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Chapter 3

Modeling Default Correlation with Applications to CDO Pricing

If all else fails, immortality can always be assured by adequate error.
John Kenneth Galbraith (1908 – 2006)
Money: Whence It Came, Where It Went (1975)

In this chapter we propose an intuitive, practical, Lévy-based, dynamic default correlation model, with applications to CDO pricing. To develop such a model, we first capture the marginal default probabilities relying on a dynamic structural Variance Gamma (VG) process. Then we impose a dynamic correlation structure on the individual obligors, of which the marginals are in line the VG-based model for the single names. This correlation model is constructed by decomposing the VG processes into two components: an individual component and a common component; the former affects just one obligor, whereas the latter has impact on the entire market.

The key advantages of our correlation structure are that it is intuitive, dynamic, and allows for easy calibration to the market since the underlyings are market observables, for which there is ample data available. In case of a homogeneous basket of obligors, the evaluation of our model is as easy as a copula-based approach. In case of a non-homogeneous basket, the computations can still be done in a fast and accurate way. An important role is played by recently developed techniques for numerical Laplace transform inversion. Our approach is backed up by various numerical experiments.

\footnote{This chapter is based on joint work with Peter den Iseger and Michel Mandjes and has been submitted as \cite{44}.}
3.1 Introduction

This chapter presents a simple, yet versatile structural correlation model that enables dynamic evaluation, in semi-closed-form, of multi-name credit derivatives. The standard example of such a derivative is the multi-name Collateralized Debt Obligation (CDO), which can be seen as a product that involves a basket of single-name products, viz. Credit Default Swaps (CDSs). Here these CDSs can be seen as contracts that ensure protection against default of an underlying reference credit: in case of a default the protection seller pays the protection buyer a given cash amount, but if there is no default before maturity, then the buyer pays the seller a fixed rate. A CDO is a basket of CDSs which are split into different risk classes, or tranches, which are paid sequentially from the most senior tranche to the most subordinate (and generally unsecured) tranche. Hence, tranches with a first claim on the assets of the asset pool are called senior tranches, whereas tranches with either a second claim or no claim at all are called junior notes. Interest and principal payments are made in order of seniority, so that junior tranches offer higher coupon payments (and interest rates) or lower prices to compensate for additional default risk. Usually, the more senior-rated tranches have the highest ratings: for example, senior tranches may be rated AAA, AA or A, while a junior, unsecured tranche may be rated BB.

Valuing CDOs is highly non-trivial, and in fact the imperfections of the existing pricing schemes are considered one of the reasons of the credit crisis of 2008-2009. In this crisis, it became apparent that the commonly used schemes were not capable of pricing CDOs accurately, as they did not capture all correlations adequately: they left a substantial amount of risk in the market unpriced. In this chapter we develop a semi-analytical algorithm for CDO pricing, which includes a novel approach to incorporate the correlation between the underlying obligors. Before pointing out the specific contributions of our work in greater detail, we first briefly sketch the state of the art, and point out the shortcomings of the existing methods.

Broadly speaking, in the literature two types of credit modeling are distinguished: intensity-based models and structural models. Intensity-based models describe defaults by means of an exogenous jump process. In these models defaults are not triggered by basic market observables, but by an exogenous component which does not depend on information from the market. In this sense, one could say that there is no economic rationale behind defaults. There is a vast body of litera-
ture on intensity-based models; to name a few practical models that calibrate well to CDS data: Duffie and Singleton [47] and Lando [76]; see also Brigo and Alfonsi [21]. More recently, Joshi and Stacey [70] proposed a new type of intensity-based model, in which the default intensity is represented by a gamma process, where the correlation between the individual obligors results from the dependence between these gamma processes. In many models, copulas are used to introduce correlation, as first proposed by Li [83], which, primarily owing to its tractability, quickly became the industry standard. In such models an important role is played by the Gaussian copula, but, because of its inherently thin tails, such models leave a substantial portion of risk unpriced. For more background on intensity-based models we refer the reader to e.g. [16, Chapter 8], and the references therein.

In structural models, dating back to the seminal work of Merton [92], the default of a firm is linked to its ability to pay back its debt. The advantage of structural models, as opposed to intensity-based models, is that they explicitly model the underlying economic processes. In Merton’s model defaults can only occur at the maturity of a bond, but this restrictive assumption was later relaxed by taking into account the entire sample path of the underlying assets by using so-called first-passage-time models. There are numerous references, e.g. Black and Cox [18], Longstaff and Schwartz [84], Leland and Toft [81], and Leland [80]. The main drawback of these models lies in the fact that they are of diffusion-type. Firstly, they do not yield positive credit spreads in the very short run: they tend to underestimate the instantaneous default probability (due to the thin tails), and therefore predict that the term structure of credit spreads should always start at zero and slope upward for firms that are not in financial difficulty, while in reality credit spreads exhibit flat or even downward sloping rates. Secondly, these relatively simple diffusion-type models exhibit a poor spread fit over longer maturities. This is due to the fact that one is always faced with the tradeoff between, on the one hand, a parsimonious parameterization which cannot guarantee consistency across option strikes and expirations, but which does lead to a more manageable model, and, on the other hand, a model that can capture the whole market at the price of unstable parameters and, very likely, unrealistic forward dynamics.

In order to add a volatility skew while retaining tractability, various authors proposed asset processes with jumps, e.g. Zhou [109], Hilberink and Rogers [62], Cariboni and Schoutens [27], Madan and Haluk [88], and Albrecher et al. [2], leading to convincing results, particularly for a specific class of Lévy processes,
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viz. Variance Gamma (VG) processes, [2, 27, 88]. The proposed models, however, lack one important feature: observable and intuitive dependency among a large number of obligors. Zhou [109] presents an analytic result for the default correlation between only two firms. Hilberink and Rogers [62] focus on the behavior of the credit spread in the very short run of a single firm. Cariboni and Schoutens [27] present a working model for the pricing of CDSs (univariate), but not for CDOs (multivariate). Madan and Haluk [88], although using the intuitively appealing VG model, assume that default payouts are independently and identically distributed across time and state, which is not in line with what is observed in the market (where the underlying reference obligors of a CDO, whose number is in the order of 125, are highly correlated).

In order to adequately price CDOs, it is clear that one should use models that incorporate the correlation between obligors in a sound way. More specifically, those models should capture the correlation between a large number of reference obligors, and in addition they should be capable of calibrating to the market. In the last few years a number of new structural models have been proposed that deal with the disadvantages of the earlier structural models and the Gaussian copula model mentioned above. These new types of structural models mainly rely on the following economic ideas. Firstly, since credit events are rare, it is important to accurately model the tails of the distributions involved. Since the tails of the normal distribution are too light to properly describe the market, it was proposed to use jump processes, which result in heavier tails and show a better performance; in particular, the aforementioned VG model gained popularity. Secondly, in order to accurately describe credit market reality, it is important to distinguish between defaults occurring due to global market and name-specific reasons. It is stressed that distinguishing between these global and specific shocks is crucial for modeling the correlation between the single-name entities, and may have a profound effect on the resulting prices and credit spreads.

Luciano and Schoutens in [86] proposed a multivariate VG process with a global random time change to model financial assets which incorporate jumps, skew, kurtosis and stochastic volatility. However, they do not incorporate idiosyncratic jumps and they do not attempt to price CDOs. Although the above models properly describe the jump nature of credit events, they fail to capture the effect of common and individual credit factors. Baxter [11] introduced a new family of structural models applicable to a large class of correlated credit products which
deals with both the jump and correlation issues. A key notion there is the distance
to-default of the reference obligors, which is used to assess the probability that
there is a default before some given future time $T$. An important feature of
all of the models is that they also describe the correlation between single-name
distance-to-default processes. This is achieved by decomposing the credit events
(process jumps) into events common for all obligors and name-specific events.
The main drawback of his approach, however, is that only one random variable
is used to model correlation, similar to the Gaussian copula model [83].

Jäckel [66] developed several similar discrete-time models for pricing more exotic
variants of credit products, e.g. multi-callable CDOs. The main building blocks of
the model are the correlation structure as suggested by Baxter [11] and stochastic
differential equations for the credit default hazard rate. Then a large homogeneous
portfolio approximation is used to find the default probabilities of each
obligor, as well as the prices of credit products over any given time step, by solving
a discrete integro-differential equation, which is numerically quite demanding.
Fiorani et al. [49] present a parsimonious structural model with purely dis-
continuous assets that takes into account correlation. However, in their model a
default can only occur at the end of the time horizon $T$, so that the setup reduces
to investigating a bond of maturity $T$, and looking at time $T$ whether or not the
bond can be paid back. In other words, their model does not cover defaults be-
fore time $T$, thus reducing the practical relevance significantly. Albrecher et al. [2]
assume a large homogeneous portfolio, i.e., all obligors have the same default
level, notional amount and recovery rate. Then a limiting argument is used to de-
termine the distribution of the CDO. While this approach gives valuable qualita-
tive insights, due to the unrealistic assumptions it is of limited practical relevance.

In view of the deficiencies of the existing models, the goal of this chapter is to
develop a practical model for pricing multi-name credit derivatives, including a
parsimonious yet intuitive correlation structure. Importantly, the resulting model
remains numerically tractable. In our approach, we start by modeling the indi-
vidual default probabilities using a Lévy-based approach: the distance-to-default
obeys a Variance Gamma (VG) process. Then, as a second step, a suitably chosen
correlation structure is imposed on the individual obligors. This is done by de-
composing the VG processes in two components: an individual component and a
common component. Our structure can be interpreted such that a fraction $(1 - \rho)$
of the time obligor $k$ is exposed to individual shocks, and a fraction $\rho$ of the time to
common shocks (this is a consequence of the independent increments property of Lévy processes). A VG process consists of two components, a gamma component (the subordinator that represents a random time change) and a Brownian component. We condition on the common gamma component and the minimum of the Brownian component. The underlying Brownian motions as well as the subordinators can be calibrated to the market (so that the marginal default probabilities coincide with the probabilities in the single-name model). The key advantage of our correlation structure is that it is intuitive, dynamic, and allows for easy calibration to the market since the underlyings are market observables, for which there is ample data available. The evaluation of our model is fast and accurate, and therefore does not impose any restrictions on numerical computations. This method relies on the computation of convolutions to calculate a recursion, which can be efficiently done using numerical Laplace transform inversion. Furthermore, in case of a homogeneous basket of obligors, our model is as easy as the Copula approach of Li \cite{83}. In more detail, the most significant contributions of this chapter are:

- The correlation structure we impose on the structural reference obligors is more intuitive than the one used in the intensity-based copula approach: we impose it on the underlying reference obligors, rather than on default intensities. Furthermore, in case of a homogeneous basket, our model is as easy as the copula model.

- Furthermore, our correlation structure is stochastic, i.e. our model describes the well known phenomenon of the joint event of high volatilities and high correlations of CDS spreads.

- We impose correlation on observable market prices, such as CDS spreads (of which there is ample data available), while intensity-based methods (e.g. the copula approach) impose correlation on the default levels (of which evidently hardly any data is available).

- Our model can be easily calibrated to the market; this procedure is fast and accurate, and relies on the computation of convolutions to calculate a recursion, which can be efficiently and accurately done using the novel Laplace inversion algorithm developed by den Iseger \cite{40}.

- Also, our model is very flexible, in the sense that by changing the parameters of the model, we can obtain many different shapes of the loss distribution of the CDO basket.
We can choose the univariate processes (i.e., the distance to default for the individual obligors) freely, and each univariate process can be separately calibrated to the CDS term structure. The calibration of the univariate processes can be separated from the calibration of the correlation structure. This makes calibration significantly more straightforward. One can choose, for example, to estimate the correlation periodically, and calibrate the univariate processes separately to CDS data.

In our model it is possible to model the correlation of each process separately, and therefore the correlation structure is both flexible and realistic (in that it gives a good fit with the actual prices).

We do not use a ‘European’ approximation for the defaults, but we model the actual defaults. That is, we do not approximate the path-dependent event of a hit of a certain threshold in a time interval, by the hit of a threshold at a fixed time point.

