Essays on mathematical and computational finance: With a view towards applied probability
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In this chapter we present the Drift Adjustment Method Laplace Algorithm (DAMLA) for efficiently and accurately calculating conditional expectations of Ornstein Uhlenbeck (OU) type processes. It is a novel approach in the sense that we can efficiently and accurately calculate conditional expectations, in high dimensions, at each time point in each scenario. Broadly speaking, our method can be used in Monte Carlo simulations as well as in quadrature methods. Applications are numerous. We use DAMLA for the simultaneous hedging of the interest and equity risk of a pension fund. More specifically, we price and determine the Greeks of an equity-linked swaption, commonly called hybrid option, to hedge pension fund risk. We find that, by combining the Drift Adjustment Method (DAM) with numerical Laplace transform inversion, i.e. the Drift Adjustment Method Laplace Algorithm (DAMLA), computational speed is greatly enhanced as compared to direct calculation (Monte Carlo). We can price and hedge a portfolio of 100,000 equity-linked swaptions in under 5 minutes on a standard PC with errors of the order $10^{-12}$. Finally, we will extend DAMLA to models driven by arbitrary Lévy processes, and show we can exactly proceed as in the diffusion case.

4.1 Introduction

In this chapter we present a novel and efficient approach for the calculation of conditional expectations. It is novel in the sense that we can efficiently and accu-
rately calculate conditional expectations, in high dimensions, at each time point in each scenario. We only need the underlying process to be an Ornstein-Uhlenbeck (OU) process. There is a vast literature about calculating conditional expectations and sensitivities (Greeks). Without giving an exhaustive list, we briefly touch upon some methods in the literature we believe are worthwhile mentioning.

Among the most well-known methods for calculating sensitivities in Monte Carlo simulation are finite difference type methods. However, it is a well known fact that for non-smooth or discontinuous payoffs the finite difference method is very inaccurate. In practise, the payoff of a portfolio of financial instruments almost always non-smooth. Classic papers on the finite difference method are e.g. Brennan & Schwartz [20], and Hull & White [63]. For an excellent discussion of finite difference type methods in finance see e.g. Glasserman [54].

A method that overcomes the smoothness problem of finite difference type methods is the Likelihood Ratio Method, as proposed by Broadie and Glasserman [24] (see e.g. [54] for an extensive discussion on this topic). This method avoids the problematic differentiation of the payoff function by differentiating the density function. Calculating conditional expectations also follows straightforwardly with this method. However, the downside is that one explicitly has to know the density function of the underlying process. This is no problem in case of OU processes driven by a Wiener process, but when we generalize to arbitrary Lévy processes we cannot use this method any more since we have no explicit expression for the density of the driving process.

A theory that encompasses both the Likelihood Ratio Method for calculating conditional expectations and sensitivities, is the Malliavin Calculus approach. Broadly speaking, the Malliavin calculus is a complex theory that consists of constructing and exploiting differentiable structures on abstract probability spaces. Its use in finance also avoids the problematic differentiation of the payoff function for calculating sensitivities and does not need an explicit expression for the density of the underlying process. However, this theory is so generic that calculating sensitivities of widely used products like Asians and swaptions is already a daunting task. A standard reference for the Malliavin calculus is [97]. For the application of the Malliavin calculus in finance good references are [13], [50], [51] and [99].
The Drift Adjustment Method Laplace Algorithm (DAMLA) that we propose is somewhat in between the two methods mentioned above. Our method is not as generic as the Malliavin Calculus approach, but on the other hand we do not need the complex machinery. By putting some structure on the underlying process, we develop a powerful method for the efficient computation of conditional expectations. The strength of the method lies in the fact that we can efficiently and accurately determine conditional expectations and sensitivities, in high dimensions, at each time point in each scenario, in linear time. We note that the adding of extra structure on the underlying process, i.e. it being an OU process, is not a very big restriction. We only need that the risk factors can be modeled by OU processes. It is not necessary that the economic variables are OU processes, but that these variables are functions of these risk factors. It is even not necessary to know the explicit form of these functions. We can simply multiply the quantities of interest with the so-called DAMLA weights produced by our method.

The rest of this chapter is structured as follows. Section 4.2 introduces the Drift Adjustment Method (DAM), i.e. the basic method underlying DAMLA. As a first application we will use DAM to determine different conditional expectations of plain vanilla options. Section 4.3 contains applications of DAM to Monte Carlo. We will price and hedge the interest rate risk of a pension fund in an asset liability management (ALM) scenario model. Subsequently, in Section 4.4 we present DAMLA: the Drift Adjustment Method Laplace Algorithm. DAMLA is basically DAM combined with the Fast Laplace Transform Inversion Algorithm, as developed by Den Iseger in [40]. In Section 4.5 we apply DAMLA to numerical methods. More specifically, we will price and hedge the total funding ratio of a pension fund by means of a so-called equity-linked swaption. Section 4.6 presents an extension of section 4.2, in the sense that we will introduce DAMLA driven by arbitrary Lévy processes. Section 4.7 concludes. Topics untouched in the sections or lengthy proofs are stated in the various appendices.

### 4.2 The Drift Adjustment Method (DAM)

Conditional expectations are of central importance in the field of mathematical finance. In this chapter we develop a new method called the Drift Adjustment Method Laplace Algorithm (DAMLA) for efficiently and accurately calculating conditional expectations of multi-dimensional OU processes $y_t$ with dynamics

$$dy_t = -A_t dt + \Sigma dL_t$$ (4.1)
4.2. THE DRIFT ADJUSTMENT METHOD (DAM)

in which $A$ is a deterministic, invertible, mean-reversion matrix, $\Sigma$ the covariance matrix and $L_t$ a multi-dimensional Lévy process. For now we will consider the case that $L_t = W_t$, i.e. a multi-dimensional Wiener process. In section 4.6 we will consider arbitrary Lévy processes. The OU process is a Gaussian process that has bounded variance if $A$ is a stable matrix. We denote with $Q_{t,x}$ the conditional distribution of $\{y_T, T \geq t\}$, given that $y_t = x$ at time $t$. Note that under $Q_{t,x}$, the solution of (4.1) is given by

$$y_T = e^{-A(T-t)}x + \int_t^T e^{-A(T-s)}\Sigma dW^x_s,$$

where $W^x_s$ is a standard Brownian motion under $Q_{t,x}$, with $W^x_t = 0$. Hence under $Q_{t,x}$ we have that $y_T$ is Gaussian distributed with mean

$$\mu_{y_T} = e^{-A(T-t)}x,$$

and variance

$$\Omega_{y_T} = \int_t^T e^{-A(T-s)}\Omega e^{-A^*(T-s)} ds,$$

with $\Omega = \Sigma \Sigma^*$, and $\Sigma^*$ denotes as usual the transpose (adjoint) of $\Sigma$.

In this chapter we present a method to compute conditional expectations of the form

$$E_{Q_{t,x}} \phi(y_T),$$

for fixed $T$ and for a large number of different $x$, in which $\phi(y_T)$ is a $\sigma(y_T)$-measurable random variable (e.g. option payoffs). An important application is the pricing of derivatives and embedded options in an asset liability management (ALM) Monte Carlo environment. It is well-known that pricing of derivatives and embedded options is equivalent with computing a conditional expectation under the so-called risk neutral probability measure. In fact, we can use our method in two distinct ways:

1. We can compute conditional probabilities on a Monte Carlo grid by computing weighted averages over the Monte Carlo scenarios;

2. We can design numerical methods that compute all the conditional expectations at once for all Monte Carlo scenarios.
Consider now the process $y_t$ started at time $t = t$ in $x'$. Under $Q_{t,x'}$, the solution of the OU process at time $t = T$ is given by

$$y_{t,x'}^T = e^{-A(T-t)}x' + \int_t^T e^{-A(T-s)}\Sigma dW_{x'}^s,$$

where we use the superscript $t, x'$ to emphasize the dependency of $y_{t,x'}^T$ on $y_{t,x'}^t = x'$, and where $W_{x'}^s$ is a standard Brownian motion under $Q_{t,x'}$, with $W_{x'}^t = 0$.

Now let $\lambda_s, t \leq s \leq T$, be the drift adjustment to incorporate the shock $(x' - x)$ in the time interval $[t, T]$. That is, $\lambda_s$ is such that

$$\int_t^T e^{-A(T-s)}\Sigma \lambda_s ds = e^{-A(T-t)}(x' - x).$$

This is by definition equal to $y_{t,x'}^T - y_{t,x}^T$. We are only interested in the distribution of $y_T$ for different initial conditions $x$ at time $t = T$. Therefore, we choose the drift adjustment as in Equation (4.4). Using this drift adjustment, we have

$$y_{t,x'}^T = y_{t,x}^T + \int_t^T e^{-A(T-s)}\lambda_s ds.$$

The Girsanov theorem states that a change of drift is equivalent to a change of measure that is given by the Radon-Nikodým derivative, i.e. $dQ_{\lambda_t,x}^{\lambda_{t,x}}$ in our case. That is, we can make a drift adjustment $\lambda_s$ given by (4.4) on the Brownian motion $W_{x'}^s$ in such a way, that under the new probability measure $Q_{\lambda_t,x}$, the expression $W_{x'}^s = W_{x}^s - \int_t^s \lambda_s ds, t \leq s \leq T$, is again a Brownian motion. We know that

$$E_{Q_{t,x}}(\phi(y_T)) \left( \frac{dQ_{\lambda_t,x}^{\lambda_{t,x}}}{dQ_{t,x}} \right) = E_{Q_{\lambda_t,x}}(\phi(y_T)),$$

in which the Radon-Nikodým derivative is explicitly given by

$$\frac{dQ_{\lambda_t,x}^{\lambda_{t,x}}}{dQ_{t,x}} = \exp \left\{ \int_t^T \lambda_s dW_{x'}^s - \frac{1}{2} \int_t^T ||\lambda_s||^2 ds \right\},$$

and $\lambda_s$ satisfies Equation (4.4). For notational convenience we suppress the subscripts $x$ and $t$ in the remainder of the chapter, and assume that all expectations are taken under the measure $Q_{\lambda_t,x}$ unless explicitly stated otherwise.

Note that the method we present here is a type of “discrete pathwise smearing” method, which means that every shock (i.e. perturbation) of the process $y(\cdot)$ is smeared out between the discrete time points $t$ and $T$. We now want to choose $\lambda_s$ in such a way that it yields the minimum variance Radon-Nikodým derivative. This is stated in Theorem 4.1.
Theorem 4.1. Consider the constrained optimization problem

\[
\min_t \int_t^T ||\lambda_s||^2 ds
\]

subject to

\[
\int_t^T e^{-A(T-s)}\Sigma\lambda_s ds = e^{-A(T-t)}(x' - x),
\]

and suppose that

\[
\int_t^T e^{As}\Omega e^{A^*s} ds
\]

is invertible, then

\[
\lambda_{s_{opt}} = \Sigma^* e^{A^*s} \left\{ \int_t^s e^{As}\Omega e^{A^*s} ds \right\}^{-1} e^{AT}(x' - x),
\]

in which \( \Omega := \Sigma\Sigma^* \).

Proof. See Appendix 4.8.

An interesting feature to note here is that the optimal \( \lambda_{s_{opt}} \) only depends on the difference between the initial starting points \((x' - x)\). Hence, \( \lambda_{s_{opt}} \) is a function of \((x' - x)\) only. Consequently, the Radon-Nikodým derivative in Equation (4.6) depends only on this difference, which is not surprising since the process we consider is Markovian. Another salient detail is that the optimal drift adjustment is linear in the difference \((x' - x)\). In Remark 4.2 we will show that the optimal deterministic drift adjustment \( \lambda_{t_{opt}} \) is actually properly defined if \( \text{Cov}(yT) \) is invertible.

