to observe, as non-primordial physics gives a contribution of \( f_{\text{NL}} \sim 5 - 10 \) to the measured CMB signal. One will need to go beyond linear order to disentangle an eventual primordial non-Gaussian signal.

Finally, let us look at one more observable that can constrain inflationary models: the non-detection of primordial black holes. These will form if at horizon re-entry (i.e. smoothing \( \zeta \) on scales of order \( H \)) we have \( \zeta > \zeta_c \), with \( \zeta_c \sim 1 \) denoting the critical value leading to black hole formation. If one assumes that \( \zeta \) follows a Gaussian distribution (with \( \langle \zeta \rangle = 0 \)) one can express the probability of having \( \zeta > \zeta_c \) in terms of the variance \( \sigma^2 = \langle \zeta^2 \rangle \) by analyzing the Gaussian probability distribution function. This probability corresponds to the fraction of space \( b \) that can collapse to form horizon-sized black holes. We have

\[ b \equiv \int_{\zeta_c}^\infty P(\zeta) d\zeta = \int_{\zeta_c}^\infty \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\zeta^2}{2\sigma^2}} d\zeta. \] (1.62)

Therefore, a given value for \( b \) leads to a upper bound on \( \sigma^2 = \langle \zeta^2 \rangle \). Using (1.45) we find that \( \sigma^2 \) is equal, up to an order one factor, to the power spectrum:

\[ \sigma^2 = \langle \zeta(x)\zeta(x) \rangle = \int d\ln k \Delta^2(k) \simeq O(1) \Delta^2(k). \] (1.63)

Now the fraction \( b \) is constrained by Hawking evaporation and present day gravitational effects [20]. Taking a typical value of \( b = 10^{-20} \) we get an upper bound on the power spectrum [21]:

\[ \Delta^2(k) < 0.01. \] (1.64)

Compared to (1.51) this bound is very weak. However, it is in principle valid on all scales. The experimental bounds coming from the CMB are much more precise, but are valid on a much smaller range of scales. In the usual Fourier decomposition of the CMB temperature fluctuations one gets information for multipole moments up to \( l \approx 2500 \). Let us make a rough estimate and state that the largest observable scale \( l = 2 \) corresponds to the mode that left the horizon 60 e-folds before the end of inflation. Then we realize that the CMB gives only information on the modes that left between 60 and (about) 53 e-folds before the end of inflation, since \( \log(2500/2) \approx 7 \). In chapter 6 we will study models in which towards the end of inflation the power spectrum increases by many magnitudes. We will see that in such a case the non-detection of primordial black holes forms a realistic observational constraint. However, there we will need a more sophisticated analysis taking into account the non-Gaussianity of \( \zeta \)'s fluctuations and the scale dependence in the fraction \( b \).

### 1.5 Inflationary model building and the \( \eta \)-problem

To get inflation we need a scalar field rolling down a potential that is flat enough to generate 60 e-folds of inflation. This provides quite a challenge for model building, as scalar fields in general do not want
CHAPTER 1. COSMOLOGY AND INFLATION

to be light. This is familiar from the case of the Higgs field in the Standard Model. The huge gap between its mass ($O(10^2)$ GeV) and the energy scale $\Lambda$ up to which we tend to trust the theory ($O(10^{18})$ GeV) is the root of all evil. In the context of the Standard Model the problem is known as the hierarchy problem. Radiative corrections of order $\Lambda^2$ modify the bare squared Higgs mass. At the end of the day the experimenter finds a physical Higgs mass of 125-126 GeV. The apparent enormous cancellation between bare mass and radiative correction leaves many physicists uneasy.

In the context of inflation, one talks about the $\eta$-problem which is really nothing else than the hierarchy problem rearing its head again. However, the situation is a bit more problematic than in the Standard Model. First, we want $\eta$ to be small. (We will work with the “potential” definition $\eta = (V''/V) = m^2_{\text{inf}}/3H^2$). That is just equivalent to trying to get the Higgs mass far below the cut-off scale. However, now that we are trying to say something about gravity, which is unrenormalizable, we should view the Standard Model as a low-energy effective theory. It is only natural to introduce higher order operators as effective operators stemming from a UV complete theory. $\eta$ receives order one corrections by the inclusion of such operators. That is why the hierarchy problem is more severe in an inflationary context. In this section we will review several models of inflation, and study how these deal with the $\eta$-problem.