This chapter is organized as follows. In Section 3.2 we model default probabilities of single-name CDSs. Then Section 3.3 briefly recalls some CDS pricing preliminaries, including some pseudo code for pricing a CDS using the setup of Section 3.2. The next section covers a number of numerical calibration examples for single-name CDS pricing, verifying whether our model calibrates well, and comparing the output with that obtained by using the method described in [27]. Section 3.5 contains the “heart” of the chapter and presents our approach for imposing correlation on the CDO’s reference obligors. We then provide in Section 3.6 an efficient way to compute the loss distribution of the CDO basket, both in case of a homogeneous and of a non-homogenous basket. Subsequently, in Section 3.7 we present a procedure to estimate and calibrate the parameters of our model. In order to make this chapter essentially self-contained, we briefly recall some CDO pricing definitions in Section 3.8, including the pseudo code for pricing a CDO using our approach. The results of our numerical experiments are presented in Section 3.9. Finally, in Section 4.7 we conclude. Lengthy proofs and output of the numerical examples can be found in the various appendices.

3.2 Modeling Default Probabilities

In this section we present expressions for the default probabilities of single-name CDSs. As a first step, we consider the model in which the distance-to-default of
the underlying reference obligors can be modeled by a geometric Brownian motion (GBM), to point out how the theory works. In a second step we use Variance Gamma (VG) processes, which form a generalization of the Brownian case (but considerably more versatile, and therefore more suitable for our purposes). In the remainder of the chapter these single-name results are used as building blocks of the multi-name model (which we present in Section 3.5).

To start the exposition, let \( \{ X_k(t), k = 1, \ldots, K \} \) be a collection of Lévy processes that are used to measure the distance-to-default of the underlying reference obligors; in a standard CDO, the total number of underlying obligors \( K \) is usually 125. We say that the underlying reference obligor \( k \) has a default in the time interval \( [t, T) \) if and only if
\[
\min_{t \leq s < T} X_k(s) \leq x_0^{(k)},
\]

for a given default level \( x_0^{(k)} < 0 \).

Now let us define the distributions of (i) the number of defaults in the time interval \( [t, T) \), i.e., \( L(t, T) \), and (ii) the number of jumps of the default process from \( \ell \) to \( \ell + 1 \) in the time interval \( [t_j, t_{j+1}) \), i.e., \( \Delta L_\ell [t_j, t_{j+1}) \); notice that the latter random variable is either 0 or 1. We begin by introducing the binary random variable \( A_k(t, T) \) that equals 1 if and only if obligor \( k \) has a default in \( [t, T) \), i.e.,
\[
A_k(t, T) := 1 \left\{ \min_{t \leq s < T} X_k(s) \leq x_0^{(k)} \right\}.
\]

We now have that the total number of defaults in the time interval \( [t, T) \) is given by
\[
L(t, T) = \sum_{k=1}^{K} A_k(t, T).
\]

We can define \( \Delta L_\ell [t_j, t_{j+1}) \) as
\[
\Delta L_\ell [t_j, t_{j+1}) := 1\{L(0, t_j) \leq \ell\} 1\{L(0, t_{j+1}) > \ell\}.
\]

As mentioned above, the latter random variable relates to jumps of \( L \) from \( \ell \) to \( \ell + 1 \), where it is noted that the event that multiple obligors simultaneously jump to default is excluded by assumption. This is justified by real life experience.

Note that in our multi-name model, to be presented in Section 3.5, we correlate the Lévy processes \( \{ X_k(t), k = 1, \ldots, K \} \). It is this correlation that makes the
computation of the distributions of $L(t,T)$ and $\Delta L_\ell[t_j,t_{j+1}]$ challenging, as will become clear later. We will argue in Section 3.8 that these distributions enable the numerical computation of most credit products. Although this chapter features CDO pricing, this should be seen as just an example, as we are able to price many other credit products that are based on $L(t,T)$ and $\Delta L_\ell[t_j,t_{j+1}]$. In fact, we can express the distribution of $\Delta L_\ell[t_j,t_{j+1}]$ in terms of the distribution of $L(t,t_j)$ and the distribution of $L(t,t_{j+1})$.

**Lemma 3.1.** The expected number of jumps to default from $\ell$ to $\ell = \ell + 1$ in the time interval $[t_j,t_{j+1})$ is given by

$$P[\Delta L_\ell[t_j,t_{j+1}) = 1] = P(L(0,t_j) \leq \ell) - P(L(0,t_{j+1}) \leq \ell).$$

**Proof.** First note that

$$1\{\Delta L_\ell[t_j,t_{j+1}) > 0\} = 1\{L(0,t_j) \leq \ell\} \cup \{L(0,t_{j+1}) > \ell\} = 1\{L(0,t_j) \leq \ell\} - 1\{L(0,t_{j+1}) \leq \ell\}.$$ 

Where we implicitly used that we exclude the event that multiple obligors simultaneously jump to default. This implies that

$$P[\Delta L_\ell[t_j,t_{j+1}) = 1] = E1\{\Delta L_\ell[t_j,t_{j+1}) > 0\} = E1\{L(0,t_j) \leq \ell\} - E1\{L(0,t_{j+1}) \leq \ell\} = P(L(0,t_j) \leq \ell) - P(L(0,t_{j+1}) \leq \ell),$$

which yields the stated. 

The objective of this chapter is to develop a fast and accurate method to compute $P(L(t,t_j) \leq \ell)$ for $j = 1, \ldots, J$.

### 3.2.1 Geometric Brownian Motion

We now assume that the distance-to-default of the underlying reference obligors $X_k(\cdot)$ can be expressed by a Geometric Brownian Motion (GBM) of which we model the logarithm:

$$X_k(t) := a_k t + \sigma_k W_k(t);$$

(3.3)

here $a_k$ and $\sigma_k$ are the forward and volatility of obligor $k$, respectively, and $W_k(\cdot)$ is a standard Brownian motion. To make the formulae more concise, we suppress the index $k$ in the remainder of this section, since all these computations relate to
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a single obligor.

We now focus on the probability that a certain reference obligor \( X(\cdot) \) hits the default level \( x_0 \) in the time interval \([0, t)\), given that there is no default before time 0 and given that the process starts at 0, i.e.,

\[
F_t(x_0) = \mathbb{P} \left( \min_{0 \leq s \leq t} X(s) \leq x_0 \mid \min_{s \leq 0} X(s) > x_0, X(0) = 0 \right),
\]

(3.4)

with \( x_0 < 0 \). Note that without loss of generality we may consider the case of a default in \([0, t)\); we could also have considered a default in some interval \([t, T)\), but this would make formulae unnecessarily opaque. It is a standard result \(^{[69]}\) that (3.4) is explicitly given by

\[
F_t(x_0) = N \left( \frac{x_0 - at}{\sigma \sqrt{t}} \right) + e^{2a\sigma^{-2}x_0} N \left( \frac{x_0 + at}{\sigma \sqrt{t}} \right)
\]

\[
= \mathbb{P} \left( at + \sigma W(t) \leq x_0 \right) + e^{2a\sigma^{-2}x_0} \mathbb{P} \left( -at + \sigma W(t) \leq x_0 \right); \quad (3.5)
\]

here \( W(\cdot) \) is a standard Brownian motion, and \( N(\cdot) \) is the standard Normal distribution function.

**Proposition 3.2.** It holds that

\[
F_t(x_0) = \mathbb{E} \left[ \Phi(X(t) - x_0) \right],
\]

(3.6)

with

\[
\Phi(x) := (1 + e^{-2a\sigma^{-2}x}) 1 \{ x \leq 0 \}.
\]

(3.7)

**Proof.** By making a drift adjustment of \(- (2a/\sigma)t\) to the Brownian motion in the second term of the right-hand side of Equation (3.5), we obtain

\[
e^{2a\sigma^{-2}x_0} \mathbb{E} 1 \{ -at + \sigma W(t) \leq x_0 \} = e^{2a\sigma^{-2}x_0} e^{-\frac{1}{2} (2a \sigma)^2 t - 2\sigma W(t)} 1 \{ at + \sigma W(t) \leq x_0 \} = \mathbb{E} e^{2a\sigma^{-2} (at - \sigma W(t) + x_0)} 1 \{ at + \sigma W(t) - x_0 \leq 0 \}.
\]

This immediately yields the stated. \(\square\)

The following property turns out to be useful later on.

**Lemma 3.3.** \( \mathbb{E} \Phi(X(t)) = 1 \) for all \( t \).

**Proof.** Using the above transformations, it is readily verified that

\[
\mathbb{E} \Phi(X(t)) = N \left( -\frac{at}{\sigma \sqrt{t}} \right) + N \left( \frac{at}{\sigma \sqrt{t}} \right),
\]

which trivially equals 1. \(\square\)
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3.2.2 Variance Gamma

From now on, we assume that the logarithm of the distance-to-default of the \( k \)-th obligor can be measured through a Variance Gamma (VG) process \( X_k(\cdot) \). As mentioned earlier, the most substantial advantage of the VG process over geometric Brownian motion is that it is more flexible, and thus allows for better calibration while retaining a parsimonious set of parameters. Brownian motion can be thought of as a special case of a VG process, see e.g. Madan et al. in \[87\], who first introduced the use of VG in finance. Like before, we will suppress the index \( k \) since computations only involve a single product.

A VG process is a real-valued Lévy process \( X(\cdot) \) which can be obtained as a Brownian motion with drift, subjected to a random time change by a gamma process \( \Gamma(\cdot) \), to which we will often refer to as the subordinator. A gamma process \( \{\Gamma(t; v, \theta), t \geq 0\} \), or simply \( \Gamma(\cdot) \), with shape parameter \( v \) and scale parameter \( \theta \), is a Lévy process such that the defining distribution of \( X(1) \) is gamma with parameters \( (v, \theta) \). Now let \( W(\cdot) \) be a standard Brownian motion, \( \{\Gamma(t; v, \theta), t \geq 0\} \) a gamma process with \( v > 0, \theta > 0 \), and \( \sigma > 0 \) and \( a \) be real parameters. Then the VG process \( X(\cdot; \sigma, v, a) \), or simply \( X(\cdot) \), is defined as

\[
X(t) = a\Gamma(t) + \sigma W(\Gamma(t)).
\]

(3.8)

with \( \Gamma(\cdot) \perp W(\cdot) \). Note that because

\[
\sigma^* W(\Gamma(t; v, \theta)) \overset{d}{=} \sigma^* \sqrt{\theta} W(\Gamma(t; v, 1)) = \sigma W(\Gamma(t; v, 1))
\]

with \( \sigma := \sigma^* \sqrt{\theta} \), we can absorb the scale parameter \( \theta \) into the volatility parameter \( \sigma \), so that we can set, without loss of generality, \( \theta = 1 \). In case of the VG process \( X(\cdot) \), we have for the default probability, which is defined as in the GBM case but now with \( X(\cdot) \) being our VG process, the following:

\[
\bar{F}_t(x_0) = \mathbb{P}\left( \min_{0 \leq s \leq t} \{a\Gamma(s) + \sigma W(\Gamma(s))\} \leq x_0 \mid \min_{s \leq 0} X(s) > x_0, X(0) = 0 \right).
\]

Similar to the GBM case, we now replace \( \bar{F}_t(x_0) \) by \( F_t(x_0) \) for further computations in this chapter:

\[
F_t(x_0) = \mathbb{P} [a\Gamma(t) + \sigma W(\Gamma(t)) \leq x_0] + e^{2a\sigma^2 x_0} \mathbb{P} [-a\Gamma(t) + \sigma W(\Gamma(t)) \leq x_0],
\]

(3.9)

with \( W(\cdot) \) a standard Brownian motion, independent of \( \Gamma(\cdot) \). It is noted that due to the jumps of the Gamma process, \( F_t(x_0) \) and \( \bar{F}_t(x_0) \) do not exactly coincide.
However, for CDO pricing purposes, when using a fine time grid in the numerical experiments, the resulting match is sufficiently close. See e.g. [77] for a discussion on limiting results from $\bar{F}_t(x_0)$ to $F_t(X_0)$.

Using the same arguments as in Section 3.2.1 we have the following proposition.

**Proposition 3.4.** It holds that

$$F_t(x_0) = \mathbb{E}[\Phi(X(t) - x_0)],$$

with $\Phi(x)$ defined as in Equation (3.7).

