Essays on mathematical and computational finance: With a view towards applied probability
Gruntjes, P.A.G.J.M.

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Chapter 1

Introduction

Trade is of all times; since the beginning of civilization people traded goods for other goods or money. It is known that derivative contracts existed during the times of the Babylonians and ancient Greeks. However, the first organized market for securities started in the seventeenth century in Holland at the Amsterdam Stock Exchange. It was founded by the Dutch East India Company (Vereenigde Oostindische Compagnie, or “VOC”) in 1602 and is considered the oldest in the world. During the so-called tulip mania there was huge speculation in options on tulips bulbs, which were traded while the bulbs were still in the ground. The tulip mania was the first well documented financial bubble in world history. At a certain point, tulips bulbs were as valuable as an Amsterdam canal house. The first official futures and options exchange opened in the middle of the nineteenth century in Chicago; the Chicago Board of Trade. However, it wasn’t until the opening of the Chicago Board Options Exchange in 1973 that the global options trade really got a boost. The combination of mathematical and technological development ensured that the derivatives market evolved into the multi-trillion dollar market that it is today.

Before accurate option pricing methodologies were available, investors and speculators relied on heuristic methods and their predictions of the future to establish some kind of crude option price. The first scientific attempt of pricing options began with the brilliant French mathematician Louis Bachelier. In his PhD thesis Théorie de la Spéculation he had worked out already 5 years before Albert Ein-

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1The market, unwittingly, obeys a law which governs it: the Law of Probability. Final scentence of Théorie de la Spéculation.
stein’s famous (second) 1905 Annus Mirabilis paper the distribution function of what is now known as the Wiener process. However, the crucial insight came more than seventy years later when Fisher Black and Myron Scholes and Robert Merton published their seminal papers on the pricing of options on stocks in 1973. Before that time, all methods suffered the same fundamental shortcoming; risk premia were not dealt with in a correct manner. Black, Scholes, and Merton proved that, under certain conditions, the risk of an option can be completely hedged by dynamically investing in the underlyings of the option. Assuming there are no arbitrage opportunities in financial markets, the price of such an option has to be the same as the value of the investment strategy with which one replicates this option, i.e. risk neutral pricing.

With the concept of replication, a whole array of more exotic options can be evaluated, which hugely increased the need and importance of financial mathematics. Products that were once deemed exotic are quite common these days; even finding their way into retail products like mortgages and investment notes. So common even, that investment banks began to pool mortgages on a large scale, and actively started trading them on the stock exchange. This so-called securitization is an economically logical step; investing always implies an exchange of risk and return, and via securitization of mortgages investment banks could hedge their own mortgage portfolios. However, because of the huge demand (and initial return) for these so-called Collateralized Debt Obligations (CDOs), more and more subprime and other non-prime mortgage debt was put into CDO assets. This in itself is not a problem, as long as the credit ratings change accordingly. This was not the case, and another bubble was created. When mortgage owners massively started defaulting on their claims in 2007, the subprime mortgage crisis was a fact. The whole CDO market collapsed and real estate prices plunged. The financial havoc that came afterwards was felt throughout the world.

One of the crucial requirements of any option pricing model is that the price of a structured product is sensible. That is, within the option pricing model, plain vanilla products, i.e. simple products that are actively traded, should be accurately priced to reflect their market prices. Bubbles and crashes made it painfully clear that this was not the case in many instances. In the CDO market, where the pricing model of Li was the industry standard, practitioners used a static Gaussian copula model with its thin tails to price complex structures. This left a substantial

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2Latin for “extraordinary year”.
amount of risk in the market unpriced. In the FX and equity markets, the rise and fall of the Long Term Capital Management (LTCM)\textsuperscript{3} hedge fund made it clear that the assumption of normally distributed asset returns in the Black Scholes model, \cite{black1973pricing, scholes1973pricing}, is much too restrictive. Black Swan events\textsuperscript{4} occurred quite regularly. These and other examples made practitioners more aware of the assumptions underlying their financial models, and the demand for more sophisticated models grew.

