Bounded rationality and learning in market competition

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This thesis promotes the use of bounded rationality in economic models. The assumption of perfect rationality often imposes high informational and computational burden on economic agents and predictions based on this assumption are not in line with observed behavior in some cases. Models of bounded rationality may better explain actual behavior in such situations.

In the thesis we consider market models where firms are boundedly rational: they do not know the demand for their product and they use different learning methods to determine the optimal price, they have incorrect beliefs about their competitors’ behavior or they do not make use of all the available information. We investigate how bounded rationality affects the market outcome and what the possible welfare effects are.

Dávid Kopányi (1985) holds an MSc degree in Economics from the Corvinus University of Budapest and an MPhil degree in Economics from the Tinbergen Institute. After graduating he joined the Center for Nonlinear Dynamics in Economics and Finance at the University of Amsterdam as a PhD student. Currently he is a Research Fellow at the University of Nottingham. His main interests include bounded rationality, learning methods, industrial organization, dynamical systems and agent-based modeling.
Bounded Rationality and Learning in Market Competition
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Bounded Rationality and Learning in Market Competition

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Now that I have finished writing my dissertation and my journey in Amsterdam has come to an end, this is a good moment to take a look back at the last 5 years. Actually, I should go back in time even further. In 2007 I spent a semester at the University of Amsterdam as an exchange student. One of the courses I followed, Non-linear Economic Dynamics, has particularly arisen my interest. Not only the topic was extremely interesting, the lecturer, Mikhail Anufriev, made the material and the lectures even more enjoyable. Some of the MPhil students of the Tinbergen Institute also followed this course and this is how I got to know about the TI. Pretty much these experiences attracted me towards TI and my research group CeNDEF.

After finishing our Master’s program at the Corvinus University of Budapest, my girlfriend\textsuperscript{1} Anita and I applied for the MPhil program of the TI. Both of us got accepted and we moved to Amsterdam in 2009. The two years of the program (especially the first one) was very demanding but it was also fun thanks to the fellow students we got to know, especially Boris, Łukasz, Mark, Matze, Nadine and Tomasz. I would also like to thank Adriaan, Arianne, Ester and Judith for the help and support we received from TI.

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\textsuperscript{1}Her status was updated to wife in 2010.
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\textsuperscript{2}Although it is not correct technically, I consider him as a Hungarian friend.
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Chapter 1

Introduction

The main purpose of economic models is to provide a framework in which the effect of different factors on the economic outcome can be analyzed. Will competition lead to an efficient allocation? Should the government intervene in the market in order to protect consumers? Which policy instruments should be used to reach the desired outcome? In order to get adequate answers to such questions, it is important to understand how actual markets work and how to model markets properly.

Mainstream economic models are built upon the assumption of rationality. This means that agents perfectly know the environment in which they interact with each other (or they have correct beliefs about it), they have correct beliefs about the actions of other agents and they have the cognitive and computational abilities to determine the optimal action given their preferences and beliefs.¹ Under these assumptions, agents do not make systematic errors and the rational choice is the predicted outcome. In game theory, standard equilibrium concepts such as Nash equilibrium, Bayesian Nash equilibrium and sequential equilibrium rely heavily on the rationality assumption. Players can calculate their best response function and they have correct (or consistent beliefs) about the actions of the other players. The same holds true for the field of industrial organization: firms as well as consumers are assumed to be rational.

¹Note that rationality does not impose restrictions on preferences. Thus, we do not consider behavior as irrational if agents pursue alternative objectives than simply maximizing their own payoff.
An often used argument in favor of the rationality assumption is that agents learn to behave rationally. Even if agents do not know what the optimal (or equilibrium) action is, reasonable learning methods will lead to the equilibrium, therefore the rationality assumption can be applied. However, there are many different learning methods and different methods may lead to different outcomes. Therefore the learning argument is not necessarily valid for supporting the rationality assumption. Another argument, due to Friedman (1953), is that irrational agents will be driven out of the market by rational agents. This argument sounds quite reasonable: irrational agents make bad choices, therefore they will lose their wealth. Note, however, that there are different degrees of (ir)rationality therefore comparing fully rational and fully irrational agents might not be the most relevant case. The bounded rationality approach assumes that even though agents are not perfectly rational, they make a reasonable choice subject to their abilities, information and possibilities. Friedman’s argument is not necessarily true if we compare perfectly rational and boundedly rational agents. In fact, Schaffer (1989) shows that it is typically not the profit maximizers that have the highest survival rate in a natural selection type framework that Friedman used in his argument. This is explained by spiteful behavior: by deviating from profit maximization, a firm can increase its chance of survival by choosing an action that hurts itself but hurts its competitors even more. Schipper (2009) gives another example where firms that simply imitate the behavior of the best-performing firm make a larger profit in equilibrium than the more sophisticated optimizers. Moreover, Haltiwanger and Waldman (1985) illustrate that boundedly rational agents can affect the equilibrium outcome more than proportionally to their weight in the population. This emphasizes that bounded rationality must be taken into account.\(^2\)

The assumption of rationality can be quite demanding informationally as well as computationally. In a market model, consumers should know which goods are supplied on the market, they need to know all the relevant characteristics of these goods (price and quality for example) and they must be able to choose the combination of goods that maximizes their utility.\(^3\) Simi-

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\(^2\) Conlisk (1996) overviews and criticizes other typical arguments in favor of perfect rationality.

\(^3\) Of course, there is a tradeoff between informational and computational burdens. There are models where consumers are not assumed to know all the prices or qualities of goods and they need to search to gather these pieces of information. Finding the Nash equilibrium in these models is much more demanding computationally as
larly, firms should perfectly know how the demand for their good depends on the price or production choice of their competitors and they should have correct beliefs about their competitors’ choice. These assumptions are quite strong. Agents might not know the market environment perfectly, they might not understand the market mechanism correctly and they might not have the computational abilities to determine the optimal action.

There is ample empirical evidence showing that agents’ behavior is not necessarily in line with predictions under the rationality assumption. Conlisk (1996) provides a good overview of such results. Armstrong and Huck (2010) survey further experimental evidence as well as real-world examples supporting the relevance of bounded rationality. Since models of bounded rationality may describe actual behavior better than perfect rationality in some situations, bounded rationality should be taken into account in theoretical market models. Therefore in this thesis we consider market models where firms are boundedly rational and we investigate the effects of bounded rationality on the market outcome.

**Bounded rationality**

Bounded rationality concerns modeling behavior that deviates from perfect rationality. In models under perfect rationality, agents’ decisions are derived from a maximization problem such as utility or profit maximization. In complex environments this task can become very complicated and agents might not have the cognitive or computational abilities to solve their optimization problem. In his survey paper, Ellison (2006) distinguishes three approaches to bounded rationality. The first one is a rule-of-thumb approach in which agents follow simple and reasonable rules for decision making instead of solving optimization problems. In the second approach agents explicitly take cognitive costs into account in their optimization problem, whereas the third approach incorporates behavioral biases found in the psychological and economic literatures.  

\footnote{Strictly speaking, we do not consider the second approach as a form of bounded rationality. If there are bounds on rationality then solving an even more complicated optimization problem does not seem a good modeling approach. Selten (1990) also argues that bounded rationality is not related to an optimization problem.}

consumers need to calculate the price or quality distribution in equilibrium to determine the optimal decision rule.
The use of bounded rationality was first suggested by Herbert Simon. Simon (1955, 1972) points out that agents might have limited computational abilities or limited information gathering capacities, therefore they may not be able to find the optimal choice. In this situation agents might be satisficing instead of optimizing: they use a simple procedure to find an outcome that yields a payoff that they consider satisfactory. Tversky and Kahneman (1974) provide empirical evidence that simple decision rules can describe subjects’ behavior well. Of course, there may be many different decision rules that are reasonable in a given situation and agents may prefer to use different rules. In fact, Stahl (1996) finds experimental evidence both for heterogeneity among individuals and for switching to rules that performed better in the past. Moreover, agents may update the rules that they are using.

Bounded rationality has been applied in many branches of economics. Conlisk (1996) provides a good overview about the different economic applications of bounded rationality. Ellison (2006) focuses on the use of bounded rationality in industrial organization. Even though the early models concerned bounded rationality on the firm side, most models focus on boundedly rational behavior on the consumer side while firms are assumed to be rational. Spiegler (2011) also discusses several models where consumers are boundedly rational. In contrast, Armstrong and Huck (2010) give an overview of the literature on the behavioral approach on the firms’ side. Examples include satisficing behavior, imitation and alternative objectives that deviate from profit maximization. This thesis also contributes to the literature on bounded rationality on the firms’ side. Kirman and Vriend (2000) provide a model of a fish market where both the consumers and the sellers are boundedly rational. Their model captures important characteristics observed in the fish market in Marseille: price dispersion and the loyalty of buyers to sellers.

We can conclude from the literature that bounded rationality is relevant in some situations but less so in others. Even though this conclusion is not strong on its own, it clearly shows that bounded rationality should be taken into account as it is capable of describing actual behavior. In fact, it may describe actual behavior better than a model under perfect rationality.
Learning

Besides bounded rationality, another factor that can have an important effect on the market outcome is learning. Learning can be incorporated in models in different ways. Agents can learn about the market environment, about other agents’ behavior or about the optimal action. These categories are not exclusive, all three can be incorporated in the same model. Consider for example a dynamic market model where firms do not know the demand function for their product. With a certain learning method they can learn the demand conditions from the market history (using observations about prices or production levels). Once they learned the demand conditions, they might use observations about their competitors’ action to learn (predict) their action for the next period. When firms know the demand conditions and they have a prediction about their competitors’ action, they need to determine the optimal action subject to their beliefs. If they cannot calculate the best response for some reason, then they may apply a learning method to find out what the optimal choice is.

There exists a wide variety of methods for modeling learning in the economics literature, including different belief-based models, adaptive learning and evolutionary methods. Fudenberg and Levine (1998), Evans and Honkapohja (2001) and Cressman (2003) give good overviews of these classes, respectively. Different learning methods may lead to different outcomes. This is illustrated in Offerman et al. (2002), for example: they consider two imitation-based and one belief-based learning rule that lead to different market outcomes theoretically as well as in a laboratory experiment. This shows that it is essential to explicitly model the agents’ learning behavior. Furthermore, the heterogeneity of agents should also be taken into account, as promoted by Hommes (2006, 2013) for example. Agents may prefer to use different learning methods (due to differences in computational abilities, for example) for finding out what the optimal decision is. Therefore, it is important to analyze what happens in a heterogeneous

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\(^5\)Note that the last two learning categories do not exist under perfect rationality. When each agent is rational, agents do not need to learn each other’s behavior as they know it by default. Also, agents can calculate the optimal action so there is nothing to learn about it either.

\(^6\)They are not exclusive either in the sense that the same learning method can be used to learn about different objects as well.
environment and how different learning methods affect each other.

In this thesis we focus on learning about the market environment. There are many situations where firms do not have full information about the environment in which they operate. For example, they might not know how the demand for their good depends on the price they charge, how it is affected by their competitors, who their competitors are and how they act. Learning is especially important when the market is subject to a structural change (e.g. a new product is introduced or a new firm enters the market). Learning has a natural role in these situations: firms gather the information resulting from their actions, they evaluate it and take it into account when making a decision. A natural way to learn about demand conditions is to gather market data and use it to estimate a demand function. This can be modeled with least squares learning (LSL). LSL consists of two parts: estimation and a decision rule. In a given period, firms use their past observations about prices and demands to estimate the unknown parameters of a so-called perceived demand function. Then, based on the parameter estimates, they choose the action that maximizes their perceived profit. When a new observation arrives, firms update their parameter estimates, leading to (possibly) different prices.

**Thesis outline**

In this thesis we incorporate bounded rationality and learning on the supply side of the market and we analyze how this affects the market outcome. We do not deviate from rationality on the demand side and we summarize the consumers’ behavior by a demand or inverse demand function. We focus on two aspects of rationality in the analysis. First, we relax the assumption of perfectly knowing the market environment. In Chapters 2 and 3 we consider different learning methods firms may apply and we analyze the corresponding market outcome. Second, we investigate the effects of deviating from rationality through the beliefs about the actions of other firms. We consider situations where firms perfectly know the market environment but they are uncertain about their competitors’ actions. In Chapter 4 we analyze the consequences of relaxing the consistency requirement on beliefs about competitors’ actions while in Chapter 5 we
investigate how the market outcome depends on the type of information agents receive about the other agents’ actions. We apply analytical as well as numerical methods in the thesis. We analyze the models theoretically as much as possible but we have to use computer simulations and numerical methods when formal analysis is not tractable. Moreover, we analyze behavior by means of a laboratory experiment in Chapter 5.

In Chapter 2 we focus on the interaction between different learning methods. We consider a Bertrand oligopoly with heterogeneous goods where firms do not know the demand function and they can apply two learning methods to determine their price. One of the methods is a misspecified version of least squares learning, with which firms focus on their own price effect only, thus they do not take into account all the relevant variables that affect the demand for their good. Under this learning method firms typically reach a so-called self-sustaining equilibrium. This term is introduced by Brousseau and Kirman (1992) and it denotes a situation in which firms maximize their profit subject to their beliefs about the demand conditions and these beliefs are correct at the equilibrium point but not outside the equilibrium. That is, firms do not learn the true demand function correctly. The other learning method is gradient learning: firms adjust their price in the direction that gives a higher profit. This method leads to the Nash equilibrium when it converges. In the steady states of the model where both learning rules are present in the market, least squares learners are in a self-sustaining equilibrium while each gradient learner gives the best response to the prices of other firms. When firms are allowed to switch between learning rules based upon how well the rules perform, we can observe a cyclical switching that is driven by changes in the stability of gradient learning.

In Chapter 3 we further investigate the properties of misspecified least squares learning. We consider the circular road model of Salop (1979), with three firms and two types of consumers. One consumer group faces low transportation costs while the other group has high transportation costs. Firms do not know the demand structure of the model and they apply least squares learning to learn the demand conditions. In the estimation the firms use a perceived demand function that linearly depends on the prices. In contrast to Chapter 2, firms can observe the
prices of each other and they use this information in the regression. Since the true demand function is piecewise linear, perceived demand functions are correctly specified locally but not globally: firms can learn at most one linear part correctly. We prove that the model has three kinds of equilibria. Firms may reach a self-sustaining equilibrium, the Nash equilibrium or an asymmetric learning-equilibrium in which one firm focuses only on the consumers with high transportation costs whereas the other two firms serve both consumer types. Both the Nash equilibrium and the asymmetric learning-equilibrium are locally stable, therefore the model has coexisting stable equilibria. We analyze the conditions under which the different outcomes are reached.

In Chapter 4 we investigate the effects of having weakly consistent beliefs about the competitor’s action. We consider a duopoly where firms produce a homogeneous good and they set both the price and the production level of the good simultaneously. In the standard model under perfect rationality, there exist a Nash equilibrium only in mixed strategies but not in pure strategies. In our model, firms are risk averse and they hold probabilistic conjectures about the actions of the other firm. We numerically show that our model may have an equilibrium in pure strategies. Beliefs are weakly consistent in this equilibrium: the modes of the belief distributions correspond to the actual actions of the other firm. We investigate how the degree of risk aversion and the amount of uncertainty regarding the price and production level of the other firm affect the equilibrium. Our results show that a small degree of risk aversion is welfare enhancing and that welfare is higher than in the mixed-strategy equilibrium of the standard model. The chapter illustrates that having weakly consistent beliefs about other agents’ actions might lead to a substantially different model prediction than under the assumption of consistent beliefs and that a small amount of bounded rationality may be welfare enhancing.

In Chapter 5 we investigate by means of a laboratory experiment how the type of information firms receive about past choices of their competitors affects their behavior and the market outcome. We conduct a laboratory experiment in which subjects play the role of firms in the market. We vary the amount and the type of information subjects receive about the choice
of their competitors. Subjects either receive or do not receive information about production levels. When subjects are informed about quantities, they can see either the total output or firm-specific production levels. Moreover, we introduce voluntary information sharing in one part of the treatments, where subjects can choose to give information about their past production level to their competitors. Our results show that subjects use information sharing to signal their willingness to collude. Aggregate outputs tend to be lower under individual information than under aggregate information, supporting the view of competition authorities that the publication of firm-specific information has anti-competitive effects. Under perfect rationality, the equilibrium prediction is the unique Nash equilibrium of the one-shot market game. However, we can observe substantial dispersion in the market outcomes in each treatment. Therefore this chapter provides experimental evidence that the rational prediction may not describe individual behavior well.

Finally, the main findings of the thesis are summarized in Chapter 6.
Chapter 2

Learning Cycles under Competing Learning Rules

2.1 Introduction

In this chapter we relax the assumption that firms perfectly know the environment in which they operate. We consider a Bertrand oligopoly with differentiated goods where firms do not have full information about the market; in particular they do not fully know the demand function they face. We assume they may use one of two different well known learning methods for deciding on the price of their good. We analyze the interaction between the different learning methods in a heterogeneous setting where some firms apply the first method while other firms use the second one. The relevance of this analysis is that the convergence properties of a learning method might be affected by the presence of another method. For instance, a method that leads to the Nash equilibrium in a homogeneous setting might result in a different outcome in a heterogeneous environment. Furthermore, if different methods lead to different outcomes in a homogeneous setting, then it is unclear what will happen in a heterogeneous environment.

This chapter is based on Anufriev et al. (2013b).
The first method we consider is least squares learning (LSL). With this method firms approximate their demand function with a linear function that depends only on their own price. They are assumed to be myopic profit maximizers, i.e. they maximize their one-period expected profit subject to their estimated demand function. The coefficients in the approximation are updated in every period. LSL is a natural learning method in this setup: when the relation between some variables is unknown, then it makes sense to specify a regression on the variables and to use the estimated relationship for decision making. In our model the approximation the firms apply is misspecified in two ways: not all the relevant variables, i.e. the prices of the other firms, are included in the regression and the functional form is incorrect.\(^1\)

The other learning method firms can apply is gradient learning (GL). With this method firms use information about the slope of their profit function at their current price for modifying this price. GL captures the idea that firms systematically change the price of their good in the direction in which they expect to earn a higher profit. Locally stable fixed points of GL correspond to local profit maxima, therefore it is natural to use it in the setting we consider. We analyze LSL and GL for the following reasons. Both of them are reasonable methods in the environment we consider and, as we will see, they have been applied in the literature of oligopolistic markets before, although never jointly. In the model we assume that firms do not observe either the prices set by other firms or the corresponding demands. Therefore, it is an important criterion that the learning methods should use information only about the firms’ own prices and demands. Both LSL and GL are appropriate in this sense. Moreover, they result in different market outcomes in a homogeneous setting, so it is not clear what kind of outcome will be observed when some firms apply LSL while others use GL and when firms are allowed to switch between the learning methods. One method may drive the other one out of the market when endogenous switching between the methods is introduced.

\[^1\]The assumption that firms focus only on their own price effect seems very restrictive at first glance. However, the equilibrium concept in this situation is exactly the same as for a situation where firms take into account the price effect of some but not all of the other firms. Thus, the version of LSL we consider represents a whole class of LSL in the sense that they all lead to the same \textit{kind} of outcome. The linearity assumption on the functional form is not very restrictive either as other functional forms also lead to the same kind of outcome.
We address the following questions in this chapter. How do the two learning methods affect each other in a heterogeneous environment? Do the dynamical properties of the model depend on the distribution of learning methods over firms? If the properties of the methods vary with respect to this distribution, can we observe cycles in which the majority of firms apply the same learning method and later they switch to the other one? Can one method drive out the other one? We study these questions with formal analysis and with computer simulations.

We find that the learning methods we consider lead to different market outcomes in a homogeneous setting. With LSL, firms move towards a so-called self-sustaining equilibrium, as introduced by Brousseau and Kirman (1992), in which the perceived and the actual demands coincide at the price charged by the firms. The learning method does not have a unique steady state; the initial conditions determine which point is reached in the end. In contrast, if GL converges, it leads to the unique Nash equilibrium of the market structure we consider.\(^2\) However, GL may not always converge and then we observe periodic cycles or quasi-periodic dynamics.

In the steady states of a heterogeneous setting with fixed learning rules, LS learners are in a self-sustaining equilibrium in which gradient learners give the best response to the price set by all other firms. The convergence properties of GL depend on the distribution of learning methods over firms: an increase in the number of gradient learners can have a destabilizing effect. When endogenous switching between the learning rules is introduced, then stable GL does not necessarily drive out LSL. Some LS learners may earn a higher profit than they would make as a gradient learner and then they would not switch to GL. However, LSL may drive out GL when the latter is unstable. An interesting cyclical switching can occur when the convergence properties of GL change as the distribution of learning methods over firms varies. When GL converges, gradient learners typically earn more than the average profit of LS learners. This gives an incentive for LS learners to switch to GL. An increase in the number of gradient learners, however, typically destabilizes GL, resulting in low profits for gradient learners so they may

\(^2\)In general, GL may converge to local profit maxima that are not globally optimal. Bonanno and Zeeman (1985) and Bonanno (1988) call this kind of outcomes local Nash equilibria. In the market structure we consider there is a unique local profit maximum so GL leads to the Nash equilibrium if it converges.
start switching back to LSL. At some point, GL will converge again and the process may repeat itself.

With this chapter we demonstrate that different learning methods are likely to coexist and that this coexistence can have substantial consequences for the dynamical properties of the learning methods. The dynamics with coexisting learning rules are more complex than in a homogeneous environment.

The chapter is organized as follows. First we review the literature on which the chapter builds in Section 2.2. Then in Section 2.3 we present the market structure and derive the Nash equilibrium of the model. Section 2.4 discusses least squares learning and gradient learning. We analyze the steady states of a homogeneous LS-learning oligopoly both analytically and through simulations. Then we investigate the dynamical properties of the model with only gradient-learning firms. Section 2.5 combines the two learning methods in a heterogeneous setting. The learning rules are fixed in the sense that firms apply the same learning rule during the whole simulation. We analyze the model both analytically and numerically. We compare the profitability of the two learning methods as the distribution of methods over firms varies. Section 2.6 focuses on switching. We illustrate cyclical switching between the learning methods when the stability of GL changes with the number of gradient learners. Section 2.7 concludes. The proofs of the propositions are presented in Appendix 2.A.

2.2 Related literature

This chapter builds upon and contributes to several recent streams of literature on learning in economics. LSL, for example, is widely used in the economics literature. The model we consider is closely related to the model of Kirman (1983) and Brousseau and Kirman (1992). These papers analyze the properties of misspecified LSL in a Bertrand duopoly with differentiated goods under a linear demand specification. The learning method they use is misspecified as firms focus on their own price effect only. In this chapter we generalize some results of Kir-
man (1983) to the case of \( n \) firms under a nonlinear demand specification. Gates et al. (1977) consider LSL in a Cournot oligopoly. Each firm regresses its average profit on its output and uses the estimated function to determine the output for the next period. The learning method the authors consider differs from ours in two respects. First, each observation has the same weight in our model whereas firms weigh observations differently in Gates et al. (1977).\(^3\) Second, the firms’ actions are specified as the action that maximizes their one-period expected profit in the model of this chapter. In Gates et al. (1977) the next action is a weighted average of the previous action and the action that maximizes the estimated profit function. Tuinstra (2004) considers the same kind of model that is studied in this chapter. The firms use a misspecified perceived demand function but a different learning method is applied. When posting a price, firms are assumed to observe the demand for their good and the slope of the true demand function at that price. Then they estimate the demand function by a linear function that matches the demand and the slope the firms faced. For the estimation firms use only the most recent observation. Firms will then use the new estimates for determining the price in the next period. Tuinstra (2004) analyzes the dynamical properties of this model and shows that complicated dynamical behavior can occur depending on the cross-price effects and the curvature of the demand functions. Schinkel et al. (2002) analyze a model of monopolistic competition in which firms do not know the true demand conditions. Firms hold subjective beliefs about demand conditions and beliefs are updated with Bayesian learning. Beliefs are misspecified as firms focus on their own price effect only, similarly as in this chapter. The authors show that the process converges to a so-called conjectural equilibrium in which firms are maximizing their expected profit subject to their conjectures and the corresponding outcome does not induce a revision of beliefs. Analogous conditions characterize the self-sustaining equilibria of this chapter.

Arrow and Hurwicz (1960) analyze the dynamical properties of GL in a general class of \( n \)-person games. They derive conditions under which the process converges to an equilibrium and they illustrate their findings for the case of a Cournot oligopoly. Both Furth (1986) and

\(^3\)We illustrate the effects of applying a weighting function in the regression in Chapter 3.
Corchon and Mas-Colell (1996) analyze a price-setting oligopoly in which firms adjust their actions using GL. The uniqueness and the stability of equilibrium points are analyzed in these papers. In this chapter we also consider these issues although in a discrete time setting.

The previously discussed papers consider a homogeneous setting in which each agent uses the same learning method. However, it is reasonable to assume heterogeneity in the sense that agents apply different methods. Furthermore, they might switch between these methods. The switching mechanism we apply is related to reinforcement learning as in Roth and Erev (1995) and to the discrete choice model applied in Brock and Hommes (1997). In Roth and Erev (1995) agents have many possible pure strategies and each strategy has a propensity that determines the probability of the pure strategy being applied. These propensities depend on past payoffs. When a particular strategy was used in a given period, then its propensity is updated by adding the realized payoff to the previous propensity. The propensities of the strategies that were not used are not updated. The probability of a pure strategy being applied is proportional to the propensity of the strategy. We also use propensities for LSL and GL in the switching mechanism but they are updated differently than in Roth and Erev (1995): when a certain method was used, then the new propensity of that method is a weighted average of its old propensity and the current profit while the propensity of the other method remains unchanged. Furthermore, we determine the probabilities in a different way: we use the discrete-choice probabilities as in Brock and Hommes (1997). This way we can control how sensitively the firms react to differences in the performance measures. In Brock and Hommes (1997) the authors analyze a cobweb model in which agents can use either a free naive or a costly perfect foresight predictor. The authors show that endogenous switching between the predictors leads to complicated dynamical phenomena as agents become more sensitive to performance differences. Droste et al. (2002) also analyze the interaction between two different behavioral rules. They consider Cournot competition with best-reply and Nash rules. With the best-reply rule, firms give the best response to the average output of the previous period. Nash firms are basically perfect foresight firms that take into account the behavior of the best-reply firms. The model of this
chapter differs in important aspects from the setup of Droste et al. (2002). First, firms do not know the demand they face in this chapter whereas the demand functions are known in Droste et al. (2002). Second, Droste et al. (2002) basically consider social learning: the firms observe the actions of every firm and they use this information for deciding on the production level. In contrast, firms observe only their own action and the corresponding outcome in this chapter. Thus, they use individual data in the learning process instead of industry-wide variables. A consequence of this difference is that firms that apply the same learning method or behavioral rule choose the same action in Droste et al. (2002) but they typically act differently in the model we consider. Third, the switching methods are also different in the two models. Droste et al. (2002) use replicator dynamics whereas we consider a discrete choice model, augmented with experimentation.