1.5.1 Small field models

Small field models are models in which the inflation rolls over a sub-Planckian distance in field space during the last 60 e-folds of inflation. Inflation takes place around a local extremum $V_0$ of the potential. In the first models [22, 23], such potentials arise from spontaneous symmetry breaking. In this class of models the $\eta$-problem appears when one adds for example a contribution of $\delta V = V_0 \phi^2 m^2_{\text{inf}}/3H^2$ to the potential. While low-energy physics is not affected by such an extra contribution, the $\eta$-parameter gets an extra contribution of order one and inflation is spoiled. One needs to fine-tune the parameters in the problem again to get the desired $\eta \lesssim 0.01$ at the point in field space where the inflaton passes 60 e-folds before the end of inflation (the pivot scale).

1.5.2 Large field models

Large field models typically use potentials of the “chaotic” type [24]. The chaotic initial conditions prescribe that the inflaton finds itself at a super-Planckian value in field space. Its subsequent rolling back to the origin creates inflation. Typical chaotic potentials are $V(\phi) = \frac{1}{4} m^2 \phi^2$ and $V(\phi) = \frac{1}{4} \lambda \phi^4$. In a way, the $\eta$ problem is even worse now. As in the case of small field models, one has to tune $m$ or $\lambda$ to a sufficiently small value to get inflation going. Now however every higher order term in the potential of the type $\delta V = \frac{2}{4} \phi^{n+4} \frac{M^n}{M_p}$ gives a large additional contribution to $\eta$. Including more and more non-renormalizable higher dimensional operators there is no end to the necessary tuning.

1.5.3 Hybrid models

Hybrid inflation [25] is two-field inflation. Next to the inflaton field $\phi$ there are one or more so-called waterfall fields $H$, whose dynamics provide an elegant ending of inflation. During inflation, the waterfall fields are heavy and therefore stabilized in a local minimum. However, the rolling of the inflaton causes one of the waterfall fields to become unstable (tachyonic). When this waterfall field falls down the slow-roll conditions are no longer met and inflation stops.
1.6. INFLATION IN SUPERGRAVITY

1.5.4 Symmetries

In all these models, and in multifield models as well, we can invoke symmetries to get around the $\eta$-problem. We can for example work with a complex scalar field $\Phi = \phi_R + i\phi_I$. Now if the inflaton potential involves only powers of $(\Phi - \bar{\Phi})$, the field $\phi_R$ will remain massless, whatever higher order operator we include. Now, however, the challenge is to break the symmetry just “softly” enough, such that a small tilt in the potential is created, for example from radiative corrections. In the next section we will see more examples of invoking symmetries to keep the inflaton potential flat.

1.5.5 Lyth’s bound

We conclude this section with the most characteristic phenomenological difference between small and large field inflation. Lyth has shown [26] that the length $\Delta \phi$ of the inflaton’s path through field space during the last 60 e-folds is proportional to the tensor to scalar ratio $r$:

$$\frac{\Delta \phi}{M_p} = \mathcal{O}(1) \left( \frac{r}{0.01} \right)^{1/2}. \quad (1.65)$$

The detection of a primordial gravitational wave signal of order $10^{-2}$ would therefore irrevocably point out that inflation is of the large field type.

1.6 Inflation in supergravity

In the Standard Model, the hierarchy is most often (partially) solved by imposing supersymmetry. Unbroken supersymmetry generates extra radiative corrections from the sparticle loops to the Higgs mass that cancel its dependence on $\Lambda$. Needless to say, solving the hierarchy problem comes at the cost of introducing many new degrees of freedom in the model. A lot of predictive power is lost. Moreover supersymmetry has to be broken at the TeV scale at least to explain why so many extremely intensive searches have not resulted in the detection of one single sparticle.

In this section we want to study inflation models in the context of local supersymmetry, supergravity, as a study of inflation cannot leave out gravity. The main advantage of working in supergravity is that we can now explicitly compute the coefficients of higher order operators, rather than guessing their form from general dimension analysis. Also, supergravity (sugra) is the low energy limit of string theory, so one could argue that sugra inflation models are compatible with quantum gravity.

A supergravity theory is defined by its Kähler potential $K$ and scalar potential $W$, which are functions of the complex superfields present in the theory \(^8\). $K$ contains information about the field space metric. $W$ contains the superfield interactions. In this thesis we will often make use of the so-called Kähler function $G$, defined as

$$G \equiv K + \ln W. \quad (1.66)$$

In terms of $G$ the F-term scalar potential reads

$$V_F = e^G [G_I G_J G_J - 3], \quad (1.67)$$

where the sum is over all fields in the problem. In most of this thesis we will not consider the D-term contributions (from gauge interactions) to the scalar potential.

