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Chapter 2

Numerical Wiener-Hopf Evaluation

1 The best material model of a cat is another, or preferably the same, cat.
Norbert Wiener (1894 – 1964)
Philosophy of Science, with A. Rosenblueth (1945)

This chapter focuses on numerical evaluation techniques related to fluctuation theory for Lévy processes; they can be applied in various domains, e.g., in finance in the pricing of so-called barrier options. More specifically, with \( \bar{X}_t := \sup_{0 \leq s \leq t} X_s \) denoting the running maximum of the Lévy process \( X_t \), the aim is to evaluate \( \mathbb{P}(\bar{X}_t \in dx) \) for \( t, x > 0 \). The starting point is the Wiener-Hopf factorization, which yields an expression for the transform \( \mathbb{E}e^{-\alpha \bar{X}_t(\vartheta)} \) of the running maximum at an exponential epoch (with \( \vartheta^{-1} \) the mean of this exponential random variable). This expression is first rewritten in a more convenient form, and then it is pointed out how to use Laplace inversion techniques to numerically evaluate \( \mathbb{P}(\bar{X}_t \in dx) \). In our experiments we rely on the efficient and accurate algorithm developed in [40]. We illustrate the performance of the algorithm with various examples: Brownian motion (with drift), a compound Poisson process, and a jump diffusion process. In models with jumps, we are also able to compute the density of the first time a specific threshold is exceeded, jointly with the corresponding overshoot. The chapter is concluded by pointing out how our algorithm can be used in order to analyze the Lévy process’ concave majorant.

2.1 Introduction

Fluctuation theory can be considered as the branch of probability theory that studies the so-called running maximum process \( \bar{X}_t := \sup_{0 \leq s \leq t} X_s \) associated with

\(^1\)This chapter is based on joint work with Peter den Iseger and Michel Mandjes has appeared as 23.
2.1. INTRODUCTION

A stochastic process $X_t$, for $t \geq 0$. Owing to its numerous applications in e.g. finance [33], and in view of its attractive mathematical properties, special attention is paid in the literature to the case that the underlying stochastic process is a Lévy process, that is, a process with stationary independent increments.

In fluctuation theory for Lévy processes, a pivotal role is played by the so-called Wiener-Hopf factorization [61], which was, as the name suggests, pioneered by Wiener and Hopf; see [74, p. 174] for a brief historic account. It was initially meant as a method to solve systems of integral equations, and later found its use in solving two-dimensional partial differential equations and several other applications. In the context of this chapter, the Wiener-Hopf factorization describes distributional properties of various functionals of Lévy processes [14, 74, 95], such as extreme values, first passage times of given thresholds, and (in case the Lévy process has jumps) the corresponding overshoot. In particular, it provides us with an expression for the Laplace transform of the running maximum up to an exponentially distributed epoch. Fluctuation-theoretic results can be applied in various domains, e.g., in finance in the pricing of so-called barrier options.

The Wiener-Hopf factors can be explicitly computed in terms of the Lévy exponent $\phi(\cdot)$ of the underlying Lévy process in various situations. This is the case if this Lévy process is spectrally one-sided [74], that is, has either only positive jumps or only negative jumps, but in addition various other specific cases can be dealt with (for instance: phase-type jumps in one direction, and general jumps in the other direction [7, 82]). Having the Wiener-Hopf factors at our disposal, we have the Laplace transform of the running maximum and minimum up to an exponential epoch; then numerical procedures can be applied to evaluate $f_{\bar{X}}(x) := P(\bar{X}_t \in dx)$ for $t, x > 0$. A complication, however, is that for a large class of Lévy processes no explicit form for the Wiener-Hopf factors is available, notably if the jumps in both directions are not of phase-type.

In this chapter we develop a numerical approach to evaluate the density $f_{X_t}(x)$ that does not need to know the explicit form of the Wiener-Hopf factors, in the sense that the input of our algorithm is just the Lévy exponent $\phi(\cdot)$. As a first step, the transform $E e^{-s \bar{X}_t(\vartheta)}$ of the running maximum at an exponential epoch (with $\vartheta^{-1}$ the mean of this exponential random variable) is expressed in terms of $s$, $\vartheta$ and $\phi(\cdot)$ only; the resulting expression contains several Fourier transforms and inverse Fourier transforms. As a second step, the tools developed in [40] are pro-
posed to perform the necessary numerical inversions.

We then systematically assess the numerical accuracy and complexity of our numerical evaluation method. This is done by considering an extensive set of examples, covering many of the specific Lévy processes proposed in the literature. It is noted that particular Lévy processes were already dealt with before; e.g., [103, 106] focus on numerical aspects related to the spectrally-negative case (that is, the underlying Lévy process has no positive jumps). The present chapter focuses on Lévy processes without small jumps; the case of small jumps will be covered by the follow-up paper [57]. Other approaches can be found in for instance [3], as well as [8] for the specific case that the underlying Lévy process is of CGMY type [29].

In the examples reported in this chapter, we primarily concentrate on the distribution of the running maximum. In addition, we also consider an example where we analyze the time it takes to exceed some threshold, as well as the associated overshoot (in a model in which the threshold can be exceeded by a jump of the underlying Lévy process); this type of computations can be used in various financial applications, e.g. those related to the pricing of barrier options. All examples show that our method is highly efficient and accurate; in this sense, we refute a claim stated by Cont and Tankov [33, p. 371]: “Computations using Equations [the Wiener-Hopf decomposition] are not very efficient because they involve the probability distribution of \( X_t \).”. In fact, our method circumvents the need to explicitly know this distribution.