**Proof.** By making a drift adjustment of $-(2a/\sigma)\Gamma(t)$ to the Brownian motion in the second term of the right-hand side of Equation (3.9), and conditioning on the gamma process, we obtain

$$e^{2a\sigma^{-2}x_0}\mathbb{E}\left\{e^{-\frac{1}{2}(2\frac{a}{\sigma})^2\Gamma(t) - 2\frac{a}{\sigma}W(\Gamma(t))}1 \{a\Gamma(t) + \sigma W(\Gamma(t)) \leq x_0\} \left| \Gamma(t) \right. \right\}$$

$$= e^{2a\sigma^{-2}x_0}\mathbb{E}e^{-\frac{1}{2}(2\frac{a}{\sigma})^2\Gamma(t) - 2\frac{a}{\sigma}W(\Gamma(t))}1 \{a\Gamma(t) + \sigma W(\Gamma(t)) \leq x_0\}$$

$$= \mathbb{E}e^{2a\sigma^{-2}(a\Gamma(t) - \sigma W(\Gamma(t)) + x_0)}1 \{a\Gamma(t) + \sigma W(\Gamma(t)) - x_0 \leq 0\}.$$ 

This completes the proof. \qed

### 3.3 Single Name CDS Pricing Preliminaries

Before presenting a numerical example in Section 3.4 in which we price CDSs, we briefly recall some CDS pricing definitions in this section, which we took from Brigo and Mercurio [22] and adapted to our framework and notation.

Divide the time interval $[0, T]$ in $0 = t_0 < t_1 < \ldots < t_J = T$. The discount factor between 0 and $t_j$ is given by $D(0, t_j)$. The price of the so-called default-leg of a CDS can now be represented by

$$\Pi_D := (1 - R) \sum_{j=0}^{J-1} D(0, t_{j+1}) \mathbb{E} [1 \{\tau < t_{j+1}\} - 1 \{\tau < t_j\}]$$

$$= (1 - R) \sum_{j=0}^{J-1} D(0, t_{j+1}) \mathbb{E} 1 \{t_j < \tau \leq t_{j+1}\}$$

$$= (1 - R) \sum_{j=0}^{J-1} D(0, t_{j+1}) (F_{t_{j+1}}(x_0) - F_{t_j}(x_0)),$$ (3.11)
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in which \( \tau \) is the default time, \( F_t(x_0) \) the default probability in the interval \([0,t)\), and \( R \) is the so-called recovery rate. Expression (3.11) gives the discounted expected default amount in time interval \([0,T)\), which is by definition the price of the default-leg of a CDS.

Similarly, one computes the price of the so-called premium leg of a CDS as

\[
\Pi_P = P \sum_{j=0}^{J-1} D(t_j) \mathbb{E}\{\tau > t_j\} = P \sum_{j=0}^{J-1} D(t_j) \left(1 - F_t(x_0)\right),
\]

(3.12)

in which \( P \) denotes the premium that has to be paid. Equation (3.12) gives the discounted expected amount of premium paid in time interval \([0,T)\), which is by definition the price of the premium-leg of a CDS. For a protection buyer the price of a CDS is simply the difference \( P_{CDS} := \Pi_P - \Pi_D \). For a protection seller, the price is the negative of this number.

As shown in this section, a prerequisite for CDS pricing is a procedure to compute the default probabilities. Using the method described in Section 3.2, we can directly compute CDS prices. We include the pseudo code for pricing CDSs using our method; we focus here on VG as underlying Lévy process, but evidently a similar procedure can be followed in the GBM case.

**Pseudo code 3.5.** Pseudo code for pricing a CDS:

1. Calculate default probabilities using Equation (3.10);
2. Calculate the default leg and premium leg of the CDS using Equations (3.11) and (3.12), respectively;
3. Subtract the default leg from the premium leg to get the price of a protection buyer CDS, and vice versa for a protection seller.

In Section 3.4 we present numerical examples in which this pseudo code is implemented.

### 3.4 Numerical Example: Calibration of the Single Name CDS Data

In this section we present two numerical examples for the calibration of single-name CDS prices, both for the GBM case and for the VG case. Firstly, as some
sort of sanity check, we calibrate the individual processes to single name CDS that was used in Cariboni and Schoutens [27], and compare it with results obtained through our method. This data is a series of CDS term structures taken from the market on the 26th of October 2004 for a whole range of differently rated (according to Moody’s rating) companies. Secondly, we calibrate the single-name data to our own (post credit crisis) data, which consists of the time series of the entities of the investment grade (BBB or higher) iTraxx Europe Index for 3, 5, 7 and 10 year maturities on February 19th, 2010. We perform these experiments to check how well our model calibrates. The iTraxx Europe Index is composed of 125 investment grade entities from six sectors: automotive, consumers, energy, financials, industrials and TMT (telecom, media, and technology).

To enable a proper comparison, we follow [27] in calibrating the model in the least squares sense. Let \( N \) be the square root of the maturity of the term structure. Let \( \pi_n \) be the market price corresponding to the \( n \)-th maturity, and let \( \bar{\pi}_n(\sigma, v, a) \) the price resulting from the model with parameters \( \sigma, v, \) and \( a \). Then the root mean square error (RMSE) given by

\[
\text{RMSE} := \min_{\sigma, v, a} \sqrt{\frac{1}{N} \sum_{n=1}^{N} (\pi_n - \bar{\pi}_n(\sigma, v, a))^2},
\]

i.e., it minimizes the \( L^2 \) norm between theoretical and market prices. We also compute the average absolute error as a percentage of the mean CDS price (APE), which is given by

\[
\text{APE} := \frac{1}{M} \min_{\sigma, v, a} \sum_{n=1}^{N} \frac{1}{N} \left| \pi_n - \bar{\pi}_n(\sigma, v, a) \right|,
\]

with \( M \) being the mean CDS price (averaged over the various maturities).

The calibration results based on the data of [27] are given in Table 3.1 in Appendix 3.14. As expected, we find that the GBM case has problems calibrating, especially in calibrating the short end (that is, the 1 and 3 year maturities), due to the facts that the credit term structure is steep on the short end and that we have a degree of freedom less to calibrate with. We also find that the calibration results of the VG case are very similar to those of Cariboni and Schoutens in [27] in terms of parameter values and APE. However, our VG model outperforms their model in the RMSE sense for every entity, although we calibrate with one parameter (degree of freedom) less. Furthermore, when pricing the CDS term structure, the approach
of [27] requires the solution of a partial differential integral equation, while our approach relies on semi-closed form expressions. The calibration results of our own data set are given in Tables 3.2–3.6. We observe that the results are in the same order of magnitude as those based on the dataset of [27].

3.5 Correlation Structure

In Section 3.2 we presented the machinery to price single name CDSs, i.e., the univariate modeling of default probabilities for the underlying reference obligors; in Section 3.4 we gave a numerical example in which the resulting procedures were applied. The goal of this section is to develop machinery to price multi-name CDOs, i.e., multivariate modeling of default probabilities of a basket of reference obligors. This is evidently not a straightforward extension of the univariate case, as in multi-name credit modeling the way correlation is incorporated plays a crucial role. When a crash hits the market, correlations shoot up, which has far-reaching pricing consequences. For a model to be practically relevant, a correlation structure should be both intuitive and parsimonious, i.e., the dynamics should be clear and its dimensions not too large. We present our approach in this section.

We assume that the reference obligors are exposed to common and individual shocks; we were inspired by the dynamics of the correlation structure proposed by Baxter [11]. In Baxter’s setup, the structure is such that a fraction $1 - \rho$ of time the obligors are exposed to the individual shocks, and a fraction $\rho$ of the time to the common shocks. Economically, this is quite intuitive; if a market-wide event hits the reference obligors, this will affect all the processes $X_k(t)$, but if a company-specific event occurs, it will only affect the specific reference obligor. Our model is more flexible, in the sense that the fraction of time obligor $k$ is exposed to individual and common shocks is obligor-specific (and equals $1 - \rho_k$ and $\rho_k$, respectively), as we can calibrate a parameter per obligor.

There are several other advantages of our method over Baxter’s. In our setup we model the dynamics of the underlying processes, in that our correlation relates to each of the subintervals of $[0, T]$, whereas Baxter’s correlation is constant over time. Furthermore, Baxter uses a ‘European’ default approximation, which essentially means that he approximates $\mathbb{P}(\min_{0 < s \leq t} X(s) < x_0)$ by $\mathbb{P}(X(t) < x_0)$, whereas we do not resort to the approximation.
The correlation structure we present for pricing CDOs is explained below. This correlation structure fits in a rich class of general correlation structures which is described in Appendix 3.11 Our correlation model allows for fast and accurate estimation and calibration procedures, which are presented in Section 3.7.

In this section we propose a correlation structure on the individual Lévy processes, which are of the form

\[ X_k(t) = a_k \Gamma_k(t) + \sigma_k W_k(\Gamma_k(t)). \]

It turns out that the analysis becomes substantially easier when assuming \textit{a priori} \( a_k = 0 \); most notably, it will turn out that results from the paper \[65\] become applicable; the computations for \( a_k \neq 0 \) can in principle be performed, but become substantially less tractable. Notice that still a sufficiently high number of parameters is left, so that this assumption tends to have just a minor impact. Furthermore we set, \( \sigma_k = 1 \), since flexibility of the model due to \( \Gamma(\cdot) \) is sufficient for calibration purposes. Thus, we end up with

\[ X_k(t) = W_k(\Gamma_k(t)). \] (3.13)

We start by imposing a correlation structure on the VG process in (3.13) by decomposing it into two independent VG processes: one company-specific process and one market process. We expose the gamma process a fraction \( \rho_k \) of the time to common shocks and a fraction \( 1 - \rho_k \) of the time to individual shocks. The deeper properties of this correlation structure are discussed in Appendix 3.11. Due to the common component, the processes \( X_k(\cdot) \) have a specific correlation structure; the parameters \( \rho_k \) follow through calibration.

Before we introduce the specific form of the processes \( X_k(\cdot) \), we first recapitulate a number of known properties. In the first place, with \( W(\cdot) \) being a centered Brownian motion, we have

\[ W(\alpha t_1 + \beta t_2) \overset{d}{=} \sqrt{\alpha} W_1(t_1) + \sqrt{\beta} W_2(t_2), \] (3.14)

with \( W_1(\cdot) \) and \( W_2(\cdot) \) independent standard Brownian motions. In the second place, because of the stationary and independent increments property of of gamma processes, we have for a gamma process \( \Gamma_k(\cdot) \) with shape parameter \( v_k \), that

\[ \Gamma_k(t) \overset{d}{=} \Gamma_c \left( \frac{\rho_k t}{v_k} \right) + \tilde{\Gamma}_k \left( \frac{(1 - \rho) t}{v_k} \right), \] (3.15)
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where $\rho_k \in [0, 1]$ and $\Gamma_c(\cdot)$ and $\tilde{\Gamma}_k(\cdot)$ are independent standard (i.e. $\theta = 1$) gamma processes. From (3.14) and (3.15) we have that

$$W_k(\Gamma_k(t)) \overset{d}{=} \tilde{W}_c \left( \Gamma_c \left( \frac{\rho_k t}{v_k} \right) \right) + \tilde{W}_k \left( \tilde{\Gamma}_k \left( \frac{(1 - \rho_k) t}{v_k} \right) \right), \quad (3.16)$$

where $\tilde{W}_c(\cdot)$ and $\tilde{W}_k(\cdot)$ are independent standard Brownian motions. This can be seen as follows.

$$\mathbb{E} e^{-s W_k(\Gamma_k(t))} = \mathbb{E} \mathbb{E}_{\Gamma_c} \left( e^{-s \tilde{W}_c(\Gamma_c(\frac{\rho_k t}{v_k}))} \left| \Gamma_c \left( \frac{\rho_k t}{v_k} \right) \right. \right) \left. \times \mathbb{E}_{\Gamma_k} \left( e^{-s \tilde{W}_k(\tilde{\Gamma}_k(\frac{(1 - \rho_k) t}{v_k}))} \left| \tilde{\Gamma}_k \left( \frac{(1 - \rho_k) t}{v_k} \right) \right. \right) \right),$$

