Chapter 4

Extension to FLRW

All results obtained in the previous chapter assumed a Minkowski flat spacetime. As cosmologists, we should consider an expanding Friedmann-Lemaître-Robertson-Walker universe, and see how the expansion of the universe affects the effective action just obtained. That will be the purpose of this chapter, which is based on our work [4].

We now work in a FLRW background:

\[ g_{\mu\nu} = a^2(\tau) \text{Diag}(1, -1, -1, -1) = a^2(\tau)\eta_{\mu\nu}. \]  

(4.1)

We work with conformal time \( \tau \), which is related to cosmic time \( t \) via \( d\tau = \frac{dt}{a} \), as was already discussed in the introduction.

4.1 Real scalar field

Let us, again, consider the case of a real scalar field first, to warm up for the \( U(1) \) Abelian Higgs model. The action for a scalar field in a potential \( V = \frac{1}{2}m^2\phi^2 \) in arbitrary metric \( g_{\mu\nu} \) is

\[ S = \int d^4x \sqrt{-g} \left[ \frac{1}{2}g^{\mu\nu}\partial_\mu \phi \partial_\nu \phi - \frac{1}{2}m^2\phi^2 \right]. \]  

(4.2)

In conformal FLRW this becomes

\[ S = \int d^4x (-a^2) \cdot \frac{1}{2}\phi \left[ \eta^{\mu\nu}\partial_\mu \partial_\nu \phi + 2H \partial_\eta + m^2a^2 \right] \phi. \]  

(4.3)

Here we have done a partial integration and dropped the surface term:

\[ \int d^4x \sqrt{-g}g(\partial_\mu \phi)g^{\mu\nu}\partial_\nu \phi = - \int d^4x \phi \partial_\mu \left( \sqrt{-g}g^{\mu\nu}\partial_\nu \phi \right). \]  

(4.4)

In cosmic FLRW, \( g_{\mu\nu} = \text{Diag}(1, -a^2(t), -a^2(t), -a^2(t)) \), we would have found

\[ S = \int d^4x (-a^3) \cdot \frac{1}{2}\phi \left[ \partial_t^2 - \frac{\nabla^2}{a^2} + 3H \partial_t + m^2 \right] \phi. \]  

(4.5)
The equation of motion for the scalar field in conformal or cosmic FLRW follows directly from the two equations above.

We stick to the conformal FLRW metric and define \( \tilde{\phi} \equiv a \phi \).

\[
L = -\frac{1}{2} \dot{\tilde{\phi}} \left[ g^{\mu\nu} \partial_\mu \partial_\nu + m^2 a^2 - \frac{a''}{a} \right] \tilde{\phi}, \tag{4.6}
\]
with \( a' \equiv \partial_\tau a \).

In this “conformal frame” the Lagrangian looks very familiar, it is a scalar field with a time dependent shift in its mass. Therefore its effective action \( \Gamma \) follows straight from, for example, (2.71):

\[
\Gamma = \int d^4x \left[ -\frac{1}{2} \tilde{\phi} \left[ g^{\mu\nu} \partial_\mu \partial_\nu + m^2 a^2 - \frac{a''}{a} \right] \tilde{\phi} - m^2 a^2 - \frac{a''}{a} \Lambda^2 + \frac{(m^2 a^2 - \frac{a''}{a})^2}{32 \pi^2} \ln \left( \frac{\Lambda}{m} \right) \right]. \tag{4.7}
\]

In this expression the cut-off \( \Lambda \) has been taken on the magnitude of the spatial three-momentum \( \vec{k} \), for which we have

\[
\omega_k^2 - \vec{k} \cdot \vec{k} = m^2 - \frac{a''}{a}. \tag{4.8}
\]

However: what are these \( \vec{k} \)? They are comoving momenta, just because they have been defined as conjugated to the conformal \( \vec{x} \). To see this in a better way, we divide the previous equation by \( a^2 \):

\[
\left( \frac{\omega_k}{a} \right)^2 - \frac{\vec{k}}{a} \cdot \frac{\vec{k}}{a} = m^2 - \frac{a''}{a^3}. \tag{4.9}
\]

On the left hand side are now physical quantities, that have the usual norm \( m^2 \) plus a correction term. In short, we want to set the cut off at

\[
\Lambda = \frac{|\vec{k}|}{a} \tag{4.10}
\]

which means that (as \( \tilde{\Lambda} \) was a cut-off on \( |\vec{k}| \)) we have to set \( \tilde{\Lambda} = a \Lambda \). (Or in other words: the physical quantity is \( k/a \), so we integrate \( k \) up to \( a \Lambda \).) Going back to the physical frame by reinserting \( \phi \) we get

\[
\Gamma = \int dt \int d^3x \left[ -\frac{1}{2} \partial_\tau \phi \left[ g^{\mu\nu} \partial_\mu \partial_\nu + 2 H \partial_\tau + m^2 a^2 \right] \phi - a^4 \left[ \frac{m^2 - \frac{a''}{a}}{16 \pi^2} \Lambda^2 - \frac{(m^2 - \frac{a''}{a})^2}{32 \pi^2} \ln \left( \frac{a \Lambda}{m} \right) \right] \right]. \tag{4.11}
\]

This result has been known for a long time \([67, 68, 69, 70]\). If the scalar is coupled to other scalars or to fermions via e.g. a Yukawa interaction, additional scalar and fermion loops contribute \([68, 71, 72]\). In this chapter we extend these results by including a coupling to a gauge field.

N.B. Working in cosmic time, one can rewrite the action (4.5) in terms of \( \tilde{\phi} \equiv a^{3/2} \phi \), which yields

\[
L = -\frac{1}{2} \tilde{\phi} \left[ \partial_\tau^2 - \frac{\nabla^2}{a^2} + m^2 - \frac{3 a''}{2 a} - \frac{3}{4} \left( \frac{\dot{a}}{a} \right)^2 \right] \tilde{\phi}. \tag{4.12}
\]

Now the shift in the mass is different. However, this field \( \tilde{\phi} \) does not have canonically normalized kinetic terms. It is only in conformal FLRW where the action resembles the Minkowski action, and where the results of the previous chapter can be applied directly, as we did in (4.7).
4.2 Action

In section 3.2 we have computed the Lagrangian of the U(1) Abelian Higgs model in a Minkowski background. In this section we generalize the result (3.15) to FLRW. Following the previous section, we will work in conformal coordinates.