We also include an example that shows how our method can be applied in the analysis of the concave majorant of our Lévy process \( X_t \). The concave majorant is the smallest concave function majorizing the process. It turns out that the probabilities involved can be expressed in terms of the running maximum of an associated Lévy process.

The chapter is organized as follows. In Section 2.2 we first define our model and present preliminaries on the role of ‘Wiener-Hopf’ in fluctuation theory for Lévy processes. This we use to derive our expression for the transform \( \mathbb{E}e^{-s\bar{X}_{e(\vartheta)}} \) in terms of \( s, \vartheta \) and \( \phi(\cdot) \) only. Then we point out how the necessary (inverse) Fourier transforms can be performed. Section 4.3 contains a broad collection of examples, confirming the accuracy and efficiency of our approach. We also dis-
cuss the advantages and limitations of the approach. Then Section 2.4 describes how our approach facilitates the analysis of the concave majorant. Finally, Section 4.7 concludes. Numerical output can be found in the appendix.

2.2 Procedure

In this section we point out how to numerically evaluate \( f_{\bar{X}_t}(x) := \mathbb{P}(\bar{X}_t \in dx) \). We first introduce the notation, then recapitulate the main results in Wiener-Hopf theory for fluctuations of Lévy processes, and then write \( \mathbb{E}e^{-s\bar{X}_t(x)} \) in a convenient form. This enables us to develop an algorithm for fast and accurate evaluation of \( f_{\bar{X}_t}(x) \).

2.2.1 Model, notation, & preliminaries

Let \( X_t \) be a (one-dimensional) Lévy process, that is, a process with stationary and independent increments. We define \( f_{X_t}(x) \) as the density of \( X_t \) (assumed to exist), i.e., \( f_{X_t}(x) = \mathbb{P}(X_t \in dx) \). The Fourier transform associated with this Lévy process is \( \mathbb{E}e^{-sX_1} = \exp \left( -t\phi(s) \right) \); here

\[
\phi(s) = \log \mathbb{E}e^{-sX_1} = -sd + \frac{1}{2} s^2\sigma^2 + \int_{-\infty}^{\infty} (e^{-sx} - 1 + sx1_{\{|x|<1\}})\Pi(dx),
\]

(2.1)
denotes the so-called Lévy exponent, with \( d \in \mathbb{R}, \sigma \geq 0 \), and the spectral measure \( \Pi(\cdot) \), concentrated on \( \mathbb{R} \setminus \{0\} \), satisfying

\[
\int_{-\infty}^{\infty} \min\{x^2, 1\}\Pi(dx) < \infty.
\]

The triplet \((d, \sigma^2, \Pi)\) is commonly referred to as the characteristic triplet, as it uniquely defines the Lévy process.

As indicated in the introduction, we let \( \bar{X}_t := \sup_{0 \leq s \leq t} X_s \); in addition \( \underline{X}_t := \inf_{0 \leq s \leq t} X_s \). Define

\[
k(\vartheta, s) := \exp \left( -\int_0^\infty \int_{(0,\infty)} \frac{1}{t} \left( e^{-t} - e^{-\vartheta t - sx} \right) \mathbb{P}(X_t \in dx)dt \right),
\]

and

\[
\bar{k}(\vartheta, s) := \exp \left( -\int_0^\infty \int_{(-\infty,0)} \frac{1}{t} \left( e^{-t} - e^{-\vartheta t - sx} \right) \mathbb{P}(X_t \in dx)dt \right),
\]
Let \( e(\vartheta) \) be an exponentially distributed random variable with mean \( 1/\vartheta \) (for \( \vartheta > 0 \)), independent of the Lévy process under consideration. ‘Wiener-Hopf’ [74, Thm. 6.16] now factorizes the Laplace transform of the position of the Lévy process at an exponential epoch, i.e., \( \mathbb{E}e^{-sX_{e(\vartheta)}} \), as follows:

\[
\frac{\vartheta}{\vartheta + \phi(s)} = \mathbb{E}e^{-sX_{e(\vartheta)}} = \mathbb{E}e^{-s\bar{X}_{e(\vartheta)}} \mathbb{E}e^{-s\bar{X}_{e(\vartheta)}} ,
\]

where

\[
\mathbb{E}e^{-sX_{e(\vartheta)}} = \frac{k(\vartheta, s)}{k(\vartheta, 0)}, \quad \mathbb{E}e^{-s\bar{X}_{e(\vartheta)}} = \frac{\bar{k}(\vartheta, s)}{\bar{k}(\vartheta, 0)} .
\]

Having in mind that our objective is to find the density \( f_{\bar{X}_t}(\cdot) \) of the running maximum at time \( t \), we focus on \( k(\vartheta, s)/k(\vartheta, 0) \). It is noted that the running minimum can be dealt with analogously, by replacing the Lévy process \( X_t \) by \( -X_t \).

This remainder of this section is about an algorithm to numerically find the distribution of \( \bar{X}_t \) (as well as the joint distribution of \( \bar{X}_t \) and \( X_t \)). In the above, we have observed that a key role in the analysis is played by the double transform \( k(\vartheta, s)/k(\vartheta, 0) \). We first write

\[
K(\vartheta, s) := \log\left(\frac{k(\vartheta, s)}{k(\vartheta, 0)}\right)
\]

in a more convenient form (in terms of \( \vartheta, s, \) and \( \phi(\cdot) \)), and then we point out how this new form can be used to develop an algorithm for fast and accurate evaluation of the density \( f_{\bar{X}_t}(x) = \mathbb{P}(\bar{X}_t \in dx) \). Evidently, a similar procedure can be followed for \( \bar{K}(\vartheta, s) := \log(\bar{k}(\vartheta, s)/\bar{k}(\vartheta, 0)) \).