A second crucial requirement is speed. Investment banks and hedge funds that actively trade complex options need fast pricing algorithms to beat the competition. Often a more complex model prices more accurately, but it’s computational complexity is higher. This is the reason why simpler models prevail over more complex models, since practitioners often value speed over accuracy. Hence the need for more sophisticated models combined with the numerical numerical techniques that cope with this increasing complexity.

The primary goal of this monograph is therefore to contribute methodologically to the pricing of options, and their numerical evaluation. Lord \cite{lindsey2005option} argues that, to price an exotic option, one needs to:

1. choose a model that is both economically plausible as analytically tractable;
2. calibrate the model to market prices of plain vanilla options;
3. price the exotic option of the calibrated model using appropriate numerical techniques.

This monograph deals with all three steps. We will further elaborate on our contributions in the outline in Section\textsuperscript{1.6}. The tools used in this thesis to compute prices and sensitivities of complex options are rooted in probability theory, which explains the subtitle “a view towards applied probability”. Market fundamentals are modeled using stochastic differential equations (SDEs) and probabilistic techniques are used to compute conditional expectations. There is an important alternative to our approach, namely via partial differential equations (PDEs).

\textsuperscript{3}A nice thing to note is that Scholes and Merton were partners at LTCM, which at the time had the most sophisticated mathematical models for derivative pricing. Although LTCM was initially hugely successful with annualized returns of over 40% in its first years, it lost 4.6 billion in less than four months in 1998, following the Russian financial crisis. Financial intervention by the Federal Reserve (Fed) was needed and the fund collapsed.

\textsuperscript{4}Metaphor coined by Taleb in his book “fooled by randomness” \cite{taleb2007black} for highly improbable events of large magnitude and consequence, which are after the fact often inappropriately rationalized with the benefit of hindsight.
However, by virtue of the famous Feynman-Kac theorem, see e.g. [104], these approaches are essentially equivalent.

1.1 Lévy Processes

The underlying processes of virtually all models in mathematical finance are so-called Lévy processes. These are continuous stochastic processes with stationary and independent increments. More formally, a càdlàg, adapted, real-valued stochastic process $X(t)$ on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ with $X(0) = 0$, is said to be a Lévy process [14] if:

1. it has independent increments, i.e. for any $s < t$, $X(t) - X(s) \perp \mathcal{F}_s$;
2. it has stationary increments, i.e. for any $s < t$, the distribution of $X(t) - X(s)$ is equal to $X(t - s)$;
3. it is stochastically continuous; i.e. for any $t > 0$ and $\varepsilon > 0$ it holds that $\lim_{s \to t} \mathbb{P}(|X(t) - X(s)| > \varepsilon) = 0$ a.s.

The most basic and well-known examples are the Wiener process (or Brownian motion) and the Poisson process. In their standard form, these processes are too basic to describe financial markets, since they are not rich enough to model various critical features of financial markets. This is why in this thesis we will consider more general Lévy processes, e.g. compound Poisson processes and the sum of compound Poisson processes and Brownian motion. These so-called jump diffusion models were first used in finance by Merton [93] and later by Kou [73]; see e.g. [33] for an overview. An important subclass of the class of Lévy processes is subordinated (that is, time-changed) Brownian motion. These are Brownian motions in which we (randomly) scale the time axis by a subordinator to obtain a new, richer Lévy process (in the sense that it contains more parameters to model financial markets). Well known examples in mathematical finance are the variance gamma (VG) process [87], and the CGMY [29] process.

The law of every Lévy process can be uniquely defined by the so-called Lévy triplet $(\mu, \sigma, \nu)$, that represent the drift, diffusion, and jump components of the process. The drift $\mu$ is real number, the volatility $\sigma$ is non-negative, and the Lévy measure $\nu(\cdot)$, concentrated on $\mathbb{R} \setminus \{0\}$, satisfies

$$\int_{\mathbb{R}} \min(1, |x|^2) \nu(dx) < \infty.$$
The Lévy measure is not merely a theoretical concept; rather it describes the “activity” of the process. If \( \nu(\mathbb{R}) = \infty \), the process is said to have infinite activity, which essentially means that there is an infinite number of jumps on every compact interval. Popular models in mathematical finance belong to this category, e.g. the variance gamma process and the CGMY process. Conversely, when \( \nu(\mathbb{R}) < \infty \) the process is said to have finite activity. The widely used compound Poisson process falls into this category.