2.3 The market structure

we consider a market with \( n \) firms, each producing a symmetrically differentiated good and competing in prices. The demand for the product of firm \( i \) depends on the price of good \( i \) and on the average price of the other goods. The demand is given by following nonlinear function:

\[
D_i(p) = \max \left\{ \alpha_1 - \alpha_2 p_i^\beta + \alpha_3 \bar{p}_{-i}, 0 \right\}, \tag{2.1}
\]

where \( p_i \) is the price of good \( i \), \( p \) is the vector of prices and \( \bar{p}_{-i} = \frac{1}{n-1} \sum_{j \neq i} p_j \). All parameters are positive and we further assume that \( \beta, \gamma \in (0, 1) \) and \( \beta \geq \gamma \). Parameter \( \alpha_3 \) specifies the relationship between the products: for \( \alpha_3 > 0 \) the goods are substitutes whereas for \( \alpha_3 < 0 \) they are complements. In this chapter we focus on substitutes.\(^5\) The demand is decreasing and convex in the own price and it is increasing and concave in the price of the other goods. The

\(^4\)Vriend (2000) gives a clear illustration that social and individual learning can lead to substantially different outcomes.

\(^5\)We discuss results for the case of complements too but we do not report the corresponding simulations. The case of complements is discussed in more detail in Kopányi (2013a) under a linear demand specification.
market structure is fully symmetric: firms face symmetric demands and the marginal cost of production is constant, identical across firms and equal to $c > 0$.

We impose some restrictions on the parameter values which ensure that a symmetric Nash equilibrium exists.

**Assumption 2.3.1.** The parameters satisfy $\alpha_1 - \alpha_2 c^\beta + \alpha_3 c^\gamma > 0$ and $-\alpha_2 \beta c^\beta + \alpha_3 \gamma c^\gamma < 0$.

The first restriction means that the demand is sufficiently large: demands are positive when each firm sets the price equal to the marginal cost, that is $D_i(c, \ldots, c) > 0$ for each firm $i$. The second restriction ensures that the demand for a good strictly decreases if the own price as well as the average price of the other firms marginally increase in a symmetric situation where $p_i = \bar{p} - i = p > c$ (as long as $D(p) > 0$). Proposition 2.3.2 characterizes the unique Nash equilibrium of the model.

**Proposition 2.3.2.** Under Assumption 2.3.1 the model has a unique Nash equilibrium. In this equilibrium, each firm charges price $p_N$ that is the unique solution to the equation

$$
\alpha_1 - \alpha_2 p_N^\beta + \alpha_3 p_N^\gamma - \alpha_2 \beta (p_N - c) p_N^{\beta-1} = 0.
$$

(2.2)

The Nash equilibrium price exceeds the marginal cost.

Note that the Nash equilibrium is symmetric and the price is independent of the number of firms. This is due to the fact that the average price of other goods determines the demand so the number of firms does not affect the equilibrium.

Firms do not have full information about the market environment. In particular, they do not know the demand specifications, furthermore they cannot observe either the prices or the demands for the other goods. They are assumed to know their own marginal cost. Firms repeatedly interact in the environment described above. They are myopic profit maximizers: they are

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$^6$With other words, the own price effect dominates cross price effects in a symmetric situation: $\frac{\partial D_i}{\partial p_i} + \sum_{j \neq i} \frac{\partial D_i}{\partial p_j} < 0$ when $p_j = p_i$ for all $j \neq i$. 

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only interested in maximizing their one-period profit. Firms can apply one of two methods to decide on the price they ask in a given period. These methods are discussed in Section 2.4.

2.4 Learning methods

One method that firms may apply is least squares learning. With this method firms use an estimated demand function for maximizing their expected profit. The other method is gradient learning: firms use information about their marginal profit at the current price and they adjust the current price of their good in the direction in which they expect to get a higher profit. Both methods focus on the own price effect without considering the effect of the price change of other goods. Section 2.4.1 presents LSL while GL is analyzed in Section 2.4.2.

2.4.1 Least squares learning

With LSL firms use past price-quantity observations for estimating a perceived demand function and then they maximize their expected profit, given this perceived demand function. The parameter estimates determine the price they set in the next period. As new observations become available, firms update the parameter estimates and thus the price of their good.

The learning mechanism

Firm $i$ assumes that the demand for its good depends linearly on the price of the good but it does not take into account dependence on the price of other goods. The perceived demand function of firm $i$ is of the form

$$D^P_i(p_i) = a_i - b_i p_i + \varepsilon_i,$$

(2.3)

Alternatively, we could consider firms that maximize a discounted stream of profits. This is referred to as active learning, where firms take into account not only the current one-period payoff of an action but also the fact that different actions might carry different amount of information. See Kiefer and Nyarko (1989) for more details and references for active learning.
where $a_i$ and $b_i$ are unknown parameters and $\varepsilon_i$ is a random noise with mean 0. Notice that firm $i$ uses a misspecified model since the actual demand (2.1) is determined by all prices, furthermore it depends on prices in a nonlinear manner. Kirman (1983) applies the same kind of misspecified LSL in a Bertrand duopoly with differentiated goods. He argues that it is reasonable for firms to disregard the prices of other goods in an oligopolistic setting. When the number of firms is large, it requires too much effort to collect every price, so firms rather focus on their own-price effect and treat the pricing behavior of the other firms as an unobservable error.

For obtaining the coefficients of the perceived demand function the firm regresses the demands it faced on the prices it asked. All past observations are used with equal weight in the regression. Let $a_{i,t}$ and $b_{i,t}$ denote the parameter estimates observed by firm $i$ at the end of period $t$.\footnote{For simplifying notation, we denote the unknown parameters of the perceived demand function as well as the corresponding parameter estimates by $a$ and $b$. This should not be confusing as we will mainly work with the parameter estimates from now on.} These estimates are given by the standard OLS formulas (see Stock and Watson, 2003, for example):

\begin{align*}
b_{i,t} &= \frac{\left(\frac{1}{t} \sum_{\tau=1}^{t} p_{i,\tau}\right) \left(\frac{1}{t} \sum_{\tau=1}^{t} q_{i,\tau}\right) - \frac{1}{t} \sum_{\tau=1}^{t} p_{i,\tau} q_{i,\tau}}{\frac{1}{t} \sum_{\tau=1}^{t} (p_{i,\tau})^2 - \left(\frac{1}{t} \sum_{\tau=1}^{t} p_{i,\tau}\right)^2}, \quad (2.4) \\
a_{i,t} &= \frac{1}{t} \sum_{\tau=1}^{t} q_{i,\tau} + b_{i,t} \frac{1}{t} \sum_{\tau=1}^{t} p_{i,\tau}, \quad (2.5)
\end{align*}

where $q_{i,\tau}$ denotes the actual demand for good $i$ in period $\tau$: $q_{i,\tau} = D_i(p_\tau)$.

Note that even though $a_{i,t}$ and $b_{i,t}$ should be positive in order to have an economically sensible perceived demand function, the parameter estimates may be negative in some periods. Furthermore, if $a_{i,t} \leq b_{i,t} c$, then firm $i$ cannot expect to earn a positive profit since the perceived demand becomes zero already at a price that is smaller than the marginal cost $c$. In these situations we assume that the firm does not use the parameter estimates to choose a price. We will shortly specify how the firm acts in this situation.

When the aforementioned cases do not occur (that is when $a_{i,t} > b_{i,t} c > 0$), then firm $i$
determines the price for the next period by maximizing its expected profit:

$$\max_{p_{i,t+1} \geq c} E_t ((p_{i,t+1} - c) (a_{i,t} - b_i (p_{i,t+1} + \varepsilon_{i,t+1}))) = \max_{p_{i,t+1} \geq c} \{ (p_{i,t+1} - c) (a_{i,t} - b_i p_{i,t+1}) \}.$$

The objective function is quadratic in $p_{i,t+1}$ and the quadratic term has a negative coefficient. Then in period $t + 1$ the firm asks the perceived profit-maximizing price $p_{i,t+1} = a_{i,t}^2 b_{i,t} + c^2$. If the condition $a_{i,t} > b_i c > 0$ does not hold, then we assume that firm $i$ draws a price randomly.

More specifically, we augment LSL with the following rule.

**Random price rule:** When $a_{i,t} > b_i c > 0$ does not hold, then firm $i$ chooses a price randomly from the uniform distribution on the set $S = \{ p \in \mathbb{R}_+^n : p_i > c, D_i(p) > 0, i = 1, \ldots, n \}$.

Note that set $S$ is the set of price vectors for which every firm makes a positive profit. Thus, according to the random price rule, when the perceived demand function is not sensible economically (i.e. $a_{i,t} \leq 0$ or $b_i \leq 0$), then the firm asks a random price rather than applying an incorrect pricing formula. Also, the firm asks a random but not unprofitable price rather than a price that yields a certain loss.

LSL is implemented in the following way. For any firm $i$:

1. $p_{i,1}$ and $p_{i,2}$ are randomly drawn from the uniform distribution on set $S$.
2. At the end of period 2 the firm uses OLS formulas (2.4) and (2.5) to obtain the parameter estimates $a_{i,2}$ and $b_{i,2}$.
3. **a.** In period $t \geq 3$ the firm asks the price $p_{i,t} = \frac{a_{i,t-1}}{2b_{i,t-1}} + \frac{c}{2}$ if $a_{i,t-1} > b_{i,t-1} c > 0$. In every other case $p_{i,t}$ is drawn from the uniform distribution on the set $S$.
   **b.** After realizing the demand, the firm updates the coefficients of the perceived demand function using (2.4) and (2.5).

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9More properly: when firm $i$ chooses a random price, then a price vector $p$ is drawn from the uniform distribution on $S$ and firm $i$ will charge the $i$-th component of $p$. Every other firm $j$ chooses a price according to the pricing formula or, if the condition $a_{j,t} > b_{j,t} c > 0$ does not hold, firm $j$ draws another vector.
4. The process stops when the absolute price change is smaller than a threshold value $\delta$ for all firms: 
\[ \max_i \{|p_{i,t} - p_{i,t-1}|\} < \delta. \]

Notice that the learning process of other firms interferes with the firm’s own learning process. As the prices of other goods change, the demand the firm faces also changes. Although the change in the demand for good $i$ is caused not only by the change in its price, firm $i$ attributes the change in its demand solely to changes in its own price and to random noise. Therefore, the firm tries to learn a demand function that changes in every period. Learning is more complicated in the initial periods since prices are more volatile than in later periods when the learning process slows down.

**Equilibria with LSL**

Brousseau and Kirman (1992) show that the misspecified LSL we consider does not converge in general.\(^{10}\) Price changes however become smaller over time as the weight of new observations decreases. Thus, the stopping criterion we specified will be satisfied at some point and the learning mechanism stops. We will see that the resulting point is very close to a so-called self-sustaining equilibrium in which the actual and the expected demands of a firm coincide. The set of self-sustaining equilibria is infinite.

With the method described above firms use a misspecified model since the perceived demand functions (2.3) differ from the actual demand functions (2.1). Nevertheless, firms may find that the price they charge results in the same actual demand as the perceived demand function predicts. If this holds for all firms, then the model is in equilibrium since the parameter estimates of the perceived demand functions do not change and firms will ask the same price in the following period. To see that this is the case, note the following. The LS coefficients at the end of period $t$ minimize the sum of squared errors up to period $t$. If the perceived and the actual demands are equal at $p_{t+1}$, then the parameter estimates $a_{t+1}$ and $b_{t+1}$ remain the same: under $a_t$ and $b_t$ the error corresponding to the new observation is 0 and the sum of squared errors up to

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\(^{10}\)LSL may converge in many other situations. Marcet and Sargent (1989) derive conditions under which LSL converges for a wide class of models.
period $t$ is minimized. Thus, the sum of squared errors up to period $t+1$ is minimized by exactly the same coefficients. Brousseau and Kirman (1992) call this kind of equilibrium *self-sustaining equilibrium*: firms charge the optimal price (subject to their perceived demand function) and the corresponding demand coincides with the demand they expected to get, therefore the firms have no reason to believe that their perceived demand function is incorrect. Following their terminology, we refer to such equilibria as self-sustaining equilibria (SSE).\footnote{This equilibrium concept is similar to the self-confirming equilibrium in Fudenberg and Levine (1993). The difference is that agents form beliefs about the environment in a SSE whereas they form beliefs about their opponents’ actions in a self-confirming equilibrium.}

The left panel of Figure 2.1 illustrates a disequilibrium of the model. The solid line is the perceived inverse demand function\footnote{We disregard the error term $\varepsilon_i$ to simplify the presentation.}

\[ p_i = P_i^p(q_i) \equiv \frac{a_i}{b_i} - \frac{1}{b_i} q_i, \]  

the dashed line depicts the actual inverse demand function

\[ p_i = P_i(q_i, \bar{p}_{-i}) \equiv \left[ \frac{1}{\alpha_2} \left( \alpha_1 + \alpha_3 \bar{p}_{-i}^\gamma - q_i \right) \right]^{\frac{1}{\gamma}}. \]  

The downward-sloping dotted line is the perceived marginal revenue. The quantity that max-
imizes the expected profit of firm $i$ is given by the x-coordinate of the intersection of the perceived marginal revenue (MR) and the marginal cost (MC). Let $q^P_i$ denote this quantity. If the firm wants to face a demand equal to $q^P_i$, then it has to ask price $p$ which is determined by the value of the perceived inverse demand function at $q^P_i$. However, the firm might face a different demand as the actual and perceived demand functions differ. Let $q^A_i$ denote the actual demand the firm faces when its price is $p$. The left panel of Figure 2.1 shows a situation in which the expected and the actual demands are not the same. This is not an SSE of the model. In this case the firm will add the new observation $(p, q^A_i)$ to the sample and run a new regression in the next period. This new observation changes the perceived demand function and the firm will charge a different price. In contrast, the right panel of Figure 2.1 illustrates the situation when $q^P_i = q^A_i$, that is the actual and the expected demands coincide at price $p$. This constitutes an SSE (provided that the corresponding variables of the other firms also satisfy these conditions). The new observation does not change the coefficients of the perceived demand function so the firm will charge the same price in subsequent periods.

A self-sustaining equilibrium can be formally defined as follows.

**Definition 2.4.1.** Price vector $p^* = (p^*_1, \ldots, p^*_n)$ and parameter estimates $\{a^*_i, b^*_i\} (i = 1, \ldots, n)$ constitute a self-sustaining equilibrium if the following conditions hold for each firm $i$:

$$p^*_i = \frac{a^*_i}{2b^*_i} + \frac{c}{2}, \quad (2.8)$$

$$ED^P_i(p^*_i) = D_i(p^*). \quad (2.9)$$

Condition (2.8) says that firms set the price that maximizes their expected profit subject to their perceived demand function. Condition (2.9) requires that the actual and the expected demands are the same at the SSE prices. Since we have 2 independent equations and 3 variables for each firm, we can express $a^*_i$ and $b^*_i$ as a function of the SSE prices. Thus, for given prices we can find perceived demand functions such that the firms are in an SSE. Proposition 2.4.2

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13Negishi (1961) and Silvestre (1977) consider similar equilibrium conditions in a general equilibrium framework.
specifies the coefficients of the perceived demand function in terms of the SSE prices. It also describes the set of SSE prices. The proposition is proved in the Appendix.

**Proposition 2.4.2.** For given prices \( p_i^* (i = 1, \ldots, n) \) the model is in an SSE if the coefficients of the perceived demand function of firm \( i \) are given by

\[
\begin{align*}
a_i^* &= D_i(p^*) \left( 1 + \frac{p_i^*}{p_i^* - c} \right), \\
b_i^* &= \frac{D_i(p^*)}{p_i^* - c}.
\end{align*}
\]

The set of SSE prices is described by the conditions \( p_i^* > c \) and \( D_i(p^*) > 0 \), or equivalently

\[
c < p_i^* < P_i(0, \bar{p}^*_{-i}) = \left[ \frac{1}{\alpha_2} \left( \alpha_1 + \alpha_3(\bar{p}^*_{-i})^\gamma \right) \right]^{\frac{1}{\beta}}.
\]

This set is nonempty and bounded.

The values of \( a_i^* \) and \( b_i^* \) derived in Proposition 2.4.2 are in line with Proposition 3 of Kirman (1983): they reduce to the same expression for the case of a duopoly with a linear demand function and zero marginal cost. Note that the set \( S \) we use in the LS algorithm coincides with the set of SSE prices. The set of SSE prices always contains the Nash equilibrium as the Nash equilibrium prices exceed the marginal cost and the corresponding demand is positive for every firm. The left panel of Figure 2.2 depicts the set of SSE prices for the case of two firms. The figure corresponds to parameter values \( \alpha_1 = 35, \alpha_2 = 4, \alpha_3 = 2, \beta = 0.7, \gamma = 0.6 \) and \( c = 4 \). We will use these parameter values in all later simulations too. For these values the Nash equilibrium price is \( p_N \approx 17.7693 \) with corresponding profit \( \pi_N \approx 223.9148 \). The collusive price and profit are given by \( p_C \approx 21.4862 \) and \( \pi_C \approx 233.5406 \), respectively.

In Proposition 2.4.2 we characterized the set of prices that may constitute an SSE. However, nothing ensures that every point of that set will actually be reached from some initial points. In fact, Kirman (1983) derives the set of points that can be reached with some initial values for the case of two firms and linear demand specification. He shows that this set is smaller than the set
Figure 2.2: The set of SSE prices for two firms (left panel) and the end prices of simulations with initial prices drawn from the uniform distribution on the set of SSE prices (right panel). Parameter values: $\alpha_1 = 35$, $\alpha_2 = 4$, $\alpha_3 = 2$, $\beta = 0.7$, $\gamma = 0.6$, $c = 4$ and $\delta = 10^{-8}$.

Note that different firms could use different types of perceived demand functions, in general. For example, some firms could take into account the prices of (some of the) other firms. Furthermore, perceived demand functions could be nonlinear in prices. We discuss the effects of heterogeneity in the perceived demand functions in the concluding remarks.

Simulation results

To illustrate some properties of LSL we simulate the model where each firm is an LS learner. We use the aforementioned parameter values with threshold value $\delta = 10^{-8}$ in the stopping criterion. First we illustrate that firms reach a point in the set of SSE prices when there are two firms. We drew 2000 random points from the uniform distribution on the set of SSE prices and ran 1000 simulations using these points as initial prices.\textsuperscript{14} In order to save time we limited the number of periods to 10000.\textsuperscript{15} The right panel of Figure 2.2 depicts the end prices of the 1000 simulations. We observe that almost all of the final points lie in the set of SSE prices and that they do not fill the whole set. Nevertheless, there is quite a variety in final prices so

\textsuperscript{14}We need two initial points for each simulation. The first 2 points are used as initial values in the first simulation, the third and the fourth are used in the second one etc.
\textsuperscript{15}So the simulation stopped at period 10000 even if the stopping criterion was not met. Based on other simulations, this does not affect the outcome substantially.
homogeneous LSL can lead to many possible outcomes. For the case of 10 firms we observed a variety in the final prices again and that most of the final points lie in the set of SSE prices. This latter result is not robust with respect to changes in the demand parameters: when the set of SSE prices is more expanded towards high prices, then final points fall outside the set of SSE prices more often. The other finding that LSL may result in many possible outcomes is robust with respect to the demand parameters. These results remain valid even when we add a small noise to the actual demands.

Figure 2.3 illustrates typical time series of prices and profits for the case of 10 firms. Although the stopping criterion is satisfied only at period 9201, we plot only the first 20 periods as the time series do not change much after that. We observe that prices are volatile in the first few periods but then they start to settle down. In this particular simulation, end prices are between 16.3 and 19.6, while profits lie between 225.4 and 229.3. Note that profits exceed the corresponding Nash equilibrium profit 223.9. We analyzed the distribution of end prices by simulating the model with initial prices drawn from the uniform distribution on the set of SSE prices. As the number of firms increases, price dispersion becomes smaller: a higher proportion of end prices lies close to the mode of the distribution. The mode lies between the Nash equi-

\[^{16}\]Simulations show that if we do not consider the non-negativity constraint on demands, then almost all points lie within the set of SSE prices irrespective of the shape of the set.
librium and the collusive prices and it moves towards the Nash equilibrium price as the number of firms increases.

2.4.2 Gradient learning

Let us now turn to the other method that firms may apply for deciding on prices. Instead of assuming a specific form for the demand function and estimating its parameters, firms use information about the slope of their profit function.\(^{17}\) Knowing the slope at the current price, firms adjust the price of their good in the direction in which they expect to get a higher profit.

The learning mechanism

The price charged by firm \(i\) in period \(t + 1\) is given by

\[
p_{i,t+1} = \max \left\{ p_{i,t} + \lambda_i \frac{\partial \pi_i(p_t)}{\partial p_{i,t}}, c \right\}, \tag{2.10}
\]

where the derivative of the profit function is

\[
\alpha_1 - \alpha_2 \bar{p}_{-i,t}^\beta + \alpha_3 \bar{p}_{-i,t}^\gamma - \alpha_2 \beta (p_{i,t} - c)p_{i,t}^\beta - 1. 
\]

Formula (2.10) shows that the price adjustment depends on the slope of the profit function and on parameter \(\lambda_i > 0\). We assume for the rest of the paper that each firm uses the same adjustment parameter, that is \(\lambda_i = \lambda\) for each firm \(i\). In Section 2.4.2 we will see that the stability properties of this learning rule depend heavily on the value of \(\lambda\).

We augment this method with an additional rule. Note that if a firm sets a too high price for which the demand is zero, then (2.10) gives the same price for the next period since the slope of the profit function is zero at that point. However, it should be clear for the firms that the zero profit may result from charging a too high price, so it is reasonable to lower the price. Therefore, we add the following rule to GL.

\(^{17}\)For analytically calculating the slope firms would need to know the actual demand function and the prices asked by other firms. Nevertheless, with market experiments they can get a good estimate of the slope without having the previously mentioned pieces of information. Thus, it is not unreasonable to assume that firms know the slope of their profit function.
**Flat demand rule:** If \( q_{i,t-2} = q_{i,t-1} = 0 \), then \( p_{i,t} = \max \{ p_{i,t-1} - \lambda_0, c \} \).

According to this rule, if a firm faced zero demand in two consecutive periods, then it lowers its previous price by \( \lambda_0 \). This rule ensures that firms cannot get stuck in the zero profit region. We assume that \( \lambda_0 \) takes the same value as \( \lambda \) in all simulations.\(^{18}\)

GL is implemented in the following way. For every firm \( i \):

1. \( p_{i,1} \) and \( p_{i,2} \) are drawn from the uniform distribution on the set \( S \).\(^{19}\)

2. In period \( t \geq 3 \):
   - If \( D_i(p_{t-2}) \neq 0 \) or \( D_i(p_{t-1}) \neq 0 \), then \( p_{i,t} = \max \left\{ p_{i,t-1} + \lambda \frac{\partial \pi_i}{\partial p_{i,t-1}}, c \right\} \).
   - If \( D_i(p_{t-2}) = D_i(p_{t-1}) = 0 \), then the price is given by \( p_{i,t} = \max \{ p_{i,t-1} - \lambda_0, c \} \).

3. The process continues until all price changes are smaller in absolute value than a threshold value \( \delta \).

Similarly to the case of LS-learning firms, the firms’ learning processes interfere with each other. Although a firm moves in the direction that is expected to yield a higher profit, it may actually face a lower profit after the price change since the profit function of the firm changes due to the price change of other firms. Nevertheless, if GL converges, then this disturbance becomes less severe as there will be only small price changes in later periods.

**Equilibrium and local stability**

Let us now investigate the dynamical properties of GL. In the first part of the analysis we will not consider non-negativity constraints on prices and demands and we disregard the flat demand rule too. We will discuss the effects of these modifications after deriving the general features of the learning rule.

\(^{18}\)The exact value of \( \lambda_0 \) affects only the speed of return from the zero profit region, it does not affect the convergence properties of the method.

\(^{19}\)Although it would be sufficient to take one initial value for the simulations, we take two initial values so that GL would be more comparable with LSL in a heterogeneous setting. We take initial values from the same set for the same reason.
The law of motion of prices is given by

\[ p_{i,t+1} = p_{i,t} + \lambda \frac{\partial \pi_i(p_t)}{\partial p_{i,t}}. \]

The system is in a steady state if the derivative of the profit function with respect to the own price is zero for all firms. Under the demand specification we consider, this condition characterizes the Nash equilibrium, so the Nash equilibrium is the unique steady state of the model with only gradient learners.

Let us now analyze the stability of the steady state. Proposition 2.4.3 summarizes the dynamical properties of the gradient-learning oligopoly. The proof of the proposition can be found in the Appendix.

**Proposition 2.4.3.** The Nash equilibrium price \( p_N \) is locally stable in the gradient-learning oligopoly if

\[
\lambda \left\{ \alpha_2 \beta p_N^{\beta - 1} \left[ 2 + (\beta - 1) \frac{p_N - c}{p_N} \right] + \alpha_3 \gamma p_N^{\gamma - 1} \frac{1}{n - 1} \right\} < 2.
\]

When the expression on the left hand side equals 2, \( n - 1 \) eigenvalues of the Jacobian matrix of the system, evaluated at the Nash equilibrium, become \(-1\), the remaining eigenvalue lies within the unit circle.