### 2.2.2 Writing the Wiener-Hopf factors in a convenient form

Our first objective is to rewrite the function \( K(\vartheta, \alpha) \). To this end, observe that

\[
K(\vartheta, \alpha) = \int_0^\infty \int_{(0,\infty)} \frac{1}{t} (1 - e^{-\alpha x}) e^{-\vartheta t} f_{X_t}(x) dx \, dt ,
\]

where we recall that \( f_{X_t}(\cdot) \) is the density of \( X_t \). Now denote by \( \mathcal{F} \) the Fourier transform, and by \( \mathcal{F}^{-1} \) the inverse Fourier transform; in the sequel, \( \mathcal{F}[f](s) \equiv \mathcal{F}[f(\cdot)](s) \) is the Fourier transform of \( f \), evaluated in \( s \). We then have the following obvious relations:

\[
\mathcal{F}[f_{X_t}(\cdot)](s) = e^{-t\phi(s)}, \quad f_{X_t}(x) = \mathcal{F}^{-1}[e^{-t\phi(\cdot)}](x),
\]

and

\[
\mathcal{F}[g_t(\cdot)](s) = t\phi'(s)e^{-t\phi(s)}, \quad \text{with } g_t(x) = xf_{X_t}(x) .
\]

It follows that

\[
\mathcal{F}^{-1}[t\phi'(\cdot)e^{-t\phi(\cdot)}](x) = x\mathcal{F}^{-1}[e^{-t\phi(\cdot)}](x).
\]
From the above, it is concluded that
\[ K(\vartheta, \alpha) = \int_0^\infty \int_{(0, \infty)} \frac{1}{xt} \left( 1 - e^{-\alpha x} \right) e^{-\vartheta t} \mathcal{F}^{-1} \left[ t\phi'\cdot e^{-t\phi(\cdot)} \right](x) dx \, dt. \]

Using Fubini’s theorem, in conjunction with the fact that \( \mathcal{F}^{-1} \) is a linear operator, this in turn equals
\[ K(\vartheta, \alpha) = \int_{(0, \infty)} \left( \frac{1}{x} \left( 1 - e^{-\alpha x} \right) \mathcal{F}^{-1} \left[ \int_0^\infty \phi'\cdot e^{-t(\phi(\cdot) + \vartheta)} dt \right](x) \right) dx. \]

Now denote
\[ F_\vartheta(x) := \mathcal{F}^{-1} \left[ \int_0^\infty \phi'\cdot e^{-t(\phi(\cdot) + \vartheta)} dt \right](x) = \frac{1}{x} \mathcal{F}^{-1} \left[ \frac{\phi'\cdot}{\phi(\cdot) + \vartheta} \right](x). \]

By \( F_\vartheta^+(x) \) we denote \( F_\vartheta(x) \) if \( x \geq 0 \) and 0 else. With \( T_\vartheta(\cdot) \) the Fourier transform of \( F_\vartheta^+(\cdot) \), it then immediately follows that
\[ K(\vartheta, \alpha) = T_\vartheta(0) - T_\vartheta(\alpha). \]

In this way we found a compact expression purely in terms of \( \phi(\cdot) \), \( \alpha \), and \( \vartheta \). It is clear, however, that there may be numerical issues if
\[ \lim_{x \to 0} \mathcal{F}^{-1} \left[ \frac{\phi'\cdot}{\phi(\cdot) + \vartheta} \right](x) \neq 0. \]

To remedy this issue, let \( \tilde{F}_\vartheta^+(x) \) equal \( F_\vartheta(x) - e^{-x} F_\vartheta(0) \) if \( x \geq 0 \) and 0 else. Then obviously, with \( \tilde{T}_\vartheta(\cdot) \) the Fourier transform of \( \tilde{F}_\vartheta^+(\cdot) \),
\[ K(\vartheta, \alpha) = \tilde{T}_\vartheta(0) - \tilde{T}_\vartheta(\alpha) + F(0) \int_0^\infty \frac{1}{x} \left( 1 - e^{-\alpha x} \right) e^{-x} dx \]
\[ = \tilde{T}_\vartheta(0) - \tilde{T}_\vartheta(\alpha) + F(0) \log(1 + \alpha), \]
where in the last step the Frullani integral equality [Z4] Lemma 1.7 has been used.

We arrive at the following pseudocode to determine \( K(\vartheta, \alpha) = \log \mathbb{E}e^{-\alpha X_{\vartheta}(\cdot)} \), with \( e(\vartheta) \), as before, an exponentially random variable with mean \( 1/\vartheta \).

**Pseudo code 2.1.** Input: \( \vartheta, \alpha \), and \( \phi(\cdot) \). Output: \( K(\vartheta, \alpha) = \log \mathbb{E}e^{-\alpha X_{\vartheta}(\cdot)} \).
1. Compute the function \( F_\vartheta(\cdot) \);
2. Compute the function \( \tilde{T}_\vartheta(\cdot) \);
3. Set \( K(\vartheta, \alpha) = \log \mathbb{E}e^{-\alpha X_{\vartheta}(\cdot)} := \tilde{T}_\vartheta(0) - \tilde{T}_\vartheta(\alpha) + F(0) \log(1 + \alpha) \).