We introduce correlation by letting $\tilde{W}_c(\Gamma_c(\cdot))$ be the same process for all $k$ — as a result, we can interpret the effect of $\tilde{W}_c(\Gamma_c(\cdot))$ and $\tilde{W}_k(\tilde{\Gamma}_k(\cdot))$ as common and company-specific shocks onto the CDO basket, respectively. Applying (3.16), the following distributional equality holds:

$$W_k(\Gamma_k(t)) \overset{d}{=} R_k(t) \tilde{W}_c(\Gamma_k(t)) + \sqrt{1 - R_k^2(t)} \tilde{W}_k(\Gamma_k(t)),$$

with

$$R_k(t) := \frac{\Gamma_c \left( \frac{\rho_k t}{v_k} \right)}{\Gamma_c \left( \frac{\rho_k t}{v_k} \right) + \tilde{\Gamma}_k \left( \frac{(1 - \rho_k) t}{v_k} \right)}.$$

To keep our model tractable we assume, with $R_k = R_k(T)$,

$$W_k(s) \overset{d}{=} \left( R_k \tilde{W}_c(s) + \sqrt{1 - R_k^2} \tilde{W}_k(s) \right). \quad (3.17)$$

---

2Note that in formula (3.16) it could be possible that $\frac{\rho_k t}{v_k} > 1$. This would mean that we are looking into the future when evaluating. When we discuss calibration in section 3.7 we will make sure this is not the case. To make notations not unnecessarily opaque, we will not introduce this here yet.
So the dependence on \( t \) is eliminated, but \( R_k \) keeps being random. Let us now consider the event of default for obligor \( k \). Then one has

\[
\mathbb{P} \left( \min_{0 \leq s \leq T} W_k(\Gamma_k(s)) < x_0^{(k)} \right) = \mathbb{P} \left( \min_{0 \leq s \leq 1} W_k(\Gamma_k(T)s) < x_0^{(k)} \right)
\]

\[
= \mathbb{P} \left( \min_{0 \leq s \leq 1} W_k(s) < \frac{x_0^{(k)}}{\sqrt{\Gamma_k(T)}} \right), \tag{3.18}
\]

Next we substitute Equation (3.17) into Equation (3.18), and we redefine a default for obligor \( k \) by

\[
\min_{0 \leq s \leq 1} \left( R_k \tilde{W}_c(s) + \sqrt{1 - R_k^2 \tilde{W}_k(s)} \right) < \frac{x_0^{(k)}}{\sqrt{\Gamma_k(T)}};
\]

this ‘new definition’ is consistent, as we have that

\[
\mathbb{P} \left( \min_{0 \leq s \leq 1} \left( R_k \tilde{W}_c(s) + \sqrt{1 - R_k^2 \tilde{W}_k(s)} \right) < \frac{x_0^{(k)}}{\sqrt{\Gamma_k(T)}}, \Gamma_k(T) \right)
\]

\[
= \mathbb{P} \left( \min_{0 \leq s \leq T} W_k(\Gamma_k(s)) < x_0^{(k)} \bigg| \Gamma_k(T) \right).
\]

Conclude that the new model has the desired marginal default probabilities.

The decomposition of the Brownian motions and gamma processes presented in this section enables us to compute default probabilities (per obligor), conditional on the ‘common components’,

\[
W^*_c = \min_{0 \leq s \leq 1} \tilde{W}_c(s)
\]

and \( \Gamma_c(\cdot) \), as follows. We first define the conditional default probabilities \( p_k(\gamma, w) \) by

\[
\mathbb{P} \left( \min_{0 \leq s \leq 1} W_k(s) < \frac{x_0^{(k)}}{\sqrt{\Gamma_k(T)}} \bigg| \Gamma_c \left( \frac{\rho_k t}{v_k} \right) = \gamma, W^*_c = w \right) = \mathbb{E}\hat{p}_k \left( \gamma, w; \tilde{\Gamma}_k \left( \frac{(1 - \rho_k)T}{v_k} \right) \right),
\]

where the conditional probability \( \hat{p} \left( \gamma, w; \tilde{\Gamma}_k \left( \frac{(1 - \rho_k)T}{v_k} \right) \right) \) is defined by

\[
\mathbb{P} \left( \min_{0 \leq s \leq 1} \left\{ R_k \tilde{W}_c(s) + \sqrt{1 - R_k^2 \tilde{W}_k(s)} \right\} < \frac{x_0^{(k)}}{\sqrt{\gamma + \tilde{\Gamma}_k \left( \frac{(1 - \rho_k)T}{v_k} \right)}} \bigg| W^*_c = w, \tilde{\Gamma}_k \left( \frac{(1 - \rho_k)T}{v_k} \right) \right).
\]

(3.19)
with
\[ R_k := \sqrt{\frac{\gamma}{\gamma + \tilde{\Gamma}_k \left( \frac{(1-\rho_k)T}{v_k} \right)}} \]

Using the above \( \mathbb{P}(L(0, T) = k) \) can be computed through
\[
\mathbb{E} \left( \sum_{(n_1, \ldots, n_K)} \prod_{i=1}^K p_i \left( \Gamma_c \left( \frac{\rho_i T}{v_i} \right), W_c^* \right)^{n_i} \left( 1 - p_i \left( \Gamma_c \left( \frac{\rho_i T}{v_i} \right), W_c^* \right) \right)^{1-n_i} \right),
\]

where the summation is over all vectors \((n_1, \ldots, n_K)\) such that \(n_i \in \{0, 1\}\), and \(\sum_{i=1}^K n_i = k\). Consider for instance the case of two obligors. We then have for the probability of two defaults the following expression:
\[
\mathbb{E} \left( p_1 \left( \Gamma_c \left( \frac{\rho_1 T}{v_1} \right), W_c^* \right) p_2 \left( \Gamma_c \left( \frac{\rho_2 T}{v_2} \right), W_c^* \right) \right).
\]

A second straightforward example concerns the homogeneous case, that is \(v_k = v\) and \(\rho_k = \rho\) for all \(k\). Then
\[
\mathbb{P}(L(0, T) = k) = \mathbb{E} \left( \binom{K}{k} \left( p \left( \Gamma_c \left( \frac{\rho T}{v} \right), W_c^* \right)^k \left( 1 - p \left( \Gamma_c \left( \frac{\rho T}{v} \right), W_c^* \right)^{K-k} \right) \right) \right).
\]

In the next section we propose an algorithm to efficiently evaluate (3.20), also in the heterogeneous case; it relies on an explicit expression for the conditional default probability \(p_k(\gamma, w)\) that is given in Appendix 3.12.

We end this section with the following remark about the impact \(\gamma\).

**Remark 3.6.** The effect of an increase in \(\gamma\) is two-fold:

1. The volatility \(\sqrt{\gamma + \tilde{\Gamma}_k \left( \frac{(1-\rho_k)T}{v_k} \right)}\) increases.
2. The correlation \(R_k\) increases.

Hence, our model has the desired property that high volatilities go hand in hand with high correlations.

### 3.6 Determining the loss distribution of the CDO basket

The goal of this section is to devise an efficient algorithm that enables the computation of the loss distribution of a non-homogeneous CDO basket using the correlation structure introduced in Section 3.5. We propose a recursive algorithm. It
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intensively relies on the expressions regarding default of an individual obligor, conditional on the common components \( W*_c \) and \( \Gamma_c(\cdot) \). With \( i_k := \rho_k/v_k \), it is assumed (without loss of generality) that the obligors are ordered such that \( i_k \leq i_{k+1} \), for \( k = 1, 2, \ldots, K \). Let \( p_K(\gamma, w) \) be as defined before. Furthermore, define by \( A(k, n) \) the event that \( k \) obligors default out of the last \( n \) obligors (i.e., obligors \( K - n + 1 \) up to \( K \)) in \([0, t]\). Introduce also

\[
P_{k,n}(\gamma, w) = \mathbb{P} \left( A(k, n) \mid \Gamma_c \left( \frac{\rho_{K+1-n}t}{v_{K+1-n}} \right) = \gamma, W*_c = w \right).
\]

It is trivial that

\[
P_{0,1}(\gamma, w) = 1 - p_K(\gamma, w), \quad P_{1,1}(\gamma, w) = p_K(\gamma, w).
\]

Then the probability distribution of the number of defaults in \([0, t]\) is given by

\[
\mathbb{P}(L(0, t) = k) = \mathbb{E} \left[ P_{k,K} \left( \Gamma_c \left( \frac{\rho_1t}{v_1} \right), W*_c = w \right) \right].
\]

In the following we derive a backward recursion which computes for \( n = 2, \ldots, K \) the conditional probabilities \( P_{k,n}(\gamma, w), k = 0, 1, \ldots, n \). It is elementary to see that

\[
P_{k,n}(\gamma, w) = (1 - p_n(\gamma, w)) \mathbb{P} \left( A(k, n - 1) \mid \Gamma_c \left( \frac{\rho_{K+1-n}t}{v_{K+1-n}} \right) = \gamma, W*_c = w \right) + p_n(\gamma, w) \mathbb{P} \left( A(k - 1, n - 1) \mid \Gamma_c \left( \frac{\rho_{K+1-n}t}{v_{K+1-n}} \right) = \gamma, W*_c = w \right).
\]

Since

\[
\Gamma_c \left( \frac{\rho_{K+1-(n-1)t}}{v_{K+1-(n-1)}} \right) \overset{d}{=} \Gamma_c \left( \frac{\rho_{K+1-n}t}{v_{K+1-n}} \right) + \hat{\Gamma}_c \left( \frac{\rho_{K+1-(n-1)t}}{v_{K+1-(n-1)}} - \frac{\rho_{K+1-n}t}{v_{K+1-n}} \right)
\]

with \( \hat{\Gamma}_c(\cdot) \) a standard gamma process independent of \( \Gamma_c(\cdot) \), we obtain that

\[
\mathbb{P} \left( A(k, n - 1) \mid \Gamma_c \left( \frac{\rho_{K+1-n}t}{v_{K+1-n}} \right) = \gamma, W*_c = w \right) = \mathbb{E} \left[ P_{k,n-1} \left( \gamma + \hat{\Gamma}_c \left( \frac{\rho_{K+1-(n-1)t}}{v_{K+1-(n-1)}} - \frac{\rho_{K+1-n}t}{v_{K+1-n}} \right), W*_c = w \right) \right]
\]

With a similar argument we obtain that

\[
\mathbb{P} \left( A(k - 1, n - 1) \mid \Gamma_c \left( \frac{\rho_{K+1-n}t}{v_{K+1-n}} \right) = \gamma, W*_c = w \right) = \mathbb{E} \left[ P_{k-1,n-1} \left( \gamma + \hat{\Gamma}_c \left( \frac{\rho_{K+1-(n-1)t}}{v_{K+1-(n-1)}} - \frac{\rho_{K+1-n}t}{v_{K+1-n}} \right), W*_c = w \right) \right].
\]
Collecting the above findings, we have the recurrence

\[
P_{k,n}(\gamma, w) = (1 - p_n(\gamma, w)) \mathbb{E} \left[ P_{k,n-1} \left( \gamma + \hat{\Gamma}_c \left( \frac{\rho_{K+1-(n-1)} t}{v_{K+1-(n-1)}} - \frac{\rho_{K+1-n} t}{v_{K+1-n}} \right), W_c^* = w \right) \right] + p_n(\gamma, w) \mathbb{E} \left[ P_{k-1,n-1} \left( \gamma + \hat{\Gamma}_c \left( \frac{\rho_{K+1-(n-1)} t}{v_{K+1-(n-1)}} - \frac{\rho_{K+1-n} t}{v_{K+1-n}} \right), W_c^* = w \right) \right]
\]

(3.21)

It is clear that we can evaluate

\[
\mathbb{E} \left[ P_{k,n-1} \left( \gamma + \hat{\Gamma}_c \left( \frac{\rho_{K+1-(n-1)} t}{v_{K+1-(n-1)}} - \frac{\rho_{K+1-n} t}{v_{K+1-n}} \right), W_c^* = w \right) \right] = \int_0^\infty P_{k,n-1}(\gamma + y, w) \, dF_n(y)
\]

(3.22)

with

\[
F_n(y) := \mathbb{P} \left( \hat{\Gamma}_c \left( \frac{\rho_{K+1-(n-1)} t}{v_{K+1-(n-1)}} - \frac{\rho_{K+1-n} t}{v_{K+1-n}} \right) \leq y \right),
\]

i.e., (3.22) essentially represents a \textit{convolution}. This observation leads to the following algorithm.