The Lévy triplet also finds its way into the so-called Lévy exponent associated with the Lévy process \( X(t) \), through the Laplace transform:

\[
\hat{f}_{X(t)}(s) := \mathbb{E} e^{-sX(t)} = \exp (-t\phi(s)),
\]

where \( s \in i\mathbb{R} \) and

\[
\phi(s) = \log \mathbb{E} e^{-sX(t)} = -s\mu + \frac{1}{2} s^2 \sigma^2 + \int_{-\infty}^{\infty} (e^{-sx} - 1 + sx1_{|x|<1}) \nu(dx),
\]

which is the famous Lévy-Khintchine formula, and the exponent \( \phi(s) \) is called the Lévy exponent. We elaborate further on transforms and numerical issues in the following section.

### 1.2 Laplace Transforms and Numerical Inversion

As already briefly mentioned in the previous section, in this thesis we will frequently use the notion of the Laplace transform. The Laplace transform \( \hat{f}(s) \) of an integrable function \( f(t) : \mathbb{R} \rightarrow \mathbb{C} \), is defined as

\[
\hat{f}(s) = \int_{-\infty}^{\infty} e^{-st} f(t) \, dt,
\]

with \( s \in \mathbb{C} \). It is tacitly assumed that all integrals are properly defined for the chosen values of \( s \). There are several other closely related transforms, which have constants appearing in their definition. Examples are the characteristic function, defined as

\[
\int_{-\infty}^{\infty} e^{ist} f(t) \, dt,
\]

with \( s \in \mathbb{R} \) and the Fourier transform, defined as

\[
\int_{-\infty}^{\infty} e^{-2\pi ist} f(t) \, dt,
\]
again with \( s \in \mathbb{R} \). This is essentially merely a change of parameters. In this thesis we will use the Laplace as it is defined in Equation (1.1).

It is a well-known fact that the Laplace transform of a probability distribution, when it exists, uniquely determines that distribution, see e.g. [17]. The Laplace transform arises very naturally in mathematical finance. As will be argued later in Section 1.4, we note that most quantities of interest, like option prices and sensitivities, can be expressed as conditional expectations. Such conditional expectations can be calculated using numerical integration, provided we have a closed form solution of the conditional distribution function (or the density). However, this is in general not the case; for many models the density is not known in closed form, or it involves complicated special functions. In such situations, quite often results are available for the Laplace transforms of the underlying quantities of interest; with numerical techniques these transforms can be inverted to obtain results for the probability distribution. In addition, even in cases in which the density is in principle known, the use of transform-based methods still has advantages in terms of simplicity; one can work with clean, explicit formulas, although a numerical inversion needs to be performed.

If \( f_{X(t)} \) is the density of \( X(t) \), assumed to exist, e.g. the law of a stochastic process describing a financial market, we can write for the Laplace transform:

\[
\hat{f}_{X(t)}(s) = \mathbb{E} \exp(-sX(t)) = \int_{-\infty}^{\infty} e^{-sx} f_{X(t)}(x) \, dx.
\]

In case the process \( X(t) \) has no density we define, with \( F_{X(t)}(x) \) the corresponding distribution function,

\[
\hat{f}_{X(t)}(s) = \int_{-\infty}^{\infty} \exp(-sx) \, dF_{X(t)}(x)
\]

The challenge lies in inverting these transforms: we wish to compute the function \( f(t) \) from \( \hat{f}(s) \). Theoretically, there is the so called Bromwich inversion integral with which this can be done:

\[
f(t) = \frac{1}{2\pi} \int_{-i\infty}^{i\infty} e^{st} \hat{f}(s) \, ds.
\]

with \( s \in i\mathbb{R} \). However, when applied to financial markets, its practical relevance is limited since the integrand is highly oscillating and converges very slowly to
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zero. This is why most modern Laplace and Fourier inversion algorithms are based on the Poisson summation formula (PSF):

\[
\sum_{k=-\infty}^{\infty} \hat{f}(i2\pi (k + v)) e^{i2\pi(k+v)t} = \sum_{j=-\infty}^{\infty} f(j + t) e^{-i2j\pi v}.
\]