Above we mentioned that the input of the procedure consists of \( \vartheta, \alpha \), and \( \phi(\cdot) \), but, as we have seen, also \( \phi'(\cdot) \) is needed. One could either evaluate \( \phi'(\cdot) \) numerically,
or use an explicit expression for $\phi'(\cdot)$. Evidently, from a numerical standpoint the latter option is preferred.

The above pseudocode requires that procedures are available to compute transforms. More specifically, to evaluate $F_{\theta}(\cdot)$ we need a routine to perform the inverse Fourier transform, and to evaluate $\bar{T}_{\theta}(\cdot)$ we need a routine to perform the Fourier transform.

Once we have $K(\theta, \alpha)$, in order to find the density $f_{\bar{X}_t}(\cdot)$, we have to perform a double Laplace inversion (with respect to $\alpha$ and $\theta$) to

$$
\int_0^\infty \int_0^\infty e^{-\theta t} e^{-\alpha x} f_{\bar{X}_t}(x) dx \, dt = \frac{1}{\theta} \cdot \mathbb{E} e^{-\alpha X_{\theta}} = \frac{e^{K(\theta, \alpha)}}{\theta}.
$$

We comment on implementation issues in Section 2.2.4.

### 2.2.3 Applications

Above we showed how to write the transform of the random quantity $\bar{X}_{e(\theta)}$ in a form that facilitates numerical evaluation. In this section, we consider various other random quantities.

- The joint distribution of the first (upward) passage time $\tau_+(x)$ ($x > 0$) and the so-called overshoot $\sigma(x) := X_{\tau_+(x)} - x$ is given explicitly in terms of Laplace transforms [100]. For $\beta \geq 0$, and under $\mathbb{E}X_1 > 0$, the Pecherskii-Rogozin identity states

$$
\int_0^\infty e^{-\beta x} \mathbb{E} e^{-\theta \tau_+(x) - \bar{\theta}(X_{\tau_+(x)} - x)} \, dx = \frac{1}{\beta - \bar{\theta}} \left( 1 - \frac{k(\theta, \beta)}{k(\theta, \bar{\theta})} \right).
$$

(2.2)

(Realize that if the Lévy process is spectrally-negative, then the overshoot is necessarily 0). It is immediate that this expression equals

$$
\frac{1}{\beta - \bar{\theta}} \left( 1 - \frac{e^{K(\theta, \beta)}}{e^{K(\theta, \bar{\theta})}} \right).
$$

As an aside, we mention that in case we do not a priori impose the requirement $\mathbb{E}X_1 > 0$, the level $x > 0$ is not reached almost surely. As a consequence the formula (2.2) should be adapted to

$$
\int_0^\infty e^{-\beta x} \mathbb{E} \left( e^{-\theta \tau_+(x) - \bar{\theta}(X_{\tau_+(x)} - x)} 1_{\{\tau_+(x) < \infty\}} \right) \, dx = \frac{1}{\beta - \bar{\theta}} \left( 1 - \frac{k(\theta, \beta)}{k(\theta, \bar{\theta})} \right).
$$

The above expressions effectively show that, using Pseudocode 2.1, we can also evaluate the triple transform (2.2).
• Another standard result from Wiener-Hopf theory [14] concerns the joint distribution of the running maximum \( \bar{X}_{\vartheta} \) and position \( X_{\vartheta} \) after an exponential amount of time. The key result is:

\[
E e^{i(\alpha_1 + \alpha_2) \bar{X}_{\vartheta}} = E e^{i\alpha_2 (X_{\vartheta} - \bar{X}_{\vartheta})} = \frac{k(\vartheta, -i(\alpha_1 + \alpha_2))}{k(\vartheta, 0)} \frac{\bar{k}(\vartheta, i\alpha_2)}{\bar{k}(\vartheta, 0)}.
\]

It is elementary to express this quantity in terms of \( K(\cdot, \cdot) \) and \( \bar{K}(\cdot, \cdot) \), and as we are able to evaluate these, we can also evaluate the joint transform under consideration.

• We also know the joint transform of the running maximum and the corresponding epoch:

\[
E e^{-\beta G_{\vartheta}} - \alpha X_{\vartheta} = \frac{k(\vartheta + \beta, \alpha)}{k(\vartheta, 0)}.
\]  

(2.3)

Unfortunately, this transform cannot be expressed in terms of \( K(\cdot, \cdot) \) and \( \bar{K}(\cdot, \cdot) \). We now point out how to evaluate \( L(\vartheta) := \log k(\vartheta, 0) \); it is easily seen that if one can compute \( K(\cdot, \cdot) \) and \( L(\cdot) \), then one can evaluate (2.3) as well. Using ‘Fubini’ and ‘Frullani’,

\[
\log k(\vartheta, 0) = -\int_0^\infty \int_{(0,\infty)} \frac{1}{t} (e^{-t} - e^{-\vartheta t}) f_{X_t}(x) \, dx \, dt
\]

\[
= -\int_0^\infty \int_{(0,\infty)} \frac{1}{t} (e^{-t} - e^{-\vartheta t}) \mathcal{F}^{-1} [e^{-\phi(\cdot)t}](x) \, dx \, dt
\]

\[
= -\int_{(0,\infty)} \mathcal{F}^{-1} \left[ \int_0^\infty \frac{1}{t} (e^{-t} - e^{-\vartheta t}) e^{-\phi(\cdot)t} \, dt \right] (x) \, dx
\]

\[
= -\int_{(0,\infty)} \mathcal{F}^{-1} \left[ \log \left( \frac{\phi(\cdot) + \vartheta}{\phi(\cdot) + 1} \right) \right] (x) \, dx.
\]

2.2.4 Notes on the inversion techniques

As indicated above, we need algorithms to perform (multi-dimensional) Laplace transforms, and techniques for Fourier inversion and inverse Fourier inversion. In [40] and the appendix of [45] fast and accurate techniques are described. In this section we give a brief account of these.