\textbf{Pseudo code 3.7.} Pseudo code for computing \(P(L(0,t) = k)\):

1. Set \(P_{0,1}(\gamma, w) = 1 - p_K(\gamma, w)\) and \(P_{1,1}(\gamma, w) = p_K(\gamma, w)\);
2. for \(n = 2, \ldots, K\) do steps (A) and (B):
   
   (A) Compute
   \[
   \mathbb{E} \left[ P_{k,n-1} \left( \gamma + \hat{\Gamma}_c \left( \frac{\rho_{K+1-(n-1)} t}{v_{K+1-(n-1)}} - \frac{\rho_{K+1-n} t}{v_{K+1-n}} \right), w \right) \right]
   \]
   for \(k = 0, 1, \ldots, n-1\);
   
   (B) Compute \(P_{k,n}(\gamma, W_c^*)\) for \(k = 0, 1, \ldots, n\), using (3.21);
3. Compute
   \[
   \mathbb{E} \left[ P_{k,K}(\Gamma_c \left( \frac{\rho_1 t}{v_1} \right), W_c^*) \right].
   \]

It is stressed that we only have to compute convolutions to calculate these recursions, which can be efficiently done by using numerical Laplace transform inversion. We use the numerical Laplace transform inversion algorithm introduced by den Iseger [40]. The complexity of the algorithm remains low; details are given in Appendix 3.13.
3.7 Estimation and Calibration of the Parameters

In this section we discuss how to estimate and calibrate the parameters in our model. We start with the straightforward case of the calibration of the shape parameters \( v_k \). Using this as input, we show how to obtain the default levels \( x_0^{(k)} \) implied by our model. After that we describe how to estimate the correlation parameters \( \rho_k \), and we end with a discussion about how to calculate the Greeks, i.e. sensitivities.

3.7.1 Calibration of the Shape Parameters

The method to obtain the shape parameters \( v_k \) is relatively straightforward. Once we obtain CDS prices, we can bootstrap the CDS spread curve. This gives us the default probabilities, which we denote by \( P_{t_j}^k(t) \) for the period \([t, t+t_j]\) for obligor \( k \). The shape parameters can now be obtained in a straightforward fashion by minimizing the \( L^2 \) norm between market prices and model prices with respect to the parameters \( v_k \).

3.7.2 Calibration of the Default Levels

We now proceed with a technique to calibrate our model for the distribution of \( L(t, t_j) \). Note that once we have the distribution of \( L(t, t_j) \) for \( j = 1, \ldots, J \), we can price a large class of credit derivatives (as will be further explained in Section 3.8). After bootstrapping the CDS curve we have obtained the default probabilities \( P_{t_j}^k(t) \) for the period \([t, t+t_j]\) for obligor \( k \). To keep the notation light, we suppress the superscript \( t_j \) in the remainder.

Above we explained how to obtain calibrated shape parameters \( v_k \). Since the default probability is strictly decreasing in the default level \( x_k^* \), we obtain

\[
x_k^*(t) = \Phi^{-1} (P_k(t); v_k),
\]

with \( \Phi (x_k^*(t); v_k) = P_k(t) \). Using the time series of CDS spread curves as input, and applying (3.23), we obtain the time series for the default levels:

\[
x_k^*(t_j) = \Phi^{-1} (P_k(t_j); v_k).
\]

Since \( X_k \) is a realization of a Lévy process, we have that

\[
x_k^*(t_j) - x_k^*(t_{j+1}) = X_k(t_{j+1}) - X_k(t_j).
\]

Hence, with a time series of CDS spread curves as input, we can obtain a time series for the underlying process \( X_k(\cdot) \).
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3.7.3 Estimation of the correlation parameters

The correlation parameters can be estimated as follows. Observe that the covariance between two CDSs, labeled $k_1$ and $k_2$, is given by $\min\{\rho_{k_1}/v_{k_1}, \rho_{k_2}/v_{k_2}\}$. We use here that for a Brownian motion $W(\cdot)$ it holds that $\text{Cov}(W(s), W(t)) = \min\{s, t\}$. We estimate this covariance as follows:

$$\frac{1}{t_J - t_0} \sum_{j=0}^{J-1} (W_{k_1}(\Gamma_{k_1}(t_j+1)) - W_{k_1}(\Gamma_{k_1}(t_j))) (W_{k_2}(\Gamma_{k_2}(t_{j+1})) - W_{k_2}(\Gamma_{k_2}(t_j))).$$

Now note that since it holds that

$$\text{Cov}(X_{k_1}, X_{k_2}) = \min\{\rho_{k_1}/v_{k_1}, \rho_{k_2}/v_{k_2}\},$$

it holds that

$$\text{Cov}(X_{k_1}, X_{k_2}) \leq \frac{\rho_{k_1}}{v_{k_1}}$$

for all $k_1 \neq k_2$.

So we choose

$$\frac{\rho_{k_1}}{v_{k_1}} = \max_{k_2: k_2 \neq k_1} \text{Cov}(X_{k_1}, X_{k_2}).$$

To make sure we circumvent the fact that $\rho_k > 1$ (already briefly mentioned in section 3.5), we recalibrate $v_k$ under the restriction

$$v_{k_1} \leq \frac{1}{\max_{k_2: k_2 \neq k_1} \text{Cov}(X_{k_1}, X_{k_2})}.$$

Hence, we calibrate the dependency structure on the maximal correlation of obligor $k_1$ with any other obligor, for all $k_1$.

3.7.4 Computing Greeks

In the above subsection, we explained how to calibrate the $v_k$, $x_k^*$, and $\rho_k$. We now show how to determine sensitivities, i.e., Greeks, once we have the default levels. From Section 3.7.2 we know that (suppressing the dependence on time), $x_k^* = \Phi^{-1}(P_k; v_k)$. Hence

$$\frac{\partial p_k(\gamma, w)}{\partial P_k} = \frac{\partial p_k(\gamma, w)}{\partial x_k^*} \frac{\partial \Phi^{-1}(P_k; v_k)}{\partial P_k},$$

and

$$\frac{\partial p_k(\gamma, w)}{\partial s_k(t_j)} = \frac{\partial p_k(\gamma, w)}{\partial P_k} \frac{\partial P_k}{\partial s_k(t_j)},$$

with $s_k(t_j)$ the CDS spread for the expiration time $t_j$ and obligor $k$. 
3.8 CDO Pricing Preliminaries

Before presenting numerical results in Section 3.9 in which we compute different loss distributions of the CDO basket, and in order to make this chapter more or less self-contained, we briefly recall some CDO pricing definitions in this section. More precisely, we follow the CDO pricing definitions of Brigo and Mercurio in [22] which we adapted to our framework and notation.

Like before, we divide the time interval in $0 = t_0 < t_1 < \ldots < t_J = T$ and the number of defaults in the time increment $[t_j, t_{j+1})$ is given by $L(t_j, t_{j+1})$. The discount factor between $0$ and $t_j$ is given by $D(0, t_j)$. The so-called tranche of the CDO is denoted by $[A, B]$, with integers $A < B$. This represents the number of defaults in tranche $[A, B]$. The price of the so-called default-leg of a CDO can now be represented by

$$P_D = (1 - R) \sum_{j=0}^{J-1} D(0, t_{j+1}) \mathbb{E}\Phi_D(t_j, t_{j+1})$$

(3.24)

in which $R$ is the so-called recovery rate\(^3\) which in principle can be time dependent (but excluded here), and $\Phi_D$ is defined as

$$\Phi_D(t_j, t_{j+1}) := \sum_{\ell=A-1}^{B-1} \Delta L[\ell, t_{j+1})$$

(3.25)

Equation (3.25) represents the expected number of jumps to default from $\ell$ to $\ell + 1$ of a given tranche $[A, B]$ in time increment $[t_j, t_{j+1})$. In turn, Equation (3.24) represents the discounted expected number of defaults in time interval $[0, T)$ of tranche $[A, B]$, which is by definition the price of the default-leg of a CDO.

Similarly, one represent the price of the so-called premium leg of a CDO as

$$P_P = P \sum_{j=0}^{j-1} D(0, t_j) \mathbb{E}\Phi_P(t_j, t_{j+1})$$

(3.26)

in which $\Phi_P$ is defined as

$$\Phi_P(t_j, t_{j+1}) := \Delta_j \left(1 \{L(t_j, T) \leq A\} + \sum_{\ell=A+1}^{B-1} \frac{B - \ell}{B - A} 1 \{L(t_j, T) = \ell\} \right)$$

(3.27)

\(^3\)The fraction of the exposure that may be recovered through bankruptcy proceedings or some other form of settlement in the event of a default.
here $\Delta_j := t_{j+1} - t_j$. Equation (3.27) represents the amount of premium paid in a time interval $[t_j, t_{j+1})$ of tranche $[A, B]$. In turn, Equation (3.26) represents the discounted expected amount of premium paid in time interval $[0, T)$ of tranche $[A, B]$, which is by definition the price of the premium-leg of a CDO.

For a protection buyer the price of a CDO is then simply the difference $P_{CDO} = P_P - P_D$. For a protection seller, it is exactly the other way around. Usually, the premium that is agreed upon is that which equates the two.

Like in the single name case, CDOs are also priced by evaluating the default probabilities $F_t(y_0)$, but correlation between defaults plays an important role now. Using the method described in Section 3.5, we can efficiently compute CDO prices by fast and accurate evaluation of the loss distribution of the CDO basket as described in Section 3.6. We are now in a position to give a pseudo code for pricing CDOs using our method. This is done below.

**Pseudo code 3.8.** Pseudo code for Pricing a CDO:

1. Calculate the loss distribution of the CDO basket, i.e. $\mathbb{P}(L(t, T) = \ell)$, and the expected number of jumps to default from $\ell$ to $\ell + 1$, i.e. $E\Delta L_\ell [t_j, t_{j+1})$, using Pseudo code 3.7;

2. Calculate the default leg and premium leg of the CDO using Equations (3.24) and (3.26), respectively;

3. Subtract the default leg from the premium leg to get the price of a protection buyer CDS, and vice versa for a protection seller.

In Section 3.9 we present numerical examples in which this pseudo code is implemented.

### 3.9 Numerical Example: Computing CDO Loss Distributions

Since the purpose of this chapter is to present a novel approach to model dynamic default correlation, the focus is on the model (and in particular the way correlations are introduced) and their properties. However, to demonstrate the feasibility of the approach, and to provide some insights, we present here a set of numerical experiments. In these experiments, we compute loss distributions...
of the CDO basket (for different values of the parameters $\rho_k$). These indicate that our model is highly flexible, in the sense that distinctly different shapes of the distribution can be obtained by choosing the various parameters appropriately.

![Figure 3.1 Value at Risk (VaR) of the CDO basket for different values of the correlation parameter $\rho_k$. Other parameters are calibrated on the market.](image)

This effect is depicted in Figure 3.1, where we observe various shapes of the loss distribution of the Value at Risk (VaR) of the CDO basket. We see that the loss distribution of the CDO basket is extremely dependent on the value of the parameter $\rho_k$, which is precisely what we observe in practice: when a global crash hits the market, correlation shoots up. The inherent flexibility of our modeling approach makes sure that we can get the shape of the loss distribution of the CDO basket right, which is of crucial importance for pricing credit products.

### 3.10 Discussion, Conclusion and Suggestions for Further Research

In this chapter we presented an Lévy-based default correlation model, with applications to CDO pricing. We modeled marginal default probabilities using a
dynamic structural Variance Gamma (VG) model, and developed a dynamic correlation structure based on these marginal probabilities, in which the individual obligors react to common and individual shocks. The key advantage of our correlation structure is that it is intuitive, dynamic, and allows for easy calibration to the market since the underlyings are market observables, for which there is ample data available. Model evaluation is fast and accurate; recursions can be performed highly efficiently using sophisticated numerical Laplace transform inversion techniques.