Inversion algorithms that use the PSF can be found in e.g. [1] and [46]. However, the problem remains that the infinite sums of the PSF converge slowly. This is why, [1] propose to use a so-called Euler summation, but in general the convergence remains slow unless there is knowledge of the location of singularities of \(\hat{f}(s)\). A recently developed inversion algorithm that overcomes all these problems is that of den Iseger [40] that approximates the infinite sum of the Poisson summation formula by using a Gaussian quadrature rule. The algorithm proposed in [40] is a substantial improvement over earlier algorithms in the sense that

- it can handle a larger class of Laplace transforms (e.g. no knowledge of the location of discontinuities or singularities is needed);
- the algorithm only needs numerical values of the Laplace transform;
- is fast (that is, the function values \(f(k\Delta)\), with \(k = 0, \ldots, M - 1\), are computed at once, in order \(M \log M\) time);
- is of near machine precision, i.e. an error of \(10^{-13}\) or less;
- can be extended to multiple dimensions, see [40] Section 5.

1.3 Wiener-Hopf Decomposition

In this section we present the Wiener-Hopf decomposition, which features in this thesis. In finance, it can be applied to e.g. the pricing of so-called barrier options and a broad range of credit products [95]. For these financial products one is interested in the density of the running maximum process \(\tilde{X}(t) := \sup_{0 \leq s \leq t} X(s)\) and/or running minimum process \(\underline{X}(t) := \inf_{0 \leq s \leq t} X(s)\) of a Lévy process \(X(t)\) for \(t > 0\).

Now let \(\vartheta\) be an exponentially distributed random variable with mean \(1/\vartheta\) (for \(\vartheta > 0\)), independent of the Lévy process under consideration. By applying the so-called Wiener-Hopf transform, e.g. [74] Thm. 6.16, we can factorize the
Laplace transform of the position of the Lévy process at an exponential epoch, i.e. $\mathbb{E} e^{-sX(\bar{e}(\vartheta))}$, as follows:
\[
\mathbb{E} e^{-sX(\bar{e}(\vartheta))} = \mathbb{E}e^{-s\bar{X}(\bar{e}(\vartheta))} \mathbb{E}e^{-sX(\bar{e}(\vartheta))},
\]
where
\[
\mathbb{E}e^{-sX(\bar{e}(\vartheta))} = \int_{0}^{\infty} \vartheta e^{-\vartheta t} \mathbb{E}e^{-sX(t)} dt = \int_{0}^{\infty} \int_{0}^{\infty} \vartheta e^{-\vartheta t} e^{-sx} \mathbb{P}(X(t) \in dx) dt = \int_{0}^{\infty} \vartheta e^{-\vartheta t} e^{-t\phi(s)} dt = \frac{\vartheta}{\vartheta + \phi(s)}.
\]

The explicit expression for the above expectation involves a double integral and a probability density $\mathbb{P}(X(t) \in dx)$ that is often not known explicitly. The challenge lies in finding the distribution of $\bar{X}(t)$ (as well as the joint distributions, e.g. that of $\bar{X}(t)$ and $X(t)$). The running minimum can be dealt with analogously, by replacing the Lévy process $X(t)$ by $-X(t)$. As we will point out in the outline of this thesis, one of our contributions is that we use state-of-the-art numerical Laplace transform inversion as well as analytical methods to invert transforms so as to obtain the density $f_{\bar{X}(t)}(x)$.

### 1.4 Asset Pricing and Conditional Expectations

Conditional expectations are of central importance in the field of mathematical finance. Prices and sensitivities of options, for instance, can be considered as conditional expectations. In this section we will briefly elaborate on this. To start the exposition, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\mathcal{G}$ a sub-$\sigma$-algebra of $\mathcal{F}$, and $Y$ a random variable with finite expectation. The conditional expectation of $Y$ given $\mathcal{G}$, is the almost surely unique random variable $\mathbb{E}(Y|\mathcal{G})$ such that
\[
\mathbb{E}[1_{G}\mathbb{E}(Y|\mathcal{G})] = \mathbb{E}[1_{G}Y] \quad \text{for every set } G \in \mathcal{G}.
\]
If $\mathcal{G} = \sigma(X)$ for some random variable $X$, then the above becomes
\[
\mathbb{E}[1_{\{X \in A\}}\mathbb{E}(Y|X)] = \mathbb{E}[1_{\{X \in A\}}Y], \quad \text{for every set } A \in \mathcal{B}(\mathbb{R}),
\]
where we simply write...
In mathematical finance the models of interest are usually exponential Lévy models, in which the asset price is modeled as an exponential function of a Lévy process $X(t)$, i.e.,