Remark 4.2. The drift adjustment \( \lambda_{t_{opt}} \) is properly defined if the covariance matrix of \( yT \), i.e. \( \text{Cov}(yT) = \Omega \), is invertible.

Proof. See Appendix 4.8.

Applying the Girsanov theorem, we can convert the optimal drift adjustment \( \lambda_{t_{opt}} \) to the following weight (i.e. Radon-Nikodým derivative) \( \hat{\pi}(x' - x) \):

\[
\hat{\pi}(x' - x) = \exp \left\{ \int_t^T (\lambda_{s_{opt}})^* dW_s - \frac{1}{2} \int_t^T ||\lambda_{s_{opt}}||^2 ds \right\}, \tag{4.7}
\]

Recapitulating, we first (deterministically) chose a minimum variance drift adjustment \( \lambda_{t_{opt}} \). According to Girsanov’s theorem, this is equal to a change of measure given by the Radon-Nikodým derivative \( \frac{d\hat{Q}^{\lambda,x}}{dQ^{\lambda,x}} \), which is, by definition, the optimal \( \hat{\pi}(x' - x) \) induced by a deterministic drift adjustment.
We now present the main theorem of this chapter. It states that if we can find a weight (e.g. \( \hat{\pi}(x' - x) \)), we can always find a unique and optimal (i.e. minimum variance) weight by taking its conditional expectation. In this respect, consider Theorem 4.3.

**Theorem 4.3.** Let \( \Pi \) be the set of all random variables \( \pi \) satisfying

\[
E_{Q_{t,x}}(\phi(y_T)\pi) = E_{Q_{t,x}}(\phi(y_T)) \text{ for all Borel measurable functions } \phi,
\]

where \( Q_{t,x'} \) is a probability measure equivalent to \( Q_{t,x} \) on \( \sigma(y_T) \). Then there exists a \( Q_{t,x} \)-a.s. unique \( \sigma(y_T) \)-measurable element \( \pi_0 \in \Pi \), for which it holds that

\[
E_{Q_{t,x}}(\pi|y_T) = \pi_0, \quad \forall \pi \in \Pi,
\]

and this \( \pi_0 \) is a simultaneous solution to each of the optimization problems

\[
\min_{\pi \in \Pi} \text{Var}_{Q_{t,x}}(\phi(y_T)\pi),
\]

for any \( \sigma(y_T) \)-measurable random variable \( \phi(y_T) \) for which the variance is well defined.

**Proof.** See Appendix 4.8.

**Remark 4.4.** Theorem 4.3 has an obvious extension, which is obtained by replacing \( \phi(y_T) \) by \( \phi(y_{T_1}, \ldots, y_{T_n}) \) and similarly \( \sigma(y_T) \) by \( \sigma(y_{T_1}, \ldots, y_{T_n}) \), with \( t < T_1 \leq \ldots \leq T_n \). The proof, with the obvious modifications, remains almost verbatim the same.

**Remark 4.5.** Note that the set \( \Pi \) is non-empty, since \( \hat{\pi} := \hat{\pi}(x' - x) \), see Equation (4.7), belongs to it in view of Equation (4.5). Taking \( \pi = \hat{\pi} \) yields \( \pi_0 = E_{Q_{t,x}}(\hat{\pi}|y_T) \), a concrete recipe for computing \( \pi_0 \).

**Corollary 4.6.** Consider the setting of Theorem 4.3 and let \( \Pi' \) be the set of all random variables \( \pi' \) satisfying

\[
E_{Q_{t,x}}(\phi(y_T)\pi') = \frac{\partial}{\partial x'}E_{Q_{t,x'}}(\phi(y_T)) \text{ for all } \sigma(y_T) \text{-measurable random variables } \phi(y_T).
\]

Then \( \pi'_0 := \frac{\partial}{\partial x'} \pi_0 \) belongs to \( \Pi' \) and it simultaneously minimizes

\[
\min_{\pi' \in \Pi} \text{Var}_{Q_{t,x}}(\phi(y_T)\pi'),
\]

for any \( \sigma(y_T) \)-measurable random variable \( \phi(y_T) \) for which the variance is well defined.

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3It is a tacitly understood that the \( \phi \) are such that the expectations in (4.8) are well defined. Clearly this is true for bounded \( \phi \).
Proof. We assume that for any \( \pi \in \Pi \) one has the equality \( \frac{\partial}{\partial \pi} \mathbb{E}_{Q_t,\pi} \phi(y_T) = \mathbb{E}_{Q_t,\pi} \left( \phi(y_T) \frac{\partial}{\partial \pi} \pi \right) \). Then the proof is analogous to the proof of Theorem 4.3.

Remark 4.7. Note that Theorem 4.3 and Corollary 4.6 hold not only for OU processes driven by a Brownian motion, but for all OU processes.

Theorem 4.3 is a powerful theorem. It states that if one can determine a feasible weight \( \pi \), one can always find a unique and optimal weight by taking its conditional expectation. Cf. the Rao-Blackwell theorem in statistics on the optimality of estimators. Moreover, as a consequence of Corollary 4.6 not only the weight \( \pi_0 \) is optimal (i.e. yields the minimum variance and is unique), but also its derivative(s). This yields a powerful tool for calculating sensitivities and conditional expectations, as will be shown in the remainder of this chapter. Furthermore, with the same arguments it can be shown that Corollary 4.6 holds for any linear operator in stead of just the derivative operator. If we now apply Theorem 4.3 to the weight in Equation (4.7), we find the following optimal weight

\[
\pi_0 = \mathbb{E} \left[ \exp \left\{ \int_t^T (\lambda_s^{\text{opt}})^* dW_s - \frac{1}{2} \int_t^T ||\lambda_s^{\text{opt}}||^2 ds \right\} \mid y_T \right].
\]  

(4.12)

Naturally, the question arises what \( \pi_0 \) explicitly looks like. This is stated in the next theorem.

Theorem 4.8. Let \( y_T \) be as in Equation (4.2) and \( \pi \) as in Equation (4.12). Then \( \pi_0 \) is given by

\[
\pi_0 = \mathbb{E} \left[ \exp \left\{ (x' - x)^* e^{-A^* T \Sigma^{-1} Z} \right\} \right] \mathbb{E} \left[ \exp \left\{ (x' - x)^* e^{-A^* T \Sigma^{-1} Z} \right\} \mid y_T \right].
\]

(4.13)

with

\[
Z = \Sigma^{-1} (y_T - \mu_{y_T}).
\]

Furthermore, \( Z \sim N(0, I) \).

Proof. See Appendix 4.8.

Remark 4.9. Note that we now have that \( \pi_0 := \pi_0(x' - x) \), which was not clear beforehand.

We end this subsection with the following remark, which will be useful later on when doing numerical computations.
Remark 4.10. The denominator of (4.13) can be expressed more explicitly as
\[
\mathbb{E}\left( \exp \left( (x' - x)^* e^{-A^*T \Sigma^{-1} Z} \right) \right) = \exp \left( \frac{1}{2}(x' - x)e^{-A^*T \Omega^{-1} e^{-AT}(x' - x)} \right).
\]
Hence, for the optimal weight \( \pi_0(x' - x) \) we now have
\[
\pi_0(x' - x) = \frac{\exp \left( (x' - x)e^{-A^*T \Sigma^{-1} Z} \right)}{\exp \left( \frac{1}{2}(x' - x)e^{-A^*T \Omega^{-1} e^{-AT}(x' - x)} \right)}.
\]

4.2.1 Convolutions

In light of efficient numerical computation (see section 4.4), it is important to note that the optimal drift adjustment weights and conditional expectations possess a convolution structure. This is shown in the following theorem.

Theorem 4.11. The optimal weight \( \pi_0(x' - x) \) can be represented as follows.
\[
\pi_0(x' - x) = \frac{\exp \left( -\frac{1}{2}||z - Z||^2 \right)}{\exp \left( -\frac{1}{2}||Z||^2 \right)},
\]
(4.14)
in which \( z \) is defined as
\[
z := \Sigma^{-1} e^{-AT}(x' - x),
\]
and \( Z \) is again
\[
Z = \Sigma^{-1} (y_T - \mu y_T).
\]
Hence, for the conditional expectation we obtain the convolution
\[
\mathbb{E} [\phi(y_T) \pi_0(y_T, z)] = \mathbb{E} \left[ \frac{\hat{\phi}(Z)}{\exp \left( -\frac{1}{2}||Z||^2 \right)} \exp \left( -\frac{1}{2}||z - Z||^2 \right) \right],
\]
(4.15)
with
\[
\hat{\phi}(z) = \phi(\mu y_T + \Sigma z).
\]

Proof. See Appendix 4.8.1. □

For the sake of completeness and their use in the following sections, we will give the first and second order derivatives in convolution form in the following theorem.

Theorem 4.12. The first and second order derivatives of \( \pi_0(x' - x) \) with respect to \( x \), i.e. \( \pi'_0(x' - x) \) and \( \pi''_0(x' - x) \) respectively, are given by
\[
\pi'_0(x' - x) = -\Sigma^{-1} e^{-AT}(z - Z)\pi_0
\]
(4.16)
\[
\pi''_0(x' - x) = -\Sigma^{-1} e^{-AT} (I - (z - Z)(z - Z)^*) e^{-A^*T \Sigma^{-1}} \pi_0,
\]
(4.17)
with \( I \) the identity matrix.

Proof. See Appendix 4.8.1. □
4.2. THE DRIFT ADJUSTMENT METHOD (DAM)

4.2.2 The Drift Adjustment Method for Calculating Greeks

In this section we will first develop general formulae for the optimal weights of the Delta (\(\Delta\)) and Gamma (\(\Gamma\)). After that, we will derive explicit expressions for the Black-Scholes case. Hence, we are interested in what effect small perturbations of the initial condition of \(y_t\) (i.e. \(y_0\)), have on the option value \(V_0 = e^{-rt}E_Q[\phi(y_{t_1}, y_{t_2}, \ldots, y_{t_n})|y_{t_n}]\) and the Delta respectively. Using the first and second order derivatives (with respect to the initial condition) of the formula for the optimal weight, i.e. Theorem 4.12, we can relatively easily determine the Delta and Gamma.

\[
\begin{align*}
\pi_0(0) &= 1 \\
\pi'_0(0) &= \Sigma^{-1}e^{-AT}Z \\
\pi''_0(0) &= -\Sigma^{-1}e^{-AT}(I-ZZ^*)e^{-A^*T}\Sigma^{-1^*}.
\end{align*}
\]

Finally, for Delta and Gamma we obtain respectively

\[
\begin{align*}
\Delta((x' - x), T) &= E[\phi(y_T)\pi'_0(x' - x)] \\
\Gamma((x' - x), T) &= E[\phi(y_T)\pi''_0(x' - x)].
\end{align*}
\]

We make the following observations.

- A shock at time \(t\) on the process \(y(\cdot)\) is smeared out on the interval \([t, T]\) as \(e^{-A(T-t)}\Delta\). Thus delta, i.e. the first order derivative with respect to the initial condition \(x\), decreases exponentially as the maturity date increases.