The development of our model opens various new possibilities for further research. Possible topics are e.g. the computation of various risk management and performance measures on credit portfolios, estimating risk premia in credit markets, pricing credit portfolios, and evaluating correlation in credit markets over time, to name but a few.
3.11 Appendix A: Correlation Structure

In this appendix we discuss in detail the correlation structure we have imposed on the gamma processes $\Gamma_k(\cdot)$, by considering it in the more general setting of Lévy processes. To be more precise, we consider the random variables

$$X_k(t) := \int_0^t 1\{u \in A_k\} dZ(u) + \int_0^t 1\{u \in A^c_k\} dY_k(u),$$

where $Z$ and the $Y_k$ are independent Lévy processes and the $A_k$ are Borel sets and $A^c_k$ denotes the complement of the set $A_k$. Let for all subsets $S$ of \{1, 2, ..., $n$\} the number $\rho_S$, that contains all market information on correlation, be given by

$$\rho_S := \int_0^t \prod_{k \in S} 1\{u \in A_k\} du.$$

Since Lévy processes have stationary and independent increments, we obtain that for fixed $t$ the joint distribution of \{X_k(t); k = 1, ..., $n$\} only depends on the $\rho_S$. More generally, we can define random variables

$$X_k(B) := \int_B 1\{u \in A_k\} dZ(u) + \int_B 1\{u \in A^c_k\} dY_k(u)$$

for each Borel set $B$ in [0, $t$). Now let for all subsets $S$ of \{1, 2, ..., $n$\} the number $\rho^B_S$ be given by

$$\rho^B_S := \int_B \prod_{k \in S} 1\{u \in A_k\} du.$$

Again, since Lévy processes have stationary and independent increments, the joint distribution of $X_1(B), \ldots, X_n(B)$ only depends on the $\rho^B_S$.

In the following we will argue that given a correlation structure on [0, $t$), we can extend this correlation structure such that the same correlation structure holds on every Borel set of [0, $t$). That is, we can define sets $\hat{A}_k$ such that for every Borel set $B$ in [0, $t$) it holds that

$$\rho^B_S = \frac{|B|}{t} \rho_S,$$

with

$$|B| := \int_B du.$$

Now suppose that we have defined a correlation structure for $X_k(t)$ by \{A_k\} ($k \in S$), and let \([a_j, b_j]\), $j \in J$, be a partition of [0, 1). Then we can define sets $A^j_k$ such that
3. MODELING DEFAULT CORRELATION WITH APPLICATIONS TO CDO PRICING

\[ 1\{ u \in A_k^j \} = 1\{ \frac{u - a_j}{b_j - a_j} \in A_k \}. \]

We are now able to define the sets \( \hat{A}_k := \cup_j A_k^j \). For these sets we have that

\[ \int_{a_j}^{b_j} \prod_{k \in S} 1\{ u \in \hat{A}_k \} \, du = \int_{a_j}^{b_j} \prod_{k \in S} 1\{ u \in A_k^j \} \, du = \int_{a_j}^{b_j} \prod_{k \in S} 1\{ \frac{u - a_j}{b_j - a_j} \in A_k \} \, du = (b_j - a_j) \int_0^t \prod_{k \in S} 1\{ u \in A_k \} \, du. \]

Consider now a set \( B = \cup_{j \in b} [a_j t, b_j t) \), with \( b \) a subset of \( J \). Then

\[ \int_B \prod_{k \in S} 1\{ u \in \hat{A}_k \} \, du = C_B \int_0^t \prod_{k \in S} 1\{ u \in A_k \} \, du, \]

with

\[ C_B = \sum_{j \in b} (b_j - a_j) = \frac{|B|}{t}. \]

Hence we can define sets \( \hat{A}_k \) such that the correlation structure holds on every Borel set in \([0, t)\). Furthermore,

\[ \hat{X}_k (t) = \int_0^t 1\{ u \in \hat{A}_k \} \, dZ (u) + \int_0^t 1\{ u \in \hat{A}_k^c \} \, dY_k (u) \]

has the same distribution as \( X_k (t) \). In this chapter we choose the sets \( \hat{A}_k \) as \([0, \rho_k t)\) and we can compute for a given function \( h(\cdot) \) the expectation

\[ \mathbb{E} h (X_1 (t), \ldots, X_n (t)) , \]

as the expectation

\[ \mathbb{E} h (\hat{X}_1 (t), \ldots, \hat{X}_n (t)) . \]

3.12 Appendix B: The Conditional Probability of the Running Minimum

In this appendix we show how to compute the conditional default probability \( p_k (\gamma, w) \). In fact, we give an expression for

\[ p (x_1; q, x_2) = \mathbb{P} \left\{ \min_{0 \leq s \leq 1} X_1 (s) < x_1 \mid \min_{0 \leq s \leq 1} X_2 (s) = x_2 \right\} . \]
with \( X(\cdot) \) a two dimensional Brownian motion starting in the origin such and covariance matrix

\[
\Omega = \begin{bmatrix}
\text{Var}X_1(1) & \text{Cov}(X_1(1), X_2(1)) \\
\text{Cov}(X_1(1), X_2(1)) & \text{Var}X_2(1)
\end{bmatrix} = \begin{bmatrix} 1 & q \\ q & 1 \end{bmatrix};
\]

in the sequel we use the identification \( q = \sin(2\beta) \). Note that the expression for \( \overline{p}(x_1; q, x_2) \) can be immediately translated into an expression for \( p_k(\gamma, w) \) using a number of elementary substitutions, analogous to [65].

We now continue with deriving an explicit expression for \( \overline{p}(x; q, w) \). To this end, consider the process

\[
dX(t) = \Sigma dW(t),
\]

with \( W(t) \) a two dimensional standard Brownian motion (independent components!) and

\[
\Sigma = \begin{bmatrix} \cos\beta & \sin\beta \\ \sin\beta & \cos\beta \end{bmatrix},
\]

where \(|\beta| \leq \frac{\pi}{4}\). Now let, for \( j = 1, 2 \),

\[
\tau^{x_j} = \inf \{ t : X_j(t) = x_j \}, \quad W(t) = \Sigma^{-1}(X(t) - x),
\]

where

\[
\Sigma^{-1} = \frac{1}{\cos^2 \beta - \sin^2 \beta} \begin{bmatrix} \cos \beta & -\sin \beta \\ -\sin \beta & \cos \beta \end{bmatrix}.
\]

Then \( W(\cdot) \) is a two dimensional standard Brownian motion starting in \( W(0) = -\Sigma^{-1}x \). Observe that the hitting times \( \tau^{x_j} \) can be expressed alternatively as

\[
\inf \{ t : W(t) \in L_j \}, \quad \text{with } L_j \text{ denoting the lines}
\]

\[
L_1 = \left\{ y \in \mathbb{R} : \Sigma^{-1} \begin{bmatrix} 0 \\ y - x_2 \end{bmatrix} \right\}, \quad L_2 = \left\{ y \in \mathbb{R} : \Sigma^{-1} \begin{bmatrix} y - x_1 \\ 0 \end{bmatrix} \right\}.
\]

Furthermore, let \( \tau^x = \min(\tau_1^{x_1}, \tau_2^{x_2}) \), and let \((r_0, \phi_0)\) be the polar coordinates of \( W(0) = -\Sigma^{-1}x \), that is, \( W(0) = (r_0 \cos(\phi_0), r_0 \sin(\phi_0)) \). The following theorem gives an expression for

\[
\pi(x, q, t) := \mathbb{P}(\tau^x \geq t) = \mathbb{P} \left( \min_{0 \leq s \leq t} X_1(s) \geq x_1, \min_{0 \leq s \leq t} X_2(s) \geq x_2 \right).
\]

**Theorem 3.9.** The probability \( \pi(x, q, t) \) is given by

\[
\sqrt{\frac{2}{\pi}} \frac{r_0}{\sqrt{t}} \exp \left( -\frac{r_0^2}{4t} \right) K \left( \frac{r_0}{\sqrt{t}}, \phi_0, \alpha \right),
\]

(3.28)
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with \( \alpha = \pi/2 + \sin^{-1}(q) \), and where

\[
K (r_0, \phi_0, \alpha) = \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin \left( \frac{(2n-1) \pi \phi_0}{\alpha} \right) \left( I_{\frac{(2n-1) \pi}{2 \alpha}} - \frac{r_0^2}{4} \right) + I_{\frac{(2n-1) \pi}{2 \alpha}} + \frac{r_0^2}{4},
\]

with \( I_v(\cdot) \) the modified Bessel function of order \( v \).

**Remark 3.10.** This result corrects a result of Iyengar [65]. The proof is identical as the proof given there, but identity (9) in his paper has to be

\[
\int_{0}^{\infty} e^{-\beta^2} I_v (\alpha t) \, dt = \frac{\sqrt{\pi}}{2} e^{\alpha^2 / 8} \beta I_v^2 \left( \frac{\alpha^2}{8 \beta} \right),
\]

and we thus obtain the result of the above theorem.

It is clear that Thm. 3.9 gives us all the information needed to compute \( \bar{p} (x; q, w) \). The remainder of his appendix explains in greater detail how this computation can be performed.

**Lemma 3.11.** It holds that

\[
\hat{p} (x, t) := \mathbb{P} \left( \min_{0 \leq s \leq t} X_2 (s) \geq x_2 \mid \min_{0 \leq s \leq t} X_1 (s) = x_1 \right) = -\frac{\partial \pi (x, q, t)}{\partial x_1} \frac{1}{p_1 (x_1, t)},
\]

with

\[
p_1 (x_1, t) \, dx_1 = \mathbb{P} \left( \min_{0 \leq s \leq t} X_1 (s) \in dx_1 \right).
\]

**Lemma 3.12.** It holds that

\[
\frac{\partial \pi (x, q, t)}{\partial x} = \frac{\partial \pi (x, q, t)}{\partial x_0} \frac{\partial \phi_0}{\partial x} + \frac{\partial \pi (x, q, t)}{\partial \phi_0} \frac{\partial \phi_0}{\partial x},
\]

with

\[
\begin{bmatrix}
\frac{\partial r_0}{\partial x_1} & \frac{\partial \phi_0}{\partial x_1} \\
\frac{\partial r_0}{\partial x_2} & \frac{\partial \phi_0}{\partial x_2}
\end{bmatrix} = \begin{bmatrix}
\cos (\phi_0) & -\sin (\phi_0) \\
\sin (\phi_0) & \cos (\phi_0)
\end{bmatrix},
\]

where

\[
\frac{\partial \pi (x, q, t)}{\partial r_0} = \left( \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{t}} - \frac{1}{2t \sqrt{2 \pi \sqrt{t}}} r_0 \right) \exp \left( -\frac{r_0^2}{4t} \right) K \left( \frac{r_0}{\sqrt{t}}, \phi_0, \alpha \right) + \frac{2r_0}{\sqrt{2 \pi t}} \exp \left( -\frac{r_0^2}{4t} \right) \frac{\partial}{\partial r_0} K \left( r_0, \phi_0, \alpha \right);
\]

\[
\frac{\partial \pi (x, q, t)}{\partial \phi_0} = \sqrt{\frac{2}{\pi}} \frac{r_0}{\sqrt{t}} \exp \left( -\frac{r_0^2}{4t} \right) \frac{\partial}{\partial \phi_0} K \left( r_0, \phi_0, \alpha \right).
\]
3.12. APPENDIX B: THE CONDITIONAL PROBABILITY OF THE RUNNING MINIMUM

The partial derivatives of $K$ with respect to $r_0$ and $\phi_0$ are given by, respectively,

$$
\sum_{n=1}^{\infty} \frac{1}{2n-1} \sin \left( \frac{(2n-1)\pi \phi_0}{\alpha} \right) \frac{\partial}{\partial r_0} \left( I_{(2n-1)\pi \alpha} - \frac{1}{2} \left( \frac{r_0^2}{4} \right) + I_{(2n-1)\pi \alpha} + \frac{1}{2} \left( \frac{r_0^2}{4} \right) \right)
$$

and

$$
\frac{\pi}{\alpha} \sum_{n=1}^{\infty} \cos \left( \frac{(2n-1)\pi \phi_0}{\alpha} \right) \left( I_{(2n-1)\pi \alpha} - \frac{1}{2} \left( \frac{r_0^2}{4} \right) + I_{(2n-1)\pi \alpha} + \frac{1}{2} \left( \frac{r_0^2}{4} \right) \right),
$$

where

$$
\frac{\partial}{\partial r_0} \left( I_{(2n-1)\pi \alpha} - \frac{1}{2} \left( \frac{r_0^2}{4} \right) + I_{(2n-1)\pi \alpha} + \frac{1}{2} \left( \frac{r_0^2}{4} \right) \right)
$$

equals

$$
\frac{r_0}{2\sqrt{t}} \left( I_{\frac{(2n-1)\pi \alpha}{2\alpha}} - \frac{1}{2} \left( \frac{r_0^2}{4t} \right) + I_{\frac{(2n-1)\pi \alpha}{2\alpha}} + \frac{1}{2} \left( \frac{r_0^2}{4t} \right) \right).
$$

Proof. The claims can be verified by elementary algebra. For the computation of $\frac{\partial}{\partial r_0} K$ we have used the identity

$$
\frac{\partial I_v(x)}{\partial x} = \frac{I_{v-1}(x) + I_{v+1}(x)}{2}.
$$


Remark 3.13. The function $K (r_0, \phi_0, \alpha)$ can be efficiently and accurate computed with the Poisson Summation quadrature formula of e.g. den Iseger [40]:

$$
\sum_{j=1}^{m} A_j \frac{\pi}{\lambda_j} \sin \left( \frac{\lambda_j \phi_0}{\alpha} \right) \left( I_{\frac{\lambda_j \alpha}{2\alpha}} - \frac{1}{2} \left( \frac{r_0^2}{4} \right) + I_{\frac{\lambda_j \alpha}{2\alpha}} + \frac{1}{2} \left( \frac{r_0^2}{4} \right) \right),
$$

with $\{\lambda_j\}$ and $\{A_j\}$ given quadrature nodes and weights. In a similar way we can evaluate the partial derivatives of $K$ with respect to $r_0$ and $\phi_0$. The above approximation yields highly accurate results; see e.g. den Iseger [40, 41].