According to Proposition 2.4.3, the steady state is locally stable if parameter \( \lambda \) is sufficiently small. The steady state loses stability through a degenerate flip bifurcation: multiple eigenvalues exit the unit circle through the value \(-1\). In general, many different cycles with period multiple of 2 could be created with this kind of bifurcation.\(^{20}\) We numerically investigate the occurring dynamics in the next section. Note that the coefficient of \( \lambda \) in the stability condition is decreasing in \( n \) as \( p_N \) is independent of \( n \). Thus, an increase in the number of firms has a stabilizing effect.

So far we have not considered the effect of the constraints \( p_i \geq c, D_i(p) \geq 0 \) and the flat

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\(^{20}\)More details about degenerate flip bifurcations can be found in Mira (1987), Bischi et al. (2000) and Bischi et al. (2009).
demand rule. For discussing these effects let us first consider a linear demand function. In that case the system is linear so there are three kinds of possible dynamics if we do not consider any constraints: convergence to a steady state, to a 2-cycle or unbounded divergence. Unbounded divergence is no longer possible when we impose the constraints on prices and demands. These constraints and the flat demand rule drive prices back towards the region where the demands are positive. Therefore, we may observe periodic cycles with a period higher than 2, quasi-periodic or aperiodic dynamics for high values of $\lambda$.

In the nonlinear setting we consider, the non-negativity constraint on prices must be imposed since a negative price would yield a complex number as demand. The effect of the constraints and the flat demand rule is the same as for a linear demand function: they exclude the possibility for unbounded divergence, we observe periodic cycles, quasi-periodic dynamics or aperiodic time series instead.

Similarly to LSL, firms could use different types of GL as well. This can be implemented with different adjustment parameter $\lambda$ across firms. Heterogeneity in parameters of the learning algorithms is well established by laboratory experiments in different settings, see e.g. Erev et al. (2010) and Anufriev et al. (2013a). In the concluding remarks we discuss the effect of individual heterogeneity with respect to the adjustment parameter $\lambda$.

**Simulation results**

We run simulations for illustrating the possible dynamics of the model with only gradient learners. We use the same parameter values as before. Figure 2.4 illustrates typical time series of prices: convergence to the Nash equilibrium for $\lambda = 0.8$ in panel (a), convergence to a 2-cycle for $\lambda = \lambda^* \approx 0.9391$ in panel (b), quasi-periodic dynamics for $\lambda = 0.9344$ in panel (c) and aperiodic dynamics for $\lambda = 1$ in panel (d). These patterns can occur for different demand parameters too but for different values of $\lambda$.

In line with Proposition 2.4.3, we observe convergence to the Nash equilibrium price when $\lambda$ is sufficiently small. Starting with initial prices in a small neighborhood of the Nash equilib-
Figure 2.4: Typical time series of prices in an oligopoly with 10 gradient learners for different values of $\lambda$. Other parameter values: $\alpha_1 = 35$, $\alpha_2 = 4$, $\alpha_3 = 2$, $\beta = 0.7$, $\gamma = 0.6$, $c = 4$ and $\delta = 10^{-8}$. Nash equilibrium price: $p_N \approx 17.7693$.

For higher values of $\lambda$ we observe periodic cycles, quasi-periodic or aperiodic dynamics.

It turns out from simulations that the lower $\lambda$ is, the larger the range of initial prices for which convergence can be observed in simulations. So when the steady state is locally stable and $\lambda$ is close to the bifurcation value, then we observe convergence only for a small set of initial prices. When initial prices lie outside of this set, then we observe periodic cycles, quasi-periodic or aperiodic dynamics.
2.5 Heterogeneous oligopoly with fixed learning rules

In this section we combine the learning methods discussed in Sections 2.4.1 and 2.4.2 and we consider the case of a heterogeneous oligopoly in which some firms use least squares learning while others apply gradient learning. Firms use a fixed learning method and they cannot change the rule they use. We will see that the main features of the two methods remain valid even in the heterogeneous setting: when $\lambda$ is sufficiently small, then LS learners get close to an SSE in which gradient learners give the best response to the prices set by the other firms.

2.5.1 Steady states and stability

Consider a market with $n_L$ LS learners and $n - n_L$ gradient learners where $0 < n_L < n$. Let us assume without loss of generality that the first $n_L$ firms are the LS learners. We discussed in Section 2.4.1 that the steady states of an LS-learning oligopoly are characterized by a self-sustaining equilibrium. The same conditions must hold for LS learners in a steady state of a heterogeneous oligopoly: their actual and perceived demands must coincide at the price they ask (given the prices of other firms), otherwise they would update their perceived demand function and the price of their good would change in the next period. At the same time, the slope of the profit function of gradient learners must be zero in a steady state otherwise the price of their good would change. Proposition 2.5.1 characterizes the steady state of the heterogeneous oligopoly with fixed learning rules. We leave the proof to the reader, it can be proved with very similar steps as in the proof of Proposition 2.3.2.

**Proposition 2.5.1.** In a steady state of the system, LS learners are in an SSE and gradient learners give the best response to the prices set by other firms. The price $p_G$ set by gradient learners is characterized by

$$\alpha_1 - \alpha_2 p_G^\beta + \alpha_3 \left[ \frac{1}{n-1} \left( \sum_{s=1}^{n_L} p_s^* + (n - n_L - 1) p_G \right) \right]^\gamma - \alpha_2 \beta (p_G - c) p_G^{\beta-1} = 0,$$

where $p_s^*$ denotes the price of LS learner $s$. 

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Later we will illustrate with numerical analysis that there is a unique solution \( p_G \) to the above equation for any \( \sum_{s=1}^{n_L} p_s^* \). Since, at a steady state, gradient learners give the best-response price, steady states are similar to a Stackelberg oligopoly outcome with LS learners as leaders and gradient learners as followers. It is, however, not a real Stackelberg outcome because LS learners do not behave as real leaders since they do not take into account the reaction function of gradient learners when setting the price of their good. Nevertheless, LS learners may accidentally earn a higher profit than gradient learners in a steady state.\(^{21}\)

Let us now turn to the stability of the steady states. As LS learners always settle down at a certain price since the weight of a new observation decreases as the number of observations increases, stability depends mainly on the dynamical properties of GL. Proposition 2.5.2 presents these properties. The proof can be found in the Appendix.

**Proposition 2.5.2.** Under the assumption that LS learners have reached their steady state price, the dynamical properties of GL are as follows. For \( 0 < n_L < n - 1 \) the price \( p_G \) set by gradient learners is locally stable if \( 0 < \lambda M_1 < 2 \) and \( 0 < \lambda M_2 < 2 \), where

\[
M_1 = \alpha_2 \beta p_G^{\beta-1} \left[ 2 + (\beta - 1) \frac{p_G - c}{p_G} \right] - \alpha_3 \gamma \left( \frac{n_L \bar{p}^* + (n - n_L - 1) p_G}{n - 1} \right)^{\gamma-1} \frac{n - n_L - 1}{n - 1},
\]

\[
M_2 = \alpha_2 \beta p_G^{\beta-1} \left[ 2 + (\beta - 1) \frac{p_G - c}{p_G} \right] + \alpha_3 \gamma \left( \frac{n_L \bar{p}^* + (n - n_L - 1) p_G}{n - 1} \right)^{\gamma-1} \frac{1}{n - 1},
\]

and \( \bar{p}^* = \frac{1}{n_L} \sum_{s=1}^{n_L} p_s^* \) is the average LS price.

For \( n_L = n - 1 \) the price set by the gradient learner is locally stable if

\[
\lambda \alpha_2 \beta p_G^{\beta-1} \left[ 2 + (\beta - 1) \frac{p_G - c}{p_G} \right] < 2.
\]

\(^{21}\)Gal-Or (1985) shows that there is a second mover advantage in a price-setting duopoly with substitute goods. Under a linear demand specification, this result can be extended to a higher number of firms too. We expect this to hold also for the nonlinear demand specification we consider when the demand functions are not too far from the linear case. However, since LS learners do not charge the optimal leaders’ price, this deviation from the optimal price may hurt the gradient learners even more and they may earn a lower profit than the LS learners.
Note that the previous proposition concerns the stability of the price set by gradient learners and not those of the steady states. Although LS learners get close to an SSE and the price set by gradient learners is locally stable for low values of $\lambda$, we cannot say that the steady state is locally stable. A small perturbation of a steady state leads to different LS prices and this changes the best-response price too. If, however, the LS prices remained the same, then gradient learners would return to the best-response price after a small perturbation. Note further that the proposition establishes sufficient conditions for local stability only. Thus, we might not observe convergence to a steady state for large perturbations of gradient learners’ price.

The distribution of learning methods over firms affects the stability of the price set by gradient learners as $n_L$ appears in the aforementioned stability conditions. It is, however, not clear analytically how stability changes with respect to $n_L$ because a change in $n_L$ affects the average LS price $\bar{p}^*$, which can take many different values. For further analyzing this issue, we use numerical calculations. First we check the direct effect of $n_L$ on stability and then we analyze how an increase in $n_L$ affects the average LS price.

Although LS prices are unknown, we can make use of the fact that the set of SSE prices is bounded: the minimal SSE price is $c$ and the maximal SSE price $\hat{p}$ is implicitly defined by

$$\alpha_1 + \alpha_3 \hat{p}^\gamma - \alpha_2 \hat{p}^\beta = 0 \quad \text{(as shown in the proof of Proposition 2.4.2).}$$

Thus, we have $c \leq \bar{p}^* \leq \hat{p}$. Taking values for $\bar{p}^*$ from this range, we can calculate $p_G, M_1$ and $M_2$ numerically. The left panel of Figure 2.5 shows that $p_G$ is unique (given the average LS price and the number of LS learners). Note that there is a value of $\bar{p}^*$ for which the best-response price is the same irrespective of the number of LS learners. This price equals the Nash equilibrium price since the best response to the Nash equilibrium price is the Nash equilibrium price itself.

It turns out from the calculations that only $M_2$ is relevant for stability: $M_1$ is always positive and $M_1 < M_2$ as $\alpha_3 > 0$. Using (2.12) we can calculate for any $\bar{p}^*$ and any $n_L$ the threshold value of $\lambda$ for which the gradient learners’ price loses stability. Using these threshold values, we depict the stability-instability region for different values of $n_L$ in the right panel of Figure 2.5.
We can see from the graph that for a given average LS price, the region of stability is increasing (decreasing) in \( n_L \) if the average LS price is larger (smaller) than the Nash equilibrium price provided that there are more than one gradient learners (i.e. \( n_L < n - 1 \)).\(^{22}\) Thus, for a fixed average LS price, an increase in the number of LS learners has a (de)stabilizing effect if the average LS price is larger (smaller) than the Nash equilibrium price and if there are at least 2 gradient learners. For \( n_L = n - 1 \) the stability condition becomes different. When the average LS price exceeds the Nash equilibrium price, then the change in the stability region is still monotonic, but it is no longer monotonic when the average LS price is lower than the Nash equilibrium price.

As the number of LS learners changes, the average LS price changes too. Since \( \bar{p}^* \) may change in any direction, we cannot say unambiguously whether a change in \( n_L \) has a stabilizing or a destabilizing effect on the price set by gradient learners. For analyzing how the average LS price changes as \( n_L \) varies, we run 1000 simulations for each value of \( n_L \) between 1 and 9 with initial prices drawn from the set of SSE prices. We used \( \lambda = 0.937 \) in these simulations: for this value of \( \lambda \) the convergence property of GL changes as \( n_L \) varies. Our results show

\(^{22}\)Remember that there is a different stability condition for the case \( n_L = n - 1 \).
that the average LS price exceeds the Nash equilibrium price in 64 – 71% of the cases for the different number of LS learners and that the average LS price does not vary much (5.47%-7.12% compared to the Nash equilibrium price of $p_N \approx 17.7693$) as the number of LS learners varies. Based on these findings we conclude that an increase in the number of LS learners has typically (but not necessarily) a stabilizing effect on GL.

The stability analysis becomes much simpler for the case of complements ($\alpha_3 < 0$). In that case it is easy to see that $0 < M_2 < M_1$ so the relevant stability condition becomes $\lambda M_1 < 2$. Numerical calculations show that the value of $M_1$ monotonically decreases in the number of LS learners irrespective of the average LS price. Thus, an increase in the number of LS learners has a stabilizing effect on the best response price when the average LS price is fixed. Overall, the relation between stability and the distribution of learning methods over firms is stronger for complements than for substitutes.

### 2.5.2 Simulation results

First we simulate the model for 2 firms with firm 1 as gradient learner and firm 2 as LS learner. We used $\lambda = \lambda_0 = 0.5$ in the simulations. The price set by the gradient-learning firm is locally stable for this choice of $\lambda$. We run 1000 simulations with initial prices drawn from the uniform distribution on the set of SSE prices. Figure 2.6 depicts the end prices and the set of SSE prices. The LS learner indeed gets close to an SSE in almost all cases: 99.7% of the points lie in the set of SSE prices. The structure of the end points also confirms that the gradient learner gives the best response price: the points lie close to the reaction curve of the gradient learner.

\(^{23}\)We obtained these results in the following way. Let $\bar{p}_{i,j}$ denote the average LS price in simulation $i$ with $j$ LS learners, where $i = 1, \ldots, 1000$ and $j = 1, \ldots, 9$. We used the same initial prices in the simulations across different values of $j$. For analyzing the range in which the average LS price varies, we considered the minimal and the maximal average LS price over the different number of LS learners for each $i$: $\min_j \bar{p}_{i,j}$ and $\max_j \bar{p}_{i,j}$. This gave the interval $[\min_j \bar{p}_{i,j}, \max_j \bar{p}_{i,j}]$ in which the average LS price varies for a given $i$ as the number of LS learners changes. We obtained an interval for each of the 1000 cases this way. Then we considered the length of these intervals $\left(\max_j \bar{p}_{i,j} - \min_j \bar{p}_{i,j}\right)$ and calculated the mean and the standard deviation of them over the 1000 runs. The 95% confidence interval of the length is [0.9723, 1.2656].
Figure 2.6: The end points of the simulations with firm 1 as gradient learner and firm 2 as LS learner. Parameter values: $\alpha_1 = 35$, $\alpha_2 = 4$, $\alpha_3 = 2$, $\beta = 0.7$, $\gamma = 0.6$, $c = 4$, $\lambda = \lambda_0 = 0.5$ and $\delta = 10^{-8}$.

Figure 2.7: The average LS and gradient profits (with 95% confidence interval) (left panel) and the percentage of runs in which gradient learners earn a higher average profit. Parameter values: $\alpha_1 = 35$, $\alpha_2 = 4$, $\alpha_3 = 2$, $\beta = 0.7$, $\gamma = 0.6$, $c = 4$ and $\lambda = \lambda_0 = 0.937$.

Figure 2.7 compares the profitability of the two learning methods. The left panel shows the average LS and gradient profits (with 95% confidence interval) for different numbers of LS learners. For drawing this graph, we simulated the model 1000 times for each number of LS learners with initial prices drawn from the uniform distribution on the set of SSE prices. We let each simulation run for 2000 periods and for each firm we considered the average of its profits over the last 100 periods as the profit of the firm in the given simulation. Thus, for the case

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242000 periods are typically enough for profits to converge when GL converges. We take the average over the last 100 periods in order to get a better view on the profitability of the methods. When GL converges, then profits do not vary much in the last periods. When the price is unstable, gradient profits change more or less periodically, so averaging over the last few profits describes the profitability of the method better than considering only the last
of \(k\) LS learners, we had \(1000k\) observations for LS profits and \(1000(10 - k)\) observations for gradient profits. We calculated the average and the standard deviation of these values separately for LS and gradient learners. The confidence interval is calculated as \(\text{mean} \pm 2\text{stdev}/\sqrt{1000k}\) and \(\text{mean} \pm 2\text{stdev}/\sqrt{1000(10 - k)}\) for LS and gradient learners respectively. The left panel shows that GL yields significantly lower average profit than LSL when the number of LS learners is low. In contrast, it gives significantly higher profits when the number of LS learners is high enough.

The right panel of Figure 2.7 depicts for each number of LS learners the percentage of the 1000 simulations in which the average gradient profit was larger than the average LS profit. The graph shows that GL becomes more profitable than LSL more often as the number of LS learners increases. Since profitability is closely related to the convergence properties of GL, this illustrates that an increase in the number of LS learners has typically a stabilizing effect.\(^{25}\)

Based on this change in the stability of GL, we conjecture a cyclical switching between the learning methods when firms are allowed to choose which method they want to apply. Conjecture 2.5.3 summarizes our expectation. In the following section we will investigate if cyclical switching occurs.

**Conjecture 2.5.3.** When firms are sensitive to profit differences, changes in the convergence properties of GL may lead to cyclical switching between the learning rules. When GL converges, LS learners have an incentive to switch to GL as it typically yields a higher profit. This increase in the number of gradient learners, however, may destabilize the best-response price, resulting in lower gradient profits. Then firms switch to LSL, so GL may converge again and the cycle may repeat itself.

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\(^{25}\)When GL does not converge, then it gives low average profit as the price fluctuates between too low and too high values. Therefore, when the average gradient profits are high, the best-response price must be locally stable and GL must converge.
2.6 Endogenous switching between learning mechanisms

We introduce competition between the learning rules in this section. We extend the model by allowing for endogenous switching between the two methods: firms may choose from the two learning rules in each period. For deciding about the rules, firms take into account their performance: the probability of choosing a specific method is positively related to the past profits realized while using that method. Section 2.6.1 specifies the switching mechanism, the simulation results are discussed in Section 2.6.2. The simulations confirm that the cyclical switching we conjectured may occur.

2.6.1 The switching mechanism

The switching mechanism is based on reinforcement learning as in Roth and Erev (1995) and it is related to the discrete choice model as in Brock and Hommes (1997). The mechanism is augmented with experimentation too. Every firm $i$ has a performance measure for each of the rules. These measures determine the probability of firm $i$ applying a certain rule. Performances depend on past realized profits. Let $l_i(t)$ denote the performance of LSL (GL) perceived by firm $i$ at the end of period $t$. The performance measure for LSL is updated in each period in the following way:

$$l_i(t) = \begin{cases} 
(1 - w)l_i(t-1) + w\pi_i(t) & \text{if firm } i \text{ used LSL in period } t \\
l_i(t-1) & \text{otherwise}
\end{cases}$$

where $w \in (0, 1]$ is the weight of the latest profit in the performance measure. The performance of GL is updated analogously. The initial performances are the first profits that were realized using the method in question for each firm. Thus, performance measures are basically weighted averages of past profits realized by the given method where weights decay geometrically.

These performance measures determine the probability of applying a learning method in the
following way. Firm \( i \) applies LSL in period \( t + 1 \) with probability

\[
P_{i,t+1}^{LS} = (1 - 2\eta) \frac{1}{\exp[\omega (g_{i,t} - l_{i,t})] + 1} + \eta, \tag{2.13}
\]

where \( \omega \geq 0 \) measures how sensitive the firms are to differences in the performance measures and \( \eta \) determines the probability of experimentation. The higher \( \omega \) is, the higher the probability of applying the method with the higher performance. For \( \omega = 0 \) firms choose both methods with 50\% probability. When \( \omega = +\infty \), then firms choose the method with the higher performance with probability \( 1 - \eta \). The interpretation of (2.13) is that the choice is based on the performance difference between the methods with probability \( 1 - 2\eta \) and the firm randomizes with equal probabilities between the methods with probability \( 2\eta \).

The model with endogenous switching is implemented as follows.

1. \( p_{i,1} \) and \( p_{i,2} \) are drawn from the uniform distribution on the set \( S \), for each \( i \).

2. In period 3, \( k \) randomly chosen firms apply LSL, the other firms use GL. LS and gradient prices are determined by the learning mechanisms discussed in Section 2.4.

3. In period 4:
   a. Firms try the other method: all LS learners switch to GL and vice versa. Prices are determined by the two learning mechanisms. The initial performances are \( l_{i,4} = \pi_{i}^{LS} \) and \( g_{i,4} = \pi_{i}^{grad} \).
   b. Firms choose a method for the following period: firm \( i \) applies LSL in period 5 with probability \( P_{i,5}^{LS} \).

4. In period \( t \geq 5 \):
   a. Prices are determined by the two learning mechanisms. The performance measures \( l_{t} \) and \( g_{t} \) are updated.
   b. Firm \( i \) chooses LSL for period \( t + 1 \) with probability \( P_{i,t+1}^{LS} \).

\( \pi_{i}^{LS} (\pi_{i}^{grad}) \) denotes the profit of firm \( i \) that was earned while using LSL (GL) in period 3 or 4.
5. The process stops when a predefined number of periods $T$ is reached.

In the simulations of the following section we use $w = 0.5$, $\omega = 25$ and $\eta = 0.005$. We simulate the model for $T = 10000$ periods.

Erev et al. (2010) find evidence for inertia in subjects’ choices in market entry experiments: subjects tend to choose the same action unless there is a substantial drop in the corresponding payoff. Anufriev and Hommes (2012) also find substantial evidence of inertia in forecasting behavior by estimating the individual learning model of heuristic switching on data from learning to forecast experiments. Inertia could be incorporated in the switching mechanism, we discuss its effects in the concluding remarks.

### 2.6.2 Learning cycles

First we shortly discuss the results of simulations when the convergence properties of GL do not change as the distribution of learning methods over firms varies and then we illustrate cyclical switching. When GL always converges (i.e. for any number of LS learners), then LSL need not be driven out even if the firms are very sensitive to performance differences. Some firms may earn a high LS profit and they apply LSL not only due to experimentation but in many consecutive periods. However, the number of LS learners is typically low. In contrast, when GL diverges fast and firms are sufficiently sensitive to performance differences, then GL is driven out by LSL. Firms apply GL only due to experimentation, each firm uses LSL in almost every period. When firms are less sensitive to differences in the performance measures, then GL is used more often but LSL is applied in the vast majority of the periods.

Now let us consider the case when the convergence properties of GL change as the distribution of learning methods over firms varies. First we illustrate cyclical switching in a duopoly because it is easier to see what drives the firms’ switching behavior when the number of firms is low. Then we show that cyclical switching occurs for higher number of firms too. We use the same demand and cost parameters as before. Figure 2.8 depicts typical time series of prices and the corresponding performance measures for the case of two firms. We used $\lambda = 0.85$ and
Figure 2.8: Cyclical switching in a duopoly. Time series of prices (upper panel), the performance measures of firm 1 (middle panel) and firm 2 (lower panel). Parameter values: $\alpha_1 = 35$, $\alpha_2 = 4$, $\alpha_3 = 2$, $\beta = 0.7$, $\gamma = 0.6$, $c = 4$, $\lambda = \lambda_0 = 0.85$, $k = 1$, $w = 0.5$, $\omega = 25$ and $\eta = 0.005$. 

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$k = 1$ in the simulation. GL is stable for this value of $\lambda$ only if one firm uses it. In the first third of the illustrated periods firm 1 uses mainly LSL while firm 2 is a gradient learner. Firm 1 tries GL in one period but it immediately switches back to LSL as the latter performs better. This change in the price of firm 1 drives away the price of firm 2 from the best-response price and it takes a few periods until the gradient learner reaches the optimal price again. Later firm 1 tries GL again and this induces a change in prices after which the firm becomes a gradient learner. When both firms apply GL, prices start an oscillating divergence. At some point the performance of GL becomes worse than that of LSL and firm 1 switches back to LSL. This ends the first oscillating part. GL, however, becomes more profitable again for firm 1 and another oscillating part starts. This part ends in the same way: firm 1 switches back to LSL after which the price set by firm 2 starts to converge. The last oscillating part starts by firm 2 switching to LSL. The price set by firm 2 decreases which yields a lower profit for firm 1. Because of this firm 1 switches to GL.

Cyclical switching can occur for a higher number of firms too. Figure 2.9 illustrates this for 10 firms with $\lambda = 0.95$ and $k = 5$. We can observe both diverging and converging phases for gradient learners which shows that the stability of the method changes. This change is related to the number of LS learners. In the first periods, the number of LS learners is high and we observe that the gradient learners’ prices converge. Then some LS learners switch to GL, which is reflected in the drop in $n_L$. As the time series of prices shows, GL becomes unstable. After that the number of LS learners starts increasing gradually until it reaches the level $n_L = 9$, for which the gradient price shows a converging pattern again. Then firms start switching to GL again, which destabilizes the price. We have found evidence for cyclical switching for the case of complements too.

Note that cyclical switching may occur only if the value of parameter $\lambda$ is such that GL converges when there are few gradient learners and it diverges otherwise. For any parameter values that satisfy Assumption 2.3.1, we can find values of $\lambda$ for which the gradient learners’ price is locally stable when the number of gradient learners is low and unstable otherwise, for a
given average LS price. Note, however, that this change in stability does not ensure that cyclical switching occurs: local stability does not imply that convergence occurs for any initial values. Nevertheless, for any parameter values that satisfy Assumption 2.3.1, there exist values of $\lambda$ for which cyclical switching may in general occur, but it may be harder to find such values of $\lambda$ for some parameter values than for others.
2.7 Concluding remarks

In this chapter we have relaxed the assumption that firms have complete knowledge about their market environment and we introduced learning in the model. Due to the richness of possible learning methods, firms may prefer to use different ways of learning about their environment. We demonstrate that several learning methods can coexist, in the sense that there is no clear winner in the profit-driven evolutionary competition between the methods, even when one of the methods is structurally misspecified. We stress that this coexistence may have a substantial effect on the dynamical properties of the learning methods and that the dynamics with heterogeneity in learning methods is much more complex than under homogeneous learning.

In this chapter we have analyzed the interaction between least squares learning and gradient learning in a Bertrand oligopoly with differentiated goods where firms do not know the demand specification and they use one of the two methods for determining the price of their good. These learning methods have been widely used for modeling learning behavior in oligopolistic markets, but mainly in a homogeneous setup. The methods that we have chosen are not atypical in the sense that other learning methods may lead to similar results: best response learning, for example, would yield similar outcomes in the current model as stable GL.