- Gamma, i.e. the second order derivative with respect to the initial condition \(x\), is smeared out on the interval \([t, T]\) as \(||\pi''_0||\) as \(||e^{-A(T-t)}|| ||e^{-A^*(T-t)}||\). This since \(||\pi''_0||\) is bounded by \(||e^{-A(T-t)}|| ||M|| ||e^{-A(T-t)}||\) for some constant matrix \(M\)

- The effect of the mean reversion matrix \(A\) is that the larger the smallest eigenvalue of \(A\), the easier the smearing of the shock. Hence, \(A\) determines the effect of a shock on time \(t\) at the end date \(T\).

Hence, we may conclude that finding Delta and Gamma is merely a matter of taking the derivative with respect to the parameter of interest, which boils down to classical calculus. Furthermore, note that the Drift Adjustment Method can be represented as “Payoff × Weight”.
Example 4.13. We will now apply the drift adjustment method (DAM) to a concrete example; the calculation of the Delta and Gamma in the Black-Scholes setting. In this model, under the risk-neutral measure \( \mathbb{Q}_0 \), the stock price dynamics are represented by the following time-homogeneous SDE

\[
dS_t = rS_t dt + \sigma S_t dW_t. \tag{4.18}
\]

Suppose we want to determine the Delta (\( \Delta \)) and Gamma (\( \Gamma \)) for a European call option. Using DAM, we can determine Delta as follows. We consider \( y_t := \log(S_t) \), in which \( S_t \) is given by (4.18), we have

\[
dy_t = \left( r - \frac{1}{2} \sigma^2 \right) dt + \sigma dW_t.
\]

Furthermore, in the Black-Scholes framework we have that the mean reversion matrix \( A = 0 \) and the covariance matrix \( \Omega = \sigma^2 \). Hence, for the weights for the Delta and Gamma we obtain respectively (note that we want the sensitivity wrt \( S_0 \) and we assumed the underlying process to be \( y_t = \log(S_t) \))

\[
\pi'_0(x' - x) = \frac{W[t, T]}{\sigma(T-t)S_0}, \tag{4.19}
\]

and

\[
\pi''_0(x' - x) = \left( \frac{(W[t, T])^2}{\sigma(T-t)} - \frac{1}{\sigma} - W[t, T] \right) \frac{1}{S_0^2 \sigma(T-t)}, \tag{4.20}
\]

in which \( W[t, T] := W(T) - W(t) \).

Proof. See Appendix 4.8.2 \qed

Hence, for a European option with payoff \( \phi(y_T) \), the Drift Adjustment Method (DAM) Delta and Gamma are given by

\[
\Delta(x, T) = \mathbb{E} [\phi(y_T)\pi'_0(x' - x)] \tag{4.21}
\]

\[
= \mathbb{E} \left[ \phi(y_T)\frac{W[t, T]}{\sigma(T-t)S_0} \right]
\]

and

\[
\Gamma(x, T) = \mathbb{E} [\phi(y_T)\pi''_0(x' - x)] \tag{4.22}
\]

\[
= \mathbb{E} \left[ \phi(y_T) \left( \frac{(W[t, T])^2}{\sigma(T-t)} - \frac{1}{\sigma} - W[t, T] \right) \frac{1}{S_0^2 \sigma(T-t)} \right]
\]

respectively.
As a final remark we want to note that the optimal DAM weights in the Black-Scholes model are equal to the Malliavin weights (see e.g.\[59\] for an excellent discussion about the applications of the Malliavin calculus in finance). This holds because the Malliavin weights in the Black-Scholes Model are, like the DAM weights, $\sigma(y_T)$-measurable. As a consequence of Theorem 4.3 these weights are unique and optimal. Hence, the Malliavin weights and the Girsanov drift adjustment weights are necessarily equal.

### 4.3 Application I: Hedging a Pension Fund’s Interest Rate Risk

Among the most important risks for a pension fund (PF) is their exposure to interest rate risk. For a decrease in the long-term interest rate increases the value of a PF’s liabilities. In this section we show how to price and hedge the interest rate risk of a pension fund using the Drift Adjustment Method.

Standard numerical methods known in the literature only calculate prices and Greeks at time $t = 0$. However, with the method we developed in this chapter, we can price and determine the Greeks over the *whole* Monte Carlo grid, i.e. *not only* on $t = 0$ but on *all* time points in each scenario. In this section we devise an interest rate hedge in the form of a swaption, and show how to price and hedge on the whole Monte-Carlo grid. In this way we can get a good insight into the long-term implications of an interest rate drop.

To start the exposition, let $T_i$ with $i \in \{1, \ldots, n\}$ be the payment dates of the underlying swap and note that the price of a zero coupon discount bond at time $t$ that pays out unity at maturity $T_n$ is given in a multi-factor Hull-White model by

$$P(t, T) = \frac{P(0, T)}{P(0, t)} \exp \left( \hat{A}_{t,T} - B_{T-t} y_t \right)$$

(4.23)

(see Appendix 4.9 for an exposition on the model), in which the dynamics of the process $y_t$ under the forward measure $\mathbb{T}$ are given by

$$dy_t = -A y_t dt + \Sigma dW_t$$

with $y_t = x$.

It can be shown that (see Appendix 4.10) the value of a receiver swaption at time $t$ can be written as

$$v^{rec} = P(t, T_0) \mathbb{E}_\mathbb{T} \left( \sum_{j=1}^{n} \hat{w}_j \exp \left( B_{T_j-T_0} Z \right) - 1 \right)^+$$

(4.24)
We can calculate the expectation in Equation (4.24) using numerical integration methods, but this procedure is efficient only if the integrand is a smooth function. It is well known that the integrand in our case, i.e. the payoff of the swap at maturity, is not a smooth function. Hence, numerical integration techniques cannot be applied to Equation (4.24) in this form. To overcome this problem, we introduce the so-called Jamshidian trick, see [68], which can be used to smoothen the payoff function of a swaption. Basically, this trick entails rewriting an option on a portfolio of bonds as a portfolio of options on bonds. This is done in Appendix 4.11. What we end up with is an expression for the value of a receiver swaption at time $t$ that looks as follows:

$$
\sum_{|k|} A_{|k|} (V_{rec}(Z_1; \lambda_{|k|})) 1 \{ Z_1 \geq \alpha(\lambda_{|k|}) \} = \sum_{|k|} A_{|k|} \sum_{j=1}^n \tilde{w}_j e^{B_j^* \lambda_{|k|}} \left[ e^{B_j^* Z_1} - e^{B_j^* \alpha(\lambda_{|k|})} \right]^+,
$$

where $A_{|k|} = \prod_{j=2}^m a_{kj}$ and $\lambda_{|k|} = (\lambda_{k_2}, ..., \lambda_{km})$ with $a_k$ the quadrature weights and $\lambda_k$ the quadrature points. Furthermore, $Z_i$ represents a standard normal random variable. Note that the term $\left[ e^{B_j^* Z_1} - e^{B_j^* \alpha(\lambda_{|k|})} \right]^+$ is the payoff of an option on a bond, of whose price we have a closed formula solution available in the standard Black Scholes framework.

We now move to the application of the Drift Adjustment Method to swaptions. As mentioned, using this method we can determine the price and Greeks on all time points in all scenarios. In this respect, consider the following. We want to price a swaption not at the point $y_t = x$, but at the point $y_t = x'$. We know that the price of a swaption is a function of $y_t$, in which $y_t$ is given in differential form by

$$
dy_t = -Ay_t dt + \Sigma dW_t \quad \text{with} \quad y_t = x'.
$$

We want to calculate conditional expectations as follows

$$
E((V_{rec}(y_T))^+|y_t = x') = E((V_{rec}(y_T))^+ \pi_0((x' - x), Z_1, \ldots, Z_m)|y_t = x)
$$

First note that the optimal weight $\pi_0((x' - x), Z_1, \ldots, Z_m)$ can be represented as a product as follows:

$$
\pi_0((x' - x), Z_1, \ldots, Z_m) = \prod_{i=1}^m \pi_{0i}((x' - x), Z_i),
$$

(see Appendix 4.12) where

$$
\pi_{0i}((x' - x), Z_i) = \frac{\exp \left( -\frac{1}{2} (z_i^* - Z_i)^2 \right)}{\exp \left( \frac{1}{2} Z_i^2 \right)}
$$
4.3. APPLICATION I: HEDGING A PENSION FUND’S INTEREST RATE RISK

is of the form \( C \exp(g((x' - x), Z)) \) (cf. equations (4.14), (4.16) and (4.17)). Because \( g \) is linear in \( Z \), we can write this as a product as follows:

\[
\pi_0((x' - x), Z_1, \ldots, Z_m) = \pi_{01}((x' - x), Z_1)\pi_{02}((x' - x), Z_2, \ldots, Z_m). \tag{4.25}
\]

Note that these weights integrate to unity, since we have that

\[
\frac{\pi_0}{E\pi_0} = \frac{\pi_{01}}{E\pi_{01}} \frac{\pi_{02}}{E\pi_{02}}. \tag{4.26}
\]

Furthermore, let us introduce the shorthand notation

\[
\hat{A}_{|k|} = A_{|k|} \prod_{i=2}^{m} \pi_{0i}((x' - x), \lambda_{k_i}).
\]

Hence, we can write

\[
\mathbb{E}_{Z_m} \mathbb{E}_{Z_{m-1}} \cdots \mathbb{E}_{Z_1} ((V^{rec}(yT))^+ \pi_0((x' - x), Z_1, \ldots, Z_m) =
\begin{pmatrix}
\sum_{k} \hat{A}_{|k|} \sum_{j=1}^{n} \hat{w}_j e^{B_{1j}^* \lambda_{|k|}} \\
\end{pmatrix}
\mathbb{E}_{Z_1} \left( e^{B_{1j}^* Z_1} - e^{B_{1j}^* \alpha(\lambda_{|k|})} \right)^+ \pi_{01}((x' - x), Z_1).
\]

Observe that the weight \( \pi_{01}((x' - x), Z_1) \) is a Radon-Nikodým derivative and is given by

\[
\pi_{01}((x' - x), Z_1) = \frac{\exp(z_1^* Z_1)}{\exp(\frac{1}{2} z_1^* z_1)},
\]

in which we define \( z_1 \) as the first element of

\[
z := \Sigma^{-1} e^{-AT} (x' - x).
\]

We can now use Girsanov’s theorem and apply a drift adjustment \( \gamma \) to the standard normal random variable \( Z_1 \), and obtain

\[
\mathbb{E}_{Z_1} \left( e^{B_{1j}^* Z_1} - e^{B_{1j}^* \alpha(\lambda_{|k|})} \right)^+ \pi_{01}((x' - x), Z_1) = \mathbb{E}_{Z_1} \left( e^{B_{1j}^* (\tilde{Z}_1 + \gamma) - e^{B_{1j}^* \alpha(\lambda_{|k|})}} \right)^+,
\]

with \( \gamma = z_1 \) and \( \mathbb{E}_{Z} f(x, z) := \mathbb{E} f(x, Z)|_{x=X} \) and \( \tilde{Z} \) a standard normal random variable. Hence, by applying DAM, we obtain for the price of a receiver swaption

\[
V^{rec} = \left( P(t, T_0) \sum_{k} \hat{A}_{|k|} \sum_{j=1}^{n} \hat{w}_j e^{B_{1j}^* \lambda_{|k|}} \right) \mathbb{E}_{Z_1} \left( e^{B_{1j}^* (\tilde{Z}_1 + z_1) - e^{B_{1j}^* \alpha(\lambda_{|k|})}} \right)^+.
\]

