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CANONICAL MEASURE ASSIGNMENTS

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Abstract. We work under the assumption of the Axiom of Determinacy and associate a measure to each cardinal $\kappa < \aleph_{\varepsilon_0}$ in a recursive definition of a *canonical measure assignment*. We give algorithmic applications of the existence of such a canonical measure assignment (computation of cofinalities, computation of the Kleinberg sequences associated to the normal ultrafilters on all projective ordinals).

§1. Introduction. One of the striking features of set theory under the Axiom of Determinacy is the fact that there is a full analysis of the cardinal structure for a fairly large initial segment of $\Theta := \sup\{\alpha$: there is a surjection from $\mathbb{R}$ onto $\alpha\}$, something which we cannot hope to get in the ZFC context. While almost none of the combinatorial properties of small cardinals (*e.g.*, $\aleph_2$, $\aleph_3$, $\aleph_{\omega^2}$) are fixed in ZFC, ZF + AD gives us definite combinatorial properties (*e.g.*, measurability, Jónssonness, Rowbottomness) of these cardinals, in particular below $\aleph_{\varepsilon_0}$, the supremum of the projective ordinals (defined in §2).

This structure is closely tied to an analysis of measures on the projective ordinals and the representation of cardinals as ultrapowers via these measures. In [4] this analysis is given below $\boldsymbol{\delta}^1_5$, and in [3] it is extended to all the $\boldsymbol{\delta}^1_n$. A key combinatorial ingredient in this analysis is the notion of a description, which gives a precise presentation of the cardinal structure (see also [5] for an introduction to this theory). In the paper [6] which can be seen as a companion to this one, it was shown that a certain fairly simple family of measures on $\boldsymbol{\delta}^1_3$ could be used to directly describe the cardinal structure below $\boldsymbol{\delta}^1_5$. This presentation of the cardinal structure avoided the notion of description, although the description theory was an integral part of the proofs.

Our goal here is to present a simple combinatorial framework which suffices to describe the cardinal structure below the supremum of the projective ordinals, and which also avoids the description analysis. We introduce the notion of an *ordinal algebra*, and we inductively assign measures to the elements of this algebra through two lifting operations. This gives us a comparatively simple notational framework

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for describing the cardinal structure below the $\boldsymbol{\delta}^1_n$ which is of independent interest and will also allow those not familiar with the description analysis to use many of the strong consequences of that theory.

Throughout this paper we make the following standing assumptions about the odd projective ordinals (definitions are given in § 2):

A1: Each $\boldsymbol{\delta}^1_{2n+1}$ has the strong partition property: $\boldsymbol{\delta}^1_{2n+1} \to (\boldsymbol{\delta}^1_{2n+1})^{\boldsymbol{\delta}^1_{2n+1}}$.
A2: Each $\boldsymbol{\delta}^1_{2n+1}$ is closed under ultrapowers.
A3: $\boldsymbol{\delta}^1_{2n+1} = \aleph_{e_n+1}$.†

We emphasize that we do not prove the assumptions **A1**, **A2**, or **A3** here, although we use these properties heavily. Assumption **A1** is one of the central results of the description analysis. A proof using descriptions for $\boldsymbol{\delta}^1_1 = \omega_1$ can be found in [5] (the original proof is due to Martin) and for $\boldsymbol{\delta}^1_3 = \omega_{\omega+1}$ can be found in [4]. Also, [3] extends the description theory to all levels of the projective hierarchy (though the strong partition property for the general $\boldsymbol{\delta}^1_{2n+1}$ is not explicitly proved in [3], it is implicit from the analysis of [3] and the arguments of [4]). Assumption **A2** follows from the existence of a $\boldsymbol{\Delta}^1_{2n+1}$ coding of the subsets of λ^1_{2n+1}, the cardinal predecessor of $\boldsymbol{\delta}^1_{2n+1}$. For $n = 0$ this is trivial, for $n = 1$ a proof is given in [5] (the original proof is due to Kunen), for $n = 2$ a proof is given in [4], and [3] again gives the machinery for generalizing the proof of [4] to arbitrary levels. Assumption **A3** is the main result of [3] (cf. below).

Aside from these general standing assumptions on the $\boldsymbol{\delta}^1_{2n+1}$, there is a crucial technical assumption we shall eventually make which we call the *canonicity assumption* (defined precisely later). This assumption concerns the values of certain ultrapowers of the $\boldsymbol{\delta}^1_{2n+1}$ by certain measures on these ordinals which we introduce. The canonicity assumption is perhaps not as intuitive as the above standing assumptions, but it is an important technical link between the description theory and the more simplified representation of the current paper. A proof of the canonicity assumption for the measures on $\boldsymbol{\delta}^1_3$ using the description analysis is given in [6]. With the general arguments of [3], this canonicity proof should extend to higher levels, but this will appear later. Our purpose here is show how these assumptions lead to a simpler and elegant way to describe the cardinal structure below the supremum of the projective ordinals (which is the cardinal $\aleph_{\varepsilon_0}$; see Theorem 4).

Remark on assumption **A3**. The canonicity assumption and **A1** and **A2** suffice to prove **A3** when combined with some general arguments not specifically using the description analysis. Specifically, AD + **A1** + **A2** proves a result referred to as Martin's theorem is proved in [5] (a version is also proved in [4]). This theorem, and the Kechris-Kunen-Martin homogeneous tree construction (see [5]), gives an upper-bound for $\boldsymbol{\delta}^1_{2n+1}$, namely $\boldsymbol{\delta}^1_{2n+1} \leq (\sup_\mu j_\mu(\boldsymbol{\delta}^1_{2n-1}))^+$, where the supremum refers to the ultrapower embeddings from the measures μ on $\boldsymbol{\delta}^1_{2n-1}$ arising in the homogeneous tree construction on a $\boldsymbol{\Pi}^1_{2n-1}$-complete set (our standing assumptions suffice to carry out this construction). A measure embedding theorem (the "global embedding theorem"), whose proof uses just the standing assumptions, reduces this supremum to that of measures of the form considered in this paper. For measures

†Cf. Theorem 4 (8.) for a definition of e_n.

on $\boldsymbol{\delta}^1_3$, this embedding theorem is proved in [4], and the general version is proved in [3]. The canonicity assumption computes these ultrapowers, and so gives the upper bound for $\boldsymbol{\delta}^1_{2n+1}$ (see Theorem 4). The lower bound follows from the fact all of the ultrapowers of $\boldsymbol{\delta}^1_{2n-1}$ by the measures of this paper can be coded by $\boldsymbol{\Delta}^1_{2n+1}$ prewellorderings, and the canonicity assumption which computes these ultrapowers. For the purposes of this paper, however, these arguments do not concern us, and this is why we added **A3** to the list of our standing assumption.

Aim of this paper. In this paper we would like to stress the algorithmic nature of this analysis of the cardinals below $\aleph_{\varepsilon_0}$: we describe a general recursive procedure of measure assignment (§ 5), and develop algorithms for

- computing all regular cardinals below $\aleph_{\varepsilon_0}$ (§ 6.1),
- computing the cofinalities of all cardinals below $\aleph_{\varepsilon_0}$ (§ 6.2), and
- computing the Kleinberg sequences derived from all normal measures on the projective ordinals (§ 6.3)

under the assumption that the measure assignment is canonical (the canonicity assumption).

§**2. Mathematical background.** In this paper, we shall be working in the theory $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}$. We shall say that a cardinal κ has the *strong partition property* if the partition relation $\kappa \to (\kappa)^\kappa$ holds, *i.e.*, if for every partition of $[\kappa]^\kappa$ into two blocks there is a homogeneous set of order type κ. We say that it has the *weak partition property* if for all $\alpha < \kappa$, the partition relation $\kappa \to (\kappa)^\alpha$ holds. Note that the strong and the weak partition properties cannot hold for any uncountable cardinal if we assume the Axiom of Choice AC: by a result of Erdős and Rado (*cf.* [7, Proposition 7.1]) any partition relation with infinite exponents violates AC. In practice, we actually use an equivalent variant of these definitions, which the reader can take as our official definition.

We first recall some terminology: Let α and κ be ordinals. A function $f : \alpha \to \kappa$ is continuous if and only if for all limit ordinals $\gamma < \alpha$,

$$f(\gamma) = \sup\{f(\xi)\,;\, \xi < \gamma\}.$$

The function f has *uniform cofinality* ω if there is a function $h : \omega \times \alpha \to \kappa$, which is increasing in the first argument, such that for $\gamma < \alpha$, we have

$$f(\gamma) = \sup\{h(n, \gamma) : n \in \omega\}.$$

We say a function $f : \alpha \to \kappa$ is of the *correct type* if it is increasing, everywhere discontinuous (*i.e.*, for all $\gamma < \alpha$, $f(\gamma) > \sup_{\beta<\gamma} f(\beta)$), and of uniform cofinality ω. We say $f : \alpha \to \kappa$ is of *continuous type* if it is increasing, continuous and has uniform cofinality ω at all successor ordinals (with obvious meaning).

We can now write $\kappa \xrightarrow{\text{club}} (\kappa)^\lambda$ for the statement "for every partition $\mathscr{P}$ of the functions from λ to κ of the correct type into two sets there is a club set $C \subseteq \kappa$ such that all functions $f : \lambda \to C$ of the correct type get the same color by $\mathscr{P}$". It is easy to see that if $\lambda = \omega \cdot \lambda$, then $\kappa \to (\kappa)^\lambda$ and $\kappa \xrightarrow{\text{club}} (\kappa)^\lambda$ are equivalent (*cf.* [4, p. 5] or [5, Fact 2.28]) and so we can freely switch between the two definitions for the weak and strong partition properties.

For $\lambda < \kappa$, λ regular, let us define the *λ-cofinal filter* $\mathscr{C}^\lambda_\kappa$ as the filter generated by the λ-closed unbounded sets in κ, *i.e.*,

$$A \in \mathscr{C}^\lambda_\kappa :\iff \text{there is a club set } C \subseteq \kappa \text{ such that } \{\alpha \in C : \operatorname{cf}(\alpha) = \lambda\} \subseteq A.$$

Clearly, $\mathscr{C}^\omega_{\omega_1}$ is the ordinary club filter on ω_1. As usual, we call σ-complete ultrafilters on κ *measures*, we call a measure *normal* if it is closed under diagonal intersection and *semi-normal* if it contains all club subsets of κ. If μ is a measure on ϱ and α is an ordinal, then (because of DC) the ultrapower α^ϱ/μ is wellfounded and thus isomorphic to an ordinal. We identify it with its Mostowski collapse. We call a cardinal κ *closed under ultrapowers* if for all $\varrho < \kappa$ and all measures μ on ϱ, we have that $\varrho^\varrho/\mu < \kappa$. If κ is regular, this is equivalent to the statement "for all $\varrho < \kappa$ and all measures μ on ϱ, we have that $\kappa^\varrho/\mu = \kappa$".

The weak partition property of κ implies the existence of many concrete measures on κ, as the following theorem of Kleinberg shows:

Theorem 1. Let κ be a cardinal with the weak partition property and $\lambda < \kappa$ a regular cardinal. Then $\mathscr{C}^\lambda_\kappa$ is a normal measure. In addition, if κ is not weakly Mahlo, then these are the only normal ultrafilters on κ.

Proof. [7, Theorem 28.10 & Exercise 28.11]. ⊣

In other words, the weak partition property of κ not only gives the existence of measures, but in our case (our cardinals will be below $\aleph_{\varepsilon_0}$ and thus not weakly Mahlo) also a structured pattern of all of the normal measures on κ (indexed by the regular cardinals below κ).

In addition, the strong partition property also connects to other combinatorial properties:

Definition 2. Let κ be a strong partition cardinal and μ a normal measure on κ. We then define a sequence $\langle \kappa^\mu_n : n < \omega \rangle$ as follows:

- $\kappa^\mu_0 := \kappa$ and
- $\kappa^\mu_{n+1} := (\kappa^\mu_n)^\kappa/\mu$.

This sequence is called the *Kleinberg sequence derived from μ*.

Theorem 3. Let κ be a strong partition cardinal and μ be a measure on κ. Then κ^κ/μ is a cardinal. If μ is normal, then $\kappa^\mu_1 = \kappa^\kappa/\mu$ is a measurable cardinal, and all κ^μ_n are Jónsson cardinals.