Using the conformal FLRW metric (4.1), we get for the nonzero connections

\[ \Gamma^i_{i0} = \Gamma^i_{0i} = \Gamma^0_{i0} = \Gamma^0_{ii} = H. \] (4.13)

Here we defined \( H = \dot{a}/a \), analogous to the usual definition in coordinate time \( H = \dot{a}/a \). We can again decompose the charged scalar field into a real and imaginary part,

\[ \Phi(x) = \frac{1}{\sqrt{2}} \left( \phi(\tau) + h(\tau, \vec{x}) + i\theta(\tau, \vec{x}) \right), \] (4.14)

with \( \phi(\tau) \) the time dependent classical background field.

For the action we can directly generalize (3.5):

\[ S = \int d^4x \mathcal{L}, \quad \mathcal{L} = \mathcal{L}_{\text{gauge-kin}} + \mathcal{L}_{\text{higgs-kin}} + \mathcal{L}_{\text{pot}} + \mathcal{L}_{\text{gaugefixing}} + \mathcal{L}_{\text{ghost}}, \]

\[ = -\frac{1}{4} g^{\mu\nu} g^{\alpha\beta} F_{\mu\nu} F_{\alpha\beta} + g^{\mu\nu} D_\mu \Phi (D_\nu \Phi)^\dagger - V(\Phi \Phi^\dagger) - \frac{1}{2\xi} G^2 + \frac{a}{a} \xi \mathcal{G}. \] (4.15)

The gauge-fixing function \( G \) is now given by \( G = G = g^{\mu\nu} \nabla_\mu A_\nu - \xi g(\phi(\tau) + h)\theta \). Note that \( \nabla_\mu g^{\mu\nu} = 0 \) (because of metric compatibility), and thus \( g^{\mu\nu} \nabla_\mu A_\nu = \nabla_\mu g^{\mu\nu} A_\nu \) and there is no ambiguity.

To make explicit all factors of the scale factor we now write \( g^{\mu\nu} = a^{-2} \eta^{\mu\nu} \) with \( \eta^{\mu\nu} \) the Minkowski metric. If we again rescale fields as we did in the previous section, we get to the conformal frame, in which the metric is Minkowski. Before we needed \( \phi \equiv a\phi \), now we define

\[ \hat{\phi}_\alpha = a\phi_\alpha, \quad \hat{V} = a^4 V(\hat{\phi}), \] (4.16)

with \( \phi_\alpha = \{ \phi, h, \theta, \eta \} \) the scalars in the theory. We denote all mass scales in these comoving coordinates with a hat.

The hatted fields are canonically normalized in the comoving frame, just as the field \( \tilde{\phi} \) in the previous section. Since the gauge field kinetic terms are conformally invariant, there is no rescaling of the gauge field. The comoving fields feel a potential \( \hat{V} \). All the comoving quantities map directly to the equivalent set-up in Minkowski, and we can use the usual Minkowski machinery to calculate Feynman diagrams, as we showed in the previous section for the case of a real scalar field. In the expressions below, all indices are raised and lowered using the Minkowski metric.

The action (4.15) is expanded in quantum fluctuations around the background. Here we state the results at each order; for details see appendix D. The classical action that contains no quantum fields reads now (D.11)

\[ S^{(0)} = \int d^4x \left\{ \frac{1}{2}(\dot{\phi})^2 - \frac{1}{2}(\partial_\tau \phi)^2 + \frac{a''}{2} \phi^2 - a^4 V \right\}. \] (4.17)

At first order in the quantum field we get (D.12)

\[ S^{(1)} = \int d^4x \left\{ - \tilde{h} \left( \left( \partial^2 - \frac{a''}{a} \right) \phi + a^3 V \phi \right) \right\}. \] (4.18)
The second order action, from which we will derive the propagators (plus some explicit two point interactions), is given by (D.13)

\[
S^{(2)} = \frac{1}{2} \int d^4x \left\{ A_\mu \left[ (\partial_\nu \partial^\nu + g^2 \phi^2 \eta^{\mu\nu} - (1 - \frac{1}{\xi}) \partial^\mu \partial^\nu \right] A_\nu \\
+ \frac{2}{\xi} \left[ A_0 (H' - 2H) A_0 - A_0 2H \partial^\mu A_\mu \right] \\
- \tilde{\theta} (\partial^2 - \frac{a''}{a} + a^2 V_{\theta \theta} + \xi g^2 \phi^2 \tilde{\theta}) - 4a A_0 \tilde{\theta} \left( \partial_\tau - \frac{a'}{a} \right) \phi_{c1} \\
- \hbar (\partial^2 - \frac{a''}{a} + a^2 V_{hh}) \hbar - 2\hbar \left[ \partial^2 - \frac{a''}{a} + \xi g^2 \phi^2 \right] \phi_{c1} \right\}.
\]

(4.19)

Here we see that the expansion of the universe does not only shift all masses. It also creates a degeneracy between the masses of \( A_0 \) and \( A_1 \), and moreover it introduces a new interaction between \( A_0 \) and \( A_1 \).

For the third order action we get (D.14)

\[
S^{(3)} = \int d^4x \left\{ - S_{\alpha\beta\gamma} a V_{\alpha\beta\gamma} \phi_{c1} + 2g A^4 \tilde{\theta} \left( \partial_\mu - \frac{a'}{a} \partial_\mu \right) \hbar + g^2 (A^2 - \xi \phi^2 - 2\xi \hbar) \phi_{c1} \hbar \right\},
\]

(4.20)

with \( \phi_{\alpha} = \{ h, \theta \} \), and \( S_{\alpha\beta\gamma}(\phi) \) symmetry factors following from a Taylor expansion of \( V(\Phi^1 \Phi) \) around \( \Phi = \phi_{c1} \).

Note that in the Minkowski limit (where the conformal frame reduces to Minkowski spacetime) the results (4.17), (4.18), (4.19) and (4.20) reduce to what we found in (3.15).

The Feynman rules that follow from this action are in figure 4.1.