As indicated above, we need techniques to perform Laplace inversion. The method described in [40] is in the spirit of the (by now classical) approaches developed earlier [1,46], in the sense that it relies on the Poisson summation formula. This Poisson summation formula relates an infinite sum of Laplace transform values.
to the $z$-transform of the function values $f(k\Delta)$, with $k = 0, \ldots, M - 1$, that we wish to evaluate, from which the $f(k\Delta)$ can be computed relying on the well-known fast Fourier transform [34].

A first complication is that the above-mentioned infinite sum tends to converge slowly. For this reason, [1] propose to use a so-called Euler summation, but in general the convergence remains prohibitively slow unless there is knowledge of the location of singularities. One of the contributions of [40] is to approximate the infinite sum by a finite sum by using a Gaussian quadrature. The resulting algorithm is a substantial improvement over earlier algorithms in the sense that (i) it can handle a larger class of Laplace transforms (e.g., no knowledge of the location of discontinuities or singularities is needed), (ii) the algorithm only needs numerical values of the Laplace transform, is fast (that is, the function values $f(k\Delta)$, with $k = 0, \ldots, M - 1$, are computed at once, in order $M \log M$ time), and is (in the one-dimensional case) of nearly machine precision, (iii) can be extended to multiple dimensions [40] Section 5). It is stressed that that last feature is of crucial importance in the setting we are considering in this chapter (having transforms that are at least 2-dimensional).

Recall that the main objective of this chapter is to find a procedure for fast and accurate evaluation of the density $f_{\bar{X}_t}(x)$ of the running maximum, for $x, t > 0$. A significant advantage of the use of the algorithm of [40] is that, for any $t > 0$, it returns essentially the whole distribution of $\bar{X}_t$ at once, that is, $f_{\bar{X}_t}(x)$ for all $x > 0$ (on a grid). This is in contrast with many algorithms proposed earlier, which return $f_{\bar{X}_t}(x)$ for just a single value of $x$ at the time.

The second prerequisite is a fast and accurate technique for performing Fourier inversion and inverse Fourier inversion. The toolbox, presented in the appendix of [45], essentially consists of three routines; a crucial role is played by piecewise polynomials. We omit precise implementation details here, and refer to [45] instead.

(A) Given the values of the transform $\mathcal{F}[f]$ at a prescribed set of points $s_1, \ldots, s_M$, a piecewise polynomial $\mathcal{P}[f]$ is computed for the function $f$. This piecewise polynomial can be efficiently evaluated for any argument.

(B) Given a piecewise polynomial $\mathcal{P}[f]$ of the function $f$, the values of the transform $\mathcal{F}[f]$ are computed at a prescribed set of points $s_1, \ldots, s_M$. This algorithm is the exact inverse of (A).
Given the function values of \( f \) at a pre-described set of points \( t_1, \ldots, t_M \), the piecewise polynomial \( \mathcal{P}[f] \) is computed.

The specific piecewise polynomials in the toolbox are piecewise Legendre polynomials, that is, a linear combination of Legendre polynomials. We can now write down the ‘implementation version’ of Pseudocode 2.1:

**Pseudo code 2.2.** Input: \( \vartheta, \alpha, \) and \( \phi(\cdot) \). Output: an approximation of \( K(\vartheta, \alpha) = \log \mathbb{E}e^{-\alpha X_{e(\vartheta)}} \).

1. Compute a piecewise polynomial for the function
   \[
   \mathcal{F}^{-1} \left[ \frac{\phi'(\cdot)}{\phi(\cdot) + \vartheta} \right](\cdot)
   \]
   by using Algorithm (A);
2. Compute a piecewise polynomial \( \mathcal{P}[F_{\vartheta}] \) for the function \( F_{\vartheta}(\cdot) \) by using Algorithm (C);
3. Set \( \mathcal{P}[F_{\vartheta}](x) := 0 \) for \( x < 0 \); call the resulting piecewise polynomial \( \mathcal{P}^+[F_{\vartheta}] \).
4. Compute a piecewise polynomial \( \mathcal{P}[\bar{T}_{\vartheta}] \) for the function
   \[
   \mathcal{F} \left[ \mathcal{P}^+[F_{\vartheta}] \right](\cdot)
   \]
   by using Algorithm (B).
5. Set \( K(\vartheta, \alpha) := \mathcal{P}[\bar{T}_{\vartheta}](0) - \mathcal{P}[\bar{T}_{\vartheta}](\alpha) + \mathcal{P}[F_{\vartheta}](0) \cdot \log(1 + \alpha). \)

As discussed earlier, the resulting values for \( K(\vartheta, \alpha) \) can then be inverted (using two-dimensional Laplace inversion) to obtain the density of \( \bar{X}_t \).

### 2.3 Numerical Applications

In this section we present a number of representative numerical applications. As a kind of ‘base-case’ we start with Brownian motion, for which the computations can be done explicitly. For the other examples, we have other techniques to validate the numerics. Some examples just involve the running maximum, while others also cover the time at which the maximum was attained, and hitting times of a given threshold.