$$S(t) = S(0) \exp(X(t)).$$

The price of an option, with expiration date $T$, evaluated at time $t < T$, to be denoted by $V(t)$, can generally be expressed as a conditional expectation, as follows:

$$V(t) = \mathbb{E}_Q \left( \frac{M(t)}{M(T)} V(T) \mid \mathcal{F}_t \right),$$

where $\mathcal{F}_t$ is the filtration, $\mathbb{E}_Q$ is the conditional expectation taken under the risk neutral measure $Q$ (to be discussed in Section 1.5), and $M(t)$ is the money account process. Alternatively, we can compute $V(t)$ as a conditional expectation under the forward measure $Q^T$, i.e.

$$V(t) = D(t,T) \mathbb{E}_{Q^T} \left[ V(T) \mid \mathcal{F}_t \right]$$

with $D(t,T)$ the discount factor between $t$ and $T$ (that we will not detail here).

Now let the Laplace transform under the forward measure $Q^T$ of the log stock price at time $T$, i.e. $X(T)$, be given by $\hat{f}_{X(T)}$. As a stylized example, suppose that we want to price a vanilla call option at time $t < T$. The price is given by

$$C(t) = D(t,T) K \mathbb{E}_{Q^T} \left[ (\exp(X(T)) - K)^+ \mid \mathcal{F}_t \right]$$

Rewriting, we obtain

$$C(t) = D(t,T) K \mathbb{E}_{Q^T} \left[ (\exp(- (k - X(T))) - 1)^+ \mid \mathcal{F}_t \right]$$

$$= D(t,T) K \int_k^\infty (\exp(-(k-x)) - 1) dF_{X(T)}(x),$$

with $k = \log(K)$. From the above, it immediately follows that the call price $C(t)$ is (up to a multiplicative constant) a convolution of the function

$$\left( e^{-x} - 1 \right) 1_{\{x \geq 0\}}$$

and $dF_{X(T)}$, where $F_{X(T)}$ is the distribution of $X(T)$ under the forward measure $Q^T$ and conditional on $\mathcal{F}_t$. Since the Laplace transform of (1.4) is given by $-\frac{1}{s} \frac{1}{1+s}$
we obtain for the Laplace transform of the call price $C(t)$

$$\hat{C}_t(s) = -D(t, T) \frac{1}{s} \frac{1}{1 + s}$$

We conclude that, if we have a transform inversion algorithm at our disposal, it facilitates us the computation of the value of the option. The example illustrates the importance of fast and accurate inversion techniques in option pricing.

### 1.5 Martingales, Measures and Girsanov’s Theorem

This thesis frequently uses the notions of martingales, measures, and Girsanov’s theorem. In this section we will touch upon these topics and explain their use in finance.

Delbaen and Schachermayer [39] argue that the only reasonable risk neutral process driving a contingent claim should be a (semi) martingale. They employ martingales to model financial markets and prove the no-arbitrage principle, under increasingly relaxed assumptions.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a filtered probability space, and consider an adapted stochastic process $X(t)$, with $0 \leq t \leq T$. If

i. $\mathbb{E}_\mathbb{P}(|X(t)|) < \infty$ for every $0 \leq t \leq T$

ii. $\mathbb{E}[X(t)|\mathcal{F}(s)] = X(s)$ for every $0 \leq s \leq t \leq T$

we say this process is a martingale.

Of course, the important fact is (ii); the other is a technical condition. Loosely speaking, a martingale is a fair game. That is, a game in which it is not possible to win (or lose) on average. Viewed in financial terms, it says so much as that the best prediction of tomorrow’s price is today’s price. Martingales are of critical importance in mathematical finance. Under the no arbitrage condition, discounted price processes are martingales under a measure with respect to which the process is expected to stay the same, i.e. the martingale or risk neutral measure.