Proof. The first claim is a result of Martin's proved in [4, Theorem 7.1]. The second claim is part of Kleinberg's analysis of strong partition cardinals from [9]. ⊣

The *projective ordinals* play an important role in the descriptive set theory of the projective sets (*cf.* [11, § 7D], [7, § 30], and [8]). Moreover, the fact that the odd projective ordinals $\boldsymbol{\delta}^1_{2n+1}$ have the strong partition property is central to the analysis of cardinals and measures below their supremum. Recall they are defined by:

$$\boldsymbol{\delta}^1_n := \sup\{\xi : \xi \text{ is the length of a prewellordering of } \omega^\omega \text{ in } \boldsymbol{\Delta}^1_n\}.$$

The Cabal has developed an intricate theory of the combinatorics of the projective ordinals summarized in the following fact:

THEOREM 4. Let n be a natural number. Then:

1. (Kunen, Martin 1971) $\boldsymbol{\delta}^1_{2n+2} = (\boldsymbol{\delta}^1_{2n+1})^+$,
2. (Kunen, Martin 1971) all $\boldsymbol{\delta}^1_n$ are measurable,
3. (Kunen, Martin, Solovay 1971) $\boldsymbol{\delta}^1_2 = \aleph_2$, $\boldsymbol{\delta}^1_3 = \aleph_{\omega+1}$, and $\boldsymbol{\delta}^1_4 = \aleph_{\omega+2}$,
4. (Martin 1971) $\boldsymbol{\delta}^1_1 \to (\boldsymbol{\delta}^1_1)^{\boldsymbol{\delta}^1_1}$,
5. (Kechris 1974) for all n, $\boldsymbol{\delta}^1_{2n+1}$ is a successor of a cardinal of cofinality ω,
6. (Kunen 1971) the ω-cofinal measure $\mathscr{C}^{\omega}_{\boldsymbol{\delta}^1_{2n+1}}$ is a normal measure on $\boldsymbol{\delta}^1_{2n+1}$ with
$$\boldsymbol{\delta}^1_{2n+1}{}^{\boldsymbol{\delta}^1_{2n+1}}/\mathscr{C}^{\omega}_{\boldsymbol{\delta}^1_{2n+1}} = \boldsymbol{\delta}^1_{2n+2} = (\boldsymbol{\delta}^1_{2n+1})^+,$$
7. (Jackson, Martin 1980) $\boldsymbol{\delta}^1_3{}^{\boldsymbol{\delta}^1_3}/\mathscr{C}^{\omega_1}_{\boldsymbol{\delta}^1_3} = \aleph_{\omega\cdot 2+1}$ and $\boldsymbol{\delta}^1_3{}^{\boldsymbol{\delta}^1_3}/\mathscr{C}^{\omega_2}_{\boldsymbol{\delta}^1_3} = \aleph_{\omega^\omega+1}$, and these two cardinals are measurable.
8. (Jackson 1985) Let $\boldsymbol{e}_0 := 0$ and $\boldsymbol{e}_{n+1} := \omega^{(\omega^{\boldsymbol{e}_n})}$ (*i.e.*, $\boldsymbol{e}_n$ is a exponential ω-tower of height $2n-1$). Then for every $n \in \omega$,
$$\boldsymbol{\delta}^1_{2n+1} = \aleph_{\boldsymbol{e}_n+1},$$
and all odd projective ordinals have the strong partition property and are closed under ultrapowers.

PROOF. A proof of all parts except for the last two can be found in [8]. Item (7) and the $n = 2$ case of (8) can be found in [4, Chapter 7]. The general case of (8) is in [3]. ⊣

§3. Ordinal algebras and measure assignments.

3.1. Ordinal algebras. We introduce the notion of an *ordinal algebra* which will be the fundamental object we shall use to describe the cardinal structure below the projective ordinals.

DEFINITION 5. An *ordinal algebra* $\mathfrak{A}_{\mathfrak{V}}$ is a free associative left-distributive algebra with operations $\oplus$, $\otimes$ over a set of generators $\mathfrak{V}$. Also, $\mathfrak{A}_\alpha$ is the ordinal algebra over the wellordered set of generators $\{\mathsf{V}_\beta\,;\,\beta < \alpha\}$.

By "free associative left-distributive algebra" we mean the standard construction from universal algebra (cf., e.g., [2, § 24]); for the sake of completeness, let us spell out what this means precisely:

Let $\mathscr{L}_{\mathfrak{V}}$ be the language that has the elements of $\mathfrak{V}$ as variables (which we call the generators), and with binary function symbols $\oplus$ and $\otimes$. The *free algebra* $\mathfrak{F}_{\mathfrak{V}}$ over the set of generators $\mathfrak{V}$ is the set whose elements are formal terms in the language $\mathscr{L}_{\mathfrak{V}}$. We define two binary operations on this set which we also denote (through a slight abuse of notation) as $\oplus$ and $\otimes$. Given terms s and t in the free algebra, we let $s \oplus t$ be the formal term $(s \oplus t)$. Note that the $\oplus$ on the left-hand side denotes the binary operation on the free algebra, while the $\oplus$ on the right-hand side denote the formal binary function symbol in the language. The binary operation $\otimes$ on the free algebra is defined in exactly the same way. To get the free left-distributive associative algebra $\mathfrak{A}_{\mathfrak{V}}$ over $\mathfrak{V}$ we identify elements of $\mathfrak{F}_{\mathfrak{V}}$ by the relations corresponding to left-distributivity and associativity for the two operations in the usual algebraic manner. More precisely, we say terms s, t of $\mathfrak{F}_{\mathfrak{V}}$ are *equivalent*, $s \sim t$, provided there is a sequence of terms $s = s_0, s_1, \ldots, s_n = t$ such that for each consecutive pair s_i, s_{i+1}, one of these can be obtained from the other by taking a subterm (in the usual sense) u of one of these and replacing it by a term v to get the other. Here u has the form of the left/right-hand side of one of the three following equations, and v has the form of the corresponding right/left-hand side:

$$(a \oplus (b \oplus c)) = ((a \oplus b) \oplus c),$$
$$(a \otimes (b \otimes c)) = ((a \otimes b) \otimes c),$$
$$(a \otimes (b \oplus c)) = ((a \otimes b) \oplus (a \otimes c)).$$

The elements of $\mathfrak{A}_{\mathfrak{V}}$ are the equivalence classes $[s]_\sim$ of the terms in $\mathfrak{F}_{\mathfrak{V}}$. Also, $\mathfrak{A}_{\mathfrak{V}}$ has the natural operations induced from $\mathfrak{F}_{\mathfrak{V}}$, i.e., $[s]_\sim \oplus [t]_\sim = [(s \oplus t)]_\sim$ and likewise for $\otimes$. It is easy to check, as usual, that these operations are well-defined on $\mathfrak{A}_{\mathfrak{V}}$, the operations $\oplus$ and $\otimes$ on $\mathfrak{A}_{\mathfrak{V}}$ are associative, and $\otimes$ left-distributes over $\oplus$.

We say a term t of the language $\mathscr{L}_{\mathfrak{V}}$ *represents* the element $a \in \mathfrak{A}_{\mathfrak{V}}$ if $a = [t]_\sim$. By definition, every element of $\mathfrak{A}_{\mathfrak{V}}$ is represented (non-uniquely) by a term of $\mathscr{L}_{\mathfrak{V}}$.

For the ordinal algebras $\mathfrak{A}_\alpha$ generated by a wellordered set of generators $\{\mathsf{V}_\beta ; \beta < \alpha\}$, we define a function o from $\mathfrak{A}_\alpha$ onto an ordinal $\mathrm{ht}(\mathfrak{A}_\alpha)$ which we call the *height* of $\mathfrak{A}_\alpha$. We shall have for $\alpha < \beta$ that the o function on $\mathfrak{A}_\beta$ extends the o function on $\mathfrak{A}_\alpha$. To begin, we define $o(\mathsf{V}_0) = 0$. Suppose we have defined o on all of $\mathfrak{A}_\alpha$. Then set $o(\mathsf{V}_\alpha) = \mathrm{ht}(\mathfrak{A}_\alpha) = \sup\{o(t) + 1 \colon t \in \mathfrak{A}_\alpha\}$ and extend o to $\mathfrak{A}_{\alpha+1}$ by $o(s \oplus t) = o(s) + o(t)$, $o(s \otimes t) = o(s) \cdot o(t)$ (ordinal addition and multiplication). By construction, o is a homomorphism from $\mathfrak{A}_\alpha$ to the ordinals with ordinal addition and multiplication.

Let us consider the simplest ordinal algebras as examples. For this, we introduce a notation for finitely iterated sums and products:

$$\mathsf{V} \otimes n := \mathsf{V} \underbrace{\oplus \cdots \oplus}_{n} \mathsf{V}, \text{ and}$$
$$\mathsf{V}^{\otimes n} := \mathsf{V} \underbrace{\otimes \cdots \otimes}_{n} \mathsf{V}.$$

($\alpha = 1$): If $\mathfrak{V} = \{\mathsf{V}_0\}$, then $o(\mathsf{V}_0) = 0$, so $o(\mathsf{V}_0 \oplus \mathsf{V}_0) = 0$, $o(\mathsf{V}_0 \otimes \mathsf{V}_0) = 0$, etc., so $\mathrm{ht}(\mathfrak{A}) = 1$.

($\alpha = 2$): If $\mathfrak{V} = \{\mathsf{V}_0, \mathsf{V}_1\}$, then $o(\mathsf{V}_0) = 0$ and $o(\mathsf{V}_1) = 1$, so $o(\mathsf{V}_1 \otimes n) = n$, and thus $\mathrm{ht}(\mathfrak{A}) = \omega$.

($\alpha = \omega$): Here we use ω generators $\mathfrak{V} = \{\mathsf{V}_0, \mathsf{V}_1, \mathsf{V}_2, \ldots\}$. So, $o(\mathsf{V}_0) = 0$, $o(\mathsf{V}_1) = 1$, and $o(\mathsf{V}_2) = \omega$. Then $o(\mathsf{V}_2^{\otimes n}) = \omega^n$, and thus $o(\mathsf{V}_3) = \omega^\omega$. Likewise, $o(\mathsf{V}_4) = \omega^{\omega^2}$, $o(\mathsf{V}_5) = \omega^{\omega^3}$, etc. So, $\mathrm{ht}(\mathfrak{A}_\omega) = \omega^{\omega^\omega}$.

PROPOSITION 6.

$$o(\mathsf{V}_\alpha) = \mathrm{ht}(\mathfrak{A}_\alpha) = \begin{cases} 1 & \alpha = 1, \\ \omega^{\omega^{\alpha-2}} & 1 < \alpha < \omega, \\ \omega^{\omega^\alpha} & \alpha \geq \omega. \end{cases}$$

PROOF. An easy induction on α. Suppose that $\mathrm{ht}(\mathfrak{A}_\alpha) = \omega^{\omega^\alpha}$. By definition, $o(\mathsf{V}_\alpha) = \omega^{\omega^\alpha}$. Also, $o(\mathsf{V}_\alpha^{\otimes n}) = (\omega^{\omega^\alpha})^n = \omega^{\omega^\alpha \cdot n}$. So,

$$\mathrm{ht}(\mathfrak{A}_{\alpha+1}) = \sup_n \omega^{(\omega^\alpha \cdot n)} = \omega^{\omega^{\alpha+1}}. \qquad \dashv$$

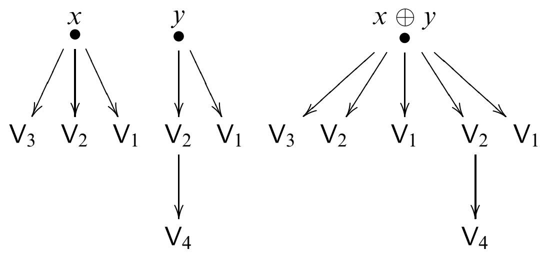

FIGURE 1. Adding the trees for $x = \mathsf{V}_3 \oplus \mathsf{V}_2 \oplus \mathsf{V}_1$ and $y = (\mathsf{V}_4 \otimes \mathsf{V}_2) \oplus \mathsf{V}_1$.

3.2. Ordinal algebras as algebras of trees. For the rest of the paper, we shall think of the elements of ordinal algebras as finite trees whose nodes are labeled with generators. In the following, we shall make this correspondence precise and prove that these algebras of trees correspond exactly to our ordinal algebras.

A finite tree T is a finite subset of $\omega^{<\omega}$ closed under initial segments. In the context of trees, we write $\bullet$ for the empty sequence (i.e., the root of the tree) and refer to the set of terminal nodes of T as $\mathrm{tn}(T)$. Trees come with a natural order on the set of successors of each node in the tree (if $t_0 := t^\frown n$ and $t_1 := t^\frown m$ are both in T, then we say that $t_0 < t_1$ if and only if $n < m$). The natural notion of isomorphism of trees is that of bijections preserving the tree structure and the order: a function $f : T \to S$ is called an *isomorphism* if f is a length-preserving bijection, $t \subseteq s$ if and only if $f(t) \subseteq f(s)$, and $f(t^\frown n)(\mathrm{lh}(t)) < f(t^\frown m)(\mathrm{lh}(t))$ if and only if $n < m$.