#### 2.3.1 Experiments

**Example 1: Brownian motion with drift.** In this example we let \( X_t = \sigma W_t + dt \), where \( W_t \) is a standard Brownian motion. In this case the distribution function of
the running maximum $\bar{X}_t$ is explicitly known \cite[p. 49]{60}:

$$p_t(x \mid d, \sigma) := \mathbb{P}(\bar{X}_t \leq x) = 1 - \Phi_N\left(\frac{-x + dt}{\sigma \sqrt{t}}\right) - e^{2dx/\sigma^2} \Phi_N\left(\frac{-x - dt}{\sigma \sqrt{t}}\right), \quad (2.4)$$

with $\Phi_N(\cdot)$ denoting the distribution function of a standard Normal random variable; the density $f_t(x)$ obviously follows by differentiation. We chose the parameters $\sigma = d = 1$, and performed the numerics for $t = 1$. Our algorithm took about 0.2 s on a standard PC, with a maximum error in the order of $10^{-12}$; see Table 2.1. The running time of the algorithm is nearly insensitive in the number of values $x$ for which the density $f_{\bar{X}_t}(x)$ is evaluated.

**Example 2: Compound Poisson process with Normally distributed jumps.** This process is neither spectrally positive nor spectrally negative (and both the upward and downward jumps are not of phase type) — in this situation we do not have an explicit expression of $\mathbb{E} e^{-\alpha \bar{X}_{\epsilon(\cdot)}}$ in terms of the Lévy exponent $\phi(\cdot)$, so that we have to resort to the approach developed in this chapter.

The output of the numerical experiments can be tested by the following procedure (applicable to any compound Poisson process). Let $(J_i)_{i \in \mathbb{N}}$ be the sequence of i.i.d. jumps (with support contained in $[0, \infty)$) with distribution function $G(\cdot)$, and $S_i$ their partial sums. Supposing the jumps arrive according to a Poisson process with rate $\Lambda$, we know that the number of jumps has a Poisson distribution with mean $\Lambda t$. Conditioning on $n$ jumps, we wish to compute

$$F_n(x) := \mathbb{P}\left(\max_{i=1,\ldots,n} S_i \leq x\right),$$

as this would bring us in the position to evaluate $\mathbb{P}(\bar{X}_t \leq x)$ by computing

$$\sum_{n=0}^{\infty} e^{-\Lambda t} \frac{(\Lambda t)^n}{n!} F_n(x).$$

To this end, define

$$F_n(x, y) := \mathbb{P}\left(\max_{i=1,\ldots,n} S_i \leq x, S_n \leq y\right).$$

It is elementary to see that these $F_n(x, y)$ obey the following recursion

$$F_{n+1}(x, y) = \int_{-\infty}^{x} G(x - z) dF_n(x, z).$$
This is essentially a convolution operation, and as a consequence all evaluations of this type can be done fast and accurately (with almost machine precision) with the algorithms developed in [40]. In our experiments we took the standard Normal distribution, the Poisson arrival rate \( \Lambda \) and time \( t \) equal to 1. The performance is again excellent: the running times and errors are in the same order as in the Brownian motion example, or even slightly better; see Table 2.2.

Example 3: Jump-diffusion model. Here we consider an example taken from Nguyen-Ngoc and Yor [95, Example A.1]. There the following jump diffusion model is considered:

\[
X_t = dt + \sigma W_t + \delta N_t, \quad d := \left( r - \frac{\sigma^2}{2} - \Lambda \zeta \right), \quad \delta := \ln(1 + \zeta),
\]

in which \( W_t \) is a standard Brownian motion and \( N_t \) an independent Poisson process with intensity \( \Lambda \) and jumps of deterministic size \( \delta < 0 \); this is the solution of a specific stochastic differential equation [95 Eqn. (2)]. We took the same parameters as in [95], namely \( r = 0.03, \sigma = 0.2, \Lambda = 0.1, \) and \( \zeta = -0.3 \). In the experiments, we evaluate the hitting probability \( \mathbb{P}(\bar{X}_t > x) \). The numerical results are displayed in Table 2.3. We find that our results match those in [95], while the running time is roughly as low as in the previous examples.

Example 4: joint distribution of first passage time & overshoot. In our last example we consider the Pecherskii-Rogozin identity, see Section 2.2.3. More specifically, we want to compute the joint density of the first passage time \( \tau_+(x) \) and the overshoot \( \sigma(x) \). In this example we consider the situation that the underlying Lévy process is a compound Poisson process; we adopt the notation of Example 2. To validate the numerical output, we have used the following alternative computation technique. Realize that the joint distribution function of the first passage time and the overshoot obeys

\[
\mathbb{P}(\tau_+(x) \leq t, \sigma(x) > y) = \sum_{n=0}^{\infty} e^{-\Lambda t} \frac{\Lambda t^n}{n!} \sum_{k=1}^{n} F_k(x, y),
\]

where the empty sum is defined as 0, and \( F_k(x, y) := \mathbb{P}(\max_{i=1, \ldots, k-1} S_i \leq x, S_k > x + y) \). As before, a recursive scheme can be set up to compute the \( F_k(x, y) \). Figure 2.1 plots \( \mathbb{P}(\tau_+(x) \leq 1, \sigma(x) \in du) \), for various values of \( x \) and \( u \). We took standard Normally distributed jumps, and Poisson arrival rate \( \Lambda \) and time \( t \) equal to 1. The running time of the algorithm was again about 0.2 s on a standard PC and the maximum error we observed was of the order \( 10^{-12} \).
A similar procedure can be used for, e.g., the joint distribution of the running maximum and the epoch at which this maximum is attained.