To price derivatives under the risk neutral measure, we need to change measure from the real life probability measure to the risk neutral probability measure. Therefore, we will now discuss the change of numéraire theorem. A numéraire
is the unit of account in which other assets are denominated. One can think of the numéraire as being the currency of a country, but this notion holds more abstractly as well. Any asset with strictly positive prices for all \( t \in [0, T] \) is called a numéraire. In this respect, consider the following. The value of a derivative at time \( t > 0 \), i.e. \( V(t) \), can be stated as:

\[
V(t) = \mathbb{E}_{Q^N} \left( \frac{N(t)}{N(T)} X(T) \bigg| \mathcal{F}_t \right)
\]

in which the numéraire is the process \( N(\cdot) \). Under an equivalent martingale measure \( Q^M \), not necessarily the money account measure, it holds that

\[
V(t) = \mathbb{E}_{Q^M} \left( \frac{N(t)}{N(T)} \left( \frac{dQ^N}{dQ^M} \right) X(T) \bigg| \mathcal{F}_t \right) / \mathbb{E}_{Q^M} \left( \frac{dQ^N}{dQ^M} \bigg| \mathcal{F}_t \right)
\]

with

\[
\frac{dQ^N}{dQ^M} = \frac{N(T) M(0)}{M(T) N(0)}.
\]

For a formal proof see Geman et al. [52]. The proof goes along the same lines as described above. The random variable \( \frac{dQ^N}{dQ^M} \) is the Radon-Nikodým derivative of \( Q^N \) w.r.t. \( Q^M \) on \( \mathcal{F}_T \); an almost surely positive random variable with \( \mathbb{E}_{Q^M} \left( \frac{dQ^N}{dQ^M} \bigg| \mathcal{F}_t \right) = 1 \).

In option pricing, the choice of numéraire is usually the money account process, i.e.

\[
M(s) = \exp \left( \int_0^s r(u) du \right).
\]

The question remains how this change of measure actually looks like. Here Girsanov’s theorem comes into play. It can be stated as follows.

Let \( W(t), 0 \leq t \leq T, \) be a Brownian motion on a filtered probability space \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}) \). Furthermore, let \( \Theta(t), 0 \leq t \leq T, \) be an adapted process. Define the Radon-Nikodým process \( \frac{dP}{d\mathbb{P}}(t) = Z(t) \) as

\[
Z(t) = \exp \left\{ - \int_0^t \Theta(u) dW(u) - \frac{1}{2} \int_0^t \Theta^2(u) du \right\}
\]
and

\[ \tilde{W}(t) = W(t) + \int_0^t \Theta(u)du, \]

and we assume that\(^5\)

\[ \mathbb{E} \exp \left( \frac{1}{2} \int_0^T \Theta^2(u)du \right) < \infty. \]

Then, under the risk neutral measure \(Q\), defined by \(\frac{dQ}{dP} = Z(T)\) the process \(\tilde{W}(t), 0 \leq t \leq T\) is a Brownian motion. For a formal proof, see e.g. Øksendal \([98]\).

In colloquial terms, a change of measure can be seen as a change of drift. We showed that we can switch from the real world probability measure \(P\) to the risk neutral measure \(Q\) by making a drift adjustment.

Because finance is a random environment, functions of stochastic processes arise very naturally in many situations. It is not immediately clear how to evaluate the behavior of a stochastic process derived as a function of an Itô process. To this end, we need a key result from stochastic calculus known as Itô’s Lemma. It can be stated as follows.

Let \(X(t)\) be an Itô process given by

\[ dX(t) = \mu(t)dt + \sigma(t)dW(t), \]  \hspace{1cm} (1.2)

with \(\mu \in \mathbb{R}\) and \(\sigma \in \mathbb{R}_+\) are the (possibly time dependent) drift and variance respectively. Furthermore, let \(f(t, x) \in C^{1,2}([0, \infty) \times \mathbb{R})\). Then, \(Y(t) = g(t, X(t))\) satisfies

\[ dY(t) = \frac{\partial g}{\partial t}(t, X(t))dt + \frac{\partial g}{\partial x}(t, X(t))dX(t) + \frac{1}{2} \frac{\partial^2 g}{\partial x^2}(t, X(t))d[X, X](t), \]

where \([X, X]\) denotes the quadratic variation of \(X(t)\). When \(dX(t)\) is given by (1.2), \(d[X, X](t) = \int_0^T \sigma^2(s)ds\). For a proof, see e.g. Øksendal \([98]\).