Along the same lines we can develop expressions for the Delta and Gamma of a receiver swaption. Note that for the first and second order derivatives of \( \pi_0(x' - x) \)
with respect to $x$, i.e. $\pi'_0(x' - x)$ and $\pi''_0(x' - x)$ respectively, we have in this case (cf. Equations (4.16) and (4.17))

\[
\begin{align*}
\pi'_0 ((x' - x), Z) &= -\Sigma^{-1} e^{-AT} \pi_z \\
\pi''_0 ((x' - x), Z) &= \Sigma^{-1} e^{-AT} \pi_z e^{-A^* T} \Sigma^{-1}
\end{align*}
\]

with

\[
\begin{align*}
\pi_z &= (z - Z) \pi_0 \\
\pi_{zz} &= -(I_m - (z - Z_1)(z - Z_1)^*) \pi_0.
\end{align*}
\]

Note that we drop the argument $(x' - x)$ in the weights to make equations less opaque. Multiplying $\pi'_0(x' - x)$ with the swaption payoff we obtain for Delta of a receiver swaption

\[
\Delta^{rec}(x' - x) = \mathbb{E}_{Z_m} \mathbb{E}_{Z_{m-1}} \ldots \mathbb{E}_{Z_1} ((V^{rec}(y_T))^{+} \pi'_0((x' - x), Z_1, \ldots, Z_m)
\]

\[
= \frac{\partial P(t, T_0)}{\partial x} - e^{-AT} \Sigma^{-1} P(t, T_0) \sum_k \hat{A}_k^{\Delta} \sum_{j=1}^n \hat{w}_j e^{B_j^* \lambda_k}
\]

\[
\times \mathbb{E}_{Z_1} \left( e^{B^*_1 Z_1} - e^{B^*_1 \alpha(\lambda_k)} \right)^+ (z_1 - Z_1) \pi_{01}((x' - x), Z_1)
\]

in which $\hat{A}_j^{\Delta}$ is defined as

\[
\hat{A}_j^{\Delta} := A_j \pi_{zz}((x' - x), \lambda_{k_j}).
\]

Applying Girsanov’s theorem now yields that

\[
\mathbb{E}_{Z_1} \left( e^{B^*_1 Z_1} - e^{B^*_1 \alpha(\lambda_{k_1})} \right)^+ (z_1 - Z_1) \pi_{01}((x' - x), Z_1)
\]

equals

\[
-\mathbb{E} \left( e^{B^*_1 (Z_1 + z_1)} - e^{B^*_1 \alpha(\lambda_{k_1})} \right)^+ Z_1.
\]

Explicit, but very lengthy expressions for the gamma can be obtained in exactly the same way as for the delta and are therefore omitted.

We now present some numerical results for a portfolio of swaptions by applying the DAM as described above. Figure 4.1 depicts graphs for the price and Delta,
4.3. APPLICATION I: HEDGING A PENSION FUND’S INTEREST RATE RISK

in a one factor model, for a scenario set of 100,000 swaption payoffs. Before evaluating the results, we note that the value of a receiver swap (see Appendix 4.10) given the maturity date \( T_n \) at time \( T_0 \) with a fixed rate (leg) \( K \), can be stated as

\[
V_{\text{rec}} = K \sum_{j=1}^{n} P(T_0, T_j) - (1 - P(T_0, T_n)).
\]

The value of a forward starting receiver swap that starts exchanging payments at \( T_i \) and ends at \( T_n \) is given by

\[
V_{\text{rec}}^{T_i} = K \sum_{j=i+1}^{n} P(T_0, T_j) - (P(T_0, T_i) - P(T_0, T_n)). \quad (4.28)
\]

In financial markets, swaps are not quoted for different fixed rates \( K \), but \( K \) is only quoted for each swap such that the present value of the swap equals zero. This rate is called the (par) swap rate. Now let us denote the swap rate for \([T_i, T_n]\) by \( y_{i,n} \). Solving (4.28) for \( K = y_{i,n} \) such that \( V_{\text{rec}} = 0 \) yields

\[
y_{i,n} = \frac{P(T_0, T_i) - P(T_0, T_n)}{\sum_{j=i+1}^{n} P(T_0, T_j)}. \]

The term in the denominator is known as the present value of a base point or \( P_{V01} \), i.e.

\[
P_{V01} = \sum_{j=i+1}^{n} P(T_0, T_j).
\]

Hence, we can now denote the value of a payer of receiver swap with a different fixed leg \( K \) as

\[
V_{\text{rec}}^{T_i} = (K - y_{i,n})P_{V01}
\]

\[
V_{\text{pay}}^{T_i} = (y_{i,n} - K)P_{V01}
\]

Hence, the value of a receiver and payer swaption becomes respectively

\[
v_{\text{rec}}^{T_i} = (K - y_{i,n})^+P_{V01} \quad (4.29)
\]

\[
v_{\text{pay}}^{T_i} = (y_{i,n} - K)^+P_{V01}
\]

Evaluating the results in figure 4.1, we can see from Equation (4.29) that if the 35-year interest rate rises, the price of a receiver swap goes down, hence the price of the receiver swaption also goes down. For the Delta we thus find a positive, convex function. Similar, but more complicated figures can be obtained for multi factor models. In these cases, care must be taken with the interpretation, since there is an interplay between effects of underlying interest rates and we only observe the net effect. For expository purposes we only present figures for the 1 factor case here.
4. THE DRIFT ADJUSTMENT METHOD LAPLACE ALGORITHM

Figure 4.1 Price and Delta a portfolio of Receiver Swaptions in a 1 Factor Model with Parameters $K = 4.5\%$ and $T = 3$ written on a 30 Year Swap. We use DAM and a Sample of $N = 100,000$ Swaptions.

4.4 DAMLA: The Drift Adjustment Method Laplace Algorithm

In this section we show how we can improve the computational efficiency of DAM by combining it with the numerical Laplace transform inversion algorithm as developed by den Iseger in [40]. We thus obtain DAMLA: the Drift Adjustment Method Laplace Algorithm. To start the exposition, remember (cf. Equation 4.15) that we want to calculate

$$E(V_T)_{\pi_0}(y_T, x) = E\left[ \frac{\hat{V}(Z)}{\exp\left(-\frac{1}{2}\|Z\|^2\right)} \exp\left(-\frac{1}{2}\|z_j - Z\|^2\right) \right]$$

for $j = 1, \ldots, N_x$.

Numerically, this looks as follows

$$V_t(z_j) = \sum_{k=1}^{N_Z} \frac{\hat{V}(Z^{(k)})}{\exp\left(-\frac{1}{2}\|Z^{(k)}\|^2\right)} \exp\left(-\frac{1}{2}\|z_j - Z^{(k)}\|^2\right) (\hat{w}_k, Z^{(k)}), \quad (4.30)$$

in which $(\hat{w}_k, Z^{(k)})$ is some quadrature rule, and $N_Z$ is large. We need to calculate expression (4.30) for many $z_j$. However, direct calculation of this sum is undesirable since the computational complexity is $O(N_Z N_x)$, i.e. it is very expensive in terms of computation time. However, we can calculate (4.30) efficiently with the
use of Laplace transforms. Bearing this in mind, let us define convolution weights $w_k$ as

$$w_k := \frac{\hat{V}(Z^{(k)})}{\exp\left(-\frac{1}{2}\|Z^{(k)}\|^2\right)} \hat{w}_k,$$

and the function $f(x)$ by

$$f(x) := \exp\left(-\frac{1}{2}\|x\|^2\right).$$

We can now rewrite Equation (4.30) as

$$g(x) = \sum_{k=1}^{m} w_k f(x - Z^{(k)}) \quad \text{for} \quad x = z_j. \quad (4.31)$$

Expression (4.31) is known as a (discrete) convolution. From Laplace transform theory we know that

$$\int e^{-sx} f(x - Z^{(k)}) dx = e^{-sZ^{(k)}} \int e^{-s(x-Z^{(k)})} f(x - Z^{(k)}) d(x - Z^{(k)}) = e^{-sZ^{(k)}} \hat{f}(s). \quad (4.32)$$

Imposing a Laplace transform with respect to $x$ to Equation (4.31) one obtains

$$\hat{g}(s) = \sum_{k=1}^{N_Z} w_k e^{-sZ^{(k)}} \hat{f}(s). \quad (4.33)$$

To calculate these Laplace transforms efficiently, we will use the Laplace transform algorithm of Den Iseger as developed in [40]. There he derives a Gaussian quadrature rule to approximate the Legendre expansion of the characteristic function. The Laplace transform inversion algorithm of Den Iseger calculates the Laplace transform of a function of the form $g$ in Equation (4.31), and efficiently inverts this Laplace transform numerically. This algorithm has computational complexity of $O(N_x + N_Z)$, which is a huge computational improvement as compared to direct calculation.

A good alternative for the Laplace algorithm of Den Iseger would be the Fast Gauss Transform (FGT) of Broadie and Yamamoto (see [25]), but this algorithm only works for Gaussian processes. However, the Laplace algorithm of Den Iseger works for more general processes, and we would like to make the theory we present here as general as possible. For other transform-based approaches with an eye on financial applications, see for instance Geman & Yor [53] and Craddock,
4. THE DRIFT ADJUSTMENT METHOD LAPLACE ALGORITHM


Other known methods in the literature are basically numerical integration techniques that evaluate point by point. If one has to evaluate 100,000 points for example, then one needs to go through the algorithm 100,000 times, hence computation time explodes. Another powerful property of the Laplace transform algorithm of Den Iseger that we would like to mention, is that the algorithm only needs numerical values of the Laplace transform and not the explicit functional form of $f(z_j - Z^{(k)})$. This will prove immensely useful when we will consider arbitrary Lévy processes in Section 4.6.

A step by step efficient numerical implementation plan for pricing and hedging swaptions using DAMLA can be outlined as follows.

**Pseudo code 4.14.** Pseudo-code for Pricing and Hedging Swaptions using DAMLA.

1. We first start by rewriting Equation (4.27) as follows

   $\mathbb{E}_{Z_m} \mathbb{E}_{Z_{m-1}} \cdots \mathbb{E}_{Z_1} \left[ \left( V^{rec}(y_T) \right)^{+} \pi_0((x'_j - x), Z_1, \ldots, Z_m) \right]$

   $= \mathbb{E}_{Z_m} \mathbb{E}_{Z_{m-1}} \cdots \mathbb{E}_{Z_1} \left[ \left( V^{rec}(Z_1, Z_2, \ldots, Z_m) \right)^{+} \frac{\exp\left(-\frac{1}{2}||Z||^2\right)}{f(z_j - Z)} \right].$

2. Using a Hermite quadrature rule (see Appendix 4.9) with weights $\hat{w}|_k|$, evaluate $Z_m, Z_{m-1}, \ldots, Z_2$ at the quadrature points $\lambda|_k|$.

3. Integrate $Z_1$ on $[\alpha(\lambda|_k|), \infty)$ using a double exponential (DE) quadrature rule (for this rule, see e.g. Broadie and Yamamoto in [25]). This yields the quadrature weights $w^{DE}_{|k|,j}$ and quadrature points $u_{|k|,j}$ of the DE rule.

4. Calculate the quadrature weights for the Laplace transform inversion algorithm of Den Iseger defined by

   $w_k := \frac{V^{rec}(\lambda|_k|, u_{|k|,j})}{\exp\left(-\frac{1}{2}||\lambda|_k||^2\right)} \frac{w^{DE}_{|k|,j}}{\exp\left(-\frac{1}{2}||u_{|k|,j}||^2\right)}.$

5. By inverting Equation (4.33) by means of the Den Iseger algorithm, we obtain the following convolution structure

   $g(z_j) = \sum_{k=1}^{N_Z} w_k f \left\{ u_{|k|,j}, \lambda|_k| - z_j \right\},$

   which we wanted to calculate.
When we apply DAMLA to the swaption example of section 4.3, we obtain the results depicted in tables 4.1 and 4.2.