For notational simplicity, we shall restrict our attention to a subclass of nice trees: We call a tree *tidy* if for all $s \in T$, the set $\{n\,;\, s^\frown n \in T\}$ is a natural number. If T is a tidy tree, let $N_T := \{n\,;\, \langle n\rangle \in T\}$. We call $\langle N_T - 1\rangle$ the *trailing node* of T. For every tree there is a unique tidy tree isomorphic to it.

Given a set X of labels, we call a pair $\langle T, \ell\rangle$ an X*-labelled tree* if T is a tidy tree and $\ell : T\backslash\{\bullet\} \to X$. The mentioned notion of isomorphism extends to a notion of isomorphism between labelled trees if we demand preservation of labels (i.e., $\ell(f(t)) = \ell(t)$) in addition.

For a tree T, we define its *shift by n* as $\mathrm{shift}_n(T) := \{t\,;\, \exists k \exists s(k^\frown s \in T \wedge t = (k+n)^\frown s)$ which produces a tree isomorphic to T. Obviously, if $\langle T, \ell\rangle$ was an X-labelled tree, then there is a unique labelling ℓ^* that makes $\langle \mathrm{shift}_n(T), \ell^*\rangle$ isomorphic to $\langle T, \ell\rangle$ as a labelled tree. *Par abus de langage*, we denote this labelling by ℓ as well.

Similarly, if $s \in \omega^{<\omega}$ and T is a tree, then we define its *push to s* by $\mathrm{push}_s(T) := \{t\,;\, \exists t^* \in T(t = s^\frown t^*)\}$. Note that this is not a tree in our sense because it is not closed under initial segments (no proper initial segment of s is an element of $\mathrm{push}_s(T)$), but if T' is a tree with $s \in \mathrm{tn}(T')$, then $T' \cup \mathrm{push}_s(T)$ is a tree. Again, if $\langle T, \ell\rangle$ was a labelled tree, there is a unique function $\ell^* : \mathrm{push}_s(T)\backslash\{s\} \to X$ that preserves the labels on T and we'll refer to it as ℓ.

If T and S are tidy trees, we define their sum and product as follows:

$$T \oplus S := T \cup \mathrm{shift}_{N_T}(S), \text{ and}$$

$$T \otimes S := S \cup \bigcup_{s \in \mathrm{tn}(S)} \mathrm{push}_s(T).$$

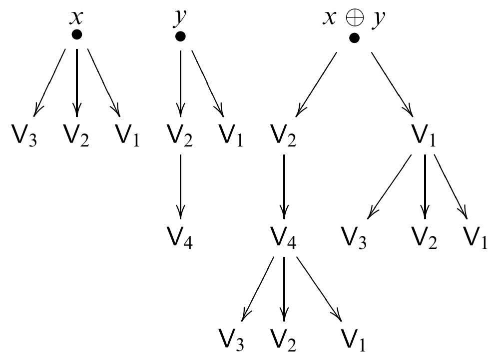

FIGURE 2. Multiplying the trees for $x = \mathsf{V}_3 \oplus \mathsf{V}_2 \oplus \mathsf{V}_1$ and $y = (\mathsf{V}_4 \otimes \mathsf{V}_2) \oplus \mathsf{V}_1$.

In the case of labelled trees, we take the unions of the labels of the constituents. Examples of addition and multiplication of trees can be seen in Figures 1 and 2. It is clear that the defined operations are associative and left-distributive. The algebra of finite tidy X-labeled trees with these operations will be denoted by $\mathfrak{T}_X$.

We shall now define two maps: one that associates to each term $s \in \mathscr{L}_{\mathfrak{V}}$ a tidy $\mathfrak{V}$-labelled tree $\mathsf{tree}(s)$; and one that associates to each tidy $\mathfrak{V}$-labelled tree $\langle T, \ell \rangle$ a particularly nice term $\mathsf{term}(T, \ell)$ in $\mathscr{L}_{\mathfrak{V}}$; the latter terms will be called *canonical terms*.

DEFINITION 7. The set of *canonical terms* in the language $\mathscr{L}_{\mathfrak{V}}$ is the set of terms defined inductively as follows. The canonical terms will be divided into generator terms, additive terms, and multiplicative terms. Every $\mathsf{V} \in \mathfrak{V}$ is a generator term. If $s_0, \dots, s_{n-1}$ are canonical terms, each of which is a generator term or a multiplicitive term, then $(\cdots(s_0 \oplus s_1) \oplus \cdots) \oplus s_{n-1})$ is an additive canonical term. Finally, if s is a canonical term and V is a generator, then $s \otimes \mathsf{V}$ is a multiplicative canonical term.

LEMMA 8. Every term in $\mathscr{L}_{\mathfrak{V}}$ is equivalent to a canonical term.

PROOF. Inductively, it is enough to show that if s and t are canonical terms then $s \oplus t$ and $s \otimes t$ are equivalent to canonical terms. We can write $s = s_0 \oplus \cdots \oplus s_n$ and $t = t_0 \oplus \cdots \oplus t_m$ where the s_i and t_j are generator terms or multiplicative terms (we allow n, m to be 0). Then $s \oplus t$ is equivalent by associativity of $\oplus$ to the additive term $s_0 \oplus \cdots \oplus s_n \oplus t_0 \oplus \cdots \oplus t_m$.

Now we show $s \otimes t$ is equivalent to a canonical term by induction on the length of t. If t is a generator, the result holds by definition. If $t = u \otimes \mathsf{V}$ where u is a canonical term, then $s \otimes (u \otimes \mathsf{V}) \sim (s \otimes u) \otimes \mathsf{V}$. By induction, $s \otimes u$ is equivalent to a canonical term, and the result follows. If $t = t_0 \oplus \cdots \oplus t_n$, then $s \otimes t \sim (s \otimes t_0) \oplus \cdots \oplus (s \otimes t_n)$ by left-distributivity. By induction, all of the $s \otimes t_i$ are equivalent to canonical terms. By associativity of $\oplus$ we may write this sum as an additive canonical term and we are done. ⊣

Let us introduce a useful notation for the simplest trees: we write t_V for the labeled tree $\langle T, \ell \rangle$ with $T := \{\bullet, \langle 0 \rangle\}$ and $\ell(\langle 0 \rangle) := \mathsf{V}$. If $\mathsf{V} \in \mathfrak{V}$ is a generator, we let $\mathsf{tree}(\mathsf{V}) := \mathsf{t}_\mathsf{V}$. If x and y are terms with $\mathsf{tree}(x)$ and $\mathsf{tree}(y)$ already defined, we let

$\mathsf{tree}(x \oplus y) := \mathsf{tree}(x) \oplus \mathsf{tree}(y)$ and similarly for $\otimes$. This defines a homomorphism from $\mathfrak{F}_{\mathfrak{V}}$ to $\mathfrak{T}_{\mathfrak{V}}$. Since $\mathfrak{T}_{\mathfrak{V}}$ is associative and left-distributive, the map factors through a homomorphism

$$\mathsf{tree}\colon \mathfrak{A}_{\mathfrak{V}} \to \mathfrak{T}_{\mathfrak{V}},$$

for which we use the same notation.

Next we define a map $\mathsf{term}\colon \mathfrak{T}_{\mathfrak{V}} \to \mathscr{L}_{\mathfrak{V}}$. If $\langle T, \ell\rangle = \mathsf{t}_{\mathsf{V}}$, then we define $\mathsf{term}(T, \ell) := \mathsf{V}$. Any other labelled tree $\langle T, \ell\rangle$ can be split up into a finite sum $\langle T_0, \ell_0\rangle \otimes \mathsf{t}_{\ell(\langle 0\rangle)} \oplus \cdots \oplus \langle T_n, \ell_n\rangle \otimes \mathsf{t}_{\ell(\langle n\rangle)}$ for labeled trees $\langle T_i, \ell_i\rangle$ of smaller height, so we let $\mathsf{term}(T, \ell) := \mathsf{term}(T_0, \ell_0) \otimes \ell(\langle 0\rangle) \oplus \cdots \oplus \mathsf{term}(T_n, \ell_n) \otimes \ell(\langle n\rangle)$, where we associate the terms in the sum from left to right. Note that $\mathsf{term}(T, \ell)$ is always a canonical term in the sense of Definition 7.

The functions tree and term witness that the algebra of trees and the free associative left-distributive algebra are isomorphic:

THEOREM 9. Let $\mathfrak{V}$ be a set of generators. Then $\mathfrak{A}_{\mathfrak{V}}$ is isomorphic to $\mathfrak{T}_{\mathfrak{V}}$.

PROOF. We use the previously defined maps $\mathsf{tree}\colon \mathfrak{A}_{\mathfrak{V}} \to \mathfrak{T}_{\mathfrak{V}}$ and $\mathsf{term}\colon \mathfrak{T}_{\mathfrak{V}} \to \mathfrak{A}_{\mathfrak{V}}$, and show that they witness that tree is an isomorphism.

Fix any $\mathfrak{V}$-labelled tidy tree $\langle T, \ell\rangle$. Then $\mathsf{term}(T, \ell)$ is a canonical term in $\mathfrak{F}_{\mathfrak{V}}$. A straightforward induction on the size of T shows that for every tidy $\mathfrak{V}$-labelled tree $\langle T, \ell\rangle$ we have $\mathsf{tree}(\mathsf{term}(T, \ell)) = \langle T, \ell\rangle$:

If T has $n + 1$ immediate extensions of $\bullet$, then $\langle T, \ell\rangle = \langle T_0, \ell_0\rangle \otimes \mathsf{t}_{\ell(\langle 0\rangle)} \oplus \cdots \oplus \langle T_n, \ell_n\rangle \otimes \mathsf{t}_{\ell(\langle n\rangle)}$ for trees T_i of smaller height, and

$$\begin{aligned}
\mathsf{tree}(\mathsf{term}(T, \ell)) &= \mathsf{tree}(\mathsf{term}(T_0, \ell_0) \otimes \ell(\langle 0\rangle) \oplus \cdots \oplus \mathsf{term}(T_n, \ell_n) \otimes \ell(\langle n\rangle)) \\
&= \mathsf{tree}(\mathsf{term}(T_0, \ell_0) \otimes \ell(\langle 0\rangle)) \oplus \cdots \oplus \mathsf{tree}(\mathsf{term}(T_n, \ell_n) \otimes \ell(\langle n\rangle)) \\
&= \mathsf{tree}(\mathsf{term}(T_0, \ell_0)) \otimes \mathsf{t}_{\ell(\langle 0\rangle)} \oplus \cdots \oplus \mathsf{tree}(\mathsf{term}(T_n, \ell_n)) \otimes \mathsf{t}_{\ell(\langle n\rangle)} \\
&= \langle T_0, \ell_0\rangle \otimes \mathsf{t}_{\ell(\langle 0\rangle)} \oplus \cdots \oplus \langle T_n, \ell_n\rangle \otimes \mathsf{t}_{\ell(\langle n\rangle)} = \langle T, \ell\rangle
\end{aligned}$$

by induction hypothesis and our definitions.

At this point we have shown that the map tree is an epimorphism from $\mathfrak{A}_{\mathfrak{V}}$ to $\mathfrak{T}_{\mathfrak{V}}$. It remains to show that this map is one-to-one.

By Lemma 8 and because tree is well-defined on $\mathfrak{A}_{\mathfrak{V}}$, it is enough to show that the homomorphism tree is one-to-one when restricted to the set of canonical terms of $\mathscr{L}_{\mathfrak{V}}$. We shall show for any canonical term $s \in \mathscr{L}_{\mathfrak{V}}$ that $\mathsf{term}(\mathsf{tree}(s)) = s$ by induction on the size of s.