To conclude the introduction, we give an outline of the rest of this thesis in the following section.

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\(^5\)This is the so-called Novikov condition. It ensures that the process \(Z(t)\) is a martingale.
1.6 Outline

This thesis is written in 4 chapters. The structure is as follows:

Chapter 2 focuses on numerical evaluation techniques related to fluctuation theory for Lévy processes. They can be applied in various domains, e.g., in finance in the pricing of so-called barrier options. More specifically, with \( \bar{X}_t := \sup_{0 \leq s \leq t} X_s \) denoting the running maximum of the Lévy process \( X_t \), the aim is to evaluate \( P(\bar{X}_t \in dx) \) for \( t, x > 0 \). The starting point is the Wiener-Hopf factorization, which yields an expression for the transform \( E e^{-\alpha \bar{X}_e(\theta)} \) of the running maximum at an exponential epoch. It is shown how to use Laplace inversion techniques to numerically evaluate \( P(\bar{X}_t \in dx) \). In our experiments we rely on the efficient and accurate algorithm developed in [40]. We illustrate the performance of the algorithm with various examples: Brownian motion (with drift), a compound Poisson process, and a jump diffusion process. In models with jumps, we are also able to compute the density of the first time a specific threshold is exceeded, jointly with the corresponding overshoot. The paper is concluded by pointing out how our algorithm can be used in order to analyze the Lévy process’ concave majorant.

In Chapter 3, we propose an intuitive, practical, Lévy-based, dynamic default correlation model, with applications to CDO pricing. In developing this model, we first capture the marginal default probabilities relying on a dynamic structural Variance Gamma (VG) process. Then we impose a dynamic correlation structure on the individual obligors, of which the marginals are in line the VG-based model for the single names. This correlation model is constructed by decomposing the VG processes into two components: an individual component and a common component; the former affects just one obligor, whereas the latter has impact on the entire market. The key advantages of our correlation structure are that it is intuitive, dynamic, and allows for easy calibration to the market since the underlyings are market observables, for which there is ample data available. In case of a homogeneous basket of obligors, the evaluation of our model is as easy as a copula-based approach. In case of a non-homogeneous basket, the computations can still be done in a fast and accurate way. An important role is played by recently developed techniques for numerical Laplace transform inversion developed by den Iseger [40]. Our approach is backed up by various numerical experiments.

In Chapter 4, we present the Drift Adjustment Method Laplace Algorithm (DAMLA) for efficiently and accurately calculating conditional expectations of Ornstein Uh-
lenbeck (OU) type processes. It is a novel approach in the sense that we can *efficiently and accurately* calculate conditional expectations, in high dimensions, at each time point in each scenario. Broadly speaking, the method we present can be used in Monte Carlo simulations as well as in quadrature methods. Applications are numerous. We use DAMLA for the simultaneous hedging of the interest *and* equity risk of a pension fund. More specifically, we price and determine the Greeks of an equity-linked swaption, commonly called hybrid option, to hedge pension fund risk. We find that, by combining the *Drift Adjustment Method (DAM)* with numerical Laplace transform inversion, i.e. the *Drift Adjustment Method Laplace Algorithm (DAMLA)*, computational speed is greatly enhanced as compared to direct calculation (Monte Carlo). We can price and hedge a portfolio of 100,000 equity-linked swaptions in under 5 minutes with errors of the order $10^{-12}$. Finally, we will extend DAMLA to models driven by arbitrary Lévy processes, and show we can exactly proceed as in the diffusion case.

The final chapter, Chapter 5, deals with a correlated overflow model. We present a multi-dimensional overflow problem and due to the ordering of the components, explicit results are obtained. In our setting, each component behaves as a compound Poisson process with unit-sized upward jumps, decreased by a linear drift. The approach relies on a Beneš-type argumentation. That is, the idea of partitioning the overflow event with respect to the last “exceedance epoch”. It is pointed out how the results can be used in credit risk modeling.