We have analyzed four different setups. In a pure LS-learning oligopoly firms move towards a self-sustaining equilibrium in which their expected and actual demands coincide at the prices they charge. The set of SSE prices contains infinitely many points including the Nash equilibrium of the model. The initial conditions determine which point is reached in the long run. We formally prove that firms reach the Nash equilibrium when every firm applies GL and the method converges. When GL does not converge, then it leads to periodic cycles, quasi-periodic or aperiodic dynamics. In a heterogeneous oligopoly with firms applying a fixed learning method, we have analytically derived that the dynamical properties of GL depend on the distribution of learning methods over firms. Numerical analysis shows that an increase in the number of LS learners can have a stabilizing effect. When GL converges, then LS learners move towards a self-sustaining equilibrium in which gradient learners give the best response to
the prices of other firms. When endogenous switching between the learning methods is introduced in the model, then a stable GL may not always drive out LSL: some LS learners may find LSL to be more profitable for them. LSL, however, may drive out GL when the latter never converges. When the convergence properties and the profitability of GL changes as the distribution of learning methods over firms varies, a cyclical switching between the learning methods may be observed. Gradient learners tend to switch to LSL when GL does not converge and thus gives low profits. This decrease in the number of gradient learners can stabilize the method, resulting in higher profits. This can give an incentive for LS learners to switch back to GL. GL, however, may lose its stability again and the cycle may repeat itself.

The previous analysis can be extended in several ways. Observations could have different weights in the LS formulas. Since observations of the early periods are less informative about the demand function due to the volatility of the prices of other firms, it might be reasonable to introduce a weighting function that gives less weight to older observations. We will investigate the effects of applying a weighting function in Chapter 3. Erev et al. (2010) find specific behavioral regularities in market entry experiments where subjects need to make decisions based on their experience. Since firms have similar tasks in our model, implementing some of these regularities can make the learning methods and the switching behavior empirically more relevant. For example, we could consider inertia in the switching behavior: firms tend to keep their current learning method unless there is a large drop in its profitability. The results of this chapter remain valid since inertia does not affect the main driving factor of cyclical switching: the stability of GL. However, it would take a longer time to observe cyclical switching as there would be less switching in the model due to inertia.

Another factor that can be considered is individual differences within the class of learning methods. In the case of LSL, firms could use different functional forms as the perceived demand functions. Furthermore, there could be informational differences among firms: some firms may observe the prices of some other firms and then they can make use of this information in their perceived demand function. The steady state analysis of LSL suggests that firms would
still reach a self-sustaining equilibrium when the perceived demand functions are misspecified. In the case of GL, individual differences could be incorporated in the model with different values of the adjustment parameter \( \lambda \) for different firms. Preliminary analysis shows that the stability of GL depends on the distribution of adjustment coefficients in this case. If the stability condition is satisfied, then every gradient learner reaches the best response price. If, however, the stability condition is not satisfied, then none of the prices of gradient learners converges. Our simulations show that the amplitude of the price oscillation and the profitability of the method are negatively related to the value of the adjustment parameter in this latter case. Even when we implement heterogeneity in the adjustment parameter of gradient learners, we can observe switching between the learning methods. Firms with very high and very low values of \( \lambda \) almost always use LSL, whereas firms with intermediate values of \( \lambda \) keep switching between the learning methods as the convergence properties of GL vary. More formal analysis of these extensions is left for future work.

The analysis can be extended to other learning methods and different market structures as well. For instance, best-response learning, fictitious play or imitation could also be applied in the current setup. It might be interesting to analyze what happens under Cournot competition when the quantities set by firms are strategic substitutes. Moreover, learning in more complex environments where firms make not only a price or quantity choice but they also need to make investment, quality or location decisions, can be studied as well along the lines outlined in this chapter.

In the next chapter we will further analyze the properties of least squares learning. We will see that the method does not necessarily lead to the Nash equilibrium even when firms can observe the prices of all the other firms and the functional form of the perceived demand function is only slightly misspecified.
Appendix 2.A  Proofs of Propositions

The proof of Proposition 2.3.2

Proof. The profit of firm $i$ is given by $\pi_i(p_i) = (p_i - c) \left( \alpha_1 - \alpha_2 p_i^\beta + \alpha_3 \bar{p}_i^\gamma \right)$. The first-order condition with respect to $p_i$ is

$$\alpha_1 - \alpha_2 p_i^\beta + \alpha_3 \bar{p}_i^\gamma - \alpha_2 \beta (p_i - c) p_i^{\beta - 1} = 0.$$  

(2.14)

This equation needs to hold for all firms. We will show that firms choose the same price in equilibrium.

Consider two arbitrary firms $i$ and $j$ and suppose indirectly that $p_i > p_j$ in equilibrium. Let $y = \sum_{k=1}^{n} p_k - p_i - p_j$. Then the first-order conditions for firms $i$ and $j$ read as

$$\alpha_1 - \alpha_2 p_i^\beta + \alpha_3 \left( \frac{p_j + y}{n-1} \right)^\gamma - \alpha_2 \beta (p_i - c) p_i^{\beta - 1} = 0,$$

(2.15)

$$\alpha_1 - \alpha_2 p_j^\beta + \alpha_3 \left( \frac{p_i + y}{n-1} \right)^\gamma - \alpha_2 \beta (p_j - c) p_j^{\beta - 1} = 0.$$  

(2.16)

Subtracting (2.15) from (2.16) yields

$$\alpha_2 \left( p_i^\beta - p_j^\beta \right) + \alpha_3 \left[ \left( \frac{p_i + y}{n-1} \right)^\gamma - \left( \frac{p_j + y}{n-1} \right)^\gamma \right] + \alpha_2 \beta \left[ (p_i - c) p_i^{\beta - 1} - (p_j - c) p_j^{\beta - 1} \right] = 0.$$

The first two terms are positive as $p_i > p_j$ and all parameters are positive. We will now show that the last term is also positive. Let $g(x) = (x - c)x^{\beta - 1}$. This function is increasing if $x \geq c$:

$$g'(x) = \beta x^{\beta - 1} - c(\beta - 1)x^{\beta - 2} > 0$$

for $x > c(1 - \frac{1}{\beta})$. This proves that the last term is also positive as $p_i > p_j$. This, however, leads to a contradiction as positive numbers cannot add up to zero. So we must have $p_i = p_j$: firms charge the same price in a Nash equilibrium. Let $p$

---

27 We assume in this formula that demands are positive in a Nash equilibrium. Later we will see that this is indeed the case.

28 We will see later that the condition $x \geq c$ holds for the Nash equilibrium price.
denote the corresponding price. Then (2.14) gives

\[ f(p) \equiv \alpha_1 - \alpha_2 p^\beta + \alpha_3 p^\gamma - \alpha_2 \beta (p - c) p^{\beta - 1} = 0. \] (2.17)

We will now show that there is a unique solution to this equation and the corresponding price is larger than the marginal cost. According to Assumption 2.3.1, \( f(c) = \alpha_1 - \alpha_2 c^\beta + \alpha_3 c^\gamma > 0 \). Note that \( f(p) \) becomes negative for high values of \( p \):

\[ f(p) = \alpha_1 - p^\gamma \left[ \alpha_2 p^{\beta - \gamma} - \alpha_3 + \alpha_2 \beta \left( 1 - \frac{c}{p} \right) p^{\beta - \gamma} \right], \]

from which it is easy to see that \( \lim_{p \to +\infty} f(p) = -\infty \). The derivative of \( f(p) \) is \( f'(p) = -\alpha_2 \beta p^{\beta - 1} + \alpha_3 \gamma p^{\gamma - 1} - \alpha_2 \beta p^{\beta - 1} \left[ 1 + (\beta - 1) \left( 1 - \frac{c}{p} \right) \right] \). Assumption 2.3.1 ensures that the sum of the first two terms is negative. The last term is also negative when \( p > c \). Thus, \( f(p) \) is strictly decreasing in \( p \) for \( p > c \). Since \( f(p) \) is continuous, this proves that there is a unique solution to \( f(p) = 0 \). Let \( p^N \) denote the symmetric Nash equilibrium price. It follows easily from the proof that \( p^N > c \) and the demands are positive in the Nash equilibrium.

We will show that the second order condition is satisfied. Differentiating (2.14) with respect to \( p_i \) yields

\[ -2\alpha_2 \beta p_i^{\beta - 1} - \alpha_2 \beta (\beta - 1) (p_i - c) p_i^{\beta - 2} = -\alpha_2 \beta p_i^{\beta - 1} \left( 2 + (\beta - 1) \frac{p_i - c}{p_i} \right), \]

This is negative for \( p = p^N \) since the term in brackets is positive: \( \frac{p^N - c}{p^N} \in (0, 1) \) as \( p^N > c \) and \( \beta - 1 > -1 \), so \( (\beta - 1) \frac{p^N - c}{p^N} > -1 \).

**The proof of Proposition 2.4.2**

**Proof.** First we derive the coefficients of the perceived demand functions in an SSE in terms of the SSE prices and then we study which prices may constitute an SSE.
From (2.9) we get \( a^*_i = D_i(p^*) + b^*_i p^*_i \). Combining this expression with (2.8) yields

\[
b^*_i = \frac{D_i(p^*)}{p^*_i - c}.
\]  

(2.18)

Using (2.18) we can express \( a^*_i \) as

\[
a^*_i = D_i(p^*) \left(1 + \frac{p^*_i}{p^*_i - c}\right).
\]  

(2.19)

The above described values constitute an SSE only if the inverse demand functions are sensible. That is, the following conditions need to be satisfied for all firms:

\[
a^*_i > 0, 
\]  

(2.20)

\[
b^*_i > 0, 
\]  

(2.21)

\[
P^*_i(0) = \frac{a^*_i}{b^*_i} > c, 
\]  

(2.22)

\[
P_i(0, \bar{p}_{-i}) = \left[ \frac{1}{\alpha_2} \left( \alpha_1 + \alpha_3 (\bar{p}_{-i}^*)^\gamma \right)^\frac{1}{\gamma} \right] > c, 
\]  

(2.23)

\[p^*_i > c. 
\]  

(2.24)

Conditions (2.20) and (2.21) ensure that the perceived demand functions are downward-sloping with a positive intercept. Conditions (2.22) and (2.23) require that the perceived and the actual inverse demands are larger than the marginal cost at \( q_i = 0 \). Condition (2.24) specifies that the SSE prices should be larger than the marginal cost. We will show that some of these constraints are redundant.

Conditions (2.20) and (2.21) hold true if and only if \( D_i(p^*) > 0 \) and \( p^*_i > c \). Combining (2.18) and (2.19) yields

\[
\frac{a^*_i}{b^*_i} = p^*_i - c + p^*_i = 2p^*_i - c.
\]
This shows that (2.22) is equivalent to (2.24). We can express \( D_i(p^*) > 0 \) as

\[
\left[ \frac{1}{\alpha_2} \left( \alpha_1 + \alpha_3(p_{-i}^*)^\gamma \right) \right]^{\frac{1}{\beta}} > p_i^*.
\]

Combining this with \( p_i^* > c \) shows that (2.23) is satisfied. Thus, the set of SSE prices is given by \( p_i^* > c \) and \( D_i(p^*) > 0 \), or equivalently \( c < p_i^* < \left[ \frac{1}{\alpha_2} \left( \alpha_1 + \alpha_3(p_{-i}^*)^\gamma \right) \right]^{\frac{1}{\beta}} \).

This set is nonempty: the Nash equilibrium price, for example, satisfies the above condition. The maximal SSE price of firm \( i \) increases in the price of other firms. Thus, the upper bound of the SSE prices is given by the price \( \hat{p} \) for which the demand is 0 if every firm charges this price: \( \alpha_1 - \alpha_2 \hat{p}^\beta + \alpha_3 \hat{p}^\gamma = 0 \). The existence and uniqueness of this price can be shown in the same way as for the Nash equilibrium price.

**The proof of Propositions 2.4.3 and 2.5.2**

*Proof.* Let us consider a heterogeneous setting in which the first \( n_L \) firms apply LSL and the remaining \( n - n_L \) firms use GL. The proof of Proposition 2.4.3 follows from this general case by setting \( n_L = 0 \).

In the proof we will apply a lemma about the eigenvalues of a matrix that has a special structure. First we will prove this lemma and then we prove Propositions 2.4.3 and 2.5.2.

**Lemma 2.A.1.** Consider an \( n \times n \) matrix with diagonal entries \( d \in \mathbb{R} \) and off-diagonal entries \( o \in \mathbb{R} \). In case of \( n = 1 \) the matrix has one eigenvalue: \( \mu = d \). If \( n > 1 \), then there are two distinct eigenvalues: \( \mu_1 = d + (n - 1) o \) (with multiplicity 1), and \( \mu_2 = d - o \) (with multiplicity \( n - 1 \)).

*Proof.* The case \( n = 1 \) is trivial so we focus on \( n > 1 \). Let \( A \) denote the matrix in question. Due to its special structure, \( A \) can be expressed as \( A = (d - o) I_n + o 1_n \), where \( I_n \) is the \( n \)-dimensional identity matrix and \( 1_n \) is the \( n \)-dimensional matrix of ones.

First note that if \( \lambda \) is an eigenvalue of \( o 1_n \) with corresponding eigenvector \( x \), then \( x \) is an eigenvector of \( A \) for the eigenvalue \( d - o + \lambda \): if \( o 1_n x = \lambda x \), then \( A x = ((d - o) I_n + o 1_n) x = \)

52
\[(d-o)x + \lambda x = (d-o+\lambda)x.\]

It is easy to see that \(o_1\times_n\) has two distinct eigenvalues: \(o \cdot n\) with multiplicity 1 and 0 with multiplicity \(n-1\). Thus, \(A\) has two distinct eigenvalues: \(\mu_1 = d-o+o \cdot n = d+(n-1)o\) with multiplicity 1 and \(\mu_2 = d-o\) with multiplicity \(n-1\).

Now the dynamical properties of the heterogeneous oligopoly can be studied in the following way. Suppose that LS prices have settled down at some level and let \(p^*_i\) denote the price of LS learner \(i\) \((i=1,\ldots,n_L)\). Since LS prices have settled down, the law of motion of the prices set by LS learners can be approximated by \(p_{i,t+1} = p_{i,t}\) for \(i=1,\ldots,n_L\) as price changes become smaller as the number of observations increases. The law of motion of the price set by gradient learners is given by \(p_{j,t+1} = p_{j,t} + \lambda \frac{\partial \pi_j(p)}{\partial p_{j,t}}\) for \(j=n_L+1,\ldots,n\). Then the Jacobian (evaluated at the steady state) is of the following form:

\[
J = \left( \begin{array}{cc} I & 0 \\ B & A \end{array} \right),
\]

where \(I\) is the \(n_L \times n_L\) identity matrix, \(0\) is an \(n_L \times (n-n_L)\) matrix of zeros, \(B\) is an \((n-n_L) \times n_L\) matrix with all entries equal to

\[
o = \lambda \frac{\partial^2 \pi_j(p)}{\partial p_i \partial p_j} = \lambda \alpha_3 \gamma \frac{1}{n-1} \left( \sum_{s=1}^{n_L} p^*_s + (n-n_L-1)p_G \right) \gamma^{-1},
\]

and \(A\) is an \((n-n_L) \times (n-n_L)\) matrix with diagonal entries

\[
d = 1 + \lambda \frac{\partial^2 \pi_j(p)}{\partial p^2_j} = 1 - \alpha_2 \beta \lambda p_G^{\beta-1} \left( 2 + (\beta - 1) \frac{p_G - c}{p_G} \right)
\]

and off-diagonal entries equal to \(o\).

Due to its special structure, the eigenvalues of \(J\) are given by the eigenvalues of \(I\) and the eigenvalues of \(A\). The stability properties of GL are determined fully by the eigenvalues.
of A. Applying Lemma 2.A.1, the eigenvalues that determine the stability of GL are 
\[ \mu_1 = d + (n - n_L - 1) o \]
with multiplicity 1 and \( \mu_2 = d - o \) with multiplicity \( n - n_L - 1 \). If \( n - n_L = 1 \), then the unique eigenvalue is \( \mu = d \).

When \( n - n_L = 1 \), the stability condition becomes
\[
\lambda \alpha_2 \beta p_G^{\beta - 1} \left( 2 + (\beta - 1) \frac{p_G - c}{p_G} \right) < 2.
\]
The eigenvalue becomes \(-1\) at the bifurcation. When \( n - n_L > 1 \), the stability conditions
\(-1 < \mu_i < 1\) simplify to \( 0 < \lambda M_1 < 2 \) and \( 0 < \lambda M_2 < 2 \) where

\[
M_1 = \alpha_2 \beta p_G^{\beta - 1} \left[ 2 + (\beta - 1) \frac{p_G - c}{p_G} \right] - \alpha_3 \gamma \left( n_L \bar{p}^* + (n - n_L - 1) p_G \right) \frac{\gamma^{-1} n - n_L - 1}{n - 1},
\]
\[
M_2 = \alpha_2 \beta p_G^{\beta - 1} \left[ 2 + (\beta - 1) \frac{p_G - c}{p_G} \right] + \alpha_3 \gamma \left( n_L \bar{p}^* + (n - n_L - 1) p_G \right) \frac{\gamma^{-1} 1}{n - 1},
\]

and \( \bar{p}^* = \frac{1}{n_L} \sum_{s=1}^{n_L} p_s^* \) is the average LS price.

By setting \( n_L = 0 \) it is easy to see that the above expressions simplify to

\[
M_1 = \alpha_2 \beta p_N^{\beta - 1} \left[ 2 + (\beta - 1) \frac{p_N - c}{p_N} \right] - \alpha_3 \gamma p_N^{\gamma - 1},
\]
\[
M_2 = \alpha_2 \beta p_N^{\beta - 1} \left[ 2 + (\beta - 1) \frac{p_N - c}{p_N} \right] + \alpha_3 \gamma p_N^{\gamma - 1} \frac{1}{n - 1}
\]

for the case of a homogeneous gradient-learning oligopoly, where \( p_N \) is the symmetric Nash equilibrium price. Since \( \alpha_3 > 0 \), \( M_2 > M_1 \). It follows from Assumption 2.3.1 that \( \alpha_2 \beta p^{\beta - 1} > \alpha_3 \gamma p^{\gamma - 1} \) for all \( p \geq c \). This ensures that \( M_1 \) is always positive:

\[
M_1 = \alpha_2 \beta p_N^{\beta - 1} \left[ 2 + (\beta - 1) \frac{p_N - c}{p_N} \right] - \alpha_3 \gamma p_N^{\gamma - 1} > \alpha_2 \beta p_N^{\beta - 1} \left[ 1 + (\beta - 1) \frac{p_N - c}{p_N} \right] > 0
\]
since \( 1 + (\beta - 1) \frac{p_N - c}{p_N} > 0 \). Thus, the relevant stability condition in the homogeneous case is \( \lambda M_2 < 2 \). At the bifurcation value of \( \lambda \), \( n - 1 \) eigenvalues become \(-1\) while the remaining eigenvalue is positive and smaller than one. \( \square \)
Chapter 3

Coexistence of Stable Equilibria under Least Squares Learning

3.1 Introduction

In the previous chapter we have considered, amongst other things, a misspecified version of least squares learning where firms do not take into account all the relevant variables that affect the demand for their good and they use an incorrect functional form in the regression. We have seen that this learning method leads to an outcome that is unrelated to any benchmark outcome of the static one-shot model under complete knowledge about the demand structure, such as the Nash equilibrium or the collusive outcome. Moreover, firms do not learn the true demand conditions correctly. This result is general in the literature on misspecified least squares learning even if firms use a correct functional form, see Gates et al. (1977) and Brousseau and Kirman (1992) for example. On the other hand, least squares learning may converge to the rational expectations equilibrium and learn the true demand conditions in many other settings, see Marcet and Sargent (1989) and Evans and Honkapohja (2001) for example. An important condition for the success

This chapter is based on Kopányi (2014).
of the method is that agents use a correct functional form in the regression. Moreover, they have
to observe all variables that are relevant for the estimation. That is, least squares learning has
to be correctly specified both in terms of the functional form and in terms of the explanatory
variables.

In this chapter we take an intermediate step between the aforementioned branches of the
literature by assuming that agents use all the relevant variables in the regression and that the
functional form they use is correctly specified locally but not globally. This situation is of
particular interest as the two extreme cases, as we discussed, lead to substantially different
outcomes and it is unclear what kind of outcomes could be reached in this intermediate situation.
As our results show, in addition to the Nash equilibrium and the self-sustaining equilibria, a third
outcome can be reached in the model: the asymmetric learning-equilibrium. Firms correctly
learn their demand function in a neighborhood of the prices they ask and some firms charge
different prices than others in this equilibrium. As a result, some firms focus on a smaller part
of the market. This outcome was not present in previous models.

As a framework of the analysis, we consider a modified version of the circular road model
introduced by Salop (1979). Three firms produce a homogeneous good. Firms are located along
a circular road, in equidistant locations. Consumers are uniformly distributed along the circle.
When a consumer wants to buy the good, it needs to visit one of the firms. Transportation
is costly, consumers face a fixed transportation cost per distance unit. Thus, the total cost of
buying the good from a specific firm is given by the sum of the price the firm asks and the
transportation costs. Demand is inelastic, each consumer is assumed to buy exactly one unit of
the good, at the lowest possible total cost. We introduce heterogeneity on the consumer side.
There are two types of consumers, one type faces low transportation cost while the other type
faces a high one.

Firms do not know the market structure and they use LSL to learn the demand function
they face. The true demand function is piecewise linear but firms approximate it with a linear
function. Hence the approximation can be locally correct but globally incorrect as a firm can
get a correct approximation for only one of the linear parts of the true demand function. In this chapter we investigate which outcomes LSL can lead to in this situation. We analytically show that the model has three kinds of equilibria. When firms use all past observations in the estimation, LSL typically leads to a self-sustaining equilibrium.\(^1\) In this equilibrium firms choose the price that maximizes their expected profit subject to their beliefs about demand conditions and their beliefs are correct in equilibrium but they are incorrect out of equilibrium. On the other hand, when not all but only the most recent observations are used in the estimation, firms reach either the symmetric Nash equilibrium or the asymmetric learning-equilibrium. In this asymmetric learning-equilibrium two firms charge a low price and the third one asks a high price. The high-price firm attracts the high-type consumers only whereas the other two firms serve both consumer types. The intuition behind this equilibrium is that the high-price firm does not attract low-type consumers, therefore it underestimates the demand at low prices and it does not perceive it profitable to charge a lower price. We analytically investigate which conditions determine the outcome of the learning process and we run numerical simulations to evaluate how frequently the different outcomes are reached.

Least squares learning was applied in market competition in other papers as well. See Chapter 2 for an overview of the literature on misspecified LSL. Our results are in line with the findings of this literature when firms use all past observations in the regression. Tuinstra (2004) takes a similar approach as we do in the sense that he considers a perceived demand function that is locally correct but globally incorrect. In his paper, the perceived demand function is the linear approximation of the true nonlinear demand function at the current price vector (i.e. the perceived demand function matches the function value and the slope of the true demand function at the current price).\(^2\) Thus, the approximation is correct at the equilibrium point only, whereas it is correct in a neighborhood of an equilibrium in our model (in case of the Nash equilibrium and the asymmetric learning-equilibrium). Another important difference is that firms focus only

\(^1\)We have seen in Chapter 2 that the steady states of the model under least squares learning where firms focus on their own price effect only are self-sustaining equilibria. In this chapter we show that this result extends to the case when firms take into account the prices of other firms as well but the functional form is not correctly specified.

\(^2\)One of the equilibrium concepts in Silvestre (1977) is based on similar conditions.
on their own price effect in the approximation in Tuinstra’s paper while they take into account
the prices of other firms as well in our model.

The chapter is structured as follows. The circular road model is discussed in Section 3.2. In Section 3.3 we discuss least squares learning and we derive the equilibria of the model. We analyze the stability of the equilibria as well. Simulation results are reported in Section 3.4. Section 3.5 concludes. Proofs are presented in Appendix 3.A.

3.2 The circular road model

The circular road model, one of the baseline models of horizontal product differentiation, was introduced by Salop (1979). In this section we first review a simplified version of the model that is relevant for our analysis and then we introduce heterogeneity on the consumer side.

3.2.1 Homogeneous consumers

Consider the market for a homogeneous good that is produced by three firms. Firms simultaneously and independently set the price of the good. Production costs are given by the same function for each firm: \( C_i(q_i) = cq_i \) for each firm \( i \), where \( q_i \) is the production level of firm \( i \) and \( c > 0 \) is a parameter. Firms are located along a circular road, in equal distance from each other. Consumers are uniformly distributed along the circle, their mass (or equivalently the circumference of the circle) is normalized to 1.

Consumers need to visit one of the firms to purchase the good. They move along the circular road, facing a transportation cost \( s \) per distance unit. If the minimal distance between firm \( i \) and a given consumer is \( x \), then the consumer’s total cost for buying the good from firm \( i \) is \( p_i + sx \), where \( p_i \) is the price charged by firm \( i \) and \( sx \) is the total transportation cost.\(^3\) Demand is inelastic: each consumer buys exactly one unit of the good. Furthermore, consumers are assumed to buy the good at the lowest possible cost, thus from the firm for which the sum of

\(^3\)It is assumed that firms cannot price discriminate so they cannot charge different prices to consumers from different locations.
(a) Consumer $X$ is located at distance $x$ from $i$.  
(b) Consumer $Y$ is located at distance $y$ from $j$.