The one point vertex \( \tilde{\lambda}_h^+ \) in figure 4.1 follows directly from (4.18):

\[
\tilde{\lambda}_h^+ = -i \left[ (\partial^2 - \nabla^2 - (H' + H^2)) \phi_{c1} + V_{\phi_{c1}} \right] = -i \left[ a^3 \left( 3 \phi_{c1} \nabla^2 - 3h \phi_{c1} + V_{\phi_{c1}} \right) \right].
\]

(4.21)

(Note that \( H' + H^2 = \frac{a''}{a} \))

To write the next four two point vertices in figure 4.1 we first read off the time dependent masses from (4.19)

\[
\begin{align*}
\dot{m}_h^2 &= \tilde{V}_{hh} - (H' + H^2) = a^2 \left[ V_{hh} - (\dot{H} + 2H^2) \right], \\
\dot{m}_\theta^2 &= \tilde{V}_{\theta\theta} + \xi g^2 \phi^2 - (H' + H^2) = a^2 \left[ V_{\theta\theta} + \xi g^2 \phi^2 - (\dot{H} + 2H^2) \right], \\
\dot{m}_\phi^2 &= \xi g^2 \phi^2 - (H' + H^2) = a^2 \left[ \xi g^2 \phi^2 - (\dot{H} + 2H^2) \right], \\
\dot{m}_{A_\mu A_\nu}^2 &= -\eta^{\mu\nu} a^2 \left[ g^2 \phi^2 \right] = -\eta^{\mu\nu} a^2 \left[ g^2 \phi^2 \right] \equiv -\eta^{\mu\nu} a^2 m_A^2,
\end{align*}
\]

(4.22)

where we used \( H^2 = a^2 H^2 \) and \( H' = a^2 (\dot{H} + 2H^2) \). These masses we will again split up in a time independent background part, which is used to construct the propagators, and a time dependent part that shows up in the interactions in figure 4.1:

\[
\dot{m}^2(\tau) = \hat{m}^2 + \delta \hat{m}^2(\tau).
\]

(4.23)
4.2. ACTION

The split is again defined by requiring the interaction to vanish at the initial time, which we choose without loss of generality to be at \( t_0 = 0 \):

\[ \delta m^2(0) = 0. \] (4.24)

We will comment further on this split, and the resulting initial conditions, in subsection 4.4.2.

The corresponding three point vertices follow directly from (4.20) and read

\[ \hat{\lambda}^+_{h,h,h} = -i \hat{V}_{\phi_{cl} h h} = -i \partial_{\phi_{cl}} \hat{m}_h^2, \]
\[ \hat{\lambda}^+_{\theta, \theta, \theta} = -i \left( \hat{V}_{\phi_{cl} \theta \theta} + 2 \xi g^2 \phi_{cl} \right) = -i \partial_{\phi_{cl}} \hat{m}_\theta^2, \]
\[ \hat{\lambda}^+_{\eta, \eta, \eta} = -2 i \xi g^2 \phi_{cl} = -i \partial_{\phi_{cl}} \hat{m}_\eta^2, \]
\[ \hat{\lambda}^+_{A_\mu, A_\nu} = 2 i \eta^{\mu \nu} g^2 \phi_{cl} = i \eta^{\mu \nu} \partial_{\phi_{cl}} \hat{m}_A^2. \] (4.25)

Figure 4.1: Feynman rules for the Abelian Higgs model in FLRW (in the conformal frame).
Next are the extra interactions between the gauge fields, that we call $\hat{\lambda}_{A_0}^{+}$ and $\hat{\lambda}_{A_i}^{+}$, that are absent in Minkowski. From (D.13) we get

$$\delta \hat{m}_{A_0}^2 = \frac{2}{\xi} \left( H' - 2H^2 \right)$$

$$\delta \hat{m}_{A_i}^2 = \frac{2}{\xi} H \partial^i.$$  

(4.26)  

(4.27)

Finally we have the interactions between $A_0$ and $\theta$, caused by the rolling of the background field and therefore also present in Minkowski. From (4.19) and (4.20) we get

$$\delta \hat{m}_{A_0}^2 \hat{\theta} = 2g (\partial \tau - H) \hat{\phi}_{cl} = a^2 \left[ 2g \hat{\phi}_{cl} \right],$$

$$\hat{\lambda}_h A_i \hat{\theta} = 2ig (\partial + H \partial \hat{h}).$$  

(4.28)

where we did the same partial integration as described at the very end of section 3.2.

Equation (4.20) also contains a term $-2g A_i \hat{\theta} \partial^i \hat{h}$. Since the final expression of each tadpole graph is independent of the spatial coordinates, this three point interaction does not contribute to the overall result. We have checked this by explicit computation.

### 4.3 Effective equation of motion

We again choose to perform the computation on the level of the effective equation of motion $A$, so we will be computing corrections to the classical equation of motion (Weinberg’s tadpole method, we now compute $\langle h^+ \rangle$). We want to organize the computation in the “perturbative” way that we also employed in sections 2.7.1 and 3.3: time independent part of the masses in the propagators, time dependent part in the interactions. In the next section we will integrate the result to get to the effective action, just like we did at the end of section 3.3. This action is then transformed from the comoving to the physical frame, thereby obtaining the effective action.

The calculation is done in the conformal frame, in terms of hatted fields and mass scales, conformal time and momenta. For notational convenience, in this section we drop the hat on all quantities; it shall be reinstated at the end when we give the results.

The calculation is analogous to the one for a Minkowski background done in the previous chapter, but with two-point interactions (4.22) that now depend on the FLRW scale factor. This is straightforward to incorporate for the diagrams with a scalar running in the loop. There are however some new technical difficulties that come in with the gauge boson loops:

1. The mass of the temporal gauge boson gets FLRW corrections (described by the vertex $\lambda_{A_0}$) but the mass of the spatial components does not. This is possible because Lorentz symmetry is broken by the time dependent background. Consequently the diagrams with $A_0$ and $A_i$ contribute differently.

2. The off-diagonal gauge boson two point interaction is non-zero, which results in the vertex $\lambda_{A_0 A_i}$. Moreover, it contains an extra spatial derivative. This leads to new diagrams with both two and three two point insertions.
4.3. EFFECTIVE EQUATION OF MOTION

\[ 0 = A = h^+ \lambda_h^+ \Delta_{\alpha\beta} + A_{c1} + h^+ \lambda_{h\alpha\beta} \Delta_{\alpha\beta}, \]

\[ + h^+ \lambda_{h\alpha\beta} \Delta_{\alpha\beta} + A^{(2)} \]

\[ + h^+ \lambda_{h\alpha\beta} \Delta_{\alpha\beta} + A^{(3)} \]

Figure 4.2: Tree level tadpole giving the classical equation of motion and the first, second and third order diagrams respectively. The summation is over all fields, for the gauge bosons also over Lorentz indices, and over \( \pm \) at the two point vertices.

3. The formalism is set up in such a way that the two point interactions vanish at the initial time (4.24). This avoids divergences that depend on the initial conditions. We will argue in subsection 4.4.2 that this is always an allowed choice, for arbitrary initial conditions, provided the initial vacuum is chosen accordingly.