If s is a generator term the result is immediate from the definitions. Now suppose $s = s_0 \oplus \cdots \oplus s_n$ is an additive canonical term. Then $\mathsf{tree}(s) = \mathsf{tree}(s_0) \oplus \cdots \oplus \mathsf{tree}(s_n)$ by definition. Each s_i is either a generator term V_i or a multiplicative term $u_i \otimes \mathsf{V}_i$. For the sake of simplicity, let us assume that they are all of the form $u_i \otimes \mathsf{V}_i$. So, $\mathsf{tree}(s_i) = \mathsf{tree}(u_i) \otimes \mathsf{t}_{\mathsf{V}_i}$. By definition of the map term and induction we have:

$$\begin{aligned}
\mathsf{term}(\mathsf{tree}(s)) &= \mathsf{term}(\mathsf{tree}(s_0) \oplus \cdots \oplus \mathsf{tree}(s_n)) \\
&= \mathsf{term}(\mathsf{tree}(u_0) \otimes \mathsf{t}_{\mathsf{V}_0} \oplus \cdots \oplus \mathsf{tree}(u_n) \otimes \mathsf{t}_{\mathsf{V}_n}) \\
&= \mathsf{term}(\mathsf{tree}(u_0)) \otimes \mathsf{V}_0 \oplus \cdots \oplus \mathsf{term}(\mathsf{tree}(u_n)) \otimes \mathsf{V}_n \\
&= u_0 \otimes \mathsf{V}_0 \oplus \cdots \oplus u_n \otimes \mathsf{V}_n \\
&= s.
\end{aligned}$$

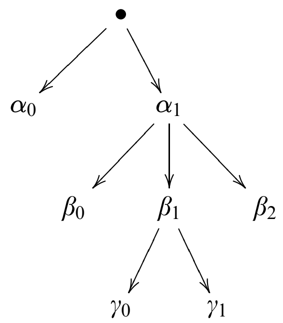

FIGURE 3. The tree representation of $\alpha_0 + (\beta_0 + (\gamma_0 + \gamma_1) \cdot \beta_1 + \beta_2) \cdot \alpha_1$.

In the third and fourth lines above we associate the terms in the sum from left to right. Finally, if $s = u \otimes \mathsf{V}$, then $\mathsf{tree}(s) = \mathsf{tree}(u) \otimes \mathsf{t}_{\mathsf{V}}$. Thus,

$$\begin{aligned}\mathsf{term}(\mathsf{tree}(s)) &= \mathsf{term}(\mathsf{tree}(u)) \otimes \mathsf{t}_{\mathsf{V}}) \\ &= \mathsf{term}(\mathsf{tree}(u)) \otimes \mathsf{V} \\ &= u \otimes \mathsf{V} = s\end{aligned}$$

by induction. $\dashv$

3.3. Measure assignments. Let us fix a set of generators $\mathfrak{V}$ and an ordinal algebra $\mathfrak{A}_{\mathfrak{V}}$. A $\mathfrak{V}$-*measure assignment* is a function $\boldsymbol{M}$ assigning measures to the generators of $\mathfrak{A}$. For each measure we say that it *has order type* ξ if it is a measure on ξ not concentrating on any smaller ordinal. The measure assignment generates an order type function $\boldsymbol{ot}$ assigning to each generator V the order type of $\boldsymbol{M}(\mathsf{V})$. We shall extend this function to a function assigning ordinals to all members of $\mathfrak{A}_{\mathfrak{V}}$ by

$$\begin{aligned}\boldsymbol{ot}(s \oplus t) &:= \boldsymbol{ot}(s) + \boldsymbol{ot}(t), \\ \boldsymbol{ot}(s \otimes t) &:= \boldsymbol{ot}(s) \cdot \boldsymbol{ot}(t)\end{aligned}$$

where the operations on the right-hand side are ordinal addition and multiplication, respectively. An example of the calculation of an order type for a more complex term can be seen in Figure 3.

Fix a term s identified with the tidy labeled tree $\langle T, \ell \rangle := \mathsf{tree}(s)$. Recall that we called $\langle N_T - 1 \rangle$ the *trailing node* of T. If v is the trailing node of T, then $\boldsymbol{ot}(\ell(v))$ is the trailing node in the ordinal representation of $\boldsymbol{ot}(s)$. In particular,

$$\mathrm{cf}(\boldsymbol{ot}(s)) = \mathrm{cf}(\boldsymbol{ot}(\ell(v))).$$

This will become important later on.

§4. Lifting and canonicity. It is our goal to analyze cardinals as ultrapowers via our measure assignments. In order to do this, we need to transform our terms into real measures on (odd) projective ordinals. We shall do this via lifting. We use two operations, the weak lift $\boldsymbol{wlift}_\kappa$ and the strong lift $\boldsymbol{slift}_\kappa$. The first uses the weak partition relation at κ and the second the strong partition relation.

If κ is an infinite cardinal and T is a finite tree with its (finite) set $\mathrm{tn}(T)$ of terminal nodes, there is a definable bijection $\ulcorner\cdot\urcorner : \kappa^{\mathrm{tn}(T)} \to \kappa$. We fix these bijections for the rest of the paper and use the same notation for all of these functions. As in the previous section, we fix a set of generators $\mathfrak{V}$, its ordinal algebra $\mathfrak{A}_{\mathfrak{V}}$, and a $\mathfrak{V}$-measure assignment $\boldsymbol{M}$.

Fix a term $x \in \mathfrak{A}$ and $\langle T, \ell\rangle = \mathsf{tree}(x)$. The ordinal $\boldsymbol{ot}(x)$ is equal to the order-type of the lexicographic ordering on the set of tuples $\langle j_0, \alpha_0, \dots, j_k, \alpha_k\rangle$ where $\langle j_0, \dots, j_k\rangle \in T$, and for $n \leq k$ we have $\alpha_n < \boldsymbol{ot}(\ell(\langle j_0, \dots, j_n\rangle))$. We frequently identify $\boldsymbol{ot}(x)$ with this set of tuples.

For each terminal node $t = \langle i_0, \dots, i_k\rangle$ of T, consider the subset $\boldsymbol{ot}(x)_t$ of $\boldsymbol{ot}(x)$ consisting of those tuples $\langle j_0, \alpha_0, \dots, j_k, \alpha_k\rangle$ with $\langle j_0, \dots, j_k\rangle = \langle i_0, \dots, i_k\rangle$. This set is naturally identified with the set (under lexicographic order) $\boldsymbol{ot}(v_{i_0}) \times \boldsymbol{ot}(v_{i_0,i_1}) \times \cdots \times \boldsymbol{ot}(v_{i_0,\dots,i_k})$, where again $v_{i_0,\dots,i_\ell} = \ell(\langle i_0, \dots, i_\ell\rangle)$. We also have the product measure

$$\mu^*(t) := \mu(i_0) \times \mu(i_0, i_1) \times \cdots \times \mu(i_0, \dots, i_k)$$

on $\boldsymbol{ot}_t(x)$, where $\mu(i_0, \dots, i_\ell)$ is the measure assigned to the variable $v_{i_0,\dots,i_\ell}$. Every function $f : \boldsymbol{ot}(x) \to \mathrm{Ord}$ induces by restriction, for each terminal node t, a function $f^t : \boldsymbol{ot}(x)_t \to \mathrm{Ord}$. The function f is, in a natural sense, the union of these subfunctions.

Notice that for each terminal node t that is a successor of the trailing node of T, we have that $\sup(f^t) = \sup(f)$.

Fix a cardinal κ with the weak partition relation and closed under ultrapowers and $x \in \mathfrak{A}_{\mathfrak{V}}$ such that $\boldsymbol{ot}(x) < \kappa$. As described above, the measure assignment determines measures $\mu^*(t)$ on $\boldsymbol{ot}(x)_t$ for each $t \in \mathrm{tn}(\mathsf{tree}(x))$.

Definition 10. We define

$$\boldsymbol{wlift}_\kappa(x) := \{A \subseteq \kappa \colon \text{there is a club set } C \subseteq \kappa \text{ such that for all } f : \boldsymbol{ot}(x) \to C \text{ of continuous type we have } \ulcorner\{[f^t]_{\mu^*(t)} : t \in \mathrm{tn}(T)\}\urcorner \in A\}$$

and call it the *weak lift* of the measure assigned to x.

Note that the notion of the weak lift is defined on $\mathfrak{A}_{\mathfrak{V}}$ using a $\mathfrak{V}$-measure assignment, but it can be defined more generally: Given any measure μ (or any labelled tree T with measures at its nodes), it makes sense to speak of the "weak lift of μ" by replacing $\mu^*(t)$ in the above definition by μ (or the appropriate measures assigned to the terminal nodes of T). For this object, we'll write $\boldsymbol{wlift}_\kappa(\mu)$. The weak partition relation on κ gives the following.

Theorem 11 (Weak Lifting Theorem). *Let κ be a weak partition cardinal closed under ultrapowers. Let $x \in \mathfrak{A}$ be such that $\boldsymbol{ot}(x) < \kappa$. Then $\boldsymbol{wlift}_\kappa(x)$ is a measure on κ.*

Proof. Exercise. A full proof of the case of trees of depth 1 can be found in [1, Theorem 10]. $\dashv$

To see a few examples, we ask the reader to check that the weak lift of a principal ultrafilter on a singleton to any weak partition cardinal κ is equal to $\mathscr{C}^\omega_\kappa$. Similarly, the weak lift of $\mathscr{C}^\omega_\lambda$ to κ is $\mathscr{C}^\lambda_\kappa$.

The strong partition relation on κ allows us to lift measures on κ to measures on the ultrapower, according to the following definition.

DEFINITION 12. Let κ be a strong partition cardinal, μ a measure on κ, and $\lambda := \kappa^\kappa/\mu$. We define a measure on λ by

$$\boldsymbol{\mathit{slift}}_\kappa(\mu) := \{A \subseteq \lambda\colon \text{there is a club set } C \subseteq \kappa \text{ such that for all } f : \kappa \to C \text{ of the correct type we have } [f]_\mu \in A\}.$$

If κ is determined by the context, we just write $\boldsymbol{\mathit{slift}}(\mu)$. The strong partition relation at κ gives immediately the following.

THEOREM 13 (Strong Lifting Theorem). Let κ be a strong partition cardinal and μ be a measure on κ. Then $\boldsymbol{\mathit{slift}}(\mu)$ is a measure on κ^κ/μ.

For notational convenience, we can combine the operations $\boldsymbol{\mathit{wlift}}$ and $\boldsymbol{\mathit{slift}}$ to a new operation that we call the *high lift of x* (here $\boldsymbol{\mathit{ot}}(x) = \varrho < \kappa$ and κ is a strong partition cardinal closed under ultrapowers):

$$\boldsymbol{\mathit{highlift}}_\kappa(x) := \boldsymbol{\mathit{slift}}(\boldsymbol{\mathit{wlift}}_\kappa(x)).$$

An important hypothesis for this paper is embodied in the following definition.

DEFINITION 14. A measure assignment is called *canonical* for the strong partition cardinal $\kappa = \aleph_{\xi+1}$ if for all $x \in \mathfrak{A}$ with $\boldsymbol{\mathit{ot}}(x) < \kappa$ we have

$$\kappa^\kappa/\boldsymbol{\mathit{wlift}}_\kappa(x) = \aleph_{\xi+o(x)+1}.$$

Moreover, for $\delta > \kappa$ closed under ultrapowers, the value of $\delta^\delta/\boldsymbol{\mathit{wlift}}_\delta(\boldsymbol{\mathit{highlift}}_\kappa(x))$ does not depend on the bijections between $\kappa^{\mathsf{tn}(\mathsf{tree}(x))}$ and κ used in the construction of $\boldsymbol{\mathit{wlift}}_\kappa(x)$.

Note that canonicity is a strong assumption and implies a lot of non-obvious claims about the behavior of sums and products of measures: e.g., while $o(\mathsf{V}_1 \oplus \mathsf{V}_2) = \omega = o(\mathsf{V}_2)$, there is no *a priori* reason that the measures associated to these two terms should be similar. Canonicity of the measure assignment ensures that they are, in the sense that they give the same ultrapowers.

To illustrate the "moreover" part of the definition, let W_1^3 be the three-fold product of the normal measure on ω_1.[1] Let μ_1 be the measure on ω_1 obtained from W_1^3 by identifying $(\omega_1)^3$ with ω_1 using the ordering $\langle\alpha_1, \alpha_2, \alpha_3\rangle <_1 \langle\beta_1, \beta_2, \beta_3\rangle$ iff $\langle\alpha_3, \alpha_1, \alpha_2\rangle <_{\mathrm{lex}} \langle\beta_3, \beta_1, \beta_2\rangle$, where $<_{\mathrm{lex}}$ denotes lexicographic ordering. Let μ_2 be defined similarly but using $\langle\alpha_1, \alpha_2, \alpha_3\rangle <_2 \langle\beta_1, \beta_2, \beta_3\rangle$ iff $\langle\alpha_3, \alpha_2, \alpha_1\rangle <_{\mathrm{lex}} \langle\beta_3, \beta_2, \beta_1\rangle$. Let $\nu_1 = \boldsymbol{\mathit{slift}}(\mu_1)$ and $\nu_2 = \boldsymbol{\mathit{slift}}(\mu_2)$. Then ν_1, ν_2 are both measures on ω_4, but are quite different and definitely non-equivalent measures. Nevertheless, ${\boldsymbol{\delta}_3^1}^{\boldsymbol{\delta}_3^1}/\boldsymbol{\mathit{wlift}}_{\boldsymbol{\delta}_3^1}(\nu_1) = {\boldsymbol{\delta}_3^1}^{\boldsymbol{\delta}_3^1}/\boldsymbol{\mathit{wlift}}_{\boldsymbol{\delta}_3^1}(\nu_2)$. In fact, the description analysis readily shows that both ultrapowers are equal to $\aleph_{\omega^{\omega^3}+1}$, which will also follow from our canonicity assumption (the measure assignment of the next section will assign to V_5 the measure $\boldsymbol{\mathit{highlift}}_{\omega_1}(\mathsf{V}_1 \oplus \mathsf{V}_1 \oplus \mathsf{V}_1)$, which is ν_1 or ν_2 depending on the bijection used, and $o(\mathsf{V}_5) = \omega^{\omega^3}$).