<table>
<thead>
<tr>
<th>Errors</th>
<th>1 Factor</th>
<th>2 Factor</th>
<th>3 Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Error Price</td>
<td>5.0500e-013</td>
<td>-5.5234e-014</td>
<td>5.3833e-013</td>
</tr>
<tr>
<td>Error Delta</td>
<td>-1.1166e-012</td>
<td>1.1702e-012</td>
<td>-1.3264e-012</td>
</tr>
<tr>
<td></td>
<td>8.4580e-011</td>
<td>4.9440e-010</td>
<td>2.2180e-010</td>
</tr>
<tr>
<td>Error Gamma</td>
<td>1.29086e-012</td>
<td>-1.5892e-012</td>
<td>1.6873e-012</td>
</tr>
<tr>
<td></td>
<td>3.3571e-011</td>
<td>1.9440e-010</td>
<td>3.3964e-010</td>
</tr>
<tr>
<td></td>
<td>4.6571e-012</td>
<td>-2.9872e-011</td>
<td>-7.8259e-010</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>5.9517e-012</td>
</tr>
</tbody>
</table>

Table 4.1 Errors of Price, Delta and Gamma of a portfolio of 100,000 Swaptions using DAMLA.

<table>
<thead>
<tr>
<th>Computation Times</th>
<th>1 Factor</th>
<th>2 Factor</th>
<th>3 Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Price</td>
<td>0.44s</td>
<td>1.66s</td>
<td>27s</td>
</tr>
<tr>
<td>Price + Delta</td>
<td>0.88s</td>
<td>4.20s</td>
<td>110s</td>
</tr>
<tr>
<td>Price + Delta + Gamma</td>
<td>1.32s</td>
<td>9.96s</td>
<td>280s</td>
</tr>
</tbody>
</table>

Table 4.2 Computation Times of the Price, Delta and Gamma of a portfolio of 100,000 Swaptions using DAMLA.

Evaluating these results, we see that the application of the full Drift Adjustment Method Laplace Algorithm yields very good results. For instance, we find that we can price and hedge 100,000 swaptions in a 3-factor model in under 5 minutes, with errors of order $10^{-10}$ or smaller. The errors are the difference between direct computation on each point, as described in section 4.3 and the speed-up computation by means of the double exponential quadrature rule of Broadie and Yamamoto and the Laplace transform inversion algorithm of Den Iseger. Note that the fact that these errors are so small is a validation for the use of transform inversion we propose in this section.
4.5 Application II: Hedging a Pension Fund’s Funding Ratio

The main objective of any pension fund is to keep their funding ratio, i.e. the fraction between their assets and liabilities, above a certain level. Normally, equity and interest rate risk are managed separately. This shifted to the new vision of managing all risks jointly by focussing on a PF’s total solvency position. Normal market fluctuations, or even sharp price drops in a single market, are no direct threat for a PF since they can be absorbed by the fund’s own buffers. However, a combination of extreme equity price falls in all markets combined with a significant decrease in long-term interest rates can make any pension fund insolvent. Therefore, PFs want to hedge against the scenario that both equity and interest rates were to turn against them simultaneously. That is, they want to hedge against the situation that large decreases in returns on equity and simultaneous decreases in long-term interest rates because of a rise in expected inflation. This would preserve the equity risk premium as much as possible.

One way to do this is to devise a so-called equity-linked swaption, commonly called hybrid option, to protect against a decrease in a PF’s funding ratio. This product resembles a normal swaption. However, a normal swaption is a pure interest hedging instrument that is commonly used by pension funds to hedge against dropping interest rates. A hybrid option on the other hand is a swaption whose strike is not fixed but depends on the return on a basket of quanto stock indices. By linking the exercise price of the hybrid option to the return on a basket of quanto stock indices, we obtain an instrument that hedges both interest rate risk and equity risk. The main difference between a regular swaption and an equity-linked swaption is that the strike of the hybrid is only known at maturity T. With a regular swaption, the exercise price is fixed. The strike of the hybrid is defined as follows:

\[ K_H = K_S - G \left( \frac{S(T)}{S(0)} - 1 \right) \]  \hspace{1cm} (4.34)

in which \( K_H \) is the strike of the hybrid, \( K_S \) the strike of the (regular) swaption, \( S(t) \) the price of the underlying basket of stocks at time \( t \), and \( G \) the gearing factor. The strike of the regular swaption is constant. The second part of Equation (4.34) assures the sensitivity to equity of the hybrid. The strike of the regular swaption is lowered with the gearing times the return on the basket between times \( t = 0 \) and \( t = T \). When we take \( G = 0 \) the payoff of the hybrid is equal to the regular
4.5. APPLICATION II: HEDGING A PENSION FUND’S FUNDING RATIO

swaption. Increasing the gearing implies that the strike of the hybrid is more dependent on the return of the equity basket, and hence the sensitivity of the value of the hybrid increases with respect to equity movements.

The payoff of the receiver hybrid at time $T$ is described by a deterministic function $\Phi$, defined by

$$
\Phi(T) = PV01 [K_H(T) - Y(T)]^+ \tag{4.35}
$$

with $Y(T)$ the swap rate at time $T$, $PV01 = \sum_k D(t, t_k)$ the discount rate of the fixings and $K_H$ the hybrid strike as in Equation (4.34).

In this chapter we focus on a receiver hybrid as an example. A regular receiver swaption ends in-the-money when the relevant swap interest rate at expiration is below a certain strike level. In this case it is wise to exercise the option. A positive gearing in combination with a negative return on the basket of stocks leads to an increase of the actual exercise price. The receiver hybrid will pay out earlier (i.e. at higher rates of the relevant swap interest rates) and offers extra protection. That extra protection compensates a possible loss in returns on the basket of stocks.

The value of an equity-linked swaption is partly determined by the return on a basket of indices. In this chapter we use four indices, that together form the basket. For the modeling of the indices we adopted a standard Black-Scholes setting of which the dynamics are given by

$$
dS_k(t) = rS_k(t)dt + \sigma_kS_k(t)dW_k(t)
$$

$$
 dB(t) = rB(t)dt
$$

In which $S_k(t)$ denotes stock index $k$, $k \in \{1, 2, 3, 4\}$, $r$ the risk-free interest rate and $W_k(t)$ a standard Brownian motion.

We implemented DAMLA for the receiver equity-linked-swaption for a five year time horizon of which the results are displayed in Figure 4.2. We used a sample of $N = 100,000$ hybrids and assumed a stylized pension fund which has invested 40% in equity, 40% in bonds and 20% in cash. The data is from 2007 before the credit crisis when this method was implemented for a PF of a large Dutch bank. In the first row of Figure 4.2 the densities of the unhedged funding ratio (UFR) are displayed, and in the second row the densities of the hedged funding ratio (HFR), for years 1–5 respectively. We see for the HFRs that the hybrid effectively cuts off the downside of the distribution by giving up some upward potential, thus mak-
ing the distribution more concentrated. Indeed, in the UFR the distribution is much more dispersed, which could make the PF go insolvent in some scenarios. Hence, we conclude that the equity-linked-swaption actually does what is designed to do.

As a final remark we want to note that in 2007, before the credit crisis, the equity-linked-swaption as described in this section was implemented for the pension fund of a large Dutch bank, and billions were saved in the credit crisis. This is a validation of the power of the hybrid as a financial instrument.

![Figure 4.2](image)

**Figure 4.2** Effect of an Equity-Linked Swaption on the Density of the Funding Ratio of a stylized PF which has invested 40% in Equity, 40% in Bonds and 20% in Cash.

### 4.6 DAMLA for Arbitrary Lévy Processes

This section is basically an extension of section 4.2 in the sense that we will extend the DAMLA to the more general class of arbitrary Lévy Processes. More
specifically, we will determine the convolution structure of DAMLA weights in case of arbitrary Lévy Processes, and show that we can proceed in exactly the same way as we did in section 4.4.

### 4.6.1 DAMLA for Lévy Processes with a Continuous Part

In this section we will assume that SDEs are not driven by a Wiener process $W_t$, but by a Lévy process with a continuous part under the risk neutral measure $Q_{t,x}$. We will denote this by $L_t = W_t + J_t$, in which $W_t$ is a standard Wiener process independent of $J_t$, a jump process. For expository purposes we did not include additional covariance terms for $W_t$ and $J_t$, since this is only a matter of scaling. Note that in this section the jump process $J_t$ can be zero. Hence, for the OU process driven by a Lévy process with continuous part, we thus have

$$y^t_x = e^{-A(T-t)}x + \int_t^T e^{-A(T-s)}\Sigma dL_s$$

Like in the diffusion case, we want to determine a feasible weight and then by taking its conditional expectation (see Theorem 4.3) compute the optimal weight. From Theorem 4.8 we know that the optimal weight (in terms of $y^t_{cont}$) for a diffusion is given by

$$\pi_0(x' - x) = \exp \left( (x' - x)^* e^{-A^*T} \Omega^{-1} y^t_{cont} \right) \frac{E \exp \left( (x' - x)^* e^{-A^*T} \Omega^{-1} y^t_{cont} \right)}{E \exp \left( B^* y^t_{cont} \right)}.$$  \hspace{1em} (4.36)

Let us define $B^*$ as follows:

$$B^* := (x' - x)^* e^{-A^*T} \Omega^{-1},$$

in which $\Omega = \Sigma \Sigma^*$. We can now represent the optimal weight in case of a diffusion as

$$\pi_0(x' - x) = \exp \left( B^* y^t_{cont} \right) \frac{E \exp \left( B^* y^t_{cont} \right)}{E \exp \left( B^* y^t_{cont} \right)}.$$  \hspace{1em} (cf. Theorem 4.3)
\[ \pi_0(x' - x) = \mathbb{E}(\pi(x' - x) | y_T) \]
\[ = \mathbb{E}\left( \frac{\exp(B^* y_T^{\text{cont}})}{\mathbb{E}\exp(B^* y_T^{\text{cont}})} | y_T \right). \]

Now observe that
\[ \mathbb{P}(y_T \leq u | y_T^{\text{cont}} = v) = \mathbb{P}(y_T^{\text{jump}} \leq u - v). \]

Hence, the conditional density of \((y_T^{\text{cont}} | y_T)\) equals
\[ f_{y_T^{\text{cont}} | y_T = u}(v) = \frac{f(u, v)}{f_{y_T}(u)} \]
\[ = \frac{f_{y_T^{\text{jump}}}(u - v) f_{y_T^{\text{cont}}}(v)}{f_{y_T}(u)}, \]

because of independence. For expository purposes we assume that the density \(f_{y_T^{\text{jump}}}(u)\) exists. However, similar calculations can be performed in a more general setting. Thus, we have
\[ \mathbb{E}\left( \frac{\exp(B^* y_T^{\text{cont}})}{\mathbb{E}\exp(B^* y_T^{\text{cont}})} \middle| y_T = u \right) = \int \left( \frac{\exp(B^* v)}{\mathbb{E}\exp(B^* y_T^{\text{cont}})} \right) \frac{f_{y_T^{\text{jump}}}(u - v) f_{y_T^{\text{cont}}}(v)}{f_{y_T}(u)} dv. \]  
(4.37)

Observe that the numerator \( \int f_{y_T^{\text{jump}}}(u - v) f_{y_T^{\text{cont}}}(v) dv \) is a convolution. Now let us define \(g_B(x)\) as
\[ g_B(x) := \frac{\exp(B^* x) f_{y_T^{\text{cont}}}(x)}{\mathbb{E}\exp(B^* y_T^{\text{cont}})}. \]  
(4.38)