We will again extract the one-loop equation of motion from the series of tadpole diagrams with one external \( h^+ \) leg depicted in figure 4.2. So we have

\[ 0 = A = A_{c1} + A^{(1)} + A^{(2)} + A^{(3)} + \text{finite} \tag{4.29} \]

where \( A_{c1} \) denotes the first tadpole diagram, without a quantum loop, that gives the classical equation of motion, and \( A^{(i)} \) stands for the tadpole diagram with \( i \) vertices.

So let us begin at the classical level. From (4.21) we find directly (remember we dropped the hat for conformal coordinates and scales):

\[ A_{c1} = i\lambda_h^+ = \partial_\mu \partial^\mu \phi_{c1} - (H'H + H^2)\phi_{c1} + V_{\phi_{c1}}. \tag{4.30} \]

and setting \( A_{c1} = 0 \) yields the classical equation of motion. Note that we are still working with the conventions defined above (2.40).

Now for the correction diagrams. We divide the calculation based on the order of the contributing graphs, which is the number of vertices in the loop of the tadpole. As discussed above, we must work to third order. Independent of this, we can distinguish three classes of diagrams depending on how they contribute to the answer. First there is the contribution that is fully analogous to the Minkowski calculation \( A_{\text{Mink}} = A_{\text{Mink}}^{(1)} + A_{\text{Mink}}^{(2)} \); the only difference is that the mass term of the scalars now depends on the scale factor. Second is \( A_{\text{mass}} = A_{\text{mass}}^{(2)} \), which arises from the extra Feynman diagrams due to the FLRW mass correction of the temporal gauge field \( \delta m^2 A_0 A_0 \); see (4.26). And finally there is \( A_{\text{mix}} = A_{\text{mix}}^{(2)} + A_{\text{mix}}^{(3)} \), the diagrams with one and two off-diagonal vertices \( \delta m^2 A_0 A_i \), connecting the temporal and spatial gauge fields (see (4.27)), also absent in Minkowski.
4.3.1 First order contribution $A^{(1)}$

The calculation of the first order diagrams proceeds analogously to the equivalent calculation in Minkowski, done in the previous section. At first order, four diagrams contribute, with $\psi_\alpha = \{h, \theta, \eta, A^\mu\}$ running in the loop. The result only depends on the time independent part of the two point interaction, as there is no vertex insertion. For each diagram the result has the same structure, given by (compare to (2.41) and (3.22))

$$ (A^{(1)}_{\text{Mink}})_\alpha = i \frac{1}{2} \lambda_{h\alpha\alpha}^+ \Delta^{++}_\alpha (x - x). \quad (4.31) $$

Just as in Minkowski the gauge loop can be expressed in terms of scalar propagators (C.3) via

$$ \eta^{\mu\nu} \Delta^{++}_{A_\mu A_\nu} (0) = -3 \Delta^{++}_A (0) - \xi \Delta^{++}_{\xi} (0). \quad (4.32) $$

The sum of all first order diagrams is, just as in the Minkowski case in (3.22) and (3.25)

$$ A^{(1)}_{\text{Mink}} = \frac{1}{2} \sum_\alpha S_\alpha (\partial_\phi m^2_\alpha) \frac{1}{4 \pi^2} \int_0^\Lambda k^2 dk \left[ \frac{1}{k} - \frac{1}{2} \frac{\bar{m}^2_\alpha}{k^3} + \ldots \right] $$

$$ = \frac{1}{16 \pi^2} \sum_\alpha S_\alpha (\partial_\phi m^2_\alpha) \left[ \Lambda^2 - \bar{m}^2_\alpha \ln(\Lambda/\bar{m}) + \text{finite} \right]. \quad (4.33) $$

In the momentum integrals here and below, the variable $k$ is the comoving momentum, $\Lambda$ is a comoving cutoff, and we have $k < \Lambda$. (Recall that graphs in this section in the comoving frame, and all quantities are actually hatted quantities. The cutoff regularisation we apply here is equivalent to a physical cutoff on physical momentum.) The sum is over $\alpha = \{h, \theta, \eta, A, \xi\}$, and $S_\alpha = \{1, 1, -2, 3, 1\}$ counting the real degrees of freedom (with a minus sign for the anticommuting ghost). Note that the factor $\partial_\phi m^2_\alpha$ is time dependent, and evaluated at $\tau$; hence $A^{(1)}_{\text{Mink}}$ is a function of $\tau$. The finite terms that we have neglected remain finite as $\Lambda \to \infty$.

4.3.2 Second order contribution $A^{(2)}$

$A^{(2)}_{\text{Mink}}$

Here we can straightaway translate the Minkowski result we found in (3.26), (3.29) and (3.33). Once we are in the conformal frame there is nothing new in the computations, we just have some shifted masses to insert. We get

$$ A^{(2)}_{\text{Mink}} = -\frac{1}{16 \pi^2} \sum_\alpha S_\alpha (\partial_\phi m^2_\alpha) \delta m^2_\alpha \ln(\Lambda/\bar{m}) - \frac{3 + \xi}{32 \pi^2} (-i \lambda_{h, \xi A_0}^+) \delta m^2_{A_0, \theta} \ln(\Lambda/\bar{m}) + \text{finite}. \quad (4.34) $$

As before: the sum is over $\alpha = \{h, \theta, \eta, A, \xi\}$, and $S_\alpha = \{1, 1, -2, 3, 1\}$.

$A^{(2)}_{\text{mass}}$ and $A^{(2)}_{\text{mix}}$

The loop with $\lambda_{A_\alpha A_\beta}$ proceeds analogously to the gauge loop computed in section 3.3. We need to set $\rho = \sigma = 0$ in (3.27). When doing the computation, sketched in appendix C.2.1, one finds that now that
the field running in the loop can only be the temporal component of the gauge field, there is an additional factor of $\frac{1}{4}$ in the final answer (compare to (3.29)). We get

$$A_{\text{mass}}^{(2)} = i \int d^4 y \frac{1}{2} \lambda^+_\alpha A_\mu A_\nu (x^0) \left[ \Delta_{A_\mu A_\nu}^{++} (x-y) \lambda^+_{\delta A_0 A_0} (y^0) \Delta_{A_0 A_\mu}^{++} (y-x) + \Delta_{A_\mu A_0}^{+-} (x-y) \lambda^+_{\delta A_0 A_0} (y^0) \Delta_{A_0 A_\mu}^{+-} (y-x) \right]$$

$$= \ldots$$

$$= - (\partial_{\mu \nu} m^2_A (\tau)) \frac{\delta m^2_{A_0 A_0} (\tau)}{64 \pi^2} (3 + \xi^2) \ln(\Lambda/\bar{m}) + \text{finite.} \quad (4.35)$$