\(^8\)In this introductory section we will take the gauge kinetic function $f$ to be canonical.
CHAPTER 1. COSMOLOGY AND INFLATION

The form of (1.67) shows that to get the positive energy density needed for inflation, we should for at least one of the superfields have $G_I \neq 0$, which means that this field has to break supersymmetry. This shows that, unlike the case for the Higgs field, supersymmetry alone cannot protect the inflaton mass. It is also clear from (1.67) that all fields are coupled to each other. Therefore it seems even more challenging to have a light inflaton field, as its mass will typically get large corrections from interactions with the other fields. These have to be heavy to be stabilized during inflation. We will study this question in chapter 5. At the other hand, this could be a blessing as well. It is often argued that the coupling of the light inflaton field to heavy other fields can be used to probe Planck scale physics from the inflationary observables. For example, in scenarios in which the inflaton makes a sharp turn through field space heavy modes can get temporarily excited, leaving small but possibly detectable features in the power spectrum [27].

In a sugra context, the $\eta$-problem is still there. If we expand the Kähler potential around $X_0$, the inflaton field value during inflation, in $\delta X = X - X_0$, we get $K = K_0 + K_{XX} |\delta X|^2 + ... = K_0 + |\Phi|^2 + ...$, with $|\Phi|$ the canonically normalized complex field. The scalar potential then gives $V_F = e^{\Phi^2} [V_0 + ...]$. (1.68)

With the inflaton some linear combination of the real and imaginary parts of $\Phi$, it is clear that the exponent in (1.68) contributes order unity: $\eta \approx 1 + ...$, which spoils inflation.

In small field sugra inflation models, the $\eta$-problem cannot be solved by introducing a symmetry that keeps the inflaton direction in field space flat. For example, when working with a Kähler potential $K = K(\Phi - \bar{\Phi})$ one sees that after Taylor expanding around the extremum and performing a Kähler transformation with an arbitrary analytical function $f$ (which leaves $G$ and $V$ invariant)

$$K \to K + 2 \text{Re } f, \quad W \to W \exp (-f),$$

the symmetry is lost. Therefore, we need to tune parameters again, just like in the non-supersymmetric case. We will see explicit examples of small field sugra inflation models in chapter 5.

In large field sugra inflation models, tuning parameters is not an option given the enormous length of the inflaton’s trajectory. However, this is the perfect environment for keeping inflaton field directions flat by introducing shift symmetries (translation symmetries). Such models will be introduced in chapter 6.

Sugra hybrid inflation, first introduced in [28], can also circumvent the $\eta$-problem by invoking a shift symmetry in the Kähler potential. An explicit model of sugra hybrid inflation will be reviewed in chapter 7.

1.7 Higgs inflation

Recently there has been a lot of interest in models that employ the Standard Model Higgs field as the inflaton [29, 30]. The beauty of this model is its simplicity: to get inflation it suffices to add a non-minimal coupling between gravity and the Higgs field. The resulting “Jordan” frame action reads (in $(+ --)$ metric, omitting the standard kinetic gauge terms)

$$S_J = \int d^4x \sqrt{-g} \left[ - \left( \frac{M^2}{2} + \xi \mathcal{H}^* \mathcal{H} \right) R(\tilde{g}_{\mu\nu}) + \tilde{g}^{\mu\nu}(D_\mu \mathcal{H})^\dagger (D_\nu \mathcal{H}) - \lambda (\mathcal{H}^\dagger \mathcal{H} - \frac{v^2}{2})^2 \right],$$

(1.70)

Note that we use the same notation for superfields and their component (scalar) fields.
where $\mathcal{H}$ denotes the complex Higgs doublet. In this frame we have the standard Mexican hat potential. To get to the “Einstein frame” in which we have canonical gravity terms one performs a conformal transformation from the Jordan frame metric $\hat{g}_{\mu\nu}$ to the Einstein frame metric $g_{\mu\nu}$:

$$g_{\mu\nu} = \omega^2 \hat{g}_{\mu\nu}. \quad (1.71)$$

After some algebra this gives

$$S_E = \int d^4x \sqrt{-g} \left[ -\frac{M_p^2}{2} R(g_{\mu\nu}) + \frac{2}{M_p^2} \left( \omega^2 \nabla_{\mu} \nabla_{\nu} \omega - 6 g^{\mu\nu} \omega \nabla_{\mu} \nabla_{\nu} \omega \right) \right], \quad (1.72)$$

with $\nabla$ the covariant derivative based on the metric $g_{\mu\nu}$. Now we see how we should pick $\omega$ in order to have a canonical gravitational term:

$$\omega^{-2} = 1 + 2 \xi \left( \frac{1}{2} H^* H - \frac{v^2}{2} \right). \quad (1.73)$$