Note that \(g_B(x)\) is a density. We now obtain for Equation (4.37) the following expression
\[ \mathbb{E}\left( \frac{\exp(B^* y_T^{\text{cont}})}{\mathbb{E}\exp(B^* y_T^{\text{cont}})} \middle| y_T = u \right) = \frac{g_B \ast f_{y_T^{\text{jump}}}(u)}{f_{y_T}(u)} \]
\[ = \frac{g_B \ast f_{y_T^{\text{jump}}}(u)}{f_{y_T^{\text{cont}}} \ast f_{y_T^{\text{jump}}}(u)}. \]  
(4.39)

Using the fact that the Laplace transform of a convolution is the product of the Laplace transforms, we obtain the optimal DAMLA weight \(\pi_0^{CL}(x' - x)\) for a Lévy process with a continuous part. This is given in the next algorithm.
Pseudo code 4.15. Algorithm for Computing DAMLA weights for Lévy Processes with Continuous Part

1. Calculate $\hat{g}_B(s) \hat{f}_{y,\text{jump}}(s) \xrightarrow{\text{invert}} g_B(x) \ast f_{y,\text{jump}}(x)$.

2. Calculate $f_{y_T}(s) \xrightarrow{\text{invert}} f_{y_T}(x)$.

3. Compute the optimal weight in case of Lévy processes with continuous part
   \[ \pi_{cL}^0(x' - x) = \frac{g_B * f_{y,\text{jump}}(u)}{f_{y_T}(u)}. \]

Note that the Laplace transform inversion in the algorithm can be done relatively easily in linear time by using the fast Laplace transform inversion algorithm of den Iseger (see [40]). Hence, we find that also in case of Lévy processes with a continuous part, the optimal weight $\pi_{cL}^0(x' - x)$ possess a convolution structure.

In section 4.4 we already noted that the fast Laplace inversion transformation algorithm of den Iseger only needs numerical values of the Laplace transform to compute the density. This proves very useful in this case, since we do not know the explicit functional form of the density of the process $y(\cdot)$. Concluding, to calculate conditional expectations (and sensitivities) of Lévy processes with a continuous part, we only need to compute its Laplace transform and we can proceed in exactly the same way as in section 4.4.

4.6.2 DAMLA for Arbitrary Lévy Processes

The theory developed so far needs a continuous diffusion part to work, i.e. a Brownian motion part. However, with the use of a limiting argument, we can show that the theory developed so far can also be applied to the larger class of arbitrary Lévy processes, viz. processes that can have an infinite activity on a finite interval. The argument is as follows. If we have a feasible weight $\pi(x)$, e.g. the optimal weight in the case of a Lévy process with continuous part $\pi_{cL}^0(x)$, we can obtain an optimal weight in the general Lévy case (possibly without continuous part) $\pi_{L}^0(x)$, by adding a fraction $\varepsilon$ times a Wiener process and then letting $\varepsilon$ approach zero. In this respect, we define the perturbed process $X_\varepsilon^t$ as

\[ dX_\varepsilon^t := -AX_\varepsilon^t dt + \Sigma dZ_t + \varepsilon dW_t, \quad (4.40) \]

in which $A$ is a deterministic mean reversion matrix, $Z_t$ an arbitrary Lévy process, and $W_t$ a standard Brownian motion, with $W \perp Z$. Note that, without loss of generality, we did not include additional variance terms, since this is merely a matter of scaling, analogous to section 4.6.1. Now note the following:
\[ X_t^\varepsilon = X_t + Y_t^\varepsilon \quad \text{with} \quad Y_t^\varepsilon = \varepsilon Y_t \quad \text{and} \quad Y_t = \int_0^t e^{-A(t-u)}dW_u. \quad (4.41) \]

We will now apply the analogue of equation (4.38) to the process \( X_t^\varepsilon \). Similar to Equation (4.36), we have for the process \( Y_t \) that

\[ B := e^{-A(T-t)} (x' - x), \]

and for the process \( Y_t^\varepsilon \) that

\[ B^\varepsilon := \frac{1}{\varepsilon^2} e^{-A(T-t)} (x' - x). \]

We thus obtain for \( g_B^\varepsilon \), i.e. the analogue of \( g_B \) in the general Lévy case, the following expression:

\[ g_B^\varepsilon (y) := \frac{\exp (B^\varepsilon y f_{Y_T^\varepsilon} (y))}{\mathbb{E} (\exp (B^\varepsilon Y_T^\varepsilon))}. \quad (4.42) \]

The optimal DAMLA weight for an OU process driven by a general Lévy process is given in in Theorem 4.16.

**Theorem 4.16.** If \( X_t \) is defined by

\[ dX_t := -AX_t dt + \Sigma dZ_t, \]

in which \( A \) is a deterministic mean reversion matrix and \( Z_t \) an arbitrary Lévy process. Then the optimal weight \( \pi_0^L (x' - x) \) for an arbitrary Lévy process is given by

\[ \pi_0^L (x' - x) = \frac{f_{X_T} \left( x_T - e^{-A(T-t)} (x' - x) \right)}{f_{X_T} (x_T)}. \quad (4.43) \]

in which \( f_{X_T} \) is the density of \( X_T \).

**Proof.** See Appendix 4.13

Intuitively, Theorem 4.16 has a nice interpretation; we project the drift adjustment made on the Wiener process \( \varepsilon W_t \) in \( X_t^\varepsilon \) on the process \( Z_t^\varepsilon \) within \( X_t^\varepsilon \). Furthermore, the weight equals the likelihood ratio weight (cf. the proof of Theorem 4.3 in Appendix 4.8).

We end this section with the following remark.
Remark 4.17. For a very small $\varepsilon$, i.e. $\varepsilon \to 0$, one has to make a large drift adjustment $\lambda$ on $W_t$ to compensate for a shock, which is equivalent to an exploding variance. To see this, consider the following:

$$\pi = \exp \left\{ \int_t^T \lambda dW_u - \frac{1}{2} \int_t^T ||\lambda||^2 du \right\}.$$ 

For the variance, only $\mathbb{E}\pi^2$ is of interest (since $\mathbb{E}(\pi) = 1$), hence consider

$$\mathbb{E}\pi^2 = \mathbb{E} \exp \left\{ 2 \int_t^T \lambda dW_u - \int_t^T ||\lambda||^2 du \right\}$$

$$= \exp \left\{ \int_t^T ||\lambda_u||^2 du \right\}.$$ 

We can see that a drift adjustment $\lambda_u \to \infty$ implies an exploding variance.

4.7 Summary, Conclusions and Suggestions for Further Research

In this chapter we developed the Drift Adjustment Method Laplace Algorithm (DAMLA) for efficiently calculating conditional expectations of Ornstein-Uhlenbeck processes. As a numerical application of the method proposed we priced and hedged both a regular swaption and an equity-linked swaption (commonly called hybrid option). We showed that our method can efficiently and accurately calculate conditional expectations in high dimensions, on all time points, in each scenario. Furthermore, we showed that the computational efficiency of the method we proposed is very efficient because of the use of transform inversion methods. We found that by applying the Laplace transform inversion algorithm of Den Iseger [40], calculating conditional expectations can be done in $O(N)$ (linear) time. Finally, we showed that our method can be extended to models driven by arbitrary Lévy processes. We showed that the DAMLA weights obtained in this case also exhibit a convolution structure, and hence we could proceed exactly as in the diffusion case. Although the method we present is a very powerful one, it does depend on the fact that the underlying process has to be of the Ornstein-Uhlenbeck type. A nice extension to the current set up is to expand the Drift Adjustment Laplace Algorithm to the more general class of affine processes.
4.8 Appendix A: Proofs Section 4.2

Proof of Theorem 4.1: For expository purposes let \( G(s) = e^{-A(T-s)}\Sigma \) and \( (x' - x) = x \). Then we have that the solution to the problem

\[
\min \int_0^t ||\lambda(s)||^2 ds
\]

subject to

\[
\int_0^t G(s)\lambda(s)ds = x
\]

is given by

\[
\lambda_{opt}(s) = G(s)^T \left[ \int_0^t (G(s)G(s)^T) ds \right]^{-1} x
\]

The proof consists of 2 parts:
1. \( \lambda_{opt} \) satisfies \( \int_0^t G(s)\lambda(s)ds = x \). This is checked readily.
2. \( \lambda(s) = \lambda_{opt}(s) + \tilde{\lambda}(s) \) with \( \int_0^t G(s)\tilde{\lambda}(s)ds = 0 \).

Now note that

\[
\int_0^t ||\lambda(s)||^2 = \int_0^t ||\lambda_{opt}(s) + \tilde{\lambda}(s)||^2 ds = \int_0^t ||\lambda_{opt}(s)||^2 ds + \int_0^t ||\tilde{\lambda}(s)||^2 ds + 2 \int_0^t \lambda_{opt}(s)^T \tilde{\lambda}(s)ds
\]

It holds that

\[
2 \int_0^t \lambda_{opt}(s)^T \tilde{\lambda}(s)ds = 2x^T \left[ \int_0^t (G(s)G(s)^T) ds \right]^{-1} \int_0^t G(s)\tilde{\lambda}(s)ds
\]

which equals zero since \( \int_0^t G(s)\tilde{f}(s)ds = 0 \). Hence, we have that

\[
\int_0^t ||\lambda(s)||^2 = \int_0^t ||\lambda_{opt}(s)||^2 + \int_0^t ||\tilde{\lambda}(s)||^2
\]

Since \( \int_0^t ||\tilde{\lambda}(s)||^2 \geq 0 \) it holds that \( \lambda_{opt} \) is given as in the theorem. \( \square \)

Proof of Remark 4.2: We will begin the proof with a well-known lemma from probability theory.
Lemma 4.18. Let $h_X$ and $h_Y$ be deterministic functions, and let the random vectors $X$ and $Y$ be given by

\[
X = \int_t^T h_X(u) \, dW_u,
\]
\[
Y = \int_t^T h_Y(u) \, dW_u.
\]

Then $(X,Y)$ are Gaussian distributed with mean 0 and covariance matrix

\[
\text{Cov}(X,Y) = \int_t^T h_X(u) h_Y^*(u) \, du.
\]

Without loss of generality we assume the process $y_T$ started in 0. Now given the fact that this process is defined by

\[
y_T = \int_0^T e^{-A(T-s)} \Sigma dW_s
\]

it follows from lemma 4.18 that

\[
\text{Cov}(y_T) = \int_0^T e^{-A(T-s)} \Omega e^{-A^*s} ds
\]

\[
= e^{-AT} \int_0^T e^{As} \Omega e^{A^*s} ds e^{-A^*T},
\]

in which $\Omega := \Sigma \Sigma^*$. Hence, if $\text{Cov}(y_T)$ is invertible, then

\[
\int_t^T e^{As} \Omega e^{A^*s} ds = e^{AT} \text{Cov}(y_T) e^{A^*T}
\]

is invertible, which proves the remark.

Proof of Theorem 4.3. In this proof we (again) suppress the dependence on $Q_{t,x}$, when writing expectations and variances. As a consequence of the Radon-Nikodým theorem, see e.g. [17], it holds that there is $Q_{t,x}$-a.s. one and only one element $\pi_0$ in $\Pi$ which is $\sigma(y_T)$-measurable.