The off-diagonal interaction $\lambda_{\delta A_0 A_1}$ contains a spatial derivative, and brings down a factor of the momentum. Since the insertion is asymmetrical, there is no more reflection symmetry, so we lose the factor of 1/2 that we had in (4.35). In appendix C.2.2 we show that the diagram is given by

$$A_{\text{mix}}^{(2)} = i \int d^4 y \lambda^+_\alpha A_\mu A_\nu (x^0) \left[ \Delta_{A_\mu A_\nu}^{++} (x-y) \lambda^+_{\delta A_0 A_0} (y^0) \Delta_{A_0 A_\mu}^{++} (y-x) + \Delta_{A_\mu A_0}^{+-} (x-y) \lambda^+_{\delta A_0 A_0} (y^0) \Delta_{A_0 A_\mu}^{+-} (y-x) \right]$$

$$= \ldots$$

$$= (\partial_{\mu \nu} m^2_A (\tau)) \frac{3H'(\tau)(1 - \xi^2)}{32 \pi^2 \xi} \ln(\Lambda/\bar{m}) + \text{finite.} \quad (4.36)$$

### 4.3.3 Third order contribution $A^{(3)}$

The third order diagrams with two two point insertions are UV finite, which can be easily checked by power counting. The only exception to this is the diagram with two off-diagonal $\lambda_{A_0 A_1}$ insertions, because each insertion contains a spatial derivative, and thus brings down a power of momentum. We thus consider the third order diagram with two mixed-interaction insertions:

$$A^{(3)}_{\text{mix}} = i \int d^4 y \int d^4 z \frac{1}{2} \lambda^+_\alpha A_\mu A_\nu (x^0) \times \sum \Delta_{A_\mu A_\nu}^{++} (x-y) \lambda^+_{\delta A_0 A_0} (y^0) \Delta_{A_0 A_\mu}^{++} (y-z) \lambda^+_{\delta A_0 A_0} (z^0) \Delta_{A_0 A_\mu}^{++} (z-x)$$

$$= i \int d^4 y \int d^4 z \frac{1}{2} i y^{\mu \nu} (\partial_{\mu \nu} m^2_A (x^0)) \times \sum \Delta_{A_\mu A_\nu}^{++} (x-y) \left( -is(a) \delta m^2_{A_0 A_0} (y^0) \right) \Delta_{A_0 A_\mu}^{++} (y-z) \times \left( -is(b) \delta m^2_{A_0 A_0} (z^0) \right) \Delta_{A_0 A_\mu}^{++} (z-x). \quad (4.37)$$

Here the factor of 1/2 comes from the reflection symmetry between the spacetime points $x$ and $y$. As we want to have two $\delta m^2_{A_0 A_0}$ insertions, the sum is over the four possibilities for the Lorentz indices:

$$(\rho, \sigma, \kappa, \tau) = (i, 0, j, 0), (0, i, 0, j), (0, i, j, 0), (i, 0, 0, j). \quad (4.38)$$

Working in the in-in formalism, the spacetime points $y$ and $z$ can be on the positive or on the negative branch, which gives four possibilities that we should sum over as well (spacetime point $x$ is always taken
on the positive branch):

\((a, b) = (++), (-+), (+-), (--)\).  

(4.39)

Of course, the choice of the branch has consequences for the sign of the Feynman rule. Therefore we used the sign function \(s(a)\) which we define as \(s(+) = 1, s(-) = -1\).

In appendix C.2.3 we show that in the end this diagram yields

\[
A^{(3)}_{\text{mix}} = (\partial_{\phi_0} m_{\Lambda}^2(\tau)) \frac{-6(1 + \xi)}{32\pi^2\xi} \ln(\Lambda/\hat{m}) + \text{finite}. 
\]

(4.40)

### 4.3.4 Summary of graphs

In the previous subsections we have computed all quadratically and logarithmically divergent contributions to the one loop equation of motion. Here we collect and summarize the results, putting the hats back on the relevant variables to indicate that we are still in the conformal frame.

The first order graphs are given by (4.33). The second order contributions are (4.34), (4.35) and (4.36), and are summarized in figure 4.3. At third order there is only one piece, given by (4.40). We now collect these terms into the three groups \(\hat{A}_{\text{Mink}}, \hat{A}_{\text{mass}}\) and \(\hat{A}_{\text{mix}}\).

The first and second order combined \(\hat{A}_{\text{Mink}} = \hat{A}^{(1)}_{\text{Mink}} + \hat{A}^{(2)}_{\text{Mink}}\) is

\[
\hat{A}_{\text{Mink}} = \frac{1}{16\pi^2} \sum\alpha S_{\alpha} \left( \partial_{\phi_{\alpha}} m_{\alpha}^2(\tau) \right) \left[ \hat{\Lambda}^2 - \hat{m}_{\alpha}^2 \ln(\hat{\Lambda}/\hat{m}) \right] - \frac{(3 + \xi)}{32\pi^2} (-i\hat{\lambda}^+_{h, A_0}) \delta m_{\alpha}^2 \ln(\hat{\Lambda}/\hat{m}). 
\]

(4.41)

This agrees with the result (3.34) found in the previous chapter. As expected, this is independent of how the two point interaction is split into a free and interacting term, since the first and second order pieces combine in the sum \(\hat{m}_{\alpha}^2 = \hat{m}_0^2 + \delta m_{\alpha}^2\). For \(A_0\) mass insertions we have the second order piece (4.35)

\[
\hat{A}_{\text{mass}} = - \left( \partial_{\phi_{\alpha}} m_{\alpha}^2(\tau) \right) \delta m_{\alpha, A_0}^2(\tau) \frac{(3 + \xi^2)}{64\pi^2} \ln(\Lambda/\hat{m}) + \text{finite}. 
\]

(4.42)

For the mixed piece we have contributions from second order (4.36) and third order (4.40), giving a total

\[
\hat{A}_{\text{mix}} = \left( \partial_{\phi_{\alpha}} m_{\alpha}^2(\tau) \right) \left( \frac{3H(1 - \xi^2)}{\xi} - \frac{6H^2(1 + \xi)}{\xi} \right) \frac{1}{32\pi^2} \ln(\Lambda/\hat{m}). 
\]

(4.43)

All factors in ((4.41), (4.42), (4.43)) that are time dependent — being the \(\hat{m}^2\)'s, \(\hat{\lambda}^+_{h, A_0}\) and \(H\) — are understood to be evaluated at \(\tau\).