![Joint distribution of the first passage time and the overshoot](image)

Figure 2.1 Joint distribution of the first passage time and the overshoot; the graph shows contour lines. \( \log \mathbb{P}(\tau_+ \leq 1, \sigma(x) \in du) \) is depicted; the hitting level is on the horizontal axis, the overshoot on the vertical axis.

### 2.3.2 Conclusions from experiments, discussion

The above experiments related to a representative set of Lévy processes: Brownian motions (with drift) and compound Poisson processes (and their combination). We focused on the distribution of the running maximum, but also on other quantities, such as first passage times and overshoots. In all experiments we also developed a technique to perform a check by explicit computations. The error was systematically in the order of \( 10^{-12} \), while the computation time was low (typically around 0.2 seconds on a standard PC). All examples showed that our method is highly efficient and accurate.

It is noted that we did not include examples with small jumps, that is, infinitely many jumps in a finite time interval. In these cases one has that \( \int_{-\infty}^{\infty} \Pi(dx) = \infty \). Small jumps complicate the analysis considerably, basically due to the fact that
the Lévy measure looks like $|x|^{\alpha}$ for some $\alpha < -1$. This case, which should be tackled with entirely different techniques, is the topic of the forthcoming paper [57].

As mentioned in the introduction, various other techniques have been proposed. In this respect we mention the recent paper [3]. In this case the jumps are approximated by their phase-type counterpart [9], so that an explicit expression for the Wiener-Hopf factors becomes available [7, 82]; then Laplace inversion, as in [40, 45], is used. This approach has obvious computational advantages (there is no need to transform functions from and to the Fourier domain), but, clearly, the harder it is to approximate the jump distribution by a phase-type distribution, the worse the accuracy. Particularly for heavy-tailed jumps (Pareto, Weibull), the fit can be poor, unless a phase-type distribution of high dimension is chosen. A significant advantage, though, is that the methodology proposed in [3] can deal with small jumps (by replacing them by an appropriately chosen Brownian motion).

## 2.4 Least concave majorant

In this section we show how our tools can be used to numerically evaluate the least concave majorant $\hat{X}_t$ of a Lévy process $X_t$ (that is, the smallest concave function majorizing $X_t$). For the Brownian case, for instance [28, 56] performed explicit calculations and derived structural insights, but so far the general Lévy case remained largely open.

The following lemma applies to any stochastic process $X_t$.

**Lemma 2.3.** Let $(\hat{X}_t)_t$ be the concave majorant of $(X_t)_t$ over the interval $[0, T]$, with $T$ possibly $\infty$. With $t \in [0, T]$, the event $\{\hat{X}_t \leq x\}$ is equivalent to

$$\left\{ \inf_{0 \leq s \leq t} \frac{x - X_s}{t - s} \geq \sup_{t \leq r \leq T} \frac{X_r - x}{r - t} \right\}.$$

**Proof.** Realize that $(x - X_s)/(t - s)$ is the slope of the line through $(t, x)$ and $(s, X_s)$, so that

$$B_- := \inf_{0 \leq s \leq t} \frac{x - X_s}{t - s}$$

is the slope of the ‘steepest’ line through $(t, x)$ that majorizes $X_s$ for any $s \in [0, t]$. Likewise,

$$B_+ := \sup_{t \leq r \leq T} \frac{X_r - x}{r - t}$$

is the slope of the line through $(t, x)$ and $(r, X_r)$ that majorizes any $X_r$ for $r \in [t, T]$. The desired equivalence follows from the fact that the line through $(t, x)$ with slope $B_-$ majorizes $X_s$ for any $s \leq t$, and the line through $(r, X_r)$ with slope $B_+$ majorizes $X_r$ for any $r > t$.
is the slope of the ‘flattest’ line through \((t, x)\) that majorizes \(X_s\) for any \(s \in [t, T]\). Then the stated follows immediately.

From the above lemma it is evident that

\[
\mathbb{P}(\hat{X}_t \leq x) = \int_b^\infty \int_{y = -\infty}^x \mathbb{P}(B_- \geq b, X_t \in dy) \mathbb{P}(B_+ \in \{dy | X_t = y\}) dy.
\]

We now point out how the probabilities (densities, respectively) in the integrand can be determined. It is straightforward to verify that

\[
\mathbb{P}(B_- \geq b, X_t \in dy) = \mathbb{P}\left(\sup_{0 \leq s \leq t} X_s - bs \leq x - bt, X_t \in dy\right). \tag{2.5}
\]

It is evident that we can evaluate (2.5) if we have, with \(Y_t := X_t - bt\), the joint transform

\[
\mathbb{E}e^{-\alpha Y_t - \beta \bar{Y}_t}, \tag{2.6}
\]

by applying inversion with respect to \(\alpha\) and \(\beta\). Realize that, due to Wiener-Hopf theory,

\[
\mathbb{E}e^{-\alpha Y_{\tau(\vartheta)} - \beta \bar{Y}_{\tau(\vartheta)}} = \mathbb{E}e^{-(\alpha + \beta) Y_{\tau(\vartheta)}} \mathbb{E}e^{-\alpha \bar{Y}_{\tau(\vartheta)}}. \tag{2.7}
\]

The factors in the right-hand side of (2.7) can be evaluated with the theory presented in Section 2.2; we then divide their product by \(\vartheta\), so that with inversion with respect to \(\vartheta\) we obtain (2.6). Then inversion with respect to \(\alpha\) and \(\beta\) yields the joint density of \(Y_t\) and \(\bar{Y}_t\), using which we can determine (2.5).