Let $\pi \in \Pi$. Since $\phi(y_T)$ is $\sigma(y_T)$-measurable, it follows from the Tower property that

\[
\mathbb{E}(\phi(y_T)\pi) = \mathbb{E}\mathbb{E}(\phi(y_T)\pi | y_T)
\]

\[
= \mathbb{E}(\phi(y_T)\mathbb{E}(\pi | y_T)).
\]
Hence $\mathbb{E}(\pi|y_T) \in \Pi$. Since $\pi_0$ is the unique $\sigma(y_T)$ measurable element in $\Pi$, $\mathbb{E}(\pi|y_T) = \pi_0$.

We now continue proving that $\pi_0$ is a simultaneous solution for each of the optimization problems (4.11) for any Borel-measurable random variable $\phi$ for which the variance is well defined. In this respect, consider the following. For all $\pi \in \Pi$ it holds that

$$Var(\phi(y_T)\pi) = Var(\mathbb{E}(\phi(y_T)\pi|y_T)) + \mathbb{E}(Var(\phi(y_T)\pi|y_T))$$
$$= Var(\phi(y_T)\mathbb{E}(\pi|y_T)) + \mathbb{E}(Var(\phi(y_T)\pi|y_T))$$
$$= Var(\phi(y_T)\pi_0) + \mathbb{E}(Var(\phi(y_T)\pi|y_T)),$$

so that

$$Var(\phi(y_T)\pi) \geq Var(\phi(y_T)\pi_0), \forall \pi \in \Pi.$$

Hence $\pi_0$ is a simultaneous solution for each of the optimization problems (4.11).

Proof of Theorem 4.8. First, using Theorem (4.1) consider the following explicit expression of $\int_t^T \lambda_u dW_u$:

$$\int_t^T \lambda_u dW_u = (x' - x)^* \left\{ \int_t^T e^{Au} \Omega e^{A^*} du \right\}^{-1} \int_t^T e^{Au} \Sigma dW_u$$
$$= (x' - x)^* \left\{ \int_t^T e^{Au} \Omega e^{A^*} du \right\}^{-1} e^{AT} \int_t^T e^{-A(T-u)} \Sigma dW_u$$
$$= (x' - x)^* \left\{ \int_t^T e^{Au} \Omega e^{A^*} du \right\}^{-1} e^{AT} y_T - (x' - x)^*$$
$$\times \left\{ \int_t^T e^{Au} \Omega e^{A^*} du \right\}^{-1} e^{At} x$$
$$= (x' - x)^* e^{-A^T} \Omega^{-1} y_T - (x' - x)^* e^{-A^T} \Omega^{-1} e^{-A(T-t)} x.$$

We see that $\pi(x' - x)$ is a function of $y_T$ only, and hence optimal (see Theorem 4.3).

Since we know that the term $-(x' - x)^* e^{-A^T} \Omega^{-1} e^{-A(T-t)} (x' - x)$ is merely a constant, and that the optimal weight has to integrate to unity, we can drop this
constant term and apply a normalization to make the computations less cumbersome. Considering this, the optimal weight can be represented as

\[ \pi_0(x' - x) = \frac{\exp\{ (x' - x)^* e^{-A^T \Omega^{-1} y_T} \}}{\mathbb{E} \exp\{ (x' - x)^* e^{-A^T \Omega^{-1} y_T} \}}. \]  \hfill (4.44)

We can further simplify this optimal weight by applying a Cholesky decomposition. In this respect, consider the following equation

\[ \exp\{ (x' - x)^* e^{-A^T \Omega^{-1} y_T} \} \propto \exp\{ (x' - x)^* e^{-A^T \Sigma^* Z} \}, \]

with \( Z \sim N(0, 1) \) and \( \Omega = \Sigma \Sigma^* \).

The expression \( \Sigma^* Z \) comes from the Cholesky decomposition of \( y_T \), i.e. \( y_T = \Sigma Z \) and \( \Omega = \Sigma \Sigma^* \), and thus the inverse equals \( \Omega^{-1} = (\Sigma^*)^{-1} \Sigma^{-1} \). Hence \( \Omega^{-1} y_T = (\Sigma^*)^{-1} \Sigma^{-1} \Sigma Z = (\Sigma^*)^{-1} Z \). Thus, the optimal weight is explicitly represented by

\[ \pi_0(x' - x) = \frac{\exp\{ (x' - x)^* e^{-A^T \Sigma^* Z} \}}{\mathbb{E} \exp\{ (x' - x)^* e^{-A^T \Sigma^* Z} \}}. \]

Which completes the proof.

\[ \Box \]

**4.8.1 Proofs Section 4.2.1**

**Proof of Theorem 4.11** Since \( \mathbb{E} e^{\alpha Z} = e^{\frac{1}{2}||\alpha||^2} \), with \( Z \sim N(0, 1) \), the denominator of (4.13) can be expressed more explicitly as

\[ \mathbb{E} \left( \exp\left( (x' - x)^* e^{-A^T \Sigma^* Z} \right) \right) = \exp\left( \frac{1}{2} (x' - x)^* e^{-A^T \Omega^{-1} e^{-A^T} (x' - x)} \right). \]

Hence, for the optimal weight \( \pi_0(x' - x) \) we now have

\[ \pi_0(x' - x) = \frac{\exp\left( (x' - x)^* e^{-A^T \Sigma^* Z} \right)}{\exp\left( \frac{1}{2} (x' - x)^* e^{-A^T \Omega^{-1} e^{-A^T} (x' - x)} \right)}. \]

To make computations less cumbersome, let us define the variable \( z \) (cf. Theorem 4.3) as

\[ z := (x' - x) \Sigma^{-1} e^{-A^T} \]

so that

\[ z^* := (x' - x)^* e^{-A^T} \Sigma^* - 1. \]

Now note that

\[ z^* Z = -\frac{1}{2} (z^* z - 2z^* Z + Z^* Z) + \frac{1}{2} z^* z + \frac{1}{2} Z^* Z. \]
Hence, for the optimal weight $\pi_0$ in convolution form we obtain

$$
\pi_0(x' - x) = \frac{\exp(z^* Z)}{\exp\left(\frac{1}{2} z^* z\right)} = \frac{\exp\left(-\frac{1}{2} ||z - Z||^2\right)}{\exp\left(-\frac{1}{2} ||Z||^2\right)}.
$$

Proof of Theorem 4.12
The first and second order derivatives of $\pi_0(x' - x)$ with respect to $z$, i.e. $\pi_z(x' - x)$ and $\pi_{zz}(x' - x)$ respectively, are given by

$$
\pi_z(x' - x) = -(z - Z)\pi_0
$$

$$
\pi_{zz}(x' - x) = -(I - (z - Z)(z - Z)^*)\pi_0.
$$

in which $I$ is the identity matrix. This implies that the first and second order derivatives of $\pi_0(x' - x)$ with respect to $x$, i.e. $\pi_0'(x' - x)$ and $\pi_0''(x' - x)$ respectively, are given by

$$
\pi_0'(x' - x) = \Sigma^{-1} e^{-AT} \pi_z
$$

$$
\pi_0''(x' - x) = \Sigma^{-1} e^{-AT} \pi_{zz} e^{-A^* T} \Sigma^{-1}^*
$$

$$
= -\Sigma^{-1} e^{-AT} (I - (z - Z)(z - Z)^*) e^{-A^* T} \Sigma^{-1}^* \pi_0.
$$

4.8.2 Proofs Section 4.2.2

Proof of Equation (4.19).

$$
\pi_{\Delta}(x' - x) = \pi_0' \frac{\partial y_0}{\partial S_0}
$$

$$
= \pi_0' \frac{1}{S_0}
$$

$$
= \frac{(\Sigma^{-1} e^{-AT}) Z}{S_0}
$$

$$
= \frac{W[t, T]}{\sqrt{T - \sigma^2 (T - t)}} \frac{1}{\sigma^2 (T - t) S_0}
$$

$$
= \frac{W[t, T]}{\sigma(T - t) S_0}.
$$
Proof of Equation (4.20).

$$\pi \Gamma (x' - x) = \frac{\pi'_0}{S_0} \frac{\partial y_0}{\partial S_0} - \frac{\pi'_0}{S_0^2}$$

$$= \frac{\pi'_0}{S_0^2} - \frac{\pi'_0}{S_0^2}$$

$$= \left( (\Sigma^{-1} e^{-AT} Z)^2 - e^{-A^T \Omega_y^{-1} e^{-AT}} - (\Sigma^{-1} e^{-AT} Z) \right) \frac{1}{S_0^2}$$

$$= \left( \left( \frac{W[t, T]}{\sigma(T-t)} \right)^2 - \frac{1}{\sigma^2(T-t)} - \frac{W[t, T]}{\sigma(T-t)} \right) \frac{1}{S_0^2}$$

$$= \left( \frac{(W[t, T])^2}{\sigma(T-t)} - \frac{1}{\sigma} - W[t, T] \right) \frac{1}{S_0^2 \sigma(T-t)}.$$

\[ \square \]

4.9 Appendix B: A Multi-Factor Hull-White Model

The state variable in the Hull-White model is given by a variable \( y_t \). The dynamics of \( y_t \) are described by

\[ dy_t = -Ay_t dt + \Sigma dW_t \quad \text{with} \quad y_0 = 0. \tag{4.45} \]

Here \( W_t \) denotes a standard Brownian motion under the nominal money account measure (defined below) and \( A \) and \( \Sigma \) are given deterministic matrices. The solution of this equation is given by

\[ y_T = e^{-A(T-t)} y_t + \int_t^T e^{-A(T-s)} \Sigma dW_s. \tag{4.46} \]

The short rate \( r_t \) depends on \( y_t \) through the relation

\[ r_t = e_0 y_t + \alpha_t, \]

with \( \alpha_t \) a deterministic function of \( t \) and \( e_0 \) a vector with ones. The money account process is given by

\[ B_t = \exp \left( \int_0^t r_s ds \right) = \exp \left( \int_0^t \alpha_s ds \right) \exp \left( \int_0^t e_0 y_s ds \right). \]

Now we define the integrated short rate process \( I_n \) as
4. THE DRIFT ADJUSTMENT METHOD LAPLACE ALGORITHM

\[ I_n := \int_t^T r_s ds \]

\[ = e_0 \int_t^T y_s ds + \int_t^T \alpha_s ds. \]

The money account measure \( \mathbb{Q} \) is implicitly defined by the condition that all tradable securities \( V_s \) are martingales under \( \mathbb{Q} \) with respect to the numéraire \( B_s \) (that is, \( V_s/B_s \) are martingales under \( \mathbb{Q} \)). This implies that the value \( P(t, T) \) at time \( t \) of a discount bond which pays 1 Euro at time \( T \) is given by

\[ P(t, T) = \mathbb{E}_\mathbb{Q} \left( \frac{B_t}{B_T} \mid \mathcal{F}_t \right) = \exp \left( - \int_t^T \alpha_s ds \right) \mathbb{E}_\mathbb{Q} \left( \exp \left( - \int_t^T e_0 y_s ds \right) \mid \mathcal{F}_t \right). \]

This implies directly

\[ \exp \left( \int_0^T \alpha_s ds \right) = \frac{\mathbb{E}_\mathbb{Q} \left( \exp \left( - \int_0^T e_0 y_s ds \right) \mid \mathcal{F}_0 \right)}{P(0, T)}. \]

Integration of equation (4.46) yields

\[ \int_t^T y_u du = \left[ I - e^{-A(T-t)} \right] A^{-1} y_t + \int_t^T \left( I - e^{-A(T-s)} \right) A^{-1} \Sigma dW_s, \quad (4.47) \]

with \( I \) the identity matrix. From this we obtain the representation

\[ P(t, T) = \frac{P(0, T)}{P(0, t)} \exp \left( \hat{A}_{t,T} - B_{T-t}^* y_t \right). \]

In which

\[ B_{T-t}^* = e_0^* A^{-1} \left( I - e^{-A(T-t)} \right), \]

and

\[ \hat{A}_{t,T} = \frac{\hat{A}}{T - t}, \]

with

\[ \hat{A} = B_{T-t}^* M_Y^{T*}(0, t) - \frac{1}{2} B_{T-t}^* \Omega_Y^{T*}(0, t) B_{T-t}. \]

The expectation \( M_Y^{T*}(t, T) \) of \( y_t \) is equal to
\[ M_Q^C(t, T) = - \left[ I - e^{-A(T-t)} \right] \hat{\Omega}e_0 + \Omega^Q_Y (A^*)^{-1} e_0 + e^{-A(T-t)}y_t, \]

and the covariance \( \Omega_Q^Y \) of \( y_t \) is equal to

\[ \Omega_Q^Y = [A \otimes I + I \otimes A]^{-1} \left( \Omega^S - (e^{-A(T-t)} \otimes e^{-A(T-t)})\Omega^S \right). \]

Both the nominal and the real curves are described by a Multi-Factor Hull-White Model, each under their own money account measure, denoted by \( Q_n \) and \( Q_r \), respectively.