**Figure 3.1:** Illustrations for the circular road with three firms.

... the price and the total transportation cost is the lowest.$^4$

To explain in more detail how demands are determined, we first focus on the competition between firms $i$ and $j$ only and discuss how their demands depend on the prices they charge. Let us consider the consumer that is located on the segment between firms $i$ and $j$, at distance $x$ from firm $i$ (see Figure 3.1a). We refer to this consumer as consumer $X$ and to the segment between consumer $X$ and firm $i$ as segment $iX$. $^5$ For consumer $X$, the total cost of buying from firm $i$ is $p_i + sx$ while the total cost of buying from firm $j$ is $p_j + s \left( \frac{1}{3} - x \right)$ since the distance between the two firms is $\frac{1}{3}$. Thus, consumer $X$ buys from firm $i$ rather than from firm $j$ when $p_i + sx < p_j + s \left( \frac{1}{3} - x \right)$. When the two total costs are equal, the consumer is said to be indifferent between the two firms. In this case the location of this *indifferent consumer* can be expressed as $x = \frac{p_j - p_i}{2s} + \frac{1}{6}$. It is easy to see that when $X$ is the indifferent consumer, the consumers that are located on segment $iX$ prefer firm $i$ to firm $j$ while those on segment $Xj$ prefer firm $j$.

For determining the demands we distinguish three cases based on the location of the indifferent consumer. First, if the indifferent consumer is located strictly between the two firms, then

---

$^4$ An alternative interpretation of the model is that the circle represents the product space and the location of consumers determines their preferences for the different products. Consumers choose a product based on the prices and on the distances from their ideal product, which corresponds to their location.

$^5$ We use the following notation for segments of the circle. Segment $ab$ denotes the segment going from point $a$ to point $b$ clockwise. Thus, segments $ab$ and $ba$ form the whole circle together.
all the consumers that are closer to firm \( i \) than \( x \) would buy from firm \( i \) rather than for firm \( j \) (and vice versa). Thus, firm \( i \) attracts \( x \) consumers while firm \( j \) attracts \( \frac{1}{3} - x \) consumers. Second, when the indifferent consumer is exactly at the location of firm \( j \), that is for \( p_j = p_i + \frac{1}{3}s \), then all the consumers on segment \( ij \) prefer firm \( i \) to firm \( j \). Let us suppose that the indifferent consumer between firms \( j \) and \( k \) lies between the two firms, at distance \( y \) from firm \( j \) (see Figure 3.1b). We call this consumer \( Y \). In this case the consumers on segment \( jY \) are indifferent between firms \( i \) and \( j \). To see this note that consumers need to pay the transportation cost for traveling to the location of firm \( j \) irrespective of which firm they will choose eventually. And at the location of firm \( j \) they are indifferent between the two firms. Indifferent consumers are traditionally assumed to choose one of the firms with equal probability, thus half of the consumers on segment \( jY \) chooses firm \( i \) while the other half chooses firm \( j \).\(^6\) Thus, firm \( j \) will face a demand of \( 0.5y \) while firm \( i \) attracts \( \frac{1}{3} + 0.5y \) consumers. Finally, when the consumer at the location of firm \( j \) strictly prefers firm \( i \) to firm \( j \), that is when \( p_j > p_i + \frac{1}{3}s \), then firm \( j \) will not attract any consumer. This small exercise already shows two important features of the model: demand functions are discontinuous and firms can drive each other out of the market.

Taking the above considerations into account, the demand firm \( i \) faces can be expressed as a function of prices in the following way. Assume without loss of generality that \( p_j \leq p_k \). Let us first consider the case when firm \( j \) does not drive firm \( k \) out of the market, i.e. \( p_j > p_k - \frac{1}{3}s \).

\(^6\)We could consider a different share of the indifferent consumers choosing firm \( i \). This would, however, not affect the results of the chapter.
In this case, the demand function of firm $i$ is given by

$$D_i(p_i, p_j, p_k) = \begin{cases} 
1 & \text{if } p_i < p_j - \frac{1}{3}s \\
\frac{11}{12} & \text{if } p_i = p_j - \frac{1}{3}s \\
\frac{1}{2} + \frac{p_i - p_k}{s} & \text{if } p_j - \frac{1}{3}s < p_i < p_k - \frac{1}{3}s \\
\frac{3}{4} + \frac{3p_i - 3p_k}{4s} & \text{if } p_i = p_k - \frac{1}{3}s \\
\frac{1}{3} + \frac{p_i + p_k - 2p_i}{2s} & \text{if } p_k - \frac{1}{3}s < p_i < p_j + \frac{1}{3}s \\
\frac{1}{12} + \frac{p_i - p_k}{4s} & \text{if } p_i = p_j + \frac{1}{3}s \\
0 & \text{if } p_j + \frac{1}{3}s < p_i 
\end{cases}$$

(3.1)

Detailed derivations are presented in Appendix 3.A. We get a different demand function when firm $k$ is driven out of the market by firm $j$. Analogous calculations as for the previous case yield the following demand function:

$$D_i(p_i, p_j, p_k) = \begin{cases} 
1 & \text{if } p_i < p_j - \frac{1}{3}s \\
\frac{11}{12} & \text{if } p_i = p_j - \frac{1}{3}s \\
\frac{1}{2} + \frac{p_i - p_k}{s} & \text{if } p_j - \frac{1}{3}s < p_i < p_j + \frac{1}{3}s \\
\frac{1}{12} & \text{if } p_i = p_j + \frac{1}{3}s \\
0 & \text{if } p_j + \frac{1}{3}s < p_i 
\end{cases}$$

Note that in both cases the demand function is discontinuous and that it consists of piecewise linear parts. This one-shot game has a unique symmetric Nash equilibrium, in which each firm charges the price $p = c + \frac{s}{3}$. See Tirole (1988), p. 283 for the proof. Having discussed how the basic model works, let us introduce heterogeneity on the consumer side.

### 3.2.2 Heterogeneous consumers

Let us consider the same market structure as before but suppose that there are two types of consumers. The types differ with respect to the transportation cost they face: low-type consumers face a unit cost of $s$ while high-type consumers pay a unit cost of $S$, where $s < S$. The
amount of consumers of each type is normalized to 1, both types are assumed to be uniformly distributed along the circular road. Firms cannot distinguish the two types, they cannot price discriminate between different consumers.\(^7\)

Similarly to the case with homogeneous consumers, firms can drive each other out of the market by choosing a sufficiently low price. Moreover, firms can also be driven out one part of the market only: it can occur that a firm attracts high-type consumers but not low-type ones. Consider for example the situation \(p_j + \frac{1}{3}S < p_i < p_j + \frac{1}{3}S\). In this case the low-type consumer that is located at the position of firm \(i\) buys from firm \(j\) rather than from firm \(i\). Consequently, firm \(i\) does not attract low-type consumers. On the other hand, the high-type consumer at the location of firm \(i\) prefers firm \(i\) to firm \(j\). Thus, in the given situation, firm \(j\) drives firm \(i\) out of the market for low-type consumers but not for high-type consumers.

Demand functions are discontinuous and consist of piecewise linear parts again. There are more parts than under homogeneous consumers since firms can be driven out of multiple subparts of market in this case. We do not report the exact formula for the demand function here as it is not important to know it for understanding the results of the chapter. The relevant linear parts of the demand function are derived in Appendix 3.A. Figure 3.2 illustrates the demand and profit functions of firm \(i\) when the other two firms do not drive each other out of the market for either consumer type. We can see that the demand function indeed consists of linear parts. There are 7 linear parts, they correspond to the following cases (as \(p_i\) increases):
1. firm \(i\) serves the whole market; 2. low-type consumers are served by firm \(i\) only, high-type consumers are served by firms \(i\) and \(j\); 3. low-type consumers are served by firm \(i\) only, high-type consumers are served by all 3 firms; 4. low-type consumers are served by firms \(i\) and \(j\), high-type consumers are served by all 3 firms; 5. both consumer types are served by all three firms; 6. firm \(i\) serves high-type consumers only, the other two firms serve both consumer types;

\(^7\)We also assume that there are no arbitrage opportunities for consumers: low-type consumers cannot buy the good on behalf of a high-type consumer (as each consumer buys exactly one unit of the good). Note that this assumption is implicitly present in the standard model with homogeneous consumers as well: each consumer can buy exactly one unit of the good, they cannot reduce the transportation costs by asking a consumer that is located closer to a given firm to buy the good for them so that they would need to travel until the location of this consumer only and not until the firm.
and 7. firm $i$ is completely driven out of the market. We can see from the profit function that the profit-maximizing decision of firm $i$ in the given situation is to drive the other two firm out of the market for the low-type consumers but not for the high-type, as the profit maximum is reached in the third case.

The model with heterogeneous consumers has a unique Nash equilibrium in pure strategies. Proposition 3.2.1 specifies the equilibrium price. The proof of the proposition is presented in Appendix 3.A.

**Proposition 3.2.1.** The Salop model with three firms and two types of consumers has a unique Nash equilibrium in pure strategies. This equilibrium is symmetric, with all three firms charging the price $p_N = \frac{2Ss}{3(S+s)} + c$ and serving both consumer types.

Note that the proposition rules out the existence of asymmetric Nash equilibria. The Nash equilibrium price is increasing in both $s$ and $S$. The intuition behind this result is the following. When transportation costs are higher, it is harder for firms to attract consumers that are located farther away from them (or equivalently, it is more costly for consumers to visit firms that are farther away from them). This reduces competition, firms gain more market power and the equilibrium price increases consequently.

It can be seen that $\frac{\partial p_N}{\partial S} < \frac{\partial p_N}{\partial s}$, that is $s$ has a larger impact on the equilibrium price than
$S$ does. To understand this result, note the following. When a transportation cost increases, firms have an incentive to increase their price since they get more market power in the given market segment. When a firm increases its price, it will lose some low-type as well as high-type consumers. Since low-type consumers are more mobile, the firm will lose more low-type consumers. Thus, it is more favorable for firms when the transportation cost of low-type consumers increases since this makes low-type consumers less mobile, resulting in a lower decrease in demand after a price increase. Thus, the equilibrium price increases more when $s$ increases.

After analyzing the static model under full information, we now turn to a dynamic model in which firms do not know the market specification and they try to learn the demand condition based on the information they receive about the market.

### 3.3 Market dynamics under learning

When firms do not know the market structure, they need to learn the demand function to find the optimal action. When firms apply least squares learning, they approximate the true demand function with a *perceived* demand function and they estimate the unknown parameters of it using past observations about price and production levels.

We assume that the only information the firms have about the market is that there are three firms in the market. Thus, they do not know either about the circular-road structure of the market or about facing different consumer types. Firms are competing with each other on the same market over time and they can observe the price charged by their competitors and the corresponding demand for their own good (but not those of their competitors). Thus, firms gather information about the market over time and they can use this information to learn about the demand for their product.

In the following subsection we specify the learning method the firms use and then we discuss the equilibria of the model under learning.
3.3.1 Least squares learning

Firms approximate the demand for their product with a linear function. The perceived demand function of firm \( i \) is given by

\[
D_i^P(p) = a_i - b_{ii}p_i + b_{ij}p_j + b_{ik}p_k + \varepsilon_i, \quad (3.2)
\]

where \( a_i \) denotes the demand intercept, \( b_{ix} \) denotes the effect of firm \( x \)'s price on the demand for firm \( i \)'s product (\( x = i, j, k \)) and \( \varepsilon_i \) is a random noise with mean 0. Parameters \( a_i \) and \( b_{ix} \) are estimated with OLS regression using observations about past prices and own-production levels.

Firms might not want to use all past observations for the estimation therefore we need to make a distinction between a firm’s observations and information set. Observations of firm \( i \) consist of the prices of all three firms and the demand firm \( i \) faces for all past periods whereas the information set contains only those observations that are used in the regression.\(^8\) The rationale behind not using all observations in the regression is that older observations might carry less information about current demand conditions than more recent ones, especially when there is a structural break in the data. Even though demand conditions are fixed in the model we consider, not using all past observations, as we will see, has important consequences for the properties of LSL.

Let us suppose that firms use the last \( \tau \) observations for the regression. Then parameter estimates for firm \( i \) are given by the standard OLS formula

\[
\beta_i = (X'_i,\tau X_i,\tau)^{-1} X'_i,\tau y_i,\tau, \quad (3.3)
\]

where \( \beta_i = (a_i, b_{ii}, b_{ij}, b_{ik})' \) is the 4 \( \times \) 1 vector of parameter estimates\(^9\), \( X_i,\tau \) is the \( \tau \times 4 \) matrix containing the price observations for the last \( \tau \) periods (explanatory variables) and \( y_i,\tau \) is the

\(^8\)Note that we use the term information set in its econometric sense and not in its game theoretical sense.

\(^9\)Similarly to Chapter 2, we denote the unknown parameters of the perceived demand function as well as the corresponding parameter estimates by the same symbol. This should not be confusing as we will mainly work with the parameter estimates from now on.
\( \tau \times 1 \) vector of the last \( \tau \) demand observations of firm \( i \) (dependent variable).\(^{10}\)

Given the parameter estimates of the perceived demand function, firm \( i \) maximizes its perceived profit \( \pi_i^P(p) = (p_i - c)D_i^P(p) \). This gives the following best-response price:

\[
p_i^{BR} = \frac{a_i + b_{ij}p_j + b_{ik}p_k}{2b_{ii}} + \frac{c}{2}.
\] (3.4)

Let us now discuss timing. At the end of period \( t \) firms have observations about all \( t \) periods. Parameter estimates are obtained by (3.3). In order to stress that parameter estimates are changing over time, we will denote the parameter estimates at the end of period \( t \) as \( a_{i,t}, b_{ii,t}, b_{ij,t} \) and \( b_{ik,t} \). Since firms are determining their prices simultaneously, they can play the best response only against the expectations they have about the prices of other firms. Thus, we have to replace \( p_j \) with \( p_{j,t+1}^e \) and \( p_k \) with \( p_{k,t+1}^e \) in (3.4), stressing again the dependence on time. We assume that firms form naive expectations, meaning that they expect other firms to charge the same price as in the previous period: \( p_{j,t+1}^e = p_{j,t} \) and \( p_{k,t+1}^e = p_{k,t} \). This leads to the following pricing formula for period \( t + 1 \):

\[
p_{i,t+1} = a_{i,t} + \frac{b_{ij,t}p_{j,t} + b_{ik,t}p_{k,t}}{2b_{ii,t}} + \frac{c}{2}.
\] (3.5)

Note that profit maximization requires \( b_{ii,t} > 0 \), that is the perceived own-price effect must be negative. Since the perceived demand functions the firms use are not correctly specified, the parameter estimate for \( b_{ii,t} \) might become negative. In this case, (3.5) does not give the perceived profit-maximizing price. Also note that when (3.5) yields a price that is lower than the marginal cost, the firm would make a negative profit (provided that it faces a positive demand). Thus, (3.5) is not applicable in this case either. In order to overcome these problems with LSL, we augment the method with the following rule.

**Random price rule:** When \( b_{ii,t} \leq 0 \) or (3.5) yields \( p_{i,t+1} < c \), then firm \( i \) chooses a price

\(^{10}\)Similar formulas apply when firms use all past observations. The only difference is that \( X \) and \( y \) then contain the prices and the corresponding demand for all past periods.
randomly from the uniform distribution on a predefined interval $I$.

Interval $I$ is specified in Section 3.4. We need to impose additional rules to overcome some numerical issues that may occur when firms do not use all observations in the regression. When prices start to settle down at a given value, there is not enough dispersion in the observations and matrix $X_{i,\tau}$ is close to being singular, resulting in imprecise parameter estimates. This can lead to extremely high prices for some periods. Since it should be clear for firms that large unexpected price changes result from the aforementioned issue, it is reasonable to assume that firms do not follow pricing rule (3.5) in this case, they rather keep their price unchanged. This leads to the following rule.

**No jump rule:** If (3.5) yields a price that is at least $K$ times higher than the price of firm $i$ in the previous period, then the firm will keep its price unchanged and charge the same price as in the previous period.\(^{11}\)

When there is not enough dispersion in the price observations, matrix $X_{i,\tau}$ can become singular, making the estimation impossible. We assume that firms keep their price unchanged when estimation is not possible.

**Impossible estimation rule:** When (3.3) is not applicable due to the singularity of $X_{i,\tau}$, then firm $i$ will keep its price unchanged and charge the same price as in the previous period.

We will elucidate the effect of these rules in the Discussion in Section 3.5. Let us now turn to the steady states of the process.

\(^{11}\)Alternatively, we could impose an upper bound on price changes as Weddepohl (1995). In that case firms would choose the highest possible price if (3.5) resulted in a too large price jump. Since large price jumps are associated with imprecise parameter estimates in the model we consider, it makes more sense not to change the price at all.
3.3.2 Equilibria under least squares learning

The system is in a steady state when neither the parameter estimates of the perceived demand functions nor the prices change. It must hold for any steady state that the true and the expected demands coincide for each firm at the given price vector \( p^* \), that is \( D_i(p^*) = ED_i^P(p^*) \) for \( i = 1, 2, 3 \). To see this, note the following. When \( D_i(p^*) = ED_i^P(p^*) \), the perceived demand function perfectly approximates the true demand function for the given price vector as the corresponding estimation error is 0. Since the parameter estimates of the perceived demand function are obtained by minimizing the sum of squared errors, this implies that the parameter estimates do not change in this case.

The same condition characterizes the self-sustaining equilibria in Brousseau and Kirman (1992) and in Chapter 2. Thus, the steady states of the model with least squares learning are self-sustaining equilibria: firms play the best response subject to their beliefs about demand conditions (i.e. the perceived demand functions) and about the prices of the other firms, and these beliefs are correct at the equilibrium price vector. Self-sustaining equilibria can be formally defined as follows.

**Definition 3.3.1.** Price vector \( p^* = (p_1^*, p_2^*, p_3^*) \) and the parameter estimates \( \{a_i^*, b_{ii}^*, b_{ij}^*, b_{ik}^*\} \) \((i, j, k = 1, 2, 3; i \neq j \neq k)\) constitute a self-sustaining equilibrium if the following conditions hold for each firm \( i \):

\[
p_i^* = \frac{a_i^* + b_{ij}^* p_j^* + b_{ik}^* p_k^*}{2b_{ii}^*} + \frac{c}{2},
\]

\[
D_i(p^*) = ED_i^P(p^*).
\]

Condition (3.6) shows that firms play the best response subject to their beliefs and (3.7) means that beliefs are confirmed in equilibrium as the actual demand is the same as the demand the firm expects to get, and the prices of the competitors are also as expected.

It can be seen from the definition that there are many different self-sustaining equilibria, thus the model has multiple steady states. Proposition 3.3.2 specifies which price vectors can
Figure 3.3: Demand and profit functions of firm $i$ in a self-sustaining equilibrium. Parameters: $s = 1$, $S = 5$ and $c = 1$. Equilibrium prices: $p^*_i = 2.0398$, $p^*_j = 2.0264$ and $p^*_k = 2.2083$.

form a self-sustaining equilibrium.

**Proposition 3.3.2.** For any price vector $p = (p_1, p_2, p_3)$ satisfying the conditions $p_i > c$ and $D_i(p) > 0$ for $i = 1, 2, 3$, there exist parameter estimates $\{a_i, b_{ii}, b_{ij}, b_{ik}\}$ ($i, j, k = 1, 2, 3; i \neq j \neq k$) such that the model is in a self-sustaining equilibrium.

Thus, prices exceed the marginal cost and each firm faces a positive demand in a self-sustaining equilibrium. Note that the condition $D_i(p) > 0$ implies that none of the firms can be driven out of the market for both types of consumers. But it is not required that each firm should attract both consumer types. In the above result, we did not take into account that $\{a_i, b_{ii}, b_{ij}, b_{ik}\}$ are not freely chosen but they result from estimation. Therefore not all the price vectors that satisfy the conditions of Proposition 3.3.2 can necessarily be reached, despite the fact that we can find parameter values for which (3.6) and (3.7) hold.

Since perceived demand functions are linear while the true demand functions are piecewise linear, firms cannot fully learn the true demand conditions: they can correctly learn the parameters of at most one linear part. Note that condition (3.7) is required to hold at the equilibrium point only, thus firms need not learn in general any linear part correctly. Panel (a) of Figure 3.3 illustrates the true and the perceived demand functions of a firm in a typical self-sustaining equilibrium. The two functions cross each other in a single point thus the firm does not learn any
linear part of the true demand function correctly. Panel (b) depicts the true and the perceived profit functions. The figure shows that in the SSE firm \( i \) maximizes its expected perceived profit but the price it chooses does not yield the true profit maximum.

Even though it is not the case typically, there are self-sustaining equilibria in which firms correctly learn the part of the true demand function on which they operate. Proposition 3.3.3 specifies these equilibria.

**Proposition 3.3.3.** The model with least squares learning has two self-sustaining equilibria in which firms correctly learn that linear part of the true demand function on which they operate. The Nash equilibrium of the game is always such an equilibrium of the learning process. When 
\[
\frac{S}{T} \geq \Sigma_1 = \frac{7 + \sqrt{89}}{4} \approx 4.1085,
\]
there also exists another equilibrium in which two firms charge
\[
p_L = \frac{11S}{12S + 15s} + c \text{ and the third firm chooses } p_H = \frac{2S^2 + 8Ss}{12S + 15s} + c.
\]
We refer to this equilibrium as asymmetric learning-equilibrium (ALE).

Figure 3.4 illustrates the demand and profit functions in the Nash equilibrium and in the asymmetric learning-equilibrium. Panels (a), (c) and (e) confirm that in both equilibria firms correctly approximate the linear part of the true demand function on which they operate. Panel (b) shows that the true profit maximum coincides with the maximum of the perceived profit function of firms in the Nash equilibrium. The same holds for the low-price firms in the ALE (see panel (d)). Note, however, that the perceived profit maximum does not correspond to the true profit maximum for the high-price firm (panel (f)). This is why the ALE is not a Nash equilibrium of the game under known demand. As panel (e) shows, the high-price firm underestimates the demand for lower prices and thus it does not perceive it more profitable to charge a lower price, even though it would yield a higher profit. It reaches a local profit maximum only.\(^{12}\)

We compare the Nash equilibrium and the ALE in Appendix 3.A. We show that \( p_N < p_L < p_H \) whenever the ALE exists. This result is in line with the fact that prices are strategic com-

\(^{12}\)The ALE can be viewed as a local Nash equilibrium since the low-price firms reach their global profit maximum while the high-price firm is in a local profit maximum only. See Bonanno and Zeeman (1985) and Bonanno (1988) for more details about this concept.
(a) Demand functions (Nash equilibrium)

(b) Profit functions (Nash equilibrium)

(c) Demand functions of low-price firms (ALE)

(d) Profit functions of low-price firms (ALE)

(e) Demand functions of the high-price firm (ALE)

(f) Profit functions of the high-price firm (ALE)

Figure 3.4: Demand and profit functions in the Nash equilibrium and in the asymmetric learning-equilibrium. Parameters: \( s = 1 \), \( S = 5 \) and \( c = 1 \).
plements in the model: the high-price firm charges a higher price than in the Nash equilibrium and this gives an incentive for the other two firms to increase their price. That is why $p_L > p_N$. Concerning profits, a low-price firm always earns a higher profit than in the Nash equilibrium. The high-price firm, however, earns a lower profit only when $\frac{S}{s}$ is low enough. When $\frac{S}{s} > 8.91$, even the high-price firm earns a higher profit than in the Nash equilibrium, therefore each firm is better-off compared to the Nash equilibrium for such values of $\frac{S}{s}$. A low-price firm earns a higher profit than the high-price firm in the ALE only when $\frac{S}{s}$ is sufficiently low. When $\frac{S}{s} > \frac{89 + 11 \sqrt{73}}{8} \approx 22.87$, the high-price firm earns a higher profit. In this case the high-price firm still underestimates the demand for low prices but the perceived profit maximum coincides with the true profit maximum. On the other hand, low-price firms perceive a relatively high slope and they underestimate the demand for high prices. Their perceived profit maximum does not coincide with the true profit maximum as it would be more profitable to charge a higher price. Finally, we compare the total profit of the three firms in the Nash equilibrium and in the ALE. We find that the total profit is always higher in the ALE.

Since prices are higher in the ALE than in the Nash equilibrium, consumers are worse-off. Moreover, welfare (measured as the total surplus) is lower. Note that for comparing the welfare in the two outcomes, we can focus on transportation costs only. The reason for this is the following. The surplus of a consumer can be measured as the net utility of consuming the good: $v - p - sx$ (or $v - p - Sx$), where $v > 0$ is the positive utility from consumption while $p + sx$ (or $p + Sx$) is the total cost of purchasing the good.\footnote{Remember that each consumer is assumed to buy the good. This implies that $v$ is assumed to be sufficiently large.} Note that the price $p$ is simply a transfer between the consumer and the firm, therefore it does not have a direct effect on welfare. Also note that total production is the same in the Nash equilibrium and in the ALE. Since the marginal cost of production is constant and equal for the firms, the difference in individual production levels does not contribute to welfare differences. Thus, from a welfare perspective, only transportation costs matter. Transportation costs are higher in the ALE than in the Nash equilibrium for two reasons. First, low-type consumers go to the low-priced firms...
only, thus some of these consumers need to travel more. Second, the high-type consumers that are indifferent between the high-price firm and one of the low-price firms, lie closer to the high-price firm than under a symmetric situation (as in the Nash equilibrium). Therefore, those high-type consumers that visit the low-price firm but would visit the other firm in a symmetric situation, travel more than in the Nash equilibrium. Thus, even though total profits are higher, welfare is lower in the ALE.

3.3.3 Stability of equilibria

As we have seen in the previous section, the model with least squares learning has three types of equilibria: a general self-sustaining equilibrium, the Nash equilibrium and the asymmetric learning-equilibrium. Next we will investigate which equilibria can be reached and which factors determine which of the equilibria is reached. It turns out that a special property of the information set plays a crucial role in this. Before defining this property, note that different price vectors may correspond to different demand conditions. For example, firm $i$ may serve both types of consumers for one price vector whereas it might serve high-type consumers only for another price vector. These price observations carry information about different structural parameters as they lie on different linear parts of the true demand function. We call price vectors in the information set of firms **aligned** when each firm serves the same consumer type(s) for each price vector. We distinguish two kinds of aligned price vectors. When all three firms serve both consumer types, we speak about **symmetrically aligned** prices. When two of the firms serve both consumer types while the third one attracts high-type consumers only, we speak about **asymmetrically aligned** prices.\(^{14}\) We define these concepts formally as follows.