[1] In the canonical measure assignment of the Section 5, this will be $\boldsymbol{\mathit{wlift}}_{\omega_1}(\mathsf{V}_1 \oplus \mathsf{V}_1 \oplus \mathsf{V}_1)$.

§5. A recursive definition of a measure assignment. In this section, we shall define a measure assignment for the algebra $\mathfrak{A}_{\varepsilon_0}$. As usual, we let $\boldsymbol{e}_0 := 0$ and $\boldsymbol{e}_{n+1} := \omega^{(\omega^{\boldsymbol{e}_n})}$; with this notation, we have that $\mathfrak{A}_{\varepsilon_0} = \bigcup_{n\in\omega} \mathfrak{A}_{\boldsymbol{e}_n}$. The assignment will be defined recursively along this union of algebras.[2] The basic idea is that the variables at a given level correspond to the high lifts of the terms (not just the variables) at the previous levels.

We start with $\mathfrak{A}_2$ generated by V_0 and V_1. We have to deal with an anomaly at the beginning: we want to have $\boldsymbol{ot}(\mathsf{V}_0) = 0$, but of course there is no measure on the empty set. For the purpose of this definition, we declare $\varnothing$ to be a measure on 0, and we define $\boldsymbol{wlift}_\kappa(\varnothing)$ to be any principal ultrafilter on $\kappa = \aleph_{\xi+1}$. Clearly, this fits well, as $o(\mathsf{V}_0) = 0$ and thus

$$\kappa^\kappa/\boldsymbol{wlift}_\kappa(\mathsf{V}_0) = \aleph_{\xi+o(\mathsf{V}_0)+1} = \aleph_{\xi+1} = \kappa.$$

We continue our definition by setting $\boldsymbol{M}(\mathsf{V}_1)$ to be the principal ultrafilter on 1 (and thus $\boldsymbol{ot}(\mathsf{V}_1) = 1$). Lifting $\boldsymbol{M}(\mathsf{V}_1)$ to $\boldsymbol{\delta}^1_1 = \aleph_1$, we get that $\boldsymbol{wlift}_{\boldsymbol{\delta}^1_1}(\mathsf{V}_1)$ is the club filter on ω_1. Again, this conforms with the canonicity requirement, as

$$\aleph_2 = \aleph_1^{\aleph_1}/\mathscr{C}_{\omega_1} = \aleph_1^{\aleph_1}/\boldsymbol{wlift}_{\boldsymbol{\delta}^1_1}(\mathsf{V}_1) = \aleph_{0+o(\mathsf{V}_1)+1}.$$

We shall now lift the measure assignment from $\mathfrak{A}_2$ to $\mathfrak{A}_\omega$, or more generally, from $\mathfrak{A}_{2+\boldsymbol{e}_n}$ to $\mathfrak{A}_{\mathrm{ht}(\mathfrak{A}_{2+\boldsymbol{e}_n})} = \mathfrak{A}_{\boldsymbol{e}_{n+1}}$.

Suppose that we have a measure assignment for $\mathfrak{A}_{2+\boldsymbol{e}_n}$ such that all order types of assigned measures are below $\boldsymbol{\delta}^1_{2n+1}$. For $\xi < 2+\boldsymbol{e}_n$, we leave $\boldsymbol{ot}(\mathsf{V}_\xi)$ and $\boldsymbol{M}(\mathsf{V}_\xi)$ unchanged. If $\xi = 2+\boldsymbol{e}_n+\eta$, where $\eta < \boldsymbol{e}_{n+1}$, then there is some term $y \in \mathfrak{A}_{2+\boldsymbol{e}_n}$ such that $\eta = o(y)$. We use the Cantor normal form of η to get a canonical representative y (this is not difficult using Proposition 6). We now define

$$\boldsymbol{M}(\mathsf{V}_\xi) := \boldsymbol{highlift}_{\boldsymbol{\delta}^1_{2n+1}}(y).$$

By definition, this means that

$$\boldsymbol{ot}(\mathsf{V}_\xi) = {\boldsymbol{\delta}^1_{2n+1}}^{\boldsymbol{\delta}^1_{2n+1}}/\boldsymbol{wlift}_{\boldsymbol{\delta}^1_{2n+1}}(y).$$

Note that in order to define $\boldsymbol{M}(\mathsf{V}_\xi)$ for $\boldsymbol{e}_n \le \xi < \boldsymbol{e}_{n+1}$ we need to know that $\boldsymbol{wlift}_{\boldsymbol{\delta}^1_{2n+1}}(y)$ is a measure on $\boldsymbol{\delta}^1_{2n+1}$, which follows from the closure of $\boldsymbol{\delta}^1_{2n+1}$ under ultrapowers. From the closure of $\boldsymbol{\delta}^1_{2n+3}$ under ultrapowers it then follows that $\boldsymbol{ot}(\mathsf{V}_\xi) < \boldsymbol{\delta}^1_{2n+3}$.

This finishes the definition of a measure assignment for $\mathfrak{A}_{\varepsilon_0}$. Again, we invite the reader to compare the recursive definition with the table in Figure 5: the first system of the table gives the values in $\mathfrak{A}_2$ (before the vertical line) and $\mathfrak{A}_\omega$ (after the vertical line) and the second system gives the values in $\mathfrak{A}_{\boldsymbol{e}_2}$ with the columns in the second system corresponding to the columns in the first system via the high lift.

To get acquainted with this definition, let us compute the values of $\boldsymbol{M}(\mathsf{V}_n)$ for $n < \omega$.

V_2: We have $\mathsf{V}_2 = \mathsf{V}_{2+\boldsymbol{e}_0+0}$, so the canonical representative y of $\eta = 0$ will be just V_0. Then $\boldsymbol{wlift}_{\boldsymbol{\delta}^1_1}(\mathsf{V}_0)$ is just the principal ultrafilter by convention. Thus, $\boldsymbol{M}(\mathsf{V}_2)$ is the strong lift of the principal ultrafilter which is the normal measure W^1_1 on ω_1.

---

[2] It may be convenient for the reader to accompany the reading of the recursive definition with the table in Figure 5 which gives all of the relevant values for terms in $\mathfrak{A}_{\omega^{\omega^\omega}}$.

V_3: Now, we have $\mathsf{V}_3 = \mathsf{V}_{2+\boldsymbol{e}_0+1}$, so $\eta = 1$ and thus our canonical y is V_1. Lifting the principal filter on 1 to ω_1 yields the normal measure W_1^1 on ω_1 as $\mathrm{W}_1^1 = \boldsymbol{wlift}_{\boldsymbol{\delta}_1^1}(\mathsf{V}_1)$. As we know, $\aleph_1{}^{\aleph_1}/\mathrm{W}_1^1 = \aleph_2$, so $\boldsymbol{ot}(\mathsf{V}_3) = \omega_2$. So, $\boldsymbol{M}(\mathsf{V}_2) = \boldsymbol{slift}(\mathrm{W}_1^1) =$ the ω-club filter on ω_2 (this is denoted S_1^1 in [4, Definition 1.3]).

V_n: For $n \geq 3$, $\mathsf{V}_n = \mathsf{V}_{2+\boldsymbol{e}_0+(n-2)}$, so $\eta = n-2$, and our term is $y = \mathsf{V}_1 \otimes (n-2)$. Also, $\boldsymbol{wlift}_{\boldsymbol{\delta}_1^1}(y) = \mathrm{W}_1^{n-2}$, the $(n-2)$-fold product of the normal measure on ω_1. If we identify $(\omega_1)^{n-2}$ with ω_1 via the ordering $(\alpha_1, \dots, \alpha_{n-2}) < (\beta_1, \dots, \beta_{n-2})$ iff $(\alpha_{n-2}, \alpha_1, \dots, \alpha_{n-3}) <_{\mathrm{lex}} (\beta_{n-2}, \beta_1, \dots, \beta_{n-3})$, then the resulting measure on ω_{n-1} is denoted S_1^{n-2} in [4, Definition 1.3]. So, using this bijection, $\boldsymbol{M}(\mathsf{V}_n) = \mathrm{S}_1^{n-2}$.

Computing the ultrapowers of $\boldsymbol{\delta}_3^1$ with the measures associated to V_2 and V_3 gives exactly the right answers by Theorem 4 (7.):

$$\boldsymbol{\delta}_3^1{}^{\boldsymbol{\delta}_3^1}/\boldsymbol{wlift}_{\boldsymbol{\delta}_3^1}(\mathsf{V}_2) = \boldsymbol{\delta}_3^1{}^{\boldsymbol{\delta}_3^1}/\mathscr{C}^{\omega_1}_{\boldsymbol{\delta}_3^1} = \aleph_{\omega\cdot 2+1} = \aleph_{\omega+o(\mathsf{V}_2)+1}, \text{ and}$$
$$\boldsymbol{\delta}_3^1{}^{\boldsymbol{\delta}_3^1}/\boldsymbol{wlift}_{\boldsymbol{\delta}_3^1}(\mathsf{V}_3) = \boldsymbol{\delta}_3^1{}^{\boldsymbol{\delta}_3^1}/\mathscr{C}^{\omega_2}_{\boldsymbol{\delta}_3^1} = \aleph_{\omega^\omega+1} = \aleph_{\omega+o(\mathsf{V}_3)+1}.$$

Rephrased in the language of measure assignments, we can interpret Kleinberg's Theorem 3 and the results from [6] as follows:

THEOREM 15 (Kleinberg). The measure assignment $\boldsymbol{M}$ on $\mathfrak{A}_2$ is canonical for $\boldsymbol{\delta}_1^1$.

THEOREM 16 (Jackson-Khafizov). The measure assignment $\boldsymbol{M}$ on $\mathfrak{A}_\omega = \mathfrak{A}_{\boldsymbol{e}_1}$ is canonical for $\boldsymbol{\delta}_3^1 = \boldsymbol{\delta}_{2\cdot 1+1}^1$.

The upper and lower bound computations underlying [3, 4, 6] indicate strongly that canonicity will hold everywhere. We call the assumption that the measure assignment defined above on $\mathfrak{A}_{\boldsymbol{e}_n}$ is canonical for $\boldsymbol{\delta}_{2n+1}^1$ (in the sense of Definition 14) the *canonicity assumption*.

§6. Applications of the canonicity assumption.

In this section we shall work under the canonicity assumption. Based on that assumption, we shall be able to give algorithms to compute the cofinalities of all cardinals below $\aleph_{\varepsilon_0}$ and the Kleinberg sequences derived from the normal ultrafilters on the odd projective ordinals.

6.1. Computation of regular cardinals. As a first step in the computation of the regular cardinals, we shall give an algorithm that identifies special variables in the set $\mathfrak{V}$ of generators. We call these variables *normal*, as they will be the ones that are assigned normal measures by our recursive assignment.

We say that V_0 and V_1 are normal. In each of the iteration steps from $\mathfrak{A}_{2+\boldsymbol{e}_n}$ to $\mathfrak{A}_{\boldsymbol{e}_{n+1}}$, we identify the following new variables as normal: for $\xi = 2 + \boldsymbol{e}_n + \eta$ for some $\eta < \boldsymbol{e}_{n+1}$, the variable V_ξ is normal if and only if $\eta = o(\mathsf{v})$ for some normal $\mathsf{v} \in \mathfrak{A}_{2+\boldsymbol{e}_n}$.