### 4.10 Appendix C: Value of a Receiver Swaption at time \( t \)

We know that under the forward measure \( T \), \( y_t \) is normally distributed as \( y_t \sim N(\mathbb{E}(y_t), \Omega_y) \) and that \( y_t = \mathbb{E}y_t + \Sigma Z \) with \( \Omega = \Sigma \Sigma^* \) and \( Z \sim N(0, 1) \) under \( T \).

Substituting (4.48) in equation (4.23) yields

\[ P(T_0, T_j) = w_j \exp(-B_{T_j-T_0} Z) \]  

(4.49)

in which \( w_j \) and \( B_{T_j-T_0} \) are defined as \( B_{T_j-T_0} = \Sigma T_0 Z \) and

\[ w_j = \frac{P(0, T_j)}{P(0, T_0)} \mathbb{E}_T \left\{ \exp(-\hat{A}_{T_0, T_j} Z) \mid F_0 \right\}. \]

Given the maturity date \( T_n \), let us now write the value \( V^{rec} \) of a receiver swap at time \( T_0 \) with a fixed rate \( K \) as

\[ V^{rec} = K \sum_{j=1}^{n} P(T_0, T_j) - (1 - P(T_0, T_n)). \]  

(4.50)

Substituting (4.49) in (4.50) now yields

\[ V^{rec} = K \sum_{j=1}^{n} w_j \exp \left( B_{T_j-T_0}^e Z \right) - (1 - w_n \exp(B_{T_n-T_0}^e Z)) \].

If we define \( \hat{w}_j = Kw_j \) and \( \hat{w}_n = (1 + K)w_n \), then the value of the receiver swap becomes

\[ V^{rec}_{T_0} = \sum_{j=1}^{n} \hat{w}_j \exp \left( B_{T_j-T_0}^e Z \right) - 1. \]

The fair price of a receiver swaption \( v_f^{rec} \), is given by

\[ v_f^{rec} = P(t, T_0)\mathbb{E}_T \left( V^{rec}_{T_0} \right)^+. \]  

(4.51)
Having written the value of a receiver swap as in equation (4.51) we can express the value of the receiver swaption at time $t$ as

$$v^{rec} = P(t, T_0) \mathbb{E}_T \left( \sum_{j=1}^{n} \hat{w}_j \exp \left( B^*_j - T_0 \right) \right) .$$

### 4.11 Appendix D: The Jamshidian Trick

To start the exposition, consider the following. Since in the multi-factor Hull-White model the matrix $Z$ is composed of $m$ independent factors $Z_1, Z_2, \ldots, Z_m$, we can (using the properties of the exponential function) rewrite the expectation in formula (4.24) by iteration as follows, in obvious notation,

$$v^{rec} = \mathbb{E}_{Z_m} \mathbb{E}_{Z_{m-1}} \ldots \mathbb{E}_{Z_1} \left( \sum_{j=1}^{n} \hat{w}_j \exp \left( B^*_1 Z_1 \right) \exp \left( B^*_2 Z_2 \right) \ldots \exp \left( B^*_m Z_m \right) - 1 \right) .$$

The value of the underlying swap is

$$V^{rec}(Z_1, Z_2, \ldots, Z_m) := \sum_{j=1}^{n} \prod_{i=1}^{m} \hat{w}_j \exp \left( B^*_i Z_i \right) - 1 .$$

(4.53)

We now condition on $Z_2, \ldots, Z_m$, i.e. we take $Z_2, \ldots, Z_m$ as fixed. Hence, equation (4.53) is strictly increasing in $Z_1$, as the vectors $B$ and $w$ have positive elements, hence the equation

$$V^{rec}(Z_1, Z_2, \ldots, Z_m) = 0$$

has a unique solution. We can now define a function $\alpha$ by

$$\alpha(Z_2, Z_3, \ldots, Z_m) : V^{rec}(\alpha, Z_2, Z_3, \ldots, Z_m) = 0 ,$$

and we have the equivalence

$$V^{rec}(Z_1, Z_2, \ldots, Z_m) \geq 0 \iff Z_1 \geq \alpha(Z_2, Z_3, \ldots, Z_m) .$$

From this we can deduce that

$$\mathbb{E}_{Z_m} \mathbb{E}_{Z_{m-1}} \ldots \mathbb{E}_{Z_1} (V^{rec})^+ = \mathbb{E}_{Z_m} \mathbb{E}_{Z_{m-1}} \ldots \mathbb{E}_{Z_1} V^{rec}(Z_1, Z_2, \ldots, Z_m) 1 \{ Z_1 \geq \alpha(Z_2, Z_3, \ldots, Z_m) \} .$$

We now approximate $\mathbb{E}_{Z_m} \mathbb{E}_{Z_{m-1}} \ldots \mathbb{E}_{Z_1}$ with a Hermite quadrature rule. Basically, a Hermite quadrature is an efficient way to numerically approximate the
integral (i.e. expectation) of an unknown function \( f(x) \). The univariate case can be stated as follows:

\[
\mathbb{E}f(Z) \approx \sum_{k=1}^{n} a_k f(\lambda_k),
\]

in which \( a_k \) are the quadrature weights and \( \lambda_k \) the quadrature points. We can extend this to the multivariate case as follows:

\[
\mathbb{E}f(Z_2, \ldots, Z_m) \approx \sum_{|k|} A_{|k|} f(\lambda_{|k|}),
\]

where \( A_{|k|} = \prod_{j=2}^{m} a_{k_j} \) and \( \lambda_{|k|} = (\lambda_{k_2}, \ldots, \lambda_{k_m}) \).

For an excellent survey of quadrature rules see e.g. [105]. Hence, for the quadrature of \( \mathbb{E}Z_m \), we obtain

\[
\sum_{|k|} A_{|k|} (V^{rec}(Z_1; \lambda_{|k|})) 1\{Z_1 \geq \alpha(\lambda_{|k|})\}
\]

\[
= \sum_{|k|} A_{|k|} \left[ V^{rec}(Z_1, \lambda_{|k|}) - V^{rec}(\alpha(\lambda_{|k|}), \lambda_{|k|}) \right] 1\{Z_1 \geq \alpha(\lambda_{|k|})\}
\]

\[
= \sum_{|k|} A_{|k|} \sum_{j=1}^{n} \hat{w}_j e^{B^*_j \lambda_{|k|} \left[ e^{B^*_j Z_1} - e^{B^*_j \alpha(\lambda_{|k|})} \right] +}. \tag{4.54}
\]

Note that we used in equation (4.54) that \( V^{rec}(\alpha(\lambda_k); \lambda_k) = 0 \).

4.12 Appendix E: Optimal Weight in Product Form

It holds that

\[
\pi_0((x' - x), Z_1, \ldots, Z_m) = \frac{\exp(z^* Z)}{\exp(\frac{1}{2} z^* z)}.
\]

Exploiting the properties of the exponential function we have that

\[
\frac{\exp(z^* Z)}{\exp(\frac{1}{2} z^* z)} = \prod_{i=1}^{m} \frac{\exp(z_i^* Z_i)}{\exp(\frac{1}{2} z_i^* z_i)}
\]

\[
= \prod_{i=1}^{m} \pi_{0i}((x' - x), Z_i),
\]

where

\[
\pi_{0i}((x' - x), Z_i) = \frac{\exp\left(-\frac{1}{2}(z_i^* - Z_i)^2\right)}{\exp(\frac{1}{2} Z_i^2)},
\]

which gives an expression for the optimal weight represented as a product.
4.13 Appendix F: Proofs Section 4.6

Proof of Theorem 4.16 To prove Theorem 4.16 according to Lévy’s theorem (see e.g. [4], page 304), it suffices to prove that the Laplace transform of \( g_{\varepsilon}B(y) \) converges to the Laplace transform of equation (4.43). Therefore, taking the Laplace transform of equation (4.42) yields

\[
\mathbb{E} \left\{ \exp \left( -s^* Y_{\varepsilon} \right) \frac{\exp (B_{\varepsilon}^* Y_{\varepsilon}^T)}{\mathbb{E} (\exp (B_{\varepsilon}^* Y_{\varepsilon}^T))} \right\}. \tag{4.55}
\]

Because \( \exp(B_{\varepsilon}^* Y_{\varepsilon}) = \exp(\frac{1}{\varepsilon} B^* \varepsilon Y) = \exp(\frac{1}{\varepsilon} B^* Y) \), we can rewrite the Laplace transform in equation (4.55) as follows

\[
\begin{align*}
\mathbb{E} \left\{ \exp \left( -s^* Y_{\varepsilon} \right) \frac{\exp (B_{\varepsilon}^* Y_{\varepsilon}^T)}{\mathbb{E} (\exp (B_{\varepsilon}^* Y_{\varepsilon}^T))} \right\} &= \mathbb{E} \left\{ \exp \left( -\varepsilon s^* Y \right) \frac{\exp (\frac{1}{\varepsilon} B^* Y)}{\mathbb{E} (\exp (\frac{1}{\varepsilon} B^* Y))} \right\} \\
&= \exp \left( \frac{1}{2} \left( \frac{1}{\varepsilon} B - s \varepsilon \right)^* \left( \frac{1}{\varepsilon} B - s \varepsilon \right) \right) \exp \left( \frac{1}{2} \left( \frac{1}{\varepsilon} B \right)^* \left( \frac{1}{\varepsilon} B \right) \right) \\
&= \exp \left( -B^* s + \frac{1}{2} \varepsilon^2 \|s\|^2 \right). \tag{4.56}
\end{align*}
\]

Now taking the limit yields

\[
\lim_{\varepsilon \downarrow 0} \exp \left( -B^* s + \frac{1}{2} \varepsilon^2 \|s\|^2 \right) = \exp (-B^* s).
\]

Hence, the Laplace transforms of the probability measures with densities \( g_{\varepsilon}B \) converge to the Laplace transform of the Dirac measure concentrated at \( B \). This implies that the weight (cf. equation 4.39)

\[
\pi_{\varepsilon} (u) = \frac{g_{\varepsilon}B * f_{X^T} (u)}{f_{X^T} (u)}
\]

converges to

\[
\pi (u) = \frac{f_{X^T} (u - B)}{f_{X^T} (u)} = \frac{f_{X^T} (u - e^{-A(T-t)} (x' - x))}{f_{X^T} (u)}
\]

with \( u = x_T \). Which completes the proof.