**Definition 3.3.4.** A set of price vectors $P \subseteq \mathbb{R}^3_+$ is called **symmetrically aligned** when all three

\(^{14}\)Note that prices could be aligned in other ways as well. For example, we could consider the case when exactly one firm attracts both types of consumers while the other two firms attract high-type consumers only. We do not consider other possibilities because they are not relevant for the equilibria of the learning process, as we have seen.
firms attract both types of consumers for all \( p \in P \): \[
|p_i - p_j| < \frac{s}{3} \quad \forall i, j = 1, 2, 3.
\]

A set of price vectors \( P \subseteq \mathbb{R}^3_+ \) is called *asymmetrically aligned* when firms \( i \) and \( j \) attract both types of consumers while firm \( k \) attracts only the high-type consumers for all \( p \in P \): \[
|p_i - p_j| < \frac{s}{3}
\]
\[
\min\{p_i, p_j\} + \frac{s}{3} < p_k < \min\{p_i, p_j\} + \frac{S}{3}.
\]

A set of price vectors \( P \subseteq \mathbb{R}^3 \) is called *not aligned* when it is neither symmetrically, nor asymmetrically aligned.

The condition \( |p_i - p_j| < \frac{s}{3} \) ensures that firms \( i \) and \( j \) do not drive each other out of the market for either consumer type. The condition \( \min\{p_i, p_j\} + \frac{s}{3} < p_k < \min\{p_i, p_j\} + \frac{S}{3} \) means that firm \( k \) is driven out of the market for low-type consumers but not for the high-type ones.

When prices are aligned, then the corresponding demand observations are consistent in the sense that they lie on the same linear part of the demand function. That is, observations carry information about the same linear demand parameters and consequently firms correctly learn the parameters that characterize the linear part of the true demand function on which they operate.

Since firms play the best response to the prices of the other firms, subject to their perceived demand function, it is important to analyze the conditions under which a set of aligned price observations remains aligned after updating the set with the best-response prices. Lemma 3.3.5 summarizes these conditions.

**Lemma 3.3.5.** When price observations are symmetrically aligned, then updating the information set with the best-response prices always results in symmetrically aligned price observations again.

When price observations are asymmetrically aligned, there are three possibilities.
1. For $\frac{S}{s} < \Sigma_1$ price observations will not be asymmetrically aligned after updating the information set with the best-response prices sufficiently many times.

2. For $\frac{S}{s} \in [\Sigma_1, \Sigma_2)$ with $\Sigma_2 = 2 + \sqrt{6} \approx 4.4495$, the updated price observations will be asymmetrically aligned if the following condition holds for the most recent price observation $p$:

$$\frac{s}{3} \left[ 2 \left( \frac{S}{s} \right)^2 - 7 \frac{S}{s} - 4 \right] + (1 + \frac{S}{s}) |p_i - p_j| + \min\{p_i, p_j\} \geq p_k.$$

3. For $\frac{S}{s} \geq \Sigma_2$, price observations always remain asymmetrically aligned after updating the information set with the best-response prices.

According to this lemma, when the information set is symmetrically aligned, then it always remains symmetrically aligned. Thus, firms will learn the true parameters of the corresponding linear part. As Proposition 3.3.3 shows, the only equilibrium that firms may reach in this situation is the Nash equilibrium. Concerning asymmetrically aligned observations, Lemma 3.3.5 says that when $\frac{S}{s}$ is not high enough, the information set will not be asymmetrically aligned eventually even if firms start with an asymmetrically aligned information set. So in this case the possible steady states of the model are a general SSE and the Nash equilibrium. For intermediate values of $\frac{S}{s}$, an extra condition is needed for ensuring that the updated information set remains asymmetrically aligned. Thus, all three steady states may exist for these values of $\frac{S}{s}$. On the other hand, an asymmetrically aligned information set always remains asymmetrically aligned by updating it with the best response prices when $\frac{S}{s}$ is high enough. Thus, the only equilibrium in this case is the asymmetric learning-equilibrium. When price observations are not aligned, then firms cannot learn the true parameters of the linear part on which they operate, consequently the only kind of steady state in the given situation is a general self-sustaining equilibrium.

Note that these results concern existence only, under specific conditions. We have not analyzed the stability of these equilibria yet. Proposition 3.3.6 summarizes the dynamical proper-
initial observations | information set | all observations | last $\tau$ observations
--- | --- | --- | ---
symm. aligned | Nash | Nash | Nash / SSE
asymm. aligned | $\frac{S}{s} \leq \Sigma_1$ | SSE | ALE / SSE
| $\Sigma_1 < \frac{S}{s} < \Sigma_2$ | ALE / SSE | ALE / Nash / SSE
| $\Sigma_2 \leq \frac{S}{s}$ | ALE | ALE
not aligned | SSE | ALE / Nash / SSE

Table 3.1: The possible outcomes of the model with least squares learning for different types of initial observations and different number of observations in the information set.

Proposition 3.3.6. Both the Nash equilibrium and the asymmetric learning-equilibrium are locally stable equilibria of the model with least squares learning.

According to the proposition, firms will reach the Nash equilibrium when initial prices are close to the Nash equilibrium price. A similar result holds for the asymmetric learning-equilibrium. Combining these considerations with Lemma 3.3.5, we can conclude that the model has coexisting locally stable steady states when $\frac{S}{s}$ is sufficiently high. Note that Proposition 3.3.6 does not cover the stability of general self-sustaining equilibria. Brousseau and Kirman (1992) show that firms do not converge to a self-sustaining equilibrium in general. To process slows down only because the weight of a new observation decreases when firms use all observations in the estimation.

Taking into account the above theoretical results, we summarize the long-run outcome of the model in Table 3.1. When firms use all observations in the estimation, then all three equilibria can occur. More specifically, when initial observations are symmetrically aligned, firms converge to the Nash equilibrium. When initial observations are asymmetrically aligned, firms reach the asymmetric learning-equilibrium when $\frac{S}{s}$ is sufficiently high. When initial observations are not aligned or if they are asymmetrically aligned but $\frac{S}{s}$ is not high enough, then firms move towards a self-sustaining equilibrium.

When only the last $\tau$ observations are used in the regression, then the Nash equilibrium and the ALE can be reached more often for the following reason: an information set which is
not aligned might become symmetrically or asymmetrically aligned as some observations drop out of the information set at some point. Firms reach the Nash equilibrium for symmetrically aligned initial prices. When initial prices are asymmetrically aligned and \( \frac{S}{s} \) is high enough, then firms converge to the asymmetric learning-equilibrium. For the other cases we cannot predict which equilibrium is reached. Our conjecture is that the information set will eventually become either symmetrically or asymmetrically aligned, so firms reach either the Nash equilibrium or the asymmetric learning-equilibrium in the end. This conjecture is based on the local stability of both the Nash equilibrium and the ALE. Since both equilibria are locally stable, we expect that observations will not jump between the different linear parts of the demand function. In this case the proportion of either the symmetrically or the asymmetrically aligned observations will increase in the information set and the information set becomes either symmetrically or asymmetrically aligned eventually. If this conjecture does not hold, then the process does not converge at all as observations keep on jumping between the different linear parts of the true demand function. This implies that a general SSE cannot be reached when only the last \( \tau \) observations are used in the regression.

In the next section we will run computer simulations to check whether our conjecture is correct. We will also investigate how often the different outcomes are reached.

3.4 Simulation results

We run simulations with 1000 different initializations. Each initialization runs until the maximal price change is smaller than the threshold value of \( 10^{-8} \), i.e. \( \max_i |p_{i,t} - p_{i,t-1}| \leq 10^{-8} \), or until period 1000 is reached. We fix the market parameters \( c = 1 \) and \( s = 1 \), and we vary the value of \( S \). Based on the theoretical results, we consider 6 different values for \( S \), for which the equilibria have different dynamical properties. Table 3.2 summarizes the values of \( S \) we consider and the corresponding prices in the Nash equilibrium and in the asymmetric learning-equilibrium. For \( S = 2 \) and \( S = 4 \) the ALE does not exist as \( S < \Sigma_1 \approx 4.1085 \). For \( S = 4.2 \) and \( S = 4.35 \) the
Table 3.2: The Nash equilibrium price and prices in the asymmetric learning-equilibrium for different values of $S$. Other parameters: $s = 1$ and $c = 1$.

<table>
<thead>
<tr>
<th>$S$</th>
<th>2</th>
<th>4</th>
<th>4.2</th>
<th>4.35</th>
<th>4.5</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_N$</td>
<td>1.4444</td>
<td>1.5333</td>
<td>1.5385</td>
<td>1.5421</td>
<td>1.5455</td>
<td>1.6061</td>
</tr>
<tr>
<td>$p_L$</td>
<td>-</td>
<td>-</td>
<td>1.7064</td>
<td>1.7121</td>
<td>1.7174</td>
<td>1.8148</td>
</tr>
<tr>
<td>$p_H$</td>
<td>-</td>
<td>-</td>
<td>2.0532</td>
<td>2.0810</td>
<td>2.1087</td>
<td>3.0741</td>
</tr>
</tbody>
</table>

ALE exists but an asymmetrically aligned information set not always remains asymmetrically aligned after updating it with the best response prices since $\Sigma_1 < S < \Sigma_2 \approx 4.495$. For the last two values of $S$ an asymmetrically aligned information set always remains asymmetrically aligned as $S > \Sigma_2$.

Concerning the parameters in the learning method, we fix $K = 5$ in the no jump rule. Whenever firms need to pick a price randomly, they use the interval $I = [c, p_H + c]$. We believe that these choices are appropriate since all the prices that are relevant for the long-run outcome of the model lie in interval $I$ and they are always smaller than $cK$ for the model parameters we use, thus the jump size is not restrictive.\textsuperscript{15} We consider different values for $\tau$ (the number of observations used in the estimation). Since there are 4 parameters to be estimated, we need at least 4 observations in the information set. We will investigate how the size of the information set affects the outcome of the model.

Since we conjectured to observe substantially different outcomes when firms use all observations compared to the case when they use the last $\tau$ observations only, we discuss the simulation results for these cases in separate sections.

### 3.4.1 Simulations with all observations

First we investigate the outcome of the model when firms use all observations for estimating the perceived demand function. In this case, firms can move towards a general SSE, they can reach the Nash equilibrium or the ALE (provided it exists). As we have shown, the latter two equilibria are reached only when the initial observations are aligned. Since initial prices are

\textsuperscript{15}Also note that our theoretical results do not depend on the rules that augment least squares learning.
Figure 3.5: Time series of prices for \( S = 2 \). Other parameters: \( s = 1 \) and \( c = 1 \).  

Drawn randomly, information sets are typically not aligned, therefore a general SSE is reached, in which firms do not approximate correctly even that linear part on which they operate.\(^{16}\)

Figure 3.5 illustrates the time series of prices in one simulation for \( S = 2 \). The figure shows that prices settle down fast and that firms charge different prices. The given simulation stopped in period 1000, the maximal difference between the true and perceived demands at the final price vector is \( 0.3 \cdot 10^{-3} \), confirming that firms move towards a self-sustaining equilibrium.

As Proposition 3.3.2 shows, many price vectors can be part of an SSE. Therefore it is worthwhile to investigate the distribution of final prices. Figure 3.6 shows histograms of the final prices over the 1000 different initializations, for different values of \( S \). The histograms show that there is substantial price dispersion and that neither the Nash-equilibrium nor the ALE provides a benchmark outcome when all observations are used. As \( S \) increases, the distribution seems to become flatter.

Table 3.3 shows descriptive statistics of the final prices for different values of \( S \). As \( S \) increases, both the average and the median prices increase.\(^{17}\) There is not much difference in the standard deviations.

In order to measure how close firms get to a self-sustaining equilibrium, we calculate the

\(^{16}\)We need \( 4 \times 3 \) initial values for each simulation. We ran numerical simulations to investigate how often initial observations are symmetrically or asymmetrically aligned. Based on 1,000,000 simulations for each value of \( S \) we considered, initial observations are symmetrically aligned in less than 0.02% of the cases whereas they are asymmetrically aligned in less than 0.77% of the cases.

\(^{17}\)Note that the upper bound of the interval for initial prices also increases.
Figure 3.6: Histogram of final prices for different values of $S$. Other parameters: $s = 1$ and $c = 1$.

<table>
<thead>
<tr>
<th>$S$</th>
<th>mean</th>
<th>median</th>
<th>stdev</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1.6060</td>
<td>1.5781</td>
<td>0.1625</td>
</tr>
<tr>
<td>4</td>
<td>1.8647</td>
<td>1.8307</td>
<td>0.2372</td>
</tr>
<tr>
<td>4.2</td>
<td>1.8836</td>
<td>1.8582</td>
<td>0.2315</td>
</tr>
<tr>
<td>4.35</td>
<td>1.9045</td>
<td>1.8787</td>
<td>0.2362</td>
</tr>
<tr>
<td>4.5</td>
<td>1.9269</td>
<td>1.9004</td>
<td>0.2451</td>
</tr>
<tr>
<td>10</td>
<td>2.6018</td>
<td>2.5203</td>
<td>0.5694</td>
</tr>
</tbody>
</table>

Table 3.3: Descriptive statistics of final prices for different values of $S$. Other parameters: $s = 1$ and $c = 1$.

The absolute difference between the actual and perceived demands at the final price vectors. The difference is 0 in an SSE. Table 3.4 shows descriptive statistics of these differences for different values of $S$. The first three rows show the mean, minimal and maximal absolute difference over individual firms whereas the last three rows report the number of initializations for which the difference is smaller than $10^{-2}$, $10^{-3}$ and $10^{-4}$ for the three firms jointly.\(^{18}\)

We can conclude from the table that differences are rather small in all cases. In almost all cases, maximal difference is at most $10^{-2}$. This confirms that firms get close to a self-sustaining equilibrium when all observations are used in the regression. We practically never observed convergence to the Nash equilibrium or to the ALE.

\(^{18}\)For comparison, the mean initial difference (i.e. in period 5) ranges from 1 to 2 for the different values of $S$ we consider.
Table 3.4: Descriptive statistics of the absolute difference between the true and perceived demands at final prices, for different values of $S$. Other parameters: $s = 1$ and $c = 1$.

<table>
<thead>
<tr>
<th>$S$</th>
<th>mean</th>
<th>min</th>
<th>max</th>
<th>diff $\leq 10^{-2}$</th>
<th>diff $\leq 10^{-3}$</th>
<th>diff $\leq 10^{-4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$5.9 \cdot 10^{-4}$</td>
<td>$2.6 \cdot 10^{-8}$</td>
<td>$4.7 \cdot 10^{-2}$</td>
<td>978</td>
<td>906</td>
<td>371</td>
</tr>
<tr>
<td>4</td>
<td>$8.9 \cdot 10^{-4}$</td>
<td>$2.2 \cdot 10^{-16}$</td>
<td>$1.4 \cdot 10^{-1}$</td>
<td>971</td>
<td>895</td>
<td>442</td>
</tr>
<tr>
<td>4.2</td>
<td>$8.0 \cdot 10^{-4}$</td>
<td>$3.9 \cdot 10^{-16}$</td>
<td>$8.2 \cdot 10^{-2}$</td>
<td>973</td>
<td>887</td>
<td>454</td>
</tr>
<tr>
<td>4.35</td>
<td>$8.2 \cdot 10^{-4}$</td>
<td>$1.1 \cdot 10^{-16}$</td>
<td>$9.0 \cdot 10^{-2}$</td>
<td>974</td>
<td>889</td>
<td>455</td>
</tr>
<tr>
<td>4.5</td>
<td>$7.9 \cdot 10^{-4}$</td>
<td>$0$</td>
<td>$7.4 \cdot 10^{-2}$</td>
<td>977</td>
<td>887</td>
<td>465</td>
</tr>
<tr>
<td>10</td>
<td>$1.0 \cdot 10^{-3}$</td>
<td>$0$</td>
<td>$9.4 \cdot 10^{-2}$</td>
<td>962</td>
<td>869</td>
<td>615</td>
</tr>
</tbody>
</table>

Figure 3.7: Time series of prices for $\tau = 4$. Parameters: $s = 1$, $S = 2$ and $c = 1$.

### 3.4.2 Simulations with the last $\tau$ observations

Next we turn to the case when information sets contain the last $\tau$ observations only. Our conjecture was that information sets become either symmetrically or asymmetrically aligned in this case and firms converge either to the Nash equilibrium or to the asymmetric learning-equilibrium. From Proposition 3.3.3 we know that the ALE does not exist for $S = 2$ and $S = 4$, thus the Nash equilibrium should always be reached for these values of $S$.

As we discussed, at least 4 observations are needed for the regression. It turns out that the process does not converge typically when firms use exactly $\tau = 4$ observations. Figure 3.7 illustrates the time series of prices in a simulation with $\tau = 4$. The figure shows that prices do not settle down at the Nash equilibrium price. They start converging towards the Nash equilibrium (already indicating that the Nash equilibrium is locally stable) but at some point they diverge.
away from it. The reason behind this is that when there is not enough dispersion in the observations, parameter estimates become imprecise and one of the firms will charge a relatively large price. When firms use 4 observations only, then the weight of a single observation is apparently large enough and the outlier observation can drive the prices far from the equilibrium.

In contrast, when firms use more observations, the weight of a single observation decreases, thus a single outlier does not drive away prices from the equilibrium that much. We indeed find convergence when the size of the information set increases. Figure 3.8 shows typical time series for $\tau = 8$. Panel (a) shows an example where prices converge to the Nash equilibrium. Prices seem to settle down at the Nash-equilibrium price after some initial oscillations. Panel (b) shows the same time series but for the last 50 periods of the simulation. It turns out that

Figure 3.8: Time series of prices for $\tau = 8$. Parameters: $s = 1$, $S = 5$ and $c = 1$. 
Table 3.5: Proportion of outcomes in the 0.001 and the 0.0001-neighborhoods (in brackets) of the Nash equilibrium (upper numbers) and the asymmetric learning-equilibrium (lower numbers) over 1000 simulations, for different values of $S$ and $\tau$. Other parameters: $s = 1$ and $c = 1$.

we do not find exact convergence but small oscillations around the Nash equilibrium. This is caused by the same numerical problem as we have for $\tau = 4$ : parameter estimates become imprecise when there is not enough variation in the observations.\textsuperscript{19} Panels (c) and (d) depict a similar pattern for the case of the ALE.

In order to investigate whether firms always converge either to a neighborhood of the Nash equilibrium or to a neighborhood of the ALE, we run 1000 simulations for each $(S, \tau)$ combination that we consider and we calculate which proportion of the final price vectors lies in a small neighborhood of the Nash equilibrium and the ALE respectively. Table 3.5 summarizes the results. The table shows 4 numbers for each $(S, \tau)$ combination. The upper values refer to the Nash equilibrium whereas the lower ones to the ALE. The numbers that are not in brackets correspond to the 0.001-neighborhood of the given equilibrium while the numbers in brackets show the proportion of final price vectors in the 0.0001-neighborhoods.\textsuperscript{20}

\textsuperscript{19}To confirm that these oscillations are due to numerical problems we run the same simulations with using the true demand coefficients when observations in the information set are aligned. In this case we always find exact convergence to one of the equilibria. These simulations serve as a theoretical benchmark only since the true coefficients are not available for firms.

\textsuperscript{20}We say that a vector $(x_1, x_2, x_3)$ lies in the $\varepsilon$-neighborhood of another vector $(y_1, y_2, y_3)$ if their Euclidean distance is smaller than or equal to $\varepsilon$:
\[ \sqrt{\sum_{i=1}^{3} (x_i - y_i)^2} \leq \varepsilon. \]
Table 3.6: Percentage of outcomes in the 0.0001-neighborhood of the Nash equilibrium and the asymmetric learning-equilibrium when the true coefficients are used for aligned price observations. Number of simulations: 1000, other parameters: $s = 1$ and $c = 1$.

The table confirms that firms almost always converge either to the Nash equilibrium or to the asymmetric learning-equilibrium. As we have discussed before, there is not exact convergence in the model, that is why not all the outcomes lie in the small neighborhood of the equilibria. Note that as $\tau$ increases, a higher proportion of the final price vectors lies in the neighborhoods that we consider. This is due to the fact that when firms use more observations in the estimation, a single outlier does not drive prices away from the equilibrium that much. The table also shows that the Nash equilibrium is reached more often as $\tau$ increases. On the other hand, the ALE becomes more dominant as $S$ increases.

To exclude the effect of the numerical problem, we run the same simulations with firms using the true parameters of the given part of the demand function when the information set is aligned. Table 3.6 summarizes the results. Since now there is exact convergence, we show the values that correspond to the 0.0001-neighborhoods only. Note that the numbers for each $(S, \tau)$ combination add up to 100%, confirming that firms always reach either the Nash equilibrium or the ALE. We again find that the Nash equilibrium is reached more often as $\tau$ increases and that firms converge to the ALE more often as $S$ increases. Note, however, that there are substantial differences in the numbers compared to Table 3.5: the Nash equilibrium is reached much more
often than before. This shows that the numerical problem that occurs when there is not enough
variation in the observations, has an important effect on which equilibrium will eventually be
reached. The results suggest that the Nash equilibrium is less stable than the ALE in the sense
that the numerical problem can drive prices from the Nash equilibrium to the ALE more often
than the other way around. In fact, panel (c) of Figure 3.8 shows a situation where prices
settle down around the Nash equilibrium initially but after a high price realization the firms
converge to the ALE. This finding can explain why firms converge more frequently to the Nash
equilibrium when \( \tau \) increases. As we discussed, when the size of the information set increases,
a single observation has a smaller effect on the parameter estimates. Therefore a high price
realization that may occur after prices have settled down around the Nash equilibrium, has a
smaller impact on the parameter estimates, therefore the best-response prices stay in the basin
of attraction of the Nash equilibrium instead of reaching the basin of attraction of the ALE.

We checked the robustness of our results with respect to the number of periods in the sim-
ulations and the number of different initializations. We focused on the case \( \tau = 8 \) and we ran
two sets of simulations with the previously used values of \( S \): one set with 10000 periods in-
stead of 1000 and another one with 10000 different initializations instead of 1000. The outcome
of these simulations is shown in Table 3.7. The results are in line with the previous ones, the
Nash equilibrium and the ALE are reached in about the same proportion of the cases as before.
Therefore we conclude that our results are robust with respect to the number of periods and to
the number of simulations.

We ran additional simulations with different values of \( K \) (the jump size in the no jump rule)
as well. The results show that the Nash equilibrium is reached more often as \( K \) decreases. The
reason for this becomes clear from panel (c) of Figure 3.8. As we can see, prices settled down
around the Nash equilibrium price initially but a large jump around period 60 moved prices
towards the ALE-prices. Thus, if we allow for smaller jumps only, prices will be driven out
from the neighborhood of the Nash equilibrium less often. But the ALE does not disappear for
smaller values of \( K \), we still observe convergence to the ALE as well.
<table>
<thead>
<tr>
<th>S</th>
<th>estimated coefficients</th>
<th>true coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>10000 periods 10000 runs</td>
<td>10000 periods 10000 runs</td>
</tr>
<tr>
<td>2</td>
<td>99.7% (98.4%) 0.0% (0.0%)</td>
<td>100% 0.0%</td>
</tr>
<tr>
<td>4</td>
<td>99.5% (97.6%) 0.0% (0.0%)</td>
<td>100% 0.0%</td>
</tr>
<tr>
<td>4.2</td>
<td>74.7% (73.7%) 25.1% (24.4%)</td>
<td>97.0% 3.0%</td>
</tr>
<tr>
<td>4.35</td>
<td>67.2% (66.0%) 32.7% (30.9%)</td>
<td>95.3% 4.7%</td>
</tr>
<tr>
<td>4.5</td>
<td>63.1% (62.1%) 36.3% (35.0%)</td>
<td>93.2% 6.8%</td>
</tr>
<tr>
<td>10</td>
<td>11.0% (10.8%) 88.8% (82.4%)</td>
<td>56.4% 43.6%</td>
</tr>
</tbody>
</table>

Table 3.7: Outcome of simulations with 10000 periods and 10000 different initializations. The proportion of outcomes in the 0.001 and the 0.0001-neighborhoods (in brackets) of the Nash equilibrium (upper numbers) and the asymmetric learning-equilibrium (lower numbers) for different values of S and \( \tau = 8 \). Other parameters: \( s = 1 \) and \( c = 1 \).

3.5 Discussion

This chapter has focused on learning about market conditions. Similarly to Chapter 2, firms apply least squares learning but now the perceived demand function is correctly specified locally and firms can observe the actions of each other. We have proved that the model has coexisting locally stable equilibria and we have shown that least squares learning can result in a suboptimal outcome for some firms even when firms use all the relevant variables in the estimation and they use a locally correct functional form.

We have considered the Salop model with 3 firms in equidistant locations and with two types of consumers, differing in their transportation costs. Firms do not know the market structure and they apply least squares learning to learn about demand conditions. They approximate the (piecewise linear) demand function with a linear perceived demand function and they maximize their profit subject to their perceived demand function. The model has three kinds of equilibria: a general self-sustaining equilibrium, the Nash equilibrium and the asymmetric learning-equilibrium. In a self-sustaining equilibrium firms approximate the true demand function correctly only in the equilibrium point but the approximation is incorrect for any other
point. In the Nash equilibrium and the ALE firms correctly learn the linear part of the true demand function on which they operate. In the ALE the high-price firm underestimates the demand for low prices and it attracts high-type consumers only. We have proved that both the Nash equilibrium and the ALE are locally stable, thus the model can have coexisting locally stable equilibria. When firms use all past observations in the approximation, then they typically reach a general SSE. On the other hand, when only the most recent observations are used, firms converge either towards the Nash equilibrium or towards the ALE. As firms use more observations in the regression (but not all observations), the Nash equilibrium is reached more often. In contrast, the ALE is reached more often as the transportation cost of high-type consumers increases.