By Proposition 6, for infinite ordinals ξ, the function o is just $\xi \mapsto \omega^{\omega^\xi}$, therefore, we can easily compute the indices of the normal variables by the following algorithm: write down the 2^{n+1} normal variables for $\mathfrak{A}_{\boldsymbol{e}_n}$, write down the values of o for these variables underneath the variables, then compute the indices of the 2^{n+1} new normal variables as $\boldsymbol{e}_{n+1} + o(\mathsf{v})$ for the values of o in your list. You can see the first three steps of the algorithm in the following table:

$\mathfrak{A}_2$		0	1		
	o	0	1		
$\mathfrak{A}_\omega$		$2 = 2 + 0$	$3 = 2 + 1$		
	o	ω	ω^ω		
$\mathfrak{A}_{\boldsymbol{e}_2}$		$\omega = \omega + 0$	$\omega + 1$	$\omega \cdot 2 = \omega + \omega$	$\omega^\omega = \omega + \omega^\omega$
	o	$\boldsymbol{e}_2 = \omega^{\omega^{\omega^\omega}}$	$\omega^{\omega^{\omega+1}}$	$\omega^{\omega^{\omega \cdot 2}}$	$\omega^{\omega^{\omega^\omega}}$

We shall prove inductively in a series of lemmas that the normal variables give rise to normal measures, first lifting a normal measure on $\varrho < \kappa$ to $\mathscr{C}_\kappa^\varrho$ by the operation ***wlift*** (Lemma 17) and then lifting the (semi-)normal measure on κ to a normal measure by the operation ***slift*** (Lemma 19):

LEMMA 17. Let κ be a strong partition cardinal closed under ultrapowers. If μ is a normal measure on $\varrho < \kappa$, then $\boldsymbol{wlift}_\kappa(\mu)$ is a normal measure.

PROOF. Recall that $\boldsymbol{wlift}_\kappa(\mu)$ is the measure on κ defined by: A has measure one if and only if there is a club set $C \subseteq \kappa$ such that for all $f : \varrho \to C$ of continuous type, we have that $[f]_\mu \in A$.

For any f of this type, if $\sup(f)$ is closed under ultrapowers then $[f]_\mu = \sup(f)$ by normality of μ. Also, for any club set $C \subseteq \kappa$ and any limit point α of C of cofinality ϱ, there is an $f : \varrho \to C$ of this type with $\sup(f) = \alpha$.

Thus, A has measure one if and only if there is a club set $C \subseteq \kappa$ such that all $\alpha \in C$ of cofinality ϱ are in A. It is well-known that the weak partition relation on κ implies that this describes a normal measure, i.e., $\boldsymbol{wlift}_\kappa(\mu)$ is the ϱ-cofinal normal measure on κ. $\dashv$

LEMMA 18. Let ν be any semi-normal measure on the strong partition cardinal κ (*i.e.*, one that contains all club subsets of κ). If $f, g : \kappa \to \kappa$ are of the correct type with $[f]_\nu < [g]_\nu$, then there are f', g' of the correct type with $[f']_\nu = [f]_\nu$, $[g']_\nu = [g]_\nu$, and $f'(\alpha) < g'(\alpha) < f'(\alpha + 1)$ for all $\alpha < \kappa$. Furthermore, $\mathrm{ran}(f') \subseteq \mathrm{ran}(f)$ and $\mathrm{ran}(g') \subseteq \mathrm{ran}(g)$.

PROOF. Define f', g' recursively by letting $f'(\alpha)$ be the least element in the range of f greater than $\sup_{\beta<\alpha} g'(\beta)$. Let $g'(\alpha)$ be the least element in the range of g which is greater than $f'(\alpha)$. Clearly there is a club set $C \subseteq \kappa$ (the points closed under g') on which $f' = f$. Since ν is semi-normal, $[f']_\nu = [f]_\nu$. If A is the ν measure one set on which $g(\alpha) > f(\alpha)$, then for $\alpha \in C \cap A$ we have $g'(\alpha) = g(\alpha)$. Thus, $[g]_\nu = [g']_\nu$ by semi-normality. $\dashv$

LEMMA 19. Let ν be any semi-normal measure on the strong partition cardinal κ. Then $\mu := \boldsymbol{slift}(\nu)$ is a normal measure.

PROOF. Let $\vartheta = \kappa^\kappa/\nu$, so μ is a measure on ϑ. Fix $F : \vartheta \to \vartheta$ which is pressing down.

Consider first the partition $\mathscr{P}_1$ where we partition pairs of functions $\langle f, g\rangle$ where $f, g : \kappa \to \kappa$ are of the correct type and $f(\alpha) < g(\alpha) < f(\alpha + 1)$ for all $\alpha < \kappa$ according to whether $[f]_\nu > F([g]_\nu)$. We claim that on the homogeneous side the stated property holds. Towards a contradiction, suppose C is club and homogeneous for the contrary side.

Fix $g \colon \kappa \to C'$ of the correct type, where C' is the set of closure points of C (*i.e.*, the $\alpha \in C$ such that α is the αth element of C). Since $F([g]_\nu) < [g]_\nu$, we may get $f : \kappa \to C$ with $F([g]_\nu) < [f]_\nu < [g]_\nu$. Let f', g' be obtained from Lemma 18. Then f', g' are of the correct type, ordered as in $\mathscr{P}_1$, and have range in C, but $[f']_\nu = [f]_\nu > F([g]_\nu) = F([g']_\nu)$, a contradiction to the definition of C. Let now C_1 be club and homogeneous for the stated side of the partition. Fix $f : \kappa \to C_1$ of the correct type and let $\delta = [f]_\nu$.

Then for any $g \colon \kappa \to C_1$ of the correct type with $[g]_\nu > \delta$ we have $F([g]_\nu) < \delta$. This follows from the definition of C_1 and Lemma 18. This shows μ is weakly normal, i.e., any pressing down function is bounded on a measure one set.

Consider next the partition $\mathscr{P}_2$ where we partition pairs $\langle f, g\rangle$ of the same type as in $\mathscr{P}_1$ but now partitioned according to whether $F([f]_\nu) \leq F([g]_\nu)$. We claim that on the homogeneous side the stated property holds. Suppose not and let C be homogeneous for the contrary side.

We can easily construct functions $f_i \colon \kappa \to C$ of the correct type such that $f_i(\alpha) < f_{i+1}(\alpha)$ and $f_i(\alpha) < f_0(\alpha+1)$ for all $i \in \omega$ and $\alpha < \kappa$. But then $F([f_0]_\nu) > F([f_1]_\nu) > \dots$, a contradiction. Fix a club set $C_2 \subseteq \kappa$ homogeneous for the stated side of $\mathscr{P}_2$.

Consider a third partition $\mathscr{P}_3$ where we partition pairs $\langle f, g\rangle$ of the same type again according to whether $F([f]_\nu) = F([g]_\nu)$. If there is a club set $C \subseteq \kappa$ homogeneous for the stated side of $\mathscr{P}_3$, then we are done since Lemma 18 implies that for any $f, g \colon \kappa \to C$ of the correct type we have $F([f]_\nu) = F([g]_\nu)$.

Suppose C_3 is homogeneous for the contrary side of $\mathscr{P}_3$. Let $C = C_1 \cap C_2 \cap C_3$. Fix $f : \kappa \to \kappa$ with $[f]_\nu > \delta$. Let $h \colon \{(\alpha, \beta) \colon \alpha < f(\beta)\} \to C$ be of uniform cofinality ω, discontinuous, and order-preserving with respect to reverse lexicographic ordering. Define a map $\pi \colon [f]_\nu \to \delta$ as follows. Let $\gamma = [g]_\nu < [f]_\nu$. Let $\pi(\gamma) = [g']_\nu$, where $g'(\beta) = h(g(\beta), \beta)$ if $g(\beta) < f(\beta)$, and $= h(0, \beta)$ otherwise. It is now easy to check that π is a well-defined, order-preserving map from $[f]_\nu$ into δ, a contradiction since $[f]_\nu > \delta$. ⊣

Theorem 20. The measure assignment from § 5 assigns normal measures to all normal variables. Consequently, ${\boldsymbol{\delta}^1_{2n+1}}^{\boldsymbol{\delta}^1_{2n+1}} / \boldsymbol{wlift}_{\boldsymbol{\delta}^1_{2n+1}}(\mathsf{v})$ is a regular cardinals for all normal variables v.

Proof. The first part of the claim follows immediately by induction from Lemmas 17 and 19. The second part follows from the fact that if there is a normal measure on λ, then λ must be regular. ⊣

Assuming that the measure assignment is canonical, we can now compute these regular cardinals easily from the table given above by looking at the row containing the o-values.

In the table in Figure 4, we list the first 32 of such cardinals (up to $\boldsymbol{\delta}^1_9$). Note that at this point we have not yet proved that these are all the regular cardinals. This will follow from the next algorithm.

6.2. Computation of the Cofinalities. In § 6.1, we singled out special variables in our algebra and proved in Theorem 20 that each of these gives rise to a regular ultrapower. In this section, we shall now reduce the computation of all cofinalities to the cofinalities associated to the normal variables. This argument will be done inductively based on the following elementary yet powerful result:

$\aleph_0$	$\boldsymbol{\delta}^1_1 = \aleph_1$	$\boldsymbol{\delta}^1_2 = \aleph_2$	$\boldsymbol{\delta}^1_3 = \aleph_{\omega+1}$
$\boldsymbol{\delta}^1_4 = \aleph_{\omega+2}$	$\aleph_{\omega\cdot 2+1}$	$\aleph_{\omega^\omega+1}$	$\boldsymbol{\delta}^1_5 = \aleph_{\omega^{\omega^\omega}+1}$
$\boldsymbol{\delta}^1_6 = \aleph_{\omega^{\omega^\omega}+2}$	$\aleph_{\omega^{\omega^\omega}+\omega+1}$	$\aleph_{\omega^{\omega^\omega}+\omega^\omega+1}$	$\aleph_{\omega^{\omega^\omega}\cdot 2+1}$
$\aleph_{\omega^{\omega^{\omega+1}}+1}$	$\aleph_{\omega^{\omega^{\omega}\cdot 2}+1}$	$\aleph_{\omega^{\omega^{\omega^\omega}}+1}$	$\boldsymbol{\delta}^1_7 = \aleph_{\omega^{\omega^{\omega^{\omega^\omega}}}+1}$
$\boldsymbol{\delta}^1_8 = \aleph_{\omega^{\omega^{\omega^{\omega^\omega}}}+2}$	$\aleph_{\omega^{\omega^{\omega^{\omega^\omega}}}+\omega+1}$	$\aleph_{\omega^{\omega^{\omega^{\omega^\omega}}}+\omega^\omega+1}$	$\aleph_{\omega^{\omega^{\omega^{\omega^\omega}}}+\omega^{\omega^\omega}+1}$
$\aleph_{\omega^{\omega^{\omega^{\omega^\omega}}}+\omega^{\omega^{\omega+1}}+1}$	$\aleph_{\omega^{\omega^{\omega^{\omega^\omega}}}+\omega^{\omega^{\omega}\cdot 2}+1}$	$\aleph_{\omega^{\omega^{\omega^{\omega^\omega}}}+\omega^{\omega^{\omega^\omega}}+1}$	$\aleph_{\omega^{\omega^{\omega^{\omega^\omega}}}\cdot 2+1}$
$\aleph_{\omega^{\omega^{(\omega^{\omega^\omega}+1)}}+1}$	$\aleph_{\omega^{\omega^{(\omega^{\omega^\omega}+\omega)}}+1}$	$\aleph_{\omega^{\omega^{(\omega^{\omega^\omega}+\omega^\omega)}}+1}$	$\aleph_{\omega^{\omega^{(\omega^{\omega^\omega}\cdot 2)}}+1}$
$\aleph_{\omega^{\omega^{(\omega^{\omega^{\omega+1}})}}+1}$	$\aleph_{\omega^{\omega^{(\omega^{\omega^{\omega}\cdot 2})}}+1}$	$\aleph_{\omega^{\omega^{\omega^{\omega^{\omega^\omega}}}}+1}$	$\boldsymbol{\delta}^1_9 = \aleph_{\omega^{\omega^{\omega^{\omega^{\omega^{\omega^\omega}}}}}+1}$

FIGURE 4. The first 32 regular cardinals.

LEMMA 21. Let μ be a measure on ϱ with $\mathrm{cf}(\varrho) = \delta$. Let κ be a weak partition cardinal closed under ultrapowers such that $\varrho < \kappa$. Then there is a cofinal embedding from $\kappa^\kappa/\mathscr{C}^\delta_\kappa$ into $\kappa^\kappa/\boldsymbol{wlift}_\kappa(\mu)$.

PROOF. For $F\colon \kappa \to \kappa$, let $\pi(F) = G$ be defined as follows: for $\alpha < \kappa$ represented by $g\colon \varrho \to \kappa$ of continuous type, let $G([g]_\mu) = F(\sup(g))$. This is well-defined since if $[g_1]_\mu = [g_2]_\mu$ and g_1, g_2 are both increasing, then $\sup(g_1) = \sup(g_2)$ (since any μ measure one set is cofinal in ϱ).