It is further noted that the evaluation of the Bessel series is relatively expensive. We have developed a procedure to reduce the number of evaluations, but we do not report on this here.
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3.13 Appendix C: Computational Details

In this appendix we explain the details for the computation of recurrence (3.21). We can write the above equation as

\[ V_{k,n}(\gamma) = (1 - p_n(\gamma, w)) V_{k,n-1} \ast h_n(\gamma) + p_n(\gamma, w) V_{k-1,n-1} \ast h_n(\gamma), \]

with \( V_{k,n}(y) \) defined in an obvious way, \( h_n(y) := f_n(-y) \) with \( f_n \) the probability density function of the random variable

\[ \hat{\Gamma}_c \left( \frac{\rho_{K+1-(n-1)t}}{v_{K+1-(n-1)}} - \frac{\rho_{K+1-n}t}{v_{K+1-n}} \right), \]

and \( V_{k,n-1} \ast h_n \) denoting the convolution

\[ V_{k,n-1} \ast h_n(\gamma) = \int_{-\infty}^{0} V_{k-1,n-1}(\gamma - y) h_n(y) \, dy. \]

The computation of \( V_{k,n}(\gamma) \) can be done recursively, by performing the following two steps:

1. Compute the convolutions \( H_{k,n-1}(\gamma) = V_{k,n-1} \ast h_n(\gamma) \), for \( k = 0, 1, \ldots, n - 1 \);

2. Compute \( V_{k,n}(\gamma) = (1 - p_n(\gamma, w)) H_{k,n-1}(\gamma) + p_n(\gamma, w) H_{k-1,n-1}(\gamma) \), for \( k = 0, 1, \ldots, n \).

For the computations of the above steps we use a toolbox for numerical inversion of Laplace transforms, developed by den Iseger [40]. This toolbox consists of the following algorithms:

(A) Given the function values of the transform \( \hat{H}(s_j) \), at a predefined set of points \( \{s_1, \ldots, s_M\} \), the algorithm computes a piecewise polynomial \( P(H) \) for the function \( H \). The piecewise polynomial \( P(H) \) can be efficiently evaluated in each point to obtain an approximation for the function values of \( H \).

(B) Given the piecewise polynomial \( P(H) \), the algorithm computes the function values \( \hat{H}(s_j) \) at the predefined set of points \( \{s_1, \ldots, s_M\} \).

(C) Given the function values of the function \( H(t_j) \), at a predefined set of points \( \{t_1, \ldots, t_M\} \), the algorithm computes the piecewise polynomial \( P(H) \).
The piecewise polynomial referred to above is a so-called piecewise Legendre polynomial. On each interval $[k\Delta, (k+1)\Delta]$, with $\Delta$ a free to choose scaling parameter, this is a linear combination of Legendre polynomials. The Legendre polynomials can be numerically stable and efficiently evaluated in each point on $[k\Delta, (k+1)\Delta]$, and hence we can numerically stable and efficiently evaluate the piecewise polynomial $P(H)$ in each point.

It is noted that algorithm (A) is precisely the inverse of Algorithm (B). The algorithms consist of a number of matrix vector multiplications, as well as a number of fast Fourier transforms (FFT). The typical running time is below 0.01 seconds on a standard PC, and the approximation error of the order of the machine precision. Algorithm (C) only consists of a number of matrix vector multiplications.

We can now compute the piecewise polynomials $P(V_{k,n})$ with the following steps.

**Pseudo code 3.14.** Pseudo code for evaluating $P(V_{k,n})$.

Input: $P(V_{k,n-1})$, for $k = 0, 1, \ldots, n - 1$.

For $k = 0, 1, \ldots, n - 1$ do

1. Compute the function values of the Laplace transform $\hat{V}_{k,n-1}(s_j)$, for $j = 1, \ldots, M$, with algorithm (A);

2. Compute $\hat{H}_{k,n-1}(s_j) = \hat{V}_{k,n-1}(s_j) \hat{h}_n(s_j)$, for $j = 1, \ldots, M$;

3. Invert $\hat{H}_{k,n-1}$ to obtain $P(H_{k,n-1})$ with algorithm (B).

For $k = 0, 1, \ldots, n$ do

1. Compute $V_{k,n}(t_j) = [1 - p_n(t_j, w)] P(H_{k,n-1})(t_j) + p_n(\gamma_j, w) P(H_{k-1,n-1})(t_j)$;

2. Compute $P(V_{k,n})$ with algorithm (C).

Given the piecewise polynomial $P(V_{k,n})$, we can evaluate $V_{k,n}$ in each point.
### Appendix D: Single Name Model Output

<table>
<thead>
<tr>
<th>Company</th>
<th>Geometric Brownian Motion</th>
<th>Variance Gamma</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>RMSE [27]</td>
<td>RMSE</td>
</tr>
<tr>
<td>Mbna Insurance</td>
<td>2.34</td>
<td>9.66</td>
</tr>
<tr>
<td>General Elec.</td>
<td>3.44</td>
<td>2.69</td>
</tr>
<tr>
<td>Wells Fargo</td>
<td>3.76</td>
<td>2.30</td>
</tr>
<tr>
<td>Citigroup</td>
<td>3.81</td>
<td>3.02</td>
</tr>
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<td>Wal-Mart</td>
<td>2.13</td>
<td>2.04</td>
</tr>
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<td>Merrill Lynch</td>
<td>2.15</td>
<td>5.68</td>
</tr>
<tr>
<td>Du Pont</td>
<td>2.07</td>
<td>1.69</td>
</tr>
<tr>
<td>American Express</td>
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<td>2.37</td>
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<td>1.69</td>
<td>6.07</td>
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<td>6.37</td>
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<td>McdonaldŠs</td>
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<td>2.66</td>
</tr>
<tr>
<td>Ford Credit Co.</td>
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<td>25.91</td>
</tr>
<tr>
<td>General Motors</td>
<td>13.49</td>
<td>29.77</td>
</tr>
<tr>
<td>Kraft Foods</td>
<td>2.86</td>
<td>3.17</td>
</tr>
<tr>
<td>Wyeth</td>
<td>7.22</td>
<td>6.38</td>
</tr>
<tr>
<td>Norfolk South.</td>
<td>3.13</td>
<td>1.94</td>
</tr>
<tr>
<td>Whirlpool</td>
<td>8.52</td>
<td>7.24</td>
</tr>
<tr>
<td>Walt Disney</td>
<td>1.17</td>
<td>3.51</td>
</tr>
<tr>
<td>Autozone</td>
<td>3.92</td>
<td>10.18</td>
</tr>
<tr>
<td>Eastman Kodak</td>
<td>8.05</td>
<td>22.16</td>
</tr>
<tr>
<td>Bombardier</td>
<td>10.62</td>
<td>41.73</td>
</tr>
</tbody>
</table>

Table 3.1 CDS Term Structure Calibration Results for GBM and VG in bps. compared to those of Cariboni and Schoutens in [27].
### Table 3.2 CDS Term Structure Calibration Results for VG in bps. based on the entities of the iTraxx Europe index on February 19th, 2010.

<table>
<thead>
<tr>
<th>Company</th>
<th>APE</th>
<th>RMSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Accor SA</td>
<td>1,18</td>
<td>0,63 %</td>
</tr>
<tr>
<td>Adecco SA</td>
<td>1,29</td>
<td>0,94 %</td>
</tr>
<tr>
<td>Aegon NV</td>
<td>2,17</td>
<td>1,48 %</td>
</tr>
<tr>
<td>Volvo AB</td>
<td>5,57</td>
<td>2,75 %</td>
</tr>
<tr>
<td>Akzo Nobel NV</td>
<td>0,04</td>
<td>0,05 %</td>
</tr>
<tr>
<td>Allianz SE</td>
<td>1,75</td>
<td>2,28 %</td>
</tr>
<tr>
<td>Alstom SA</td>
<td>1,37</td>
<td>1,04 %</td>
</tr>
<tr>
<td>Anglo American PLC</td>
<td>1,45</td>
<td>0,89 %</td>
</tr>
<tr>
<td>ArcelorMittal</td>
<td>1,50</td>
<td>0,54 %</td>
</tr>
<tr>
<td>Assicurazioni Generali SpA</td>
<td>1,15</td>
<td>1,06 %</td>
</tr>
<tr>
<td>Aviva PLC</td>
<td>2,99</td>
<td>2,71 %</td>
</tr>
<tr>
<td>AXA SA</td>
<td>2,43</td>
<td>2,32 %</td>
</tr>
<tr>
<td>Banca Monte dei Paschi di Siena SpA</td>
<td>0,61</td>
<td>0,48 %</td>
</tr>
<tr>
<td>Banco Bilbao Vizcaya Argentaria SA</td>
<td>1,88</td>
<td>1,38 %</td>
</tr>
<tr>
<td>Banco Espirito Santo SA</td>
<td>0,14</td>
<td>0,06 %</td>
</tr>
<tr>
<td>Banco Santander SA</td>
<td>1,64</td>
<td>1,22 %</td>
</tr>
<tr>
<td>Bank of Scotland PLC</td>
<td>1,73</td>
<td>0,96 %</td>
</tr>
<tr>
<td>Barclays Bank PLC</td>
<td>1,01</td>
<td>0,84 %</td>
</tr>
<tr>
<td>BASF SE</td>
<td>0,69</td>
<td>1,10 %</td>
</tr>
<tr>
<td>Bayer AG</td>
<td>0,17</td>
<td>0,27 %</td>
</tr>
<tr>
<td>Bayerische Motoren Werke AG</td>
<td>0,76</td>
<td>0,64 %</td>
</tr>
<tr>
<td>Bertelsmann AG</td>
<td>1,81</td>
<td>0,98 %</td>
</tr>
<tr>
<td>BNP Paribas</td>
<td>0,33</td>
<td>0,34 %</td>
</tr>
<tr>
<td>BP PLC</td>
<td>0,51</td>
<td>0,93 %</td>
</tr>
<tr>
<td>British American Tobacco PLC</td>
<td>0,72</td>
<td>0,98 %</td>
</tr>
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</table>
### Table 3.3