### 4.4 Effective action

The previous section found the effective one loop equation of motion. Now we want to extract an effective action from that (even if we know that formally we can set \(\varphi^+ = \varphi^-, \) for any quantum field \(\varphi, \) only at the level of the equation of motion). As we did around (3.35), we will find the effective action \(\Gamma\) from the Euler-Lagrange prescription, which now reads

\[
\mathcal{A} = \partial_{\tau} \frac{\delta \Gamma}{\delta \partial_{\tau} \phi_{cl}} - \frac{\delta \Gamma}{\delta \phi_{cl}}. 
\]

(4.44)
Figure 4.3: The second order tadpole diagrams and their corresponding mathematical expression (below each graph). These Feynman diagrams are in (conformal) coordinate space, with the left and right vertices at \( x \) and \( y \) respectively. \( x \) is always on the plus-branche, \( y \) can be on both branches, so we sum over \( \alpha = \{+,-\} \). The argument of each of the propagators is \((y-x)\) (we used that \( \Delta^- = \Delta^+ - \Delta \)). All time-dependent quantities are evaluated at \( x^0 = \tau \).

So let us again begin at the classical level. We found \( A_{\text{cl}} \) in (4.30). Reinstating the hats (we were working in the conformal frame) we find the classical action:

\[
\Gamma_{\text{cl}} = \int d^3 x d\tau \left[ -\frac{1}{2} \phi_{\text{cl}} \left( \partial^2 - \nabla^2 - \frac{a''}{a} \right) \dot{\phi}_{\text{cl}} - \dot{V} \right]
\]

\[
= \int d^3 x d\tau \sqrt{-g_{\text{conf}}} \left[ -\frac{1}{2a^2} \phi_{\text{cl}} \left( \partial^2 - \nabla^2 + 2 \frac{a'}{a} \right) \dot{\phi}_{\text{cl}} - \dot{V} \right]
\]

\[
= \int d^3 x dt \sqrt{-g_{\text{phys}}} \left[ -\frac{1}{2} \phi_{\text{cl}} \left( \partial^2 - \nabla^2 + 3H \right) \dot{\phi}_{\text{cl}} - \dot{V} \right],
\]

where we used that \( \mathcal{H}' + \mathcal{H}^2 = a''/a \), and \( H = \dot{a}/a \). In the second line we went to unhatted quantities, and in the third we changed to physical time. The measure in conformal coordinates is \( \sqrt{-g_{\text{conf}}} = a^4 \), and in physical coordinates \( \sqrt{-g_{\text{phys}}} = a^3 \). These results we had already found in (4.5) and (4.6). We can
of course also write
\[ \Gamma_{\text{cl}} = \int d^4x \sqrt{-g} \left[ \frac{1}{2} \partial_{\mu} \phi_{\text{cl}} \partial^{\mu} \phi_{\text{cl}} - V(\phi_{\text{cl}}) \right]. \] (4.46)

Now for the quantum corrections to the classical action. The relevant terms at the level of the equations of motion are summarized in section 4.3.4, and the one loop correction to the effective action is defined as
\[ \Gamma^{1-\text{loop}} = \int d^3x d\tau (\hat{L}_{\text{Mink}} + \hat{L}_{\text{mass}} + \hat{L}_{\text{mix}} + \text{finite}). \] (4.47)

All but one term in \( \hat{A}_{\text{Mink}} \) are polynomial, the exception being the \( \hat{\lambda}^+_{hA\theta} \) term. For this term, the \( \hat{\phi}_{\text{cl}} \) dependent factors are
\[ -i\hat{\lambda}^{+}_{hA\theta} \delta \hat{m}^2_{\text{AdS}} = 4g^2 \left( \hat{\phi}'_{\text{cl}} - \mathcal{H} \hat{\phi}_{\text{cl}} - \mathcal{H}^2 \hat{\phi}''_{\text{cl}} \right). \] (4.48)
This expression follows from performing Euler-Lagrange on a Lagrangian
\[ \mathcal{L} = 2g^2 \left( \hat{\phi}'_{\text{cl}}^2 - 2\mathcal{H} \hat{\phi}_{\text{cl}} \hat{\phi}'_{\text{cl}} + \mathcal{H}^2 \hat{\phi}_{\text{cl}}^2 \right) = \frac{1}{2} \delta \hat{m}^4_{\text{AdS}}. \] (4.49)

For the rest of the terms in \( \hat{A}_{\text{Mink}} \), which are polynomial in \( \hat{\phi}_{\text{cl}} \), the corresponding action is found simply by integrating with respect to \( \hat{\phi}_{\text{cl}} \) and negating. All terms in \( \hat{A}_{\text{mass}} \) and \( \hat{A}_{\text{mix}} \) are also polynomial in \( \hat{\phi}_{\text{cl}} \), so can be similarly integrated. Thus, from (4.41), (4.42) and (4.43), and using (4.49), we obtain
\[
\begin{align*}
\hat{L}_{\text{Mink}} &= -\frac{1}{16\pi^2} \sum_\alpha \mathcal{S}_\alpha \left( \hat{m}_\alpha^2 \hat{\Lambda}^2 - \frac{1}{2} \hat{m}_\alpha^4 \ln(\hat{\Lambda}/\hat{\bar{\mu}}) \right) - \frac{(3 + \xi)}{64\pi^2} \delta m^4_{\Lambda\alpha\theta} \ln(\Lambda/\bar{\mu}), \\
\hat{L}_{\text{mass}} &= -\frac{1}{32\pi^2} \hat{m}_\alpha^2 \ln(\hat{\Lambda}/\hat{\bar{\mu}}) \left( \frac{(3 + \xi)}{\xi} (2\mathcal{H}^2 - \mathcal{H}') \right), \\
\hat{L}_{\text{mix}} &= -\frac{1}{32\pi^2} \hat{m}_\alpha^2 \ln(\Lambda/\bar{\mu}) \left( \frac{3(1 - \xi)}{\xi} \mathcal{H}' - \frac{6(1 + \xi)}{\xi} \mathcal{H}^2 \right),
\end{align*}
\] (4.50)
with \( \alpha = \{ h, \theta, \eta, A, \xi \} \) and \( \mathcal{S}_\alpha = \{ 1, 1, -2, 3, 1 \} \).