Suppose $[F_1]_{\mathscr{C}^\delta_\kappa} = [F_2]_{\mathscr{C}^\delta_\kappa}$. Let $C \subseteq \kappa$ be a club set such that for all $\alpha \in C$ of cofinality δ we have $F_1(\alpha) = F_2(\alpha)$. Then for any $\beta < \kappa$ represented by a $g\colon \varrho \to C$ of continuous type we have $G_1(\beta) = F_1(\sup(g)) = F_2(\sup(g)) = G_2(\beta)$ whence $[G_1]_{\boldsymbol{wlift}_\kappa(\mu)} = [G_2]_{\boldsymbol{wlift}_\kappa(\mu)}$. Thus, π gives a well-defined map from $\kappa^\kappa/\mathscr{C}^\delta_\kappa$ into $\kappa^\kappa/\boldsymbol{wlift}_\kappa(\mu)$.

To see this is cofinal, let $G\colon \kappa \to \kappa$. For $\alpha < \kappa$ of cofinality δ, define $F(\alpha) = \sup\{G([g]_\mu)\,;\, \sup(g) = \alpha\}$, where the supremum ranges over g of continuous type. This is well-defined as κ is regular and closed under ultrapowers.

Then $\pi([F]_{\mathscr{C}^\delta_\kappa}) > [G]_{\boldsymbol{wlift}_\kappa(\mu)}$ since for all $g\colon \varrho \to \kappa$ of continuous type we have $\pi(F)([g]_\mu) = F(\sup(g)) = \sup\{G([g']_\mu)\colon\ \sup(g') = \sup(g)\} \geq G([g]_\mu)$. ⊣

As an immediate consequence, we can reduce the computation of the cofinality of $\kappa^\kappa/\boldsymbol{wlift}_\kappa(x)$ for an arbitrary term $x \in \mathfrak{A}_{\varepsilon_0}$ to the cofinalities of the basic variables:

COROLLARY 22. Let $x \in \mathfrak{A}_{\varepsilon_0}$ be a term with trailing node v such that $\ell_x(v) = \mathsf{v}$. If $o(x) < \boldsymbol{e}_{n+1}$, write $\kappa := \boldsymbol{\delta}^1_{2n+1}$ and $\lambda := \kappa^\kappa/\boldsymbol{wlift}_\kappa(\mathsf{v})$. Then

$$\mathrm{cf}(\kappa^\kappa/\boldsymbol{wlift}_{\boldsymbol{\delta}^1_{2n+1}}(x)) = \mathrm{cf}(\lambda).$$

PROOF. This is immediate from Lemma 21 keeping in mind that $\mathrm{cf}(\boldsymbol{ot}(x)) = \mathrm{cf}(\mathrm{ot}(v))$. ⊣

We shall now recursively define a function $\mathsf{nor}\colon \mathfrak{V} \to \mathfrak{V}$ assigning normal variables to arbitrary generators in the algebra. Our recursion will go along the tower of algebras $\mathfrak{A}_{2+\boldsymbol{e}_n}$ as the definition of the measure assignment in § 5.

In $\mathfrak{A}_2$, all basic variables are normal, so the function nor can just be the identity. Suppose that we have defined the function nor on $\mathfrak{A}_{2+\boldsymbol{e}_n}$ and want to extend it to $\mathfrak{A}_{\boldsymbol{e}_{n+1}}$. Each of the generators V_α of $\mathfrak{A}_{\boldsymbol{e}_{n+1}}$ was either already in $\mathfrak{A}_{2+\boldsymbol{e}_n}$ or is of the form $\mathsf{V}_{2+\boldsymbol{e}_n+\xi}$ for some $\xi < \mathrm{ht}(\mathfrak{A}_{2+\boldsymbol{e}_n})$. By the recursive measure assignment from § 5, this variable $\mathsf{V}_\alpha = \mathsf{V}_{2+\boldsymbol{e}_n+\xi}$ is linked to terms $x \in \mathfrak{A}_{2+\boldsymbol{e}_n}$ such that $o(x) = \xi$. Let x be such a term with representing tree $\langle T_x, \ell_x\rangle$ and trailing node $v \in T_x$. Then $\ell_x(v)$ is a generator of $\mathfrak{A}_{2+\boldsymbol{e}_n}$.

We can now define

$$\mathsf{nor}(\mathsf{V}_\alpha) := \mathsf{V}_{2+\boldsymbol{e}_n+o(\mathsf{nor}(\ell_x(v)))}.$$

THEOREM 23. For each generator v of $\mathfrak{A}_{\varepsilon_0}$ and every odd projective ordinal $\kappa = \boldsymbol{\delta}^1_{2n+1}$ such that $\boldsymbol{ot}(\mathsf{v}) < \kappa$, we have that

$$\mathrm{cf}(\kappa^\kappa/\boldsymbol{wlift}_\kappa(\mathsf{v})) = \mathrm{cf}(\kappa^\kappa/\boldsymbol{wlift}_\kappa(\mathsf{nor}(\mathsf{v}))).$$

PROOF. The claim is proved by induction on n. Recall that the generators v with $\boldsymbol{ot}(\mathsf{v}) < \boldsymbol{\delta}^1_{2n+1}$ are precisely those in $\mathfrak{A}_{2+\boldsymbol{e}_n}$. The case $n = 0$ is trivial as nor is the identity on the generators in $\mathfrak{A}_2$ (*i.e.*, $\mathsf{V}_0, \mathsf{V}_1$). Assume the theorem holds for n, i.e., for $\boldsymbol{\delta}^1_{2n+1}$ and $\mathfrak{A}_{2+\boldsymbol{e}_n}$, and we show it holds for $n+1$, i.e., for $\boldsymbol{\delta}^1_{2n+3}$ and $\mathfrak{A}_{\boldsymbol{e}_{n+1}}$.

Let v be a generator in $\mathfrak{A}_{\boldsymbol{e}_{n+1}}$, so $\mathsf{v} = \mathsf{V}_{2+\boldsymbol{e}_n+\xi}$ for some $\xi < \boldsymbol{e}_{n+1} = \mathrm{ht}(\mathfrak{A}_{2+\boldsymbol{e}_n})$. Fix $x \in \mathfrak{A}_{2+\boldsymbol{e}_n}$ such that $o(x) = \xi$, let v be the trailing term of $\langle T_x, \ell_x\rangle$, and $\mathsf{v}^* := \ell_x(v)$. By definition of nor, we have $\mathsf{nor}(\mathsf{v}) = \mathsf{V}_{2+\boldsymbol{e}_n+o(\mathsf{nor}(\mathsf{v}^*))}$. Let $\lambda := \boldsymbol{\delta}^1_{2n+1}$. By Corollary 22 and the induction hypothesis, we have that

$$\mathrm{cf}(\lambda^\lambda/\boldsymbol{wlift}_\lambda(x)) = \mathrm{cf}(\lambda^\lambda/\boldsymbol{wlift}_\lambda(\mathsf{v}^*)).$$

But $\lambda^\lambda/\boldsymbol{wlift}_\lambda(x) = \boldsymbol{ot}(\mathsf{v})$ and $\lambda^\lambda/\boldsymbol{wlift}_\lambda(\mathsf{v}^*) = \boldsymbol{ot}(\mathsf{nor}(\mathsf{v}))$. Now we can apply Lemma 21 (with the κ there being $\boldsymbol{\delta}^1_{2n+3}$) to finish the claim. ⊣

Using Corollary 22 and Theorem 23, we can now describe the algorithm to compute the value of $\mathrm{cof}_\kappa(x) := \mathrm{cf}(\kappa^\kappa/\boldsymbol{wlift}_\kappa(x))$ recursively for arbitrary x. Suppose that we have already computed $\mathrm{cof}_\kappa \restriction \mathfrak{A}_{2+\boldsymbol{e}_n}$ for all odd projective ordinals $\kappa \geq \boldsymbol{\delta}^1_{2n+1}$. We shall give an algorithm to compute $\mathrm{cof}_\kappa \restriction \mathfrak{A}_{\boldsymbol{e}_{n+1}}$ for all $\kappa \geq \boldsymbol{\delta}^1_{2n+3}$.

ALGORITHM 24. Given a term $x \in \mathfrak{A}_{\boldsymbol{e}_{n+1}}$, ask whether $x \in \mathfrak{A}_{2+\boldsymbol{e}_n}$ or not.

Case 1. $x \in \mathfrak{A}_{2+\boldsymbol{e}_n}$: Then $\mathrm{cof}_\kappa(x)$ has already been determined.

Case 2. $x \notin \mathfrak{A}_{2+\boldsymbol{e}_n}$:
- Find the trailing node v of $\langle T_x, \ell_x\rangle$.
- Set $\mathsf{v} := \ell_x(v)$.
- Compute $\mathsf{nor}(\mathsf{v})$.
- Then $\mathrm{cof}_\kappa(x) = \kappa^\kappa/\mathsf{nor}(\mathsf{v})$.

COROLLARY 25. Algorithm 24 correctly computes the cofinality of $\kappa^\kappa/\boldsymbol{wlift}_\kappa(x)$.

PROOF. Obvious from Corollary 22 and Theorem 23. ⊣

Let us apply the algorithm to the examples of non-normal variables given in Figure 5: V_4, $\mathsf{V}_{\omega+2}$, V_{ω^2} and $\mathsf{V}_{\omega^{\omega^2}}$.

- The variable $\mathsf{V}_4 = \mathsf{V}_{2+2}$ is highlifted from $\mathsf{V}_1 \oplus \mathsf{V}_1$ (using $\boldsymbol{\delta}^1_1$). Obviously, V_1 is the trailing term of $\mathsf{V}_1 \oplus \mathsf{V}_1$, and hence, $\mathsf{nor}(\mathsf{V}_4) := \mathsf{V}_{2+o(\mathsf{V}_1)} = \mathsf{V}_3$. Therefore,
$$\mathrm{cof}_\kappa(\mathsf{V}_4) := \kappa^\kappa/\boldsymbol{wlift}_\kappa(\mathsf{V}_3).$$
- The variable $\mathsf{V}_{\omega+2}$ is highlifted from $\mathsf{V}_1 \oplus \mathsf{V}_1$ (to $\boldsymbol{\delta}^1_3$). By the same argument, $\mathsf{nor}(\mathsf{V}_{\omega+2}) = \mathsf{V}_{\omega+1}$. Thus,
$$\mathrm{cof}_\kappa(\mathsf{V}_{\omega+2}) := \kappa^\kappa/\boldsymbol{wlift}_\kappa(\mathsf{V}_{\omega+1}).$$

- The variable V_{ω^2} is highlifted from $\mathsf{V}_2 \otimes \mathsf{V}_2$ (to $\boldsymbol{\delta}^1_3$) whose trailing term is V_2. Therefore, $\mathsf{nor}(\mathsf{V}_{\omega^2})$ is the high lift of V_2 which is $\mathsf{V}_{\omega\cdot 2}$. Thus,

$$\mathrm{cof}_\kappa(\mathsf{V}_{\omega^2}) := \kappa^\kappa / \boldsymbol{wlift}_\kappa(\mathsf{V}_{\omega\cdot 2}).$$

- Finally, the variable $\mathsf{V}_{\omega^{\omega^2}}$ is highlifted from V_4. We already computed $\mathsf{nor}(\mathsf{V}_4)$ earlier to be V_3, so $\mathsf{nor}(\mathsf{V}_{\omega^{\omega^2}})$ is the high lift of V_3 which is $\mathsf{V}_{\omega^\omega}$, and hence

$$\mathrm{cof}_\kappa(\mathsf{V}_{\omega^{\omega^2}}) := \kappa^\kappa / \boldsymbol{wlift}_\kappa(\mathsf{V}_{\omega^\omega}).$$

Corollary 25 and the canonicity assumption give an algorithm for computing the cofinality of any successor cardinal $\aleph_{\alpha+1}$ for $\alpha < \varepsilon_0$. Namely, first find the n such that $\boldsymbol{e}_n \leq \alpha < \boldsymbol{e}_{n+1}$. Let α' be such that $\alpha = \boldsymbol{e}_n + \alpha'$. Let $x \in \mathfrak{A}_{\boldsymbol{e}_n}$ be a term with $o(x) = \alpha'$. Apply Algorithm 24 to x to get the variable v (the normalization of the trailing term of x) and calculate the cofinality as $\mathrm{cf}(\aleph_{\alpha+1}) = (\boldsymbol{\delta}^1_{2n+1})^{\boldsymbol{\delta}^1_{2n+1}} / \boldsymbol{wlift}_{\boldsymbol{\delta}^1_{2n+1}}(\mathsf{v}) = \aleph_{\boldsymbol{e}_n + o(\mathsf{v})+1}$.

Corollary 26. If $\boldsymbol{\delta}^1_{2n+1} < \aleph_{\alpha+1} < \aleph_{\varepsilon_0}$, then $\mathrm{cf}(\aleph_{\alpha+1}) > \boldsymbol{\delta}^1_{2n+1}$.