In the model we have made some assumptions whose effects should be discussed. First of all, we have introduced heterogeneity on the demand side of the market. This is not an unrealistic assumption as consumers could easily differ in their transportation costs, moreover it makes the model more general. With homogeneous consumers, the true demand function is still piecewise linear so least squares learning can lead to an SSE or to the Nash equilibrium. The ALE, however, does not exist since if a firm does not attract any consumers, then it will charge a lower price eventually since the observations with zero demand will move the parameter estimates in a direction that yields a lower price. Thus, consumer heterogeneity is essential for having an asymmetric outcome.

We have augmented least squares learning with seemingly ad hoc rules. These rules are, however, quite reasonable and they improve the learning process. According to the random price rule, firms choose a random price when the estimation leads to an upward-sloping demand function or when the pricing rule gives a price that is lower than the marginal cost. In either case, the normal pricing rule does not give the profit maximum so it should not be used. When the own price effect is positive, the perceived demand function is not sensible economically. Firms should charge an infinitely high price in this case to maximize their profit. It should be clear for firms that the positive own price effect comes from some estimation problem, therefore
they should not follow the normal pricing rule. Note that it would not affect the final outcome much if firms charged a very high price in this situation. They would face zero demand in the given period, leading to new parameter estimates that give a lower price. This would affect the distribution of final prices in an SSE but we would still find convergence for the Nash equilibrium or to the ALE when firms do not use all observations in the regression. Concerning the case where the pricing formula gives a price that is lower than the marginal cost, firms are always better off by choosing a random price that yields a nonnegative profit than following the pricing formula that results in a negative profit.

We have introduced the *no jump rule* and the *impossible estimation rule* to overcome a problem when the process starts to converge. In that case there is not enough variation in the observations, leading to imprecise parameter estimates. This drives away the process from the equilibrium. It might seem as if these rules lead to an artificial stability in the model as we require firms to use the same price as in the previous period but actually these rules rule out an artificial instability. Note that this problem occurs only when the process started to converge. Thus, firms observe that prices have settled down around some values and then the new parameter estimates lead to an unexpectedly large price. First of all firms might be reluctant to make such a big price change, secondly after observing the time series of prices it should be clear that this sudden price change comes from a numerical problem, therefore it is better not to change the price. Concerning the impossible estimation rule, when parameter estimates cannot be obtained, then firms either choose a price randomly or they fix the price as we suggest. Note that a rule like the no jump rule or the impossible estimation rule is essential for having convergence in a model that is not subject to noise (e.g. demand shock) when firms do not use all past observations in the regression. If the process converged to a certain value, then estimation would not be possible since each observation would perfectly correspond to the steady state. Thus, it needs to be specified what happens when parameter estimates cannot be calculated. Keeping the price unchanged is a reasonable solution to this problem.

We have seen that in the Nash equilibrium and in the asymmetric learning-equilibrium,
perceived demand functions are correctly specified in the neighborhood of the equilibrium price. This makes these outcomes more robust than the self-sustaining equilibria in Brousseau and Kirman (1992) or in Chapter 2 in the sense that in case of an SSE a firm would discover that its perceived demand function is misspecified by choosing a slightly different price. This is not the case for the Nash equilibrium and the asymmetric learning-equilibrium.21

Let us now elaborate on whether our results could still hold in more general models. If we increase the number of firms or the number of consumer types in the Salop framework, the true demand function remains piecewise linear and thus firms may learn only one linear part correctly. Therefore firms can reach an SSE or the Nash equilibrium again. Our conjecture is that firms can reach more than one asymmetric outcomes in this situation. There are more possibilities for having asymmetric outcomes as firms may attract different consumer groups or a given consumer group can be served by a different number of firms. We expect that the same kind of outcomes can occur in different market structures as well. Consider for example a situation where firms are competing on two different markets. When the choke price is different between the markets, there will be a kink in the aggregate demand function at the lower choke price. If the functional form of the perceived demand function corresponds to the functional form of the true demand function on one (or both) of the markets, then the same kind of asymmetric equilibria may occur. Some firms might focus on one market only, while other firms might be active on both markets.

The aim of this chapter was to analyze the properties of least squares learning in a situation when firms can observe the prices of all other firms and the functional form they use in the regression is locally correct. We have found that the same kind of outcome can be reached as under an incorrect functional form or with not observing all the relevant variables (SSE), and firms can also reach the outcome that corresponds to correct functional form and full observation (the Nash equilibrium). Additionally, we have found another kind of outcome, the asymmetric

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21Note, however, that this difference is due to the different informational structure of the models. In Brousseau and Kirman (1992) and in Chapter 2 firms can observe their own actions only, whereas they have full information about the actions in our model.
learning-equilibrium, that was not found in other settings and this outcome has worse welfare properties than the Nash equilibrium outcome. Our results suggest that it is better not to use all observations in the regression as observations might correspond to different demand regions and therefore the estimation may not yield a good approximation. It might also be worthwhile to experiment with the price (by charging a lower sales price for example) every now and then as this might ensure that firms do not get locked up in a suboptimal situation. Moreover, we have seen that imposing extra rules improves the convergence properties of least squares learning by overcoming the numerical problem with estimation when there is not enough dispersion in the observations.
Appendix 3.A Proofs and derivations

Derivation of demand function (3.1)

Proof. When \( p_i < p_j - \frac{1}{3}s \), then firms \( j \) and \( k \) are driven out of the market since firm \( i \) attracts each consumer. So \( q_i = 1 \) in this case.

When \( p_i = p_j - \frac{1}{3}s \), then firm \( k \) is driven out of the market, only firms \( i \) and \( j \) attract consumers. For the given prices the consumer at the location of firm \( j \) is indifferent between the two firms. There exists another indifferent consumer, say \( Y \), on segment \( ji \). Let us suppose that consumer \( Y \) is located at distance \( y \) from firm \( i \). Then it holds for this consumer that

\[
p_i + sy = p_j + s \left( \frac{2}{3} - y \right)
\]

Using that \( p_i = p_j - \frac{1}{3}s \), we get \( y = 0.5 \). Since the consumer at the location of firm \( j \) is indifferent between the two firms, all the consumers between firm \( j \) and consumer \( Y \) are indifferent between firms \( i \) and \( j \). The amount of these consumers is \( 1 - \frac{1}{3} - \frac{1}{2} = \frac{1}{6} \). Half of them choose firm \( j \), firm \( i \) attracts all other consumers. Thus, \( q_i = \frac{11}{12} \).

When \( p_j - \frac{1}{3}s < p_i < p_k - \frac{1}{3}s \), then firm \( k \) is driven out of the market and there is one indifferent consumer on segment \( ij \) and another one on segment \( ji \). Suppose that the indifferent consumer on segment \( ji \) lies at distance \( y \) from firm \( i \). For this consumer it holds that

\[
p_i + sy = p_j + s \left( \frac{2}{3} - y \right)
\]

From which \( y = \frac{p_j - p_i}{2s} + \frac{1}{3} \). Suppose that the other indifferent consumer (on segment \( ij \)) lies at distance \( x \) from firm \( i \). A similar calculation as before yields \( x = \frac{p_j - p_i}{2s} + \frac{1}{6} \). Then firm \( i \) gets a demand of \( q_i = x + y = \frac{p_j - p_i}{2s} + \frac{1}{2} \).

When \( p_i = p_k - \frac{1}{3}s \), then all three firms are active and the consumer at the location of firm \( k \) is indifferent between firms \( i \) and \( k \). There is an indifferent consumer on segment \( ij \). Let is suppose that he is located at distance \( x \) from firm \( i \). In this case, \( x = \frac{p_j - p_i}{2s} + \frac{1}{6} \). There is another indifferent consumer on segment \( jk \). Let us suppose that he is located at distance \( y \) from firm \( k \). Similar calculations as before yield \( y = \frac{p_j - p_k}{2s} + \frac{1}{6} \). Since the consumers between the location of firm \( k \) and the location of the indifferent consumer on segment \( jk \) are indifferent between firms \( i \) and \( k \), firm \( i \) attracts half of them. In this case, the demand of firm \( i \) is given by

\[
q_i = x + \frac{1}{3} + 0.5y = \frac{3p_j - p_k - 2p_i}{4s} + \frac{7}{12}.
\]

Using that \( p_i = p_k - \frac{1}{3}s \), the previous expression simplifies
to $q_i = \frac{3}{4} + \frac{3p_j - 3p_k}{4s}$.

When $p_k - \frac{1}{3}s < p_i < p_j + \frac{1}{3}s$, then all three firms are active and there is an indifferent consumer between each 2 firms. Suppose that the indifferent consumer between firms $i$ and $j$ is located at distance $x$ from firm $i$. Then, as we calculated before, $x = \frac{p_j - p_i}{2s} + \frac{1}{6}$. Similarly, if the indifferent consumer between firms $i$ and $k$ lies at distance $y$ from firm $i$, then $y = \frac{p_k - p_i}{2s} + \frac{1}{6}$.

Thus, $q_i = x + y = \frac{p_j + p_k - 2p_i}{2s} + \frac{1}{3}$.

For $p_i = p_j + \frac{1}{3}$, the consumer at the location of firm $i$ is indifferent between firms $i$ and $j$. Let the indifferent consumer between firms $i$ and $k$ be located at distance $y$ from firm $i$. Then $y = \frac{p_k - p_i}{2s} + \frac{1}{6}$, as before. Since the consumers that are between this indifferent consumer and firm $i$, are indifferent between firms $i$ and $j$, firm $i$ attracts only half of them. Thus, $q_i = \frac{1}{2}y = \frac{p_k - p_i}{4s} + \frac{1}{12}$.

When $p_i > p_j + \frac{1}{3}$, even the consumer at the location of firm $i$ chooses firm $j$. Thus, firm $i$ is driven out of the market: $q_i = 0$.

\[\square\]

The proof of Proposition 3.2.1

**Proof.** First note that $p_i \geq c$ must hold for each firm in equilibrium. Otherwise the firm with the lowest price, say firm $j$, would always face a positive demand and would make a certain loss on each product. The firm could increase its profit by choosing a higher price for which its profit is at least 0. This can be achieved by any $p_j \geq c$.

Now we will show that each firm must face a positive demand in equilibrium. To see this suppose that firm $i$ is driven out of the whole market by firm $j$, that is $p_j < p_i - \frac{1}{3}S$. Let $c \leq p_j \leq p_k$ without loss of generality. In this case, firm $i$ can increase its profit by choosing the price $p_i = c + \varepsilon$ for a sufficiently small but positive $\varepsilon$. For this price firm $i$ will not be driven out of the market since $p_j - \frac{1}{3}S = c + \varepsilon - \frac{1}{3}S < c$ for a sufficiently small $\varepsilon$, meaning that firm $i$ can only be driven out of the market with a price that is smaller than the marginal cost. This, as we have seen, cannot occur in equilibrium.

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The condition that each firm must have a positive demand in equilibrium implies that all three firms must attract high-type consumers. Thus, equilibria can differ only in the number of firms attracting low-type consumers. There might be three possibilities: 3, 2 or 1 firm attracts low-type consumers. We investigate these cases separately.

Case 1: symmetric Nash equilibrium

When all three firms attract low-type consumers, then firm $i$ faces the following demand function:

$$D_i(p) = \frac{2}{3} + (p_j + p_k - 2p_i) \left( \frac{1}{2S} + \frac{1}{2s} \right).$$

To see this note that there is one low-type and one high-type indifferent consumer between any two firms. The low-type indifferent consumer between firms $i$ and $j$ is at the distance $x = \frac{p_j - p_i}{2s} + \frac{1}{6}$ from firm $i$. A similar formula applies for the high-type indifferent consumer and for the indifferent consumers between firms $i$ and $k$.

Firm $i$ maximizes its profit with respect to its price: $\max_{p_i} (p_i - c)D_i(p)$. The first-order conditions for firms 1, 2 and 3 respectively are

$$\frac{2}{3} + (p_2 + p_3 - 2p_1) \left( \frac{1}{2S} + \frac{1}{2s} \right) - \left( \frac{1}{S} + \frac{1}{s} \right) (p_1 - c) = 0, \quad (3.8)$$

$$\frac{2}{3} + (p_1 + p_3 - 2p_2) \left( \frac{1}{2S} + \frac{1}{2s} \right) - \left( \frac{1}{S} + \frac{1}{s} \right) (p_2 - c) = 0, \quad (3.9)$$

$$\frac{2}{3} + (p_1 + p_2 - 2p_3) \left( \frac{1}{2S} + \frac{1}{2s} \right) - \left( \frac{1}{S} + \frac{1}{s} \right) (p_3 - c) = 0. \quad (3.10)$$

Subtracting (3.9) from (3.8) yield $\frac{5}{2} (p_2 - p_1) \left( \frac{1}{S} + \frac{1}{s} \right) = 0$, from which $p_1 = p_2$. Similarly, subtracting (3.10) from (3.9) gives $p_1 = p_3$. Let $p_N$ denote this common price. Then the first-order conditions simplify to $\frac{2}{3} - \left( \frac{1}{2} + \frac{1}{S} \right) (p_N - c) = 0$, from which

$$p_N = \frac{2Ss}{3(S + s)} + c.$$

The corresponding profits are $\pi_N = \frac{2}{3} (p_N - c) = \frac{4}{9} \frac{Ss}{S + s}$.

The price vector $p = (p_N, p_N, p_N)$ constitutes a Nash equilibrium only if none of the firms has an incentive to deviate from this price unilaterally. A firm can deviate in two possible ways.
It can drive out the two other firms from the market of the low-type consumers or it can drive out the other firms from the whole market.\(^\text{22}\)

Let us first consider the case when firm 1 chooses \(p_1 \leq p_N - \frac{S}{3} \leq p_1 \leq p_N - \frac{1}{3}S\). In this case firm 1 attracts the low-type consumers and the three firms share the high-type consumers. Thus, firm 1 faces the following demand function:

\[
D_1(p) = \frac{4}{3} + \frac{p_N - p_1}{S},
\]

To find the optimal price, the following constrained optimization problem needs to be solved:

\[
\max_{p_1 \leq p_N - \frac{s}{3}} (p_1 - c) \left( \frac{4}{3} + \frac{p_N - p_1}{S} \right).
\]

The Karush-Kuhn-Tucker conditions yield

\[
\left( \frac{4}{3} + \frac{p_N - p_1}{S} - \frac{1}{S}(p_1 - c) \right) \left( p_N - \frac{s}{3} - p_1 \right) = 0.
\]

Let us suppose that \(\frac{4}{3} + \frac{p_N - p_1}{S} - \frac{1}{S}(p_1 - c) = 0\). This gives \(p_1 = \frac{2S}{3} + \frac{p_N + c}{2}\). We need to check whether the condition \(p_1 \leq p_N - \frac{S}{3}\) is satisfied.

\[
\frac{2S}{3} + \frac{p_N + c}{2} \leq p_N - \frac{s}{3}
\]

\[
0 \leq \frac{p_N - c}{2} - \frac{2S}{3} - \frac{s}{3}
\]

\[
0 \leq \frac{Ss}{3(S + s)} - \frac{2S}{3} - \frac{s}{3}
\]

\[
0 \leq Ss - 2S(S + s) - s(S + s)
\]

\[
0 \leq -2S^2 - 2Ss - s^2,
\]

where we used the formula for \(p_N\). The last condition is never satisfied so we can conclude that \(p_1^D = p_N - \frac{S}{3}\) is the optimal deviation in this case. The corresponding demand and profit are

\[
q_1^D = \frac{1}{3} \left( 4 + \frac{s}{S} \right) \quad \text{and} \quad \pi_1^D = \left( p_N - \frac{S}{3} - c \right) \frac{1}{3} \left( 4 + \frac{s}{S} \right),
\]

which simplifies to \(\pi_1^D = \frac{s}{9} \frac{S - s}{S + s} \left( 4 + \frac{s}{S} \right)\).

\(^{22}\)We do not have to consider marginal deviations from \(p_N\) since the first-order conditions imply that the local profit maximum is reached at \(p_N\).
Firm 1 does not have an incentive to deviate if $\pi_N \geq \pi^D_1$, which gives

$$\frac{4}{9} S s \geq s \frac{S - s}{9 S + s} \left(4 + \frac{s}{S}\right)$$

$$4 S \geq (S - s) \left(4 + \frac{s}{S}\right)$$

$$0 \geq -3s - \frac{s^2}{S}.$$

The last inequality is always satisfied as $S, s > 0$. Thus, this deviation is never profitable.

Now let us consider the other deviation when firm 1 drives the other firms out from the whole market. In this case $p_1 < p_N - \frac{1}{3} S$ should hold. Note, however, that $p_N - \frac{1}{3} S = \frac{25}{3} S + c - \frac{1}{3} S = \frac{1}{3} S \frac{S - s}{S + s} + c < c$ as $s - S < 0$. This means that firm 1 would have to charge a price below the marginal cost to attract every consumer, leading to negative profits.

Thus, firms do not have an incentive to deviate unilaterally from the price $p_N$. The price vector $p = (p_N, p_N, p_N)$ is the unique symmetric Nash equilibrium.

Case 2: asymmetric situation with 2 firms serving low-type consumers

Now we will show that the situation in which exactly one firm focuses only on high-type consumers, cannot constitute a Nash equilibrium. Assume without loss of generality that firm 3 charges a high price such that only high-type consumers buy from firm 3: $\min\{p_1, p_2\} + \frac{s}{3} \geq p_3 \geq \min\{p_1, p_2\} + \frac{s}{3}$. In this situation the demand functions are as follows:

$$D_1(p) = \frac{5}{6} + \frac{p_2 - p_1}{s} + \frac{p_2 + p_3 - 2p_1}{2S},$$

$$D_2(p) = \frac{5}{6} + \frac{p_1 - p_2}{s} + \frac{p_1 + p_3 - 2p_2}{2S},$$

$$D_3(p) = \frac{1}{3} + \frac{p_1 + p_2 - 2p_3}{2S}.$$

Profit maximization yields the following first-order conditions (for firms 1, 2 and 3 respec-
\[
\frac{5}{6} + \frac{p_2 - p_1}{s} + \frac{p_2 + p_3 - 2p_1}{2S} - \left( \frac{1}{s} + \frac{1}{S} \right) (p_1 - c) = 0, \quad (3.14)
\]
\[
\frac{5}{6} + \frac{p_1 - p_2}{s} + \frac{p_1 + p_3 - 2p_2}{2S} - \left( \frac{1}{s} + \frac{1}{S} \right) (p_2 - c) = 0, \quad (3.15)
\]
\[
\frac{1}{3} + \frac{p_1 + p_2 - 2p_3}{2S} - \frac{1}{S}(p_3 - c) = 0. \quad (3.16)
\]

If a Nash equilibrium exists in the given situation, it must be the solution of these first-order conditions. By subtracting (3.15) from (3.14), it can be seen that \( p_1 = p_2 \) must hold. Therefore, let \( p_1 = p_2 = p_L \) and \( p_3 = p_H \). The first-order conditions then simplify to

\[
\frac{5}{6} + \frac{p_H - p_L}{2S} - \left( \frac{1}{s} + \frac{1}{S} \right) (p_L - c) = 0, \quad (3.17)
\]
\[
\frac{1}{3} + \frac{p_L - p_H}{S} - \frac{1}{S}(p_H - c) = 0. \quad (3.18)
\]

Subtracting (3.18) from (3.17) yields

\[
\frac{p_H - p_L}{2S} = \frac{p_L - c}{5s} - \frac{1}{10}. \quad (3.19)
\]

Combining (3.17) with (3.19) gives

\[
\frac{11}{15} + \frac{p_L - c}{5s} - \left( \frac{1}{s} + \frac{1}{S} \right) (p_L - c) = 0. \]

Solving this equation for \( p_L \) yields

\[
p_L = \frac{11Ss}{12S + 15s} + c.
\]

Plugging this expression for \( p_L \) in (3.19) yields an equation that can be solved for \( p_H \). The solution simplifies to

\[
p_H = \frac{2S^2 + 8Ss}{12S + 15s} + c.
\]

Note that the previous calculations yield admissible prices only when \( p_L + \frac{1}{3}S \geq p_H \geq p_L \).
\( p_L + \frac{1}{3}s \), or equivalently \( \frac{1}{3}s \leq p_H - p_L \leq \frac{1}{3}S \). Using that

\[
p_H - p_L = \frac{2S^2 - 3Ss}{12S + 15s},
\]

the condition \( \frac{1}{3}s \leq p_H - p_L \) leads to \( 4Ss + 5s^2 \leq 2S^2 - 3Ss \), or equivalently \( 2 \left( \frac{S}{s} \right)^2 - 7 \frac{S}{s} - 5 \geq 0 \). Solving this quadratic equation gives that \( \frac{S}{s} \geq \frac{7 + \sqrt{89}}{4} \) must hold.

The condition \( p_H - p_L \leq \frac{1}{3}S \) leads to \( 12S + 15s \geq 6S - 9s \), from which \( 6S + 24s \geq 0 \). This condition is satisfied as \( S, s > 0 \). Thus, this type of asymmetric Nash equilibrium may exist only when \( \frac{S}{s} \geq \frac{7 + \sqrt{89}}{4} \).

The price vector \( p = (p_L, p_L, p_H) \) constitutes a Nash equilibrium only if none of the firms has an incentive to deviate unilaterally. Now we will show that either the high-price firm or the low-price firms can earn a higher profit by charging a different price. First, let us calculate the profits under \( p = (p_L, p_L, p_H) \). Plugging the prices in demand functions (3.11)-(3.13) yields

\[
q_1 = q_2 = q_L = \frac{11S + 11s}{12S + 15s} \quad \text{and} \quad q_3 = q_H = \frac{2S + 8s}{12S + 15s}.
\]

The corresponding profits are \( \pi_1 = \pi_2 = \pi_L = \frac{121sS + (S+s)}{(12S + 15s)^2} \) and \( \pi_3 = \pi_H = S \left( \frac{2S + 8s}{12S + 15s} \right)^2 \).

First let us suppose that the high-price firm deviates and charges \( p_{D3} = p_L \), where superscript \( D \) refers to deviation. In that case \( q_{D3} = \frac{2}{3} \) since all three firms charge the same price. The corresponding profit is \( \pi_{D3} = \frac{2}{3} \frac{11Ss}{12S + 15s} \). This deviation leads to a higher profit for firm 3 when

\[
\frac{2}{3} \frac{11Ss}{12S + 15s} > S \left( \frac{2S + 8s}{12S + 15s} \right)^2 \quad \text{or} \quad 11S(12S + 15s) > 6S^2 + 48Ss + 96s^2,
\]

\[
0 > 6S^2 - 84Ss - 69s^2. \tag{3.20}
\]

Now let us suppose that firm 1 deviates by charging \( p_{D1} = p_H \). In that case firm 1 serves the high-type consumers only so it faces a similar demand function as (3.13). Thus, its demand equals \( q_{D1} = \frac{1}{3} + \frac{p_L - p_H}{2S} = \frac{6S + 13s}{2(12S + 15s)} \) and the corresponding profit is \( \pi_{D1} = \frac{6S + 13s}{12S + 15s} \frac{S(S + 4s)}{12S + 15s} \). This
deviation leads to a higher profit for firm 1 when

\[
\frac{6S + 13s}{12S + 15s} \cdot \frac{S(S + 4s)}{12S + 15s} > \frac{121Ss(S + s)}{(12S + 15s)^2} \]

\[
(6S + 13s)(S + 4s) > 121s(S + s)
\]

\[
6S^2 - 84SS - 69s^2 > 0.
\]

Comparing conditions (3.20) and (3.21), we find that one of the firms always has an incentive to deviate whenever \(6S^2 - 84S - 69s^2 \neq 0\). Now we will show that the high-price firm has an incentive to deviate even if the previous equation holds with equality. Note that we did not consider the optimal deviation in the previous calculations. We only showed that there exists a deviation that is more profitable under certain conditions. When \(6S^2 - 84S - 69s^2 = 0\) holds, firm 3 is indifferent between charging \(p_L\) and \(p_H\) (keeping the price of the other two firms fixed):

\[
\pi_3(p_L, p_L, p_L) = \pi_3(p_L, p_L, p_H).
\]

We will now show that the marginal profit of firm 3 is not equal to 0 at \(p = (p_L, p_L, p_L)\). This implies that a marginal deviation from \(p_3 = p_L\) (in the appropriate direction) yields a strictly higher profit, thus \(p = (p_L, p_L, p_H)\) cannot be a Nash equilibrium.

The marginal profit of firm 3 at \(p = (p_L, p_L, p_L)\) can be calculated using (3.10):

\[
\frac{\partial \pi_3}{\partial p_3} \bigg|_{p=(p_L, p_L, p_L)} = \frac{2}{3} - \left( \frac{1}{S} + \frac{1}{s} \right) (p_L - c).
\]

Plugging in the formula for \(p_L\) yields \(\frac{2}{3} - \frac{S + s}{8S} \frac{11Ss}{12S + 15s}\), which simplifies to \(\frac{7S + 9s}{12S + 15s}\). This expression is always positive since \(S, s > 0\). Thus, firm 3 can get a strictly higher profit by marginally increasing its price: \(\pi_3(p_L, p_L, p_L + \varepsilon) > \pi_3(p_L, p_L, p_L)\) for a small enough \(\varepsilon > 0\). Combining the last inequality with (3.22) shows that \(p = (p_L, p_L, p_H)\) cannot be a Nash equilibrium.

Thus, we have shown that one of the firms can always get a higher profit by unilaterally changing its price. We can conclude that there does not exist an asymmetric Nash equilibrium.
in pure strategies where exactly two firms attract low-type consumers.