Now write the hatted variables in terms of their unhatted counterparts to go back to the physical frame. Use that \( \mathcal{H}' = a^2 \mathcal{H}^2 \) and \( \mathcal{H}' = a^2 (\mathcal{H} + \mathcal{H}^2) \). We can factor four powers of the scale factor out of each term. One of them is used to change conformal time into cosmic time (\( dt = ad\tau \)), the other three are swept into \( \sqrt{-\bar{g}} \). The result is
\[ \Gamma^{1-\text{loop}} = \int d^3x dt \sqrt{-\bar{g}} \left( L_{\text{Mink}} + L_{\text{mass}} + L_{\text{mix}} + \text{finite} \right), \] (4.51)
with
\[
\begin{align*}
L_{\text{Mink}} &= -\frac{1}{16\pi^2} \sum_\alpha \mathcal{S}_\alpha \left( m_\alpha^2 \Lambda^2 - \frac{1}{2} m_\alpha^4 \ln(\Lambda/\bar{\mu}) \right) - \frac{(3 + \xi)}{64\pi^2} \delta m^4_{\Lambda\alpha\theta} \ln(\Lambda/\bar{\mu}), \\
L_{\text{mass}} &= -\frac{1}{32\pi^2} m_\alpha^2 \ln(\Lambda/\bar{\mu}) \left( \frac{(3 + \xi)}{\xi} (\mathcal{H}^2 - \bar{H}) \right), \\
L_{\text{mix}} &= -\frac{1}{32\pi^2} m_\alpha^2 \ln(\Lambda/\bar{\mu}) \left( \frac{3(1 - \xi)}{\xi} (\mathcal{H} + \mathcal{H}^2) - \frac{6(1 + \xi)}{\xi} \mathcal{H}^2 \right).
\end{align*}
\] (4.52-4.54)

Here we also used that whereas \( \hat{\Lambda} \) was a conformal cutoff on conformal three-momentum (equivalent to comoving momentum), \( \Lambda \) is now a physical cutoff on physical three-momentum. In other words: \( \hat{\Lambda} = a\Lambda \), just like in the discussion around (4.10).
4.4. EFFECTIVE ACTION

When we now plug in the FLRW-corrected two point interactions (4.22), we find for the total one loop effective action (up to field independent terms)

\[
\Gamma^{1-\text{loop}} = \frac{-1}{16\pi^2} \int d^3x dt \sqrt{-g} \left[ (V_{hh} + V_{\theta\theta} + 3m_A^2) \Lambda^2 - \left( (V_{hh} - \ddot{H} - 2H^2) + (V_{\theta\theta} - \ddot{H} - 2H^2) \right)^2 + 3m_A^4 
\right. \\
\left. + 2\xi V_{\theta\theta} m_A^2 - (6 + 2\xi)g^2 \phi_{cl}^4 + 6m_A^2 \left( \ddot{H} + 2H^2 \right) \left[ \ln(\Lambda/\bar{m}) \right] \right],
\]

(4.55)

Recall that \( m_A^2 = g^2 \phi_{cl}^2 \). This result is still gauge variant, which was to be expected. Gauge invariance is only achieved on-shell, as we already discussed in section 3.5. The FLRW analogue of (3.38) reads

\[
\int d^4x \sqrt{-g} \delta m_A^4 = \int d^4x 4g^2 a^3 \phi_{cl}^2 = \int d^4x \sqrt{-g} 4m_A^2 V_{\theta\theta}. 
\]

(4.56)

We use this on the last term in (4.52) (or, in fact, on the second term in the third line of (4.55)). On-shell the result (4.55) takes the form

\[
\Gamma^{1-\text{loop}} = \frac{-1}{16\pi^2} \int d^3x dt \sqrt{-g} \left[ (V_{hh} + V_{\theta\theta} + 3m_A^2) \Lambda^2 - \left( (V_{hh} - \ddot{H} - 2H^2) + (V_{\theta\theta} - \ddot{H} - 2H^2) \right)^2 + 3m_A^4 
\right. \\
\left. - 6m_A^2 \left( V_{\theta\theta} - \ddot{H} - 2H^2 \right) \right] \frac{\ln(\Lambda/\bar{m})}{2},
\]

(4.57)

which is gauge invariant, as it should be. Introducing the notation

\[
\tilde{V}_{\alpha\alpha} \equiv V_{\alpha\alpha} - \ddot{H} - 2H^2,
\]

(4.58)

we rewrite the final result, up to field independent terms, as a straightforward generalization of (3.39):

\[
\Gamma = \int d^4x \left[ \frac{1}{2} \partial_{\mu} \phi_{cl} \partial^{\mu} \phi_{cl} - V(\phi_{cl}) 
\right. \\
\left. - \frac{\Lambda^2}{16\pi^2} \left( \tilde{V}_{hh}(t) + V_{\theta\theta}(t) + 3g^2 \phi_{cl}(t)^2 \right) 
\right. \\
\left. + \frac{\ln(\Lambda/\bar{m})}{32\pi^2} \left( \tilde{V}_{hh}(t) + \tilde{V}_{\theta\theta}(t) + 3g^4 \phi_{cl}(t)^4 - 6g^2 \phi_{cl}(t)^2 \tilde{V}_{\theta\theta}(t) \right) \right].
\]

(4.59)

4.4.1 Fermions and additional scalars

It is straightforward to add fermions and additional scalars to the calculation. If these fields are coupled to the Higgs field, and thus have a mass term dependent on \( \phi_{cl} \), they will contribute to the effective equation of motion for the background Higgs field \( \phi_{cl}(t) \) and to the effective action.
We assume the extra scalars are in a basis with canonical kinetic terms and have diagonal masses, and do not mix with $h$. Similarly, we assume the extra fermions have diagonal masses. It is easy to relax these assumptions and generalize the results.

In terms of Feynman diagrams, there are extra tadpole graphs with the additional scalars and fermions running in the loop. The calculation for additional scalars is analogous to that of the Higgs fluctuations $h$ already done, with a contribution at first and second order. The result is

$$
\Gamma_{1\text{loop}}^{(\text{scalar})} = - \frac{1}{16\pi^2} \int d^3x \sqrt{-g} \left[ V_{\chi\chi} \Lambda^2 - \left( V_{\chi\chi} - \dot{H} - 2H^2 \right) \frac{\ln(\Lambda/\hat{m})}{2} \right],
$$

where $\chi$ is the additional real scalar and $V(\chi, \phi)$ its potential.