Proof. Follows immediately from the algorithm. The result was proved using completely different methods (generic codes) by Kechris and Woodin. ⊣

To illustrate the algorithm, let us compute the cofinality of $\kappa = \aleph_{\alpha+1}$ where $\alpha = \omega^{\omega^{(\omega^{\omega^2}+\omega^{\omega\cdot 2+3})}}$ (so $\boldsymbol{\delta}^1_5 < \kappa < \boldsymbol{\delta}^1_7$). Clearly, $\boldsymbol{e}_2 < \alpha < \boldsymbol{e}_3$, and $\alpha' = \alpha$ in the notation of the previous paragraph. The term $x \in \mathfrak{A}_{\boldsymbol{e}_2}$ with $o(x) = \alpha'$ is the generator V_β, with $\beta = \omega^{\omega^2} + \omega^{\omega\cdot 2+3}$. We next compute $\mathsf{nor}(\mathsf{V}_\beta)$. The variable V_β corresponds to the high lift of the term $y = \mathsf{V}_4 \oplus (\mathsf{V}_3 \otimes \mathsf{V}_3 \otimes \mathsf{V}_2 \otimes \mathsf{V}_2 \otimes \mathsf{V}_2) \in \mathfrak{A}_{\boldsymbol{e}_1}$. The trailing variable is V_2, which is normal. Thus, $\mathsf{nor}(\mathsf{V}_\beta) = \boldsymbol{highlift}(\mathsf{V}_2) = \mathsf{V}_{\omega\cdot 2}$. So, $\mathrm{cf}(\kappa) = {\boldsymbol{\delta}^1_5}^{\boldsymbol{\delta}^1_5} / \boldsymbol{wlift}_{\boldsymbol{\delta}^1_5}(\mathsf{V}_{\omega\cdot 2}) = \aleph_{\omega^{\omega^\omega}+o(\mathsf{V}_{\omega\cdot 2})+1}$, whence

$$\mathrm{cf}\left(\aleph_{\omega^{\omega^{(\omega^{\omega^2}+\omega^{\omega\cdot 2+3})}}+1}\right) = \aleph_{\omega^{\omega^{\omega\cdot 2}}+1}.$$

6.3. Computation of the Kleinberg sequences. Under the canonicity assumption, the Kleinberg sequences can now be easily read off.

Lemma 27. If $\boldsymbol{wlift}_\kappa(\mathsf{v}) = \mathscr{C}^\lambda_\kappa$ is a normal ultrafilter on $\kappa := \boldsymbol{\delta}^1_{2n+1}$, then the Kleinberg sequence on κ derived from $\mathscr{C}^\lambda_\kappa$ is given by

$$\kappa_n^{\mathscr{C}^\lambda_\kappa} := \kappa^\kappa / \boldsymbol{wlift}_\kappa(\mathsf{v} \otimes n).$$

Proof. Taking iterated ultrapowers as in the definition of the Kleinberg sequence corresponds to taking iterated sums: it is true in general that the n-fold sum ultrapower embeds into the n-fold iterated ultrapower (a proof can be found as [1, Proposition 13]). The Ultrapower Shifting Lemma [10, Lemma 2.7] gives us an upper bound for the iterated ultrapower under assumptions on the cofinality function that follow from Corollary 26. ⊣

By Corollary 25, we know that the normal measures are all generated by normal variables, and by § 6.1, we have a simple algorithm to compute the o-values of the normal variables. Therefore, we can read off the values of the Kleinberg sequences by considering the sequence $o(\mathsf{v}) \cdot n$ for a normal variable v. As an example, we can read off the Kleinberg sequences on $\boldsymbol{\delta}^1_5$ as follows (for $n \geq 1$): $\aleph_{\omega^{\omega^\omega}+n+1}$, $\aleph_{\omega^{\omega^\omega}+\omega\cdot n+1}$, $\aleph_{\omega^{\omega^\omega}+\omega^\omega\cdot n+1}$, $\aleph_{\omega^{\omega^\omega}\cdot n+1}$, $\aleph_{\omega^{\omega^{\omega+1}}\cdot n+1}$, $\aleph_{\omega^{\omega^{\omega\cdot 2}}\cdot n+1}$, and $\aleph_{\omega^{\omega^{\omega^\omega}}\cdot n+1}$.

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Appendix A. In this appendix, we give a table (Figure 5) of all of the relevant values for our measure assignment described in § 5. We give the terms and their values of o, $\boldsymbol{ot}$ and $\boldsymbol{M}$. In the next row, we list the value of $\boldsymbol{wlift}_\kappa(x)$ for some $\kappa = \boldsymbol{\delta}^1_{2n+1}$ where $o(x) < \boldsymbol{e}_{n+1}$ (inductively, the order types of terms x with $o(x) < \boldsymbol{e}_{n+1}$ will be $< \boldsymbol{\delta}^1_{2n+1}$, so we can lift the values of the measure assignment on these terms to $\boldsymbol{\delta}^1_{2n+1}$). These first four rows of values can be computed independently of the canonicity conjecture. The following row gives the value of $\kappa^\kappa/\boldsymbol{wlift}_\kappa(x)$ for $\kappa = \boldsymbol{\delta}^1_{2n+1}$ where $o(x) < \boldsymbol{e}_{n+1}$ under the assumption of the canonicity conjecture. The last row lists whether the term is a normal variable or not. For the non-normal measures, we use the S-notation for the families of measures introduced in [3, p. 119] (the $\tilde{\mathrm{S}}^{\ell,m}_n$ measure are defined as the $\mathrm{S}^{\ell,m}_n$ measures of [3] except we use functions of continuous type instead of correct type).

The table comes in two systems: the first system lists terms from $\mathfrak{A}_\omega$, the second system lists terms from $\mathfrak{A}_{e_2}$. The two systems are linked by the operation of high lift: the columns in the second system correspond to those variables whose values for $\boldsymbol{M}$ and $\boldsymbol{ot}$ are the high lifts of the terms in $\mathfrak{A}_\omega$ in the same column of the first system.

It is clear from the construction that all terms come with information about their stage of construction. In addition, there is some descriptive set theoretic information hidden in the recursive construction that we should like to point the reader's attention to. In constructing the measure assignment of § 5, we assign the

	V_0		V_1		$\mathsf{V}_1 \oplus \mathsf{V}_1$	...	V_2	...	$\mathsf{V}_2 \otimes \mathsf{V}_2$	...	V_3	...	V_4	...
o	0		1		2	...	ω	...	ω^2	...	ω^ω	...	ω^{ω^2}	...
$\boldsymbol{ot}$	0		1		2	...	ω_1	...	${\omega_1}^2$	...	ω_2	...	ω_3	...
$\boldsymbol{M}$	$\varnothing$		p		$\bullet$ / \ p p	...	$\mathscr{C}^{\omega}_{\omega_1}$	...	$\bullet$ | $\mathscr{C}^{\omega}_{\omega_1}$ | $\mathscr{C}^{\omega}_{\omega_1}$	...	$\mathscr{C}^{\omega}_{\omega_2}$	...	S^2_1	...
$\boldsymbol{wlift}_\kappa$	p		$\mathscr{C}^{\omega}_{\kappa}$		$\mathscr{C}^{\omega}_{\kappa} \times \mathscr{C}^{\omega}_{\kappa}$	...	$\mathscr{C}^{\omega_1}_{\kappa}$	...	$\boldsymbol{wlift}_\kappa(\mathscr{C}^{\omega}_{\omega_1} \times \mathscr{C}^{\omega}_{\omega_1})$	...	$\mathscr{C}^{\omega_2}_{\kappa}$	...	$\boldsymbol{wlift}_\kappa(\mathrm{S}^2_1)$	...
ultrapower	$\aleph_{e_n+1}$		$\aleph_{e_n+2}$		$\aleph_{e_n+3}$	...	$\aleph_{e_n+\omega+1}$	...	$\aleph_{e_n+\omega^2+1}$	...	$\aleph_{e_n+\omega^\omega+1}$	...	$\aleph_{e_n+\omega^{\omega^2}+1}$	...
normal?	NORMAL		NORMAL		—	...	NORMAL	...	—	...	NORMAL	...	—	...

	V_ω	...	$\mathsf{V}_{\omega+1}$	...	$\mathsf{V}_{\omega+2}$	...	$\mathsf{V}_{\omega\cdot 2}$	...	V_{ω^2}	...	$\mathsf{V}_{\omega^\omega}$	...	$\mathsf{V}_{\omega^{\omega^2}}$	...
o	$\boldsymbol{e}_2 = \omega^{\omega^\omega}$	...	$\omega^{\omega^{\omega+1}}$	...	$\omega^{\omega^{\omega+2}}$	...	$\omega^{\omega^{\omega\cdot 2}}$	...	$\omega^{\omega^{\omega^2}}$	...	$\omega^{\omega^{\omega^\omega}}$	...	$\omega^{\omega^{\omega^{\omega^2}}}$	...
$\boldsymbol{ot}$	$\boldsymbol{\delta}^1_3$	...	$\boldsymbol{\delta}^1_4$	...	$\aleph_{\omega+3}$	...	$\aleph_{\omega\cdot 2+1}$	...	$\aleph_{\omega^2+1}$	...	$\aleph_{\omega^\omega+1}$	...	$\aleph_{\omega^{\omega^2}+1}$	...
$\boldsymbol{M}$	$\mathscr{C}^{\omega}_{\boldsymbol{\delta}^1_3}$	...	$\mathscr{C}^{\omega}_{\boldsymbol{\delta}^1_4}$	...	$\mathrm{S}^{1,2}_3$	...	$\mathscr{C}^{\omega}_{\aleph_{\omega\cdot 2+1}}$	...	$\tilde{\mathrm{S}}^{2,2}_3$	...	$\mathscr{C}^{\omega}_{\aleph_{\omega^\omega+1}}$	...	$\tilde{\mathrm{S}}^{3,2}_3$	...
$\boldsymbol{wlift}_\kappa$	$\mathscr{C}^{\boldsymbol{\delta}^1_3}_{\kappa}$	...	$\mathscr{C}^{\boldsymbol{\delta}^1_4}_{\kappa}$	...	$\boldsymbol{wlift}_\kappa(\mathrm{S}^{1,2}_3)$	...	$\mathscr{C}^{\aleph_{\omega\cdot 2+1}}_{\kappa}$	...	$\boldsymbol{wlift}_\kappa(\tilde{\mathrm{S}}^{2,2}_3)$	...	$\mathscr{C}^{\aleph_{\omega^\omega+1}}_{\kappa}$	...	$\boldsymbol{wlift}_\kappa(\tilde{\mathrm{S}}^{3,2}_3)$	...
ultrapower	$\aleph_{e_n+e_2+1}$	...	$\aleph_{e_n+\omega^{\omega^{\omega+1}}+1}$	...	$\aleph_{e_n+\omega^{\omega^{\omega+2}}+1}$	...	$\aleph_{e_n+\omega^{\omega^{\omega\cdot 2}}+1}$	...	$\aleph_{e_n+\omega^{\omega^{\omega^2}}+1}$	...	$\aleph_{e_n+\omega^{\omega^{\omega^\omega}}+1}$	...	$\aleph_{e_n+\omega^{\omega^{\omega^{\omega^2}}}+1}$	...
normal?	NORMAL		NORMAL		—	...	NORMAL	...	—	...	NORMAL	...	—	...

FIGURE 5. Table of the values of o, $\boldsymbol{ot}$, $\boldsymbol{M}$, $\boldsymbol{wlift}_\kappa$, the ultrapowers and normality for terms in $\mathfrak{A}_{\boldsymbol{e}_2}$. Here p denotes a principal ultrafilter.

measures to the new variables in $\mathfrak{A}_{\boldsymbol{e}_{n+1}}$ based on the measure assignment on terms in $\mathfrak{A}_{2+\boldsymbol{e}_n}$. One of these is slightly special: the measure assigned to $\mathsf{V}_{2+\boldsymbol{e}_n}$ itself comes from the special variable V_0 and thus is not really high lifted, but rather lifted only once. We shall say that this variable is of level $2n+1$. All of the other newly created variables are of level $2n+2$. This defines a notion of level for all generators of $\mathfrak{A}_{\varepsilon_0}$ except for V_0 and V_1. For example V_2 is of level 1, V_α, $2 < \alpha < \omega$ are of level 2, V_ω is of level 3, V_α, $\omega = \boldsymbol{e}_1 < \alpha < \boldsymbol{e}_2 = \omega^{\omega^\omega}$ are of level 4, $\mathsf{V}_{\omega^{\omega^\omega}}$ is of level 5, etc. This notion of level is connected to descriptive set theory in the following sense: the measures associated to variables of level n are typical measures occurring in the homogeneous tree construction for the complete $\boldsymbol{\Pi}^1_n$ set.

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