<table>
<thead>
<tr>
<th>Company</th>
<th>APE</th>
<th>RMSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>British Telecommunications PLC</td>
<td>1,42</td>
<td>0,92 %</td>
</tr>
<tr>
<td>Cadbury Schweppes US Finance LLC</td>
<td>0,22</td>
<td>0,44 %</td>
</tr>
<tr>
<td>Carrefour SA</td>
<td>0,36</td>
<td>0,46 %</td>
</tr>
<tr>
<td>Casino Guichard Perrachon SA</td>
<td>1,54</td>
<td>1,28 %</td>
</tr>
<tr>
<td>Centrica PLC</td>
<td>0,65</td>
<td>0,82 %</td>
</tr>
<tr>
<td>Ciba Holding AG</td>
<td>0,24</td>
<td>0,59 %</td>
</tr>
<tr>
<td>Commerzbank AG</td>
<td>0,05</td>
<td>0,06 %</td>
</tr>
<tr>
<td>Cie de Saint-Gobain</td>
<td>2,03</td>
<td>1,46 %</td>
</tr>
<tr>
<td>Compagnie Financiere Michelin</td>
<td>1,20</td>
<td>0,88 %</td>
</tr>
<tr>
<td>Compass Group PLC</td>
<td>0,82</td>
<td>1,34 %</td>
</tr>
<tr>
<td>Credit Agricole SA</td>
<td>1,58</td>
<td>1,32 %</td>
</tr>
<tr>
<td>Credit Suisse Group AG</td>
<td>1,06</td>
<td>1,13 %</td>
</tr>
<tr>
<td>Daimler AG</td>
<td>1,03</td>
<td>0,82 %</td>
</tr>
<tr>
<td>Deutsche Bank AG</td>
<td>1,19</td>
<td>1,14 %</td>
</tr>
<tr>
<td>Deutsche Lufthansa AG</td>
<td>1,74</td>
<td>0,71 %</td>
</tr>
<tr>
<td>Deutsche Post AG</td>
<td>0,94</td>
<td>1,52 %</td>
</tr>
<tr>
<td>Deutsche Telekom AG</td>
<td>0,10</td>
<td>0,12 %</td>
</tr>
<tr>
<td>Diageo PLC</td>
<td>0,23</td>
<td>0,28 %</td>
</tr>
<tr>
<td>E.ON AG</td>
<td>0,22</td>
<td>0,34 %</td>
</tr>
<tr>
<td>Edison SpA</td>
<td>0,70</td>
<td>0,61 %</td>
</tr>
<tr>
<td>EDP - Energias de Portugal SA</td>
<td>0,81</td>
<td>0,62 %</td>
</tr>
<tr>
<td>EDF SA</td>
<td>0,09</td>
<td>0,12 %</td>
</tr>
<tr>
<td>EnBW Energie Baden-Wuerttemberg AG</td>
<td>0,61</td>
<td>0,90 %</td>
</tr>
<tr>
<td>Enel SpA</td>
<td>0,55</td>
<td>0,53 %</td>
</tr>
<tr>
<td>European Aeronautic Defence and Space Co NV</td>
<td>1,28</td>
<td>1,17 %</td>
</tr>
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</table>

Table 3.3 CDS Term Structure Calibration Results for VG in bps. based on the entities of the iTraxx Europe index on February 19th, 2010.
### Table 3.4

CDS Term Structure Calibration Results for VG in bps. based on the entities of the iTraxx Europe index on February 19th, 2010.

<table>
<thead>
<tr>
<th>Company</th>
<th>APE</th>
<th>RMSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Experian Finance PLC</td>
<td>0,54</td>
<td>0,99 %</td>
</tr>
<tr>
<td>Finmeccanica SpA</td>
<td>0,22</td>
<td>0,17 %</td>
</tr>
<tr>
<td>Fortum Oyj</td>
<td>0,51</td>
<td>0,84 %</td>
</tr>
<tr>
<td>France Telecom SA</td>
<td>0,48</td>
<td>0,80 %</td>
</tr>
<tr>
<td>Gas Natural SDG SA</td>
<td>1,18</td>
<td>0,99 %</td>
</tr>
<tr>
<td>GDF Suez</td>
<td>0,18</td>
<td>0,26 %</td>
</tr>
<tr>
<td>Glencore International AG</td>
<td>0,88</td>
<td>0,42 %</td>
</tr>
<tr>
<td>Groupe Auchan SA</td>
<td>0,23</td>
<td>0,32 %</td>
</tr>
<tr>
<td>Danone</td>
<td>0,50</td>
<td>0,65 %</td>
</tr>
<tr>
<td>Hannover Rueckversicherung AG</td>
<td>1,84</td>
<td>2,27 %</td>
</tr>
<tr>
<td>Hellenic Telecommunications Organization SA</td>
<td>0,43</td>
<td>0,35 %</td>
</tr>
<tr>
<td>Henkel AG &amp; Co KGaA</td>
<td>0,59</td>
<td>1,04 %</td>
</tr>
<tr>
<td>Holcim Ltd</td>
<td>1,37</td>
<td>0,85 %</td>
</tr>
<tr>
<td>Iberdrola SA</td>
<td>0,80</td>
<td>0,76 %</td>
</tr>
<tr>
<td>Intesa Sanpaolo SpA</td>
<td>0,80</td>
<td>0,81 %</td>
</tr>
<tr>
<td>J Sainsbury PLC</td>
<td>1,35</td>
<td>1,07 %</td>
</tr>
<tr>
<td>JTI UK Finance PLC</td>
<td>0,03</td>
<td>0,06 %</td>
</tr>
<tr>
<td>Koninklijke Ahold NV</td>
<td>1,01</td>
<td>0,99 %</td>
</tr>
<tr>
<td>Koninklijke DSM NV</td>
<td>0,55</td>
<td>0,91 %</td>
</tr>
<tr>
<td>Koninklijke KPN NV</td>
<td>0,36</td>
<td>0,52 %</td>
</tr>
<tr>
<td>Koninklijke Philips Electronics NV</td>
<td>0,26</td>
<td>0,40 %</td>
</tr>
<tr>
<td>Lanxess Finance BV</td>
<td>0,77</td>
<td>0,49 %</td>
</tr>
<tr>
<td>Linde AG</td>
<td>0,53</td>
<td>0,74 %</td>
</tr>
<tr>
<td>LVMH Moet Hennessy Louis Vuitton SA</td>
<td>0,67</td>
<td>0,96 %</td>
</tr>
<tr>
<td>Marks &amp; Spencer PLC</td>
<td>1,07</td>
<td>0,67 %</td>
</tr>
</tbody>
</table>
### Table 3.5 CDS Term Structure Calibration Results for VG in bps.

<table>
<thead>
<tr>
<th>Company</th>
<th>APE</th>
<th>RMSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Metro AG</td>
<td>1.14</td>
<td>0.88 %</td>
</tr>
<tr>
<td>Muenchener Rueckversicherungs AG</td>
<td>0.89</td>
<td>1.56 %</td>
</tr>
<tr>
<td>National Grid PLC</td>
<td>0.64</td>
<td>0.70 %</td>
</tr>
<tr>
<td>Nestle SA</td>
<td>0.55</td>
<td>1.06 %</td>
</tr>
<tr>
<td>Next PLC</td>
<td>1.52</td>
<td>1.17 %</td>
</tr>
<tr>
<td>Pearson PLC</td>
<td>0.23</td>
<td>0.33 %</td>
</tr>
<tr>
<td>Portugal Telecom International Finance BV</td>
<td>0.04</td>
<td>0.03 %</td>
</tr>
<tr>
<td>PPR</td>
<td>1.94</td>
<td>1.25 %</td>
</tr>
<tr>
<td>Publicis Groupe</td>
<td>0.41</td>
<td>0.43 %</td>
</tr>
<tr>
<td>Reed Elsevier PLC</td>
<td>0.26</td>
<td>0.34 %</td>
</tr>
<tr>
<td>Repsol YPF SA</td>
<td>0.76</td>
<td>0.63 %</td>
</tr>
<tr>
<td>Rolls-Royce PLC</td>
<td>0.22</td>
<td>0.28 %</td>
</tr>
<tr>
<td>RWE AG</td>
<td>0.23</td>
<td>0.41 %</td>
</tr>
<tr>
<td>SABMiller PLC</td>
<td>1.21</td>
<td>1.21 %</td>
</tr>
<tr>
<td>Safeway Ltd</td>
<td>0.89</td>
<td>1.26 %</td>
</tr>
<tr>
<td>Sanofi-Aventis SA</td>
<td>0.13</td>
<td>0.19 %</td>
</tr>
<tr>
<td>Siemens AG</td>
<td>1.56</td>
<td>2.04 %</td>
</tr>
<tr>
<td>Societe Generale</td>
<td>1.49</td>
<td>1.29 %</td>
</tr>
<tr>
<td>Sodexo</td>
<td>0.69</td>
<td>0.87 %</td>
</tr>
<tr>
<td>Solvay SA</td>
<td>0.46</td>
<td>0.60 %</td>
</tr>
<tr>
<td>STMicroelectronics NV</td>
<td>0.53</td>
<td>0.68 %</td>
</tr>
<tr>
<td>Suedzucker International Finance BV</td>
<td>0.97</td>
<td>0.76 %</td>
</tr>
<tr>
<td>Svenska Cellulosa AB</td>
<td>0.11</td>
<td>0.11 %</td>
</tr>
<tr>
<td>Swedish Match AB</td>
<td>0.80</td>
<td>0.81 %</td>
</tr>
<tr>
<td>Swiss Reinsurance</td>
<td>2.27</td>
<td>2.27 %</td>
</tr>
</tbody>
</table>

This table presents the Calibration Results for VG in bps. based on the entities of the iTraxx Europe index on February 19th, 2010.
<table>
<thead>
<tr>
<th>Company</th>
<th>APE</th>
<th>RMSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Telecom Italia SpA</td>
<td>1.71</td>
<td>1.12%</td>
</tr>
<tr>
<td>Telefonica SA</td>
<td>0.52</td>
<td>0.50%</td>
</tr>
<tr>
<td>Telekom Austria AG</td>
<td>1.51</td>
<td>1.43%</td>
</tr>
<tr>
<td>Telenor ASA</td>
<td>0.83</td>
<td>1.01%</td>
</tr>
<tr>
<td>TeliaSonera AB</td>
<td>0.38</td>
<td>0.63%</td>
</tr>
<tr>
<td>Tesco PLC</td>
<td>0.74</td>
<td>0.74%</td>
</tr>
<tr>
<td>Royal Bank of Scotland PLC/The</td>
<td>1.34</td>
<td>0.73%</td>
</tr>
<tr>
<td>ThyssenKrupp AG</td>
<td>2.82</td>
<td>0.91%</td>
</tr>
<tr>
<td>TNT NV</td>
<td>0.64</td>
<td>0.66%</td>
</tr>
<tr>
<td>Total SA</td>
<td>0.43</td>
<td>0.77%</td>
</tr>
<tr>
<td>UBS AG</td>
<td>0.88</td>
<td>0.75%</td>
</tr>
<tr>
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<td>0.63</td>
<td>0.50%</td>
</tr>
<tr>
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<td>0.01</td>
<td>0.02%</td>
</tr>
<tr>
<td>Union Fenosa SA</td>
<td>1.33</td>
<td>1.12%</td>
</tr>
<tr>
<td>United Utilities PLC</td>
<td>0.87</td>
<td>0.96%</td>
</tr>
<tr>
<td>Vattenfall AB</td>
<td>0.01</td>
<td>0.01%</td>
</tr>
<tr>
<td>Veolia Environnement</td>
<td>0.43</td>
<td>0.50%</td>
</tr>
<tr>
<td>Vinci SA</td>
<td>0.51</td>
<td>0.53%</td>
</tr>
<tr>
<td>Vivendi</td>
<td>1.75</td>
<td>1.37%</td>
</tr>
<tr>
<td>Vodafone Group PLC</td>
<td>0.80</td>
<td>0.92%</td>
</tr>
<tr>
<td>Volkswagen AG</td>
<td>1.19</td>
<td>0.97%</td>
</tr>
<tr>
<td>Wolters Kluwer NV</td>
<td>0.65</td>
<td>1.00%</td>
</tr>
<tr>
<td>WPP 2005 Ltd</td>
<td>0.66</td>
<td>0.41%</td>
</tr>
<tr>
<td>Xstrata PLC</td>
<td>1.68</td>
<td>0.89%</td>
</tr>
<tr>
<td>Zurich Insurance Co</td>
<td>2.11</td>
<td>2.18%</td>
</tr>
</tbody>
</table>

Table 3.6 CDS Term Structure Calibration Results for VG in bps. based on the entities of the iTraxx Europe index on February 19th, 2010.