Furthermore, observe that

\[
\mathbb{P}(B_+ \leq b \mid X_t = y) = \mathbb{P}(\forall r \in [t, T]: X_r \leq x - bt + br \mid X_t = y),
\]

which due to the Markov property equals \(\mathbb{P}(\bar{Y}_{T-t} \leq x - y + br)\). This probability can be evaluated relying on the approach proposed in Section 2.2.

### 2.5 Concluding remarks

In this chapter we proposed efficient and accurate numerical evaluation techniques related to fluctuation theory for Lévy processes; fluctuation-theoretic results can be used in a wide range of application domains, e.g., in mathematical finance in the pricing of so-called barrier options. Several quantities were studied, notably the running maximum (including the time at which this maximum is attained) and the hitting time of a given level (including the corresponding overshoot). Our technique is to first rewrite the transforms of interest in terms of the
function $K(\vartheta, \alpha)$, then we developed a technique to compute $K(\vartheta, \alpha)$ in terms of $\alpha, \vartheta$, and the Lévy exponent $\phi(\cdot)$, and finally we rely on the inversion approach of [40] to obtain the densities and probabilities of interest. We performed an extensive set of numerical experiments; we reported on a number of representative examples; the Lévy processes covered Brownian motions with drift, compound Poisson processes, etc. The experiments showed that our approach works excellently, with errors typically in the order of $10^{-12}$, and computation times well below 1 second. It is stressed that fluctuation theory is an important application of the developed techniques, but that various other applications are envisaged; the chapter is concluded by pointing out how our algorithm can be used in order to analyze the Lévy process’ concave majorant.
### 2.6 Appendix A: Numerical Output

<table>
<thead>
<tr>
<th>$x$</th>
<th>Exact value</th>
<th>Our approach</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.100</td>
<td>0.981642775681878</td>
<td>0.981642775681889</td>
<td>1.1·10$^{-14}$</td>
</tr>
<tr>
<td>0.200</td>
<td>0.959808377402784</td>
<td>0.959808377402806</td>
<td>2.1·10$^{-14}$</td>
</tr>
<tr>
<td>0.300</td>
<td>0.934418330627279</td>
<td>0.934418330627315</td>
<td>3.5·10$^{-14}$</td>
</tr>
<tr>
<td>0.400</td>
<td>0.905474132623003</td>
<td>0.905474132623054</td>
<td>5.0·10$^{-14}$</td>
</tr>
<tr>
<td>0.500</td>
<td>0.873063262493356</td>
<td>0.873063262493424</td>
<td>6.8·10$^{-14}$</td>
</tr>
<tr>
<td>0.600</td>
<td>0.837361797336003</td>
<td>0.837361797336090</td>
<td>0.6·10$^{-14}$</td>
</tr>
<tr>
<td>0.700</td>
<td>0.798633285289814</td>
<td>0.798633285289922</td>
<td>1.0·10$^{-13}$</td>
</tr>
<tr>
<td>0.800</td>
<td>0.757223745024288</td>
<td>0.757223745024420</td>
<td>1.3·10$^{-13}$</td>
</tr>
<tr>
<td>0.900</td>
<td>0.713552900554567</td>
<td>0.713552900554721</td>
<td>1.5·10$^{-13}$</td>
</tr>
<tr>
<td>1.000</td>
<td>0.668102001223171</td>
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</tr>
<tr>
<td>1.100</td>
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</tr>
<tr>
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</tr>
<tr>
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<td>0.232357189191843</td>
<td>0.232357189192255</td>
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</tr>
</tbody>
</table>

**Table 2.1** Probability $X_t$ exceeds $x$ before time 1, for the standard Brownian model of Example 1, with absolute value of the error.
### Table 2.2

<table>
<thead>
<tr>
<th>$x$</th>
<th>Exact value</th>
<th>Our approach</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.100</td>
<td>0.331632985165521</td>
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<td>3.6·$10^{-13}$</td>
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<tr>
<td>0.200</td>
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<td>1.4·$10^{-13}$</td>
</tr>
<tr>
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</tr>
<tr>
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</tr>
<tr>
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</tr>
<tr>
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</tr>
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</tr>
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<tr>
<td>2.000</td>
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<td>0.037965730270409</td>
<td>3.4·$10^{-14}$</td>
</tr>
</tbody>
</table>

Table 2.2 Probability $X_t$ exceeds $x$ before time 1, for the compound Poisson process with Normally distributed jumps of Example 2, with absolute value of the error.
### 2. NUMERICAL WIENER-HOPF EVALUATION

**Table 2.3** Probability $e^{X_t}$ exceeds $H = 1.2$ before $t$, for the jump diffusion model.

<table>
<thead>
<tr>
<th>$t$ in years</th>
<th>Our approach</th>
<th>from [95]</th>
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<tr>
<td>1/12</td>
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</tr>
<tr>
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</tr>
<tr>
<td>1/4</td>
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<td>0.080</td>
</tr>
<tr>
<td>1/2</td>
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<td>0.228</td>
</tr>
<tr>
<td>3/4</td>
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<td>0.334</td>
</tr>
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</tr>
<tr>
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</tr>
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<tr>
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<tr>
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<tr>
<td>9.5</td>
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<td>0.808</td>
</tr>
<tr>
<td>10</td>
<td>0.812959374999832</td>
<td>0.813</td>
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</table>