Some more algebra then gives

$$S_E = \int d^4x \sqrt{-g} \left[ -\frac{M_p^2}{2} R(g_{\mu\nu}) + g^{\mu\nu} \left( 3 \omega^4 \frac{\xi^2}{M_p^2} \partial_{\mu} (H^* H) \partial_{\nu} (H^* H) 
+ \omega^2 (D_{\mu} H)(D_{\nu} H) - \omega^4 \lambda (H^* H - \frac{v^2}{2})^2 \right) \right]. \quad (1.74)$$

Indeed we are back to Einstein gravity. To check the form of the kinetic terms we put in $H = \frac{1}{\sqrt{2}} (\theta_1 + i \theta_2)$, $\phi_R(t) \equiv \phi + h$. As before, $\phi = \phi(t)$ is the background field, the quantum Higgs field is $h(t, \vec{x})$, $\theta_i(t, \vec{x})$ are the Goldstone bosons. For the kinetic terms we get

$$S_{E}^{(\text{kin})} = \frac{1}{\sqrt{2}} \left( \theta_1 + i \theta_2 \right), \quad (1.75)$$

with $\chi_i = (\phi_R, \theta_i)$. Now the last step is to transform to a field that has canonical kinetic terms. Considering only the background field $\phi$ we want to transform to a field $\tilde{\phi}$ defined via

$$\omega^2 \left[ \delta_{ij} + 6 \omega^2 \frac{\xi^2}{M_p^2} \phi^2 \right] \partial_{\mu} \phi \partial^{\mu} \partial_{\mu} \phi = \partial_{\mu} \tilde{\phi} \partial^{\mu} \tilde{\phi}. \quad (1.77)$$

In the small field regime $\phi < M_p/\xi$ we get $\tilde{\phi} \simeq \phi$ so the Mexican hat potential remains. In the mid field regime $M_p/\xi < \phi < M_p/\sqrt{2}$ we find $\tilde{\phi} \simeq \frac{M_p}{\sqrt{2}} \left( 1 - \frac{\sqrt{3}}{2} \right)$. In the large field regime $\phi > M_p/\sqrt{2}$, where inflation should take place we get $\tilde{\phi} \simeq M_p \left( \sqrt{6} \ln \left( 2 \frac{\phi}{M_p} \right) + \sqrt{3} \right)$. Then it follows that in this large field regime the potential written in terms of the canonically normalized field $\tilde{\phi}$ is given by

$$V(\tilde{\phi}) = \frac{\lambda M_p^4}{4 \xi^2} \left( 1 - \frac{2}{\sqrt{6}} \left( \frac{\tilde{\phi}}{M_p} - \frac{\sqrt{3}}{2} \right) \right), \quad (1.78)$$
CHAPTER 1. COSMOLOGY AND INFLATION

where we omitted the negligible the SM vev $v$. At the end of the day, the effect of the non-minimal coupling is an effective flattening of the Higgs potential. One finds that for $\xi \approx 700 - 10^4$ the resulting potential can support inflation. For all these allowed values for $\xi$, the predictions $n_s \approx 0.96$ and $r \leq 0.01$ are in perfect agreement with the Planck results (1.51) and (1.52).

A drawback of the mechanism is, apart from the usual tuning needed in non-supersymmetric inflation models, that the Higgs mass found at the LHC seems to be just too light for Higgs inflation, which requires $m_H \gtrsim 129$ GeV. However, this bound depends heavily on the top mass and might still shift in the future [31, 32]. Another problem is that in the regime $M_p/\xi < \phi < M_p/\sqrt{\xi}$ the theory becomes for a short while explicitly dependent on its UV completion [30], which goes of course quite against the spirit of the model. To cure these problems it has been suggested to implement Higgs inflation in a supergravity framework [33, 34, 35]. However, now the appealing minimality of the model is lost. Finally, a modification that includes a dilaton in the spectrum has been considered in [36]. Such a scenario has the power to explain both inflation and the late-time acceleration of the universe.

While more research on Higgs inflation is on its way, we will use the current framework as a motivation to compute in chapters 3 and 4 the effective action in models where the Higgs background field value changes in time.