Just as for the bosons, the tadpole diagrams with a fermion loop can be mapped to the calculation for Minkowski space, except that the “mass” terms now depend on the FLRW scale factor. To discuss the effective action for $\phi$ the second line we went to physical coordinates by factoring out an overall $a^4$ factor, and rewriting the $m_{\psi}^2$ in terms of derivatives with respect to physical time $t$. The first contribution to the logarithmic term incorporates the expansion of the universe. The second contribution to the logarithmic term is because the $\phi_{cl}$ field is rolling, and is also present in Minkowski spacetime.

\[\text{If the fermions are charged under gauge groups, there will be an additional gauge connection. These extra terms do not affect the effective action for } \phi_{cl}, \text{ and for simplicity we leave them out.}\]
Again we can simplify this result by going on-shell. For a fermion mass $m_\psi = \lambda \phi_{cl}$ that is linear in the Higgs field — which is the case for Yukawa interactions and also for gaugino masses in supersymmetric theories — this gives

$$\Gamma^{1\text{-loop}}_{(\text{fermion})} = \frac{1}{16\pi^2} \sum_f \int d^3x \ dt \sqrt{-g} \left[ m_\psi^2 \Lambda^2 - \frac{1}{2} \left( m_\psi^4 - m_\psi^2 \tilde{V}_{\theta\theta} \right) \ln(\Lambda/\bar{m}) \right]. \quad (4.65)$$

Here we have used, again, the background field equations and Goldstone’s theorem. $\tilde{V}_{\theta\theta}$ was defined in (4.58).

### 4.4.2 Initial conditions

Finally, we want to comment on the initial conditions that we have chosen to compute the time dependent effective action. Physically they seem rather peculiar, but we want to argue here that this does not compromise our final result.

Our interactions are time dependent, and thus we needed to define the split between a time independent mass and a time dependent two point interaction (4.24)

$$m_{\alpha\beta}(t) = \bar{m}_{\alpha\beta}^2 + \delta m_{\alpha\beta}(t), \quad \delta m_{\alpha\beta}(0) = 0. \quad (4.66)$$

We furthermore chose initial conditions for $\phi(t)$ and $a(t)$ such that the off-diagonal and Lorentz violating two point interactions vanished completely at the initial time:

$$\delta \phi_{cl}(0) = \delta \phi'_{cl}(0) = \mathcal{H}(0) = \mathcal{H'}(0). \quad (4.67)$$

These choices ensured the simplicity of the propagators. They also ensured the vanishing of the $t = 0$ boundary terms coming from integration by parts when evaluating the loop diagrams in section 4.3. If these boundary terms did not vanish, they would yield extra contributions to the final result, contributions that depend on the initial conditions, and that diverge as $t \to 0$.

Our chosen initial conditions are peculiar, and are not the ones to be used in a realistic situation. The problem in straightforwardly generalizing our calculation to arbitrary initial conditions are the two point interactions $\delta m_{\alpha_0\alpha_0}$, $\delta m_{\alpha_0\alpha_i}$, and $\delta m_{\alpha\beta}^{2\theta}$. To simplify the structure of the free action, and use the standard expressions for the propagator, we have treated them as interactions. To satisfy (4.66) then requires the initial conditions (4.67).

However, in principle there is nothing to stop us from also splitting these two point interactions into a free and interacting part, as in (4.66). Technically, this is complicated, as Lorentz symmetry is broken, and the gauge fields and Goldstone bosons all mix at the initial time. Nevertheless, in principle we can expand all fields in mode functions, as we did for the Minkowski case in sections 2.7.2 and 3.4. The mode functions then satisfy the off-diagonal mode equations (diagonalizing the equations will result in a momentum dependent diagonalization). Then (4.66) is satisfied, all terms depending on the initial conditions vanish, and the results are the same as for our choice of initial conditions (4.67).

In slightly different words, we argue that the result is independent of the initial conditions as long as we choose the initial vacuum to be that of the free theory, which is defined by the split of the quadratic term into a time independent mass and a time dependent interaction mass. That is, solve the mode equations derived from the free action with $\bar{m}_{\beta\beta}^2$, and the corresponding annihilation operators annihilate the vacuum. The different vacua, corresponding to different initial conditions, are then related by a Bogoliubov transformation. For the scalar field theory this was shown by Baacke et al. [73] (see also [74]).
CHAPTER 4. EXTENSION TO FLRW

For a $U(1)$ model we require a more general Bogoliubov transformation, with momentum and polarization dependent coefficients that mix the fields. In principle this should be straightforward, but we will not present any further details here.

In practice, choosing the initial conditions (4.67) simplifies the calculation of the free field mode functions and propagators, and eliminates boundary terms, which is why we choose it. We have argued that a full treatment of initial conditions would yield the same result, at least for the divergent corrections to the equation of motion.

4.5 Discussion

We conclude that in a FLRW universe the effective action depends in exactly the same way as in Minkowski on the masses of all scalar fields in the $U(1)$ Abelian Higgs model. The only novelty is a universal mass shift:

$$m^2 \rightarrow m^2 - (\dot{H} + 2H^2) = m^2 - \frac{1}{a^2} (\dot{H} + H^2) = m^2 - \frac{a''}{a^3}. \tag{4.68}$$

We found the same shift in the scalar mass in the effective action when we considered only the real scalar field, in (4.11). This just reflects that a scalar field feels the expansion of the universe as an extra contribution to its effective mass, which was already shown in (4.6). Therefore, we might have guessed our final answer from directly shifting all scalar mass in the final Minkowski result (3.39), but it has taken this very non-trivial computation to convince ourselves.

The effective action for an Abelian gauge theory in de Sitter space-time has been calculated by [75, 76, 77] using the Landau gauge. More recently the calculation was done in the $R_\xi$ gauge, showing gauge invariance of the effective action [78]. To obtain this result an adiabatic approximation was made which fails in the $\xi \rightarrow 0$ limit. We have extended these results to a generic FLRW spacetime and allow for the possibility of time dependence of the background field, which in a cosmological set-up can be displaced from its potential minimum. For the first time, gauge invariance in general FLRW has been shown.

Our results agree with the expressions in the literature in the appropriate limit. In the limit of a static background field and a constant Hubble parameter our results agree with [78]. In the Minkowski limit we retrieve the effective action calculated in the previous chapter, and also the effective equations of motion found earlier in [51, 54, 55, 66]. Finally, taking both a static background field and a static background we get the familiar Coleman-Weinberg potential [41].

As already mentioned, we have only calculated the UV divergent terms, as these will generically give the dominant contribution. Using a renormalization prescription, these terms (together with the wavefunction renormalization of the gauge field) suffice to derive the renormalization group equations (RGE) and find the RG improved action. An additional task left for the future is to take into full account the backreaction of the scalar on spacetime. Essentially, one must allow for spin-0 fluctuations of the metric, determine their mixing with the scalar, diagonalize to a new basis, and use this basis as the starting point of the calculation. A further generalization is to include a non-minimal coupling to gravity, so as to describe models of Higgs inflation. Finally, one could also generalize the decomposition of $\Phi$ (4.14) to allow for a time dependent classical background in the imaginary direction.