Case 3: asymmetric situation with 1 firm serving low-type consumers

Now we will show that the situation in which two firms focus only on the high-type consumers, cannot constitute a Nash equilibrium. Assume without loss of generality that firm 1 charges a low price such that it attracts every low-type consumer: \( p_1 + \frac{s}{3} \geq \{p_2, p_3\} \geq p_1 + \frac{s}{3} \). In this situation the demand functions are as follows:

\[
\begin{align*}
D_1(p) &= \frac{4}{3} + \frac{p_2 + p_3 - 2p_1}{2S}, \\
D_2(p) &= \frac{1}{3} + \frac{p_1 + p_3 - 2p_2}{2S}, \\
D_3(p) &= \frac{1}{3} + \frac{p_1 + p_2 - 2p_3}{2S},
\end{align*}
\]

with the corresponding first-order conditions for profit maximization

\[
\begin{align*}
\frac{4}{3} + \frac{p_2 + p_3 - 2p_1}{2S} - \frac{1}{S}(p_1 - c) &= 0, \quad (3.23) \\
\frac{1}{3} + \frac{p_1 + p_3 - 2p_2}{2S} - \frac{1}{S}(p_2 - c) &= 0, \quad (3.24) \\
\frac{1}{3} + \frac{p_1 + p_2 - 2p_3}{2S} - \frac{1}{S}(p_3 - c) &= 0. \quad (3.25)
\end{align*}
\]

By subtracting (3.25) from (3.24), it can be seen that \( p_2 = p_3 \) must hold. Let \( p_1 = p_L \) and \( p_2 = p_3 = p_H \). Then the first-order conditions simplify to

\[
\begin{align*}
\frac{4}{3} + \frac{p_H - p_L}{S} - \frac{1}{S}(p_L - c) &= 0, \quad (3.26) \\
\frac{1}{3} + \frac{p_L - p_H}{2S} - \frac{1}{S}(p_H - c) &= 0. \quad (3.27)
\end{align*}
\]

Subtracting (3.27) from (3.26) yields \( 1 + \frac{2}{2S}(p_H - p_L) + \frac{1}{S}(p_H - p_L) = 0 \). This equation, however, does not give an admissible solution. Since every coefficient is positive and the right hand side is 0, \( p_H < p_L \) must hold, which contradicts the assumption \( p_H \geq p_L + \frac{s}{3} \). Thus,
there exists no asymmetric pure-strategy Nash equilibrium in which exactly one firm serves the low-type consumers.

The proof of Proposition 3.3.2

Proof. To simplify notation, let $D^P_i(p) = A_i - b_{ii}p_i$, where $A_i = a_i + b_{ij}p_j + b_{ik}p_k$. Then using (3.4) the best response price is given by

$$p_i^{\text{BR}} = \frac{A_i}{2b_{ii}} + \frac{c}{2}.$$  \hspace{1cm} (3.28)

Since $p_i = p_i^{\text{BR}}$ in an SSE, the perceived demand is given by

$$D^P_i(p) = \frac{A_i - b_{ii}c}{2}.$$  \hspace{1cm} (3.29)

Note that 9 variables characterize an SSE under the simplified notation: 1 price and the 2 parameters of the perceived demand function for each firm. On the other hand, there are 6 conditions (best response price and equality of actual and perceived demands for each firm). Thus, the system of equations that characterizes an SSE might be solved, with 3 free variables. We will now show that for a given price vector $p = (p_i, p_j, p_k)$ we can find values of $\{A_i, b_{ii}\}_{i=1}^3$ such that the system is in an SSE.

From (3.28) we get $A_i = b_{ii} (2p_i - c)$. Combining this with (3.29), the perceived demand simplifies to $D^P_i(p) = b_{ii} (p_i - c)$. Since the actual and the perceived demands must coincide at price vector $p = (p_i, p_j, p_k)$, it must hold that $D_i(p) = b_{ii} (p_i - c)$, from which

$$b_{ii} = \frac{D_i(p)}{p_i - c}.$$  \hspace{1cm} (3.30)

Combining this with the previous formula for $A_i$ yields

$$A_i = \frac{D_i(p)}{p_i - c} (2p_i - c).$$  \hspace{1cm} (3.31)
Thus, for a given price vector \( p = (p_i, p_j, p_k) \), formulas (3.30) and (3.31) specify the values of \( b_{ii} \) and \( A_i \) under which the system is in an SSE.

Let us investigate which price vectors lead to an economically sensible perceived demand function. That is, we want to characterize the set of prices for which \( b_{ii} > 0 \) and \( A_i > 0 \) (i.e. the perceived demand function is downward-sloping and the “intercept” is positive).23

It follows from (3.30) that \( b_{ii} > 0 \) if and only if \( D_i(p) > 0 \) and \( p_i > c \). Under these conditions, \( A_i > 0 \) is satisfied as well.

\[ \text{The proof of Proposition 3.3.3} \]

\textbf{Proof.} We know from Proposition 3.3.2 that \( D_i(p) > 0 \) must hold for each firm. Since each firm must face a positive demand in an SSE, each firm must attract high-type consumers. This implies that there are three possible SSE in which firms correctly learn one linear part of the true demand function, depending on whether 1, 2 or 3 firms serve low-type consumers.24

When all 3 firms attract low-type consumers, then demand conditions are characterized by 
\[ D_i(p) = \frac{2}{3} + (p_j + p_k - 2p_i) \left( \frac{1}{2S} + \frac{1}{2s} \right) \] (see Case 1 in the proof of Proposition 3.2.1). The best response function can be derived from first-order conditions (3.8)-(3.10). As we have seen, these first-order conditions have a unique solution, which corresponds to the Nash equilibrium of the model with known demand. Thus, when all 3 firms serve both consumer types and firms correctly learn the corresponding linear part of the true demand function, then the Nash equilibrium is the unique steady state of the learning process.

When only 2 firms attract low-type consumers, then demand conditions are characterized by (3.11)-(3.13) (see Case 2 in the proof of Proposition 3.2.1). The corresponding best response functions can be derived from first-order conditions (3.14)-(3.16). These first-order conditions have a unique solution, in which the low-price firms charge 
\[ p_L = \frac{11Ss}{12S + 13s} + c \] and the high-

---

23For having an economically sensible perceived demand function, one might consider introducing the conditions \( a_i > 0, b_{ij} > 0 \) and \( b_{ik} > 0 \) in addition to the condition \( A_i > 0 \). Note, however, that these extra conditions do not restrict the set of admissible prices further as for a given positive \( A_i \) one can always find values for \( a_i, b_{ij} \) and \( b_{ik} \) such that \( A_i = a_i + b_{ij}p_j + b_{ik}p_k \) holds and the conditions on the signs are satisfied.

24Note that these are exactly the same cases that we analyzed in the proof of Proposition 3.2.1.
price firm asks the price \( p_H = \frac{2S^2 + 8Ss}{12S + 15s} \). We have also seen that this outcome exists only when \( \frac{S}{s} \geq \Sigma_1 \). Even though this outcome is not a Nash equilibrium of the model with known demand, it is a steady state of the learning process. The reason behind this is that firms do not know that it would be profitable to change their price unilaterally since they approximate the demand function with a linear function, implying that they do not know that they would get a much higher demand by undercutting other firms. Thus, when only 2 firms serve both consumer types and firms correctly learn the corresponding linear part of the true demand function, then the unique steady state is given by 2 firms charging \( p_L \) and 1 firm charging \( p_H \). We refer to this outcome as asymmetric learning-equilibrium.

We have seen that when only 1 firm serves the low-type consumer, then first-order conditions (3.23)-(3.25) do not yield an admissible solution. Therefore the learning process does not have a steady state in this situation.

This shows that the Nash equilibrium and the asymmetric learning-equilibrium are the only steady states in which all three firms correctly learn the linear part of the true demand function on which they operate.

\[ \square \]

**Comparison of the Nash equilibrium and the ALE**

First we show that the Nash equilibrium price is smaller than the lower price in the ALE. Comparing \( p_N \) and \( p_L \), we get that \( p_N < p_L \) if and only if \( \frac{2Ss}{S + s} < \frac{11Ss}{12S + 15s} \). This reduces to \( 0 < 9S + 3s \), which always holds since \( S, s > 0 \). We have shown before that \( p_H > p_L \) whenever the ALE exists. Thus, we have \( p_N < p_L < p_H \).

We have seen in the proof of Proposition 3.2.1 that the Nash-equilibrium profit is \( \pi_N = \frac{4Ss}{9S + s} \) while the profits in the ALE are given by \( \pi_L = \frac{121Ss(S + s)}{(12S + 15s)^2} \) and \( \pi_H = S \left( \frac{2S + 8s}{12S + 15s} \right)^2 \). The low-price firms make a higher profit than the high-price firm only if \( 121s(S + s) > 4(S + 4s)^2 \), from which \( 0 > 4 \left( \frac{S}{s} \right)^2 - 89\frac{S}{s} - 57 \). This gives \( \frac{S}{s} < \frac{89 + 11\sqrt{73}}{8} \approx 22.87 \).

The Nash-equilibrium profit is always smaller than the profit of low-price firms in the ALE:

\[ \frac{4Ss}{9S + s} < \frac{121Ss(S + s)}{(12S + 15s)^2} \] if and only if \( 4(12S + 15s)^2 < 1089(S + s)^2 \), which reduces to \( 0 < 9S + 3s \).
This inequality is always satisfied.

Next we show that the Nash-equilibrium profit is larger than the profit of the high-price firm in the ALE only if \( \frac{S}{s} \) is low enough. \( \frac{4Ss}{9S+s} > S\left(\frac{2S+8s}{12S+15s}\right)^2 \) if and only if \((4S + 5s)^2 > (S + s)(S + 4s)^2\). This is equivalent to the following inequality: \(-\left(\frac{S}{s}\right)^3 + 7\left(\frac{S}{s}\right)^2 + 16\frac{S}{s} + 9 > 0\). First we will argue that the function \( f(x) = -x^3 + 7x^2 + 16x + 9 \) has a single real root. Note that \( f \) is a cubic function, therefore it may have 1, 2 or 3 real roots. It is easy to see that \( f \) has a local maximum at \( x = \frac{7 + \sqrt{117}}{3} \approx 5.61 \) and a local minimum at \( x = \frac{7 - \sqrt{117}}{3} \approx -0.95 \). Since the function value is positive both at the local maximum and at the local minimum (around 142.51 and 0.97, respectively), the function has a unique root. Numerical calculations show that this root is around 8.91. Thus, when \( \frac{S}{s} < 8.91 \), then \( \pi_N > \pi_H \), otherwise the opposite relation holds.

Finally, we show that the total profit of firms in the Nash equilibrium is always smaller than in the ALE. Using the previous formulas, \( 3\pi_N < 2\pi_L + \pi_H \) reduces to \( 6s(4S + 5s)^2 < 121s(S + s)^2 + 2(S + 4s)^2(S + s) \). This simplifies further to \( 0 < 2S^3 + 43S^2s + 50Ss^2 + 3s^3 \). This inequality is always satisfied as \( S, s > 0 \).

The proof of Lemma 3.3.5

Proof. When price observations are aligned, estimation yields the true parameters that characterize the given linear part of the demand function. Under symmetrically aligned price observations the parameter estimates are given by \( a_i = \frac{2}{3} \), \( b_{ii} = \frac{1}{S} + \frac{1}{s} \) and \( b_{ij} = b_{ik} = \frac{1}{2S} + \frac{1}{2s} \) (see Case 1 in the proof of Proposition 3.2.1 for the corresponding demand function). Thus, using (3.4), the best response of firm \( i \) is

\[
p_i^{BR} = \frac{2}{3} + \left(\frac{1}{2S} + \frac{1}{2s}\right) \left(p_j + p_k\right) + \frac{c}{2}.
\]

Then \( |p_i^{BR} - p_j^{BR}| = \frac{1}{3} |p_j - p_i| \) for any two firms. Since price observations were symmetrically aligned, \( |p_j - p_i| < \frac{s}{3} \) holds and therefore \( |p_i^{BR} - p_j^{BR}| < \frac{s}{3} \) is also satisfied. Thus, adding the best-response prices to the price observations gives a symmetrically aligned set again. \(^{25}\)

\(^{25}\)Notice the contraction mapping feature of playing the best-response price. This implies that symmetrically
When firms \(i\) and \(j\) attract both types of consumers while firm \(k\) attracts high-type consumers only, then firms learn the following demand parameters: \(a_i = a_j = \frac{5}{6}, b_{ii} = b_{jj} = \frac{1}{2}, a_k = \frac{1}{3}, b_{kk} = \frac{1}{3}\), and \(b_{ki} = b_{kj} = \frac{1}{2}\) (see Case 2 in the proof of Proposition 3.2.1 for the corresponding demand functions). Using (3.4), the best-response prices are given by

\[
\begin{align*}
  p_{i}^{BR} &= \frac{5}{6} + \frac{1}{2} \left( \frac{1}{2S + 1} \right) p_j + \frac{1}{2} p_k + \frac{c}{2}, \\
  p_{j}^{BR} &= \frac{5}{6} + \frac{1}{2} \left( \frac{1}{2S + 1} \right) p_i + \frac{1}{2} p_k + \frac{c}{2}, \\
  p_{k}^{BR} &= \frac{1}{3} + \frac{1}{2S} (p_i + p_j) + \frac{c}{2}.
\end{align*}
\]

Then \(|p_{i}^{BR} - p_{j}^{BR}| = \left( \frac{1}{2S + 1} \right) |p_j - p_i| = \frac{2S+1}{2S+4} |p_j - p_i| < \frac{5}{6}\) since \(\frac{2S+1}{2S+4} < 1\) and \(|p_j - p_i| < \frac{5}{6}\) because price observations were asymmetrically aligned. Thus, the first condition in the definition is satisfied.\(^{26}\)

Let us suppose that \(p_i \leq p_j\) in the most recent price observation. In that case, \(p_{j}^{BR} \leq p_{i}^{BR}\) and it must hold for having asymmetrically aligned price observations that \(p_{j}^{BR} + \frac{5}{3} < p_{k}^{BR} < p_{j}^{BR} + \frac{5}{3}\).

Using the formulas above, it can be shown that

\[
p_{k}^{BR} - p_{j}^{BR} = \frac{1}{12(S + s)} \left[ 2s^2 - 3Ss - 3Sp_i + (3S + 3s)p_j - 3sp_k \right].
\]

We will first show that the condition \(p_{k}^{BR} - p_{j}^{BR} < \frac{S}{3}\) is always satisfied. Using the formula for \(p_{k}^{BR} - p_{j}^{BR}\), the condition simplifies to

\[-\frac{S}{3} \left( 2\frac{S}{s} + 7 \right) + \left( 1 + \frac{S}{s} \right) (p_j - p_i) < p_k - p_i.\]

aligned prices converge to the same value. Since prices are best response to each other, firms will reach the Nash equilibrium in this case.

\(^{26}\)Note the contraction mapping feature again, which implies that the low-price firms will reach the same price if the information set always remains asymmetrically aligned.
The left-hand side is smaller than \(-\frac{S}{3} (2\frac{S}{s} + 7) + (1 + \frac{S}{s}) \frac{s}{3}\) since \(p_j - p_i < \frac{s}{3}\). It is easy to see that this expression is always negative. On the other hand, the right-hand side is positive since \(p_k - p_i > \frac{s}{3}\). Thus, \(p_k^{BR} - p_j^{BR} < \frac{S}{3}\) is always satisfied.

Next let us consider the condition \(p_k^{BR} - p_j^{BR} > \frac{s}{3}\). Using the formula for \(p_k^{BR} - p_j^{BR}\), the condition simplifies to

\[
\frac{s}{3} \left[ 2 \left( \frac{S}{s} \right)^2 - 7 \frac{S}{s} - 4 \right] + \left( 1 + \frac{S}{s} \right) (p_j - p_i) > p_k - p_i.
\]

The left-hand side of the inequality is greater than or equal to \(\frac{s}{3} \left[ 2 \left( \frac{S}{s} \right)^2 - 7 \frac{S}{s} - 4 \right]\) as \(p_j - p_i \geq 0\). The right-hand side is smaller than \(\frac{s}{3}\) since price observations are asymmetrically aligned. Thus, a sufficient condition for (3.32) to hold is that

\[
\frac{s}{3} \left[ 2 \left( \frac{S}{s} \right)^2 - 7 \frac{S}{s} - 4 \right] \geq \frac{s}{3}.
\]

This leads to \(2 \left( \frac{S}{s} \right)^2 - 8 \frac{S}{s} - 4 \geq 0\), for which \(\frac{S}{s} \geq 2 + \sqrt{6}\) must hold. Thus, when the latter condition holds, asymmetrically aligned price observations always remain asymmetrically aligned, irrespective of the exact values in the last price observation. On the other hand, when \(\frac{S}{s} < 2 + \sqrt{6}\), condition (3.32) has to hold for the most recent price observation in order to have asymmetrically aligned price observations again.

Since price observations were asymmetrically aligned, \(\frac{s}{3} < p_k - p_i\). Thus, the following condition must hold

\[
\frac{s}{3} < \frac{s}{3} \left[ 2 \left( \frac{S}{s} \right)^2 - 7 \frac{S}{s} - 4 \right] + \left( 1 + \frac{S}{s} \right) (p_j - p_i).
\]

As we have seen, playing the best response works as a contraction mapping for the low-price firms, therefore \(p_j - p_i \to 0\) if the information set always remains asymmetrically aligned. Thus, \(\frac{s}{3} \leq \frac{s}{3} \left[ 2 \left( \frac{S}{s} \right)^2 - 7 \frac{S}{s} - 4 \right]\) must hold. This leads to \(1 \leq 2 \left( \frac{S}{s} \right)^2 - 7 \frac{S}{s} - 4\), from which \(\frac{S}{s} \geq \frac{7 + \sqrt{89}}{4}\). Thus, when the latter condition does not hold, then an asymmetrically aligned
information set cannot stay asymmetrically aligned.

The proof of Proposition 3.3.6

Proof. We will now show that both the Nash equilibrium and the ALE are locally stable equilibria. First we will describe the system in the neighborhood of the equilibria and then we show that the eigenvalues of the Jacobian are always less than 1 in absolute value. First we focus on the Nash equilibrium.

Part 1: Stability of the Nash equilibrium

When prices in the information set are symmetrically aligned, then firms learn the correct demand parameters of the linear part on which they operate. Moreover, as we have seen in Lemma 3.3.5, updating the information set with the best-response prices results in a symmetrically aligned information set again. Thus, the parameters of the perceived demand functions do not change in this case. Then the perceived demand function of firm $i$ is always given by $D_i^P(p) = \frac{2}{3} - \left(\frac{1}{S} + \frac{1}{s}\right) p_i + \left(\frac{1}{2S} + \frac{1}{2s}\right) p_j + \left(\frac{1}{2S} + \frac{1}{2s}\right) p_k$ (see Case 1 in the proof of Proposition 3.2.1 for the demand parameters of the relevant linear part). Then the next-period price of firm $i$ is given by

$$p_{i,t+1} = \frac{1}{3} \frac{Ss}{S + s} + \frac{1}{4} (p_{j,t} + p_{k,t}) + \frac{1}{2} c.$$ 

This holds for every firm $i$, therefore the Jacobian of the system is given by

$$J = \begin{pmatrix}
0 & \frac{1}{4} & \frac{1}{4} \\
\frac{1}{4} & 0 & \frac{1}{4} \\
\frac{1}{4} & \frac{1}{4} & 0
\end{pmatrix}.$$
The characteristic equation is given by

\[ k(\lambda) = -\lambda^3 + \frac{3}{16} \lambda + \frac{1}{32} = 0. \]

It is easy to see that \( k(\lambda) \) can also be expressed as \( k(\lambda) = -\left(\lambda - \frac{1}{2}\right)\left(\lambda + \frac{1}{4}\right)^2 \). Thus, the eigenvalues of the Jacobian are \( \lambda_1 = \frac{1}{2} \) and \( \lambda_2 = -\frac{1}{4} \). Both eigenvalues are smaller than 1 in absolute value, therefore the Nash equilibrium is locally stable.

Part 2: Stability of the ALE

When prices in the information set are asymmetrically aligned, then firms learn the correct demand parameters of the linear part on which they operate. Moreover, as we have seen in Lemma 3.3.5, updating the information set with the best-response prices results in an asymmetrically aligned information set again when \( S > \Sigma_2 \).\(^{27}\) Thus, the parameters of the perceived demand functions do not change in this case. Suppose that firms \( i \) and \( j \) are the low-price firms and firm \( k \) is the high-price firm. Then the perceived demand function of firm \( i \) is always given by \( D^P_i(p) = \frac{5}{6} - \left(\frac{1}{3} + \frac{1}{s}\right) p_i + \left(\frac{1}{2s} + \frac{1}{s}\right) p_j + \frac{1}{2sp_k} \) while that of firm \( k \) is \( D^P_k(p) = \frac{1}{3} - \frac{1}{3} p_k + \frac{1}{2sp_i} + \frac{1}{2sp_j} \) (see Case 2 in the proof of Proposition 3.2.1 for the demand parameters of the relevant linear part). Then the next-period price of firms \( i \) and \( k \) are given by

\[
\begin{align*}
p_{i,t+1} &= \frac{5}{12} S + \frac{1}{4} S + \frac{s}{4} p_{i,t} + \frac{1}{4} p_{j,t} + \frac{s}{2} p_{k,t} + \frac{1}{2} c, \\
p_{k,t+1} &= \frac{1}{6} S + \frac{1}{4} (p_{i,t} + p_{j,t}) + \frac{1}{2} c.
\end{align*}
\]

The next-period price of firm \( j \) is given by a similar formula as for firm \( i \), we just need to switch \( i \) and \( j \). Then the Jacobian of the system is given by

\(^{27}\)Even if \( \frac{S}{s} > \Sigma_2 \) does not hold, we can consider a sufficiently small neighborhood of the ALE for which the updated information set is asymmetrically aligned. This can be done as (3.32) holds for \((p_L, p_L, p_H)\) whenever the ALE exists.
\[ J = \begin{pmatrix} 0 & A & B \\ A & 0 & B \\ C & C & 0 \end{pmatrix}, \]

where \( A = \frac{1}{4} \frac{2S+s}{S+s}, B = \frac{1}{4} \frac{s}{S+s} \) and \( C = \frac{1}{4} \). The characteristic equation is given by

\[ k(\lambda) = -\lambda^3 + (A^2 + 2BC) \lambda + 2ABC = 0. \]

It is easy to see that \( k(\lambda) \) can also be expressed as \( k(\lambda) = -(\lambda + A)(\lambda^2 - A\lambda - 2BC) \). Thus, one eigenvalue is \( \lambda_1 = -A \). This eigenvalue is always smaller than 1 in absolute value. \( A > 0 \) since \( S, s > 0 \). \( A < 1 \) if and only if \( 2S + s < 4(S + s) \), which is always satisfied.

The other two eigenvalues are the solutions of the equation \( \lambda^2 - A\lambda - 2BC = 0 \). The discriminant is \( D = A^2 + 8BC > 0 \), so there are two real roots: \( \lambda_{2,3} = \frac{A \pm \sqrt{A^2 + 8BC}}{2} \). Root \( \lambda_2 = \frac{A + \sqrt{A^2 + 8BC}}{2} \) has the larger absolute value. Its absolute value is smaller than 1 if and only if \( \sqrt{A^2 + 8BC} < 2 - A \), from which - using that \( A < 1 - A^2 + 8BC < 4 - 4A + A^2 \). This simplifies to the condition \( A + 2BC < 1 \).

Plugging in the values for \( A, B \) and \( C \) yields \( A + 2BC = \frac{1}{4} \frac{2S+s}{S+s} + 2 \frac{1}{4} \frac{s}{S+s} \frac{1}{4} = \frac{1}{8} \frac{4S+3s}{S+s} \). This is smaller than 1 in absolute value if and only if \( 4S + 3s < 8(S + s) \), which is satisfied for any \( S, s > 0 \).

Thus, all three eigenvalues are smaller than 1 in absolute value, implying that the ALE is locally stable.
Chapter 4

Price-Quantity Competition under Strategic Uncertainty

4.1 Introduction

There are two traditional ways of modeling competition between firms producing homogeneous commodities. Cournot (1838) introduces quantity competition in which firms set the quantity of the good and the price adjusts such that the market clears. In contrast, Bertrand (1883) suggests a model in which price is the strategic variable and quantities clear the market. These two models serve as the basic framework in the literature of market competition. However, both models have their drawbacks. Under quantity competition, a market clearing mechanism is required to reach the price for which the demand equals aggregate production. In the basic model of price competition, Bertrand assumed that firms can produce any amount of the good and the output is realized immediately. Firms, however, might not be able to or might not want to\(^1\) serve the whole market at a given price. One way to address this issue is to introduce capacity constraints

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\(^1\)This can occur with convex cost functions: undercutting the price of other firms may not be profitable due to the large increase in production costs. See Dastidar (1995), for example.
in the model: firms simultaneously choose capacity levels first and they decide about the price only after observing the capacity levels. See Edgeworth (1925), Kreps and Scheinkman (1983), Gelman and Salop (1983) and Davidson and Deneckere (1986), for example. These models, however, do not take into account that production takes time: firms typically need to produce the good in advance, without knowing the exact demand they will face. Taking these considerations into account, a reasonable alternative of modeling competition is to treat both prices and quantities as strategic variables to be set simultaneously.\footnote{Judd (1996) provides an additional argument for price-quantity competition. He argues that the theoretical predictions of simpler models in which firms set price or quantity only, depend crucially on whether the strategic variables are strategic substitutes or strategic complements. Therefore, excluding either strategic variable from the analysis can have a substantial effect on the results so it is better to treat them together.} An inconvenient characteristic of simultaneous price-quantity setting is that there does not exist a Nash equilibrium in pure strategies (see Levitan and Shubik, 1978 and Maskin, 1986). Roy Chowdhury (2008), however, proposes a variation of the model that may lead to a pure-strategy Nash equilibrium. In his model firms can choose their price from discrete values only.

In this chapter we consider alternative assumptions under which a pure-strategy equilibrium may exist in a market where firms simultaneously set both the price and the production level of the good. We consider the market for a homogeneous good that is produced by two firms. Firms are risk averse and they have mean-variance preferences. They hold probabilistic conjectures about the actions of the other firm and they choose the optimal actions given their conjectures. Conjectures are, however, not entirely in line with the actual action of the other firm. Thus, in this chapter we investigate the effect of weakening the consistency requirement on beliefs. Under perfect rationality the actual actions of other firms confirm the beliefs about their actions: the conjectured action is realized in case of pure-strategy equilibria while realized actions are in line with the conjectured probability distribution in case of mixed-strategy equilibria. In our model we require beliefs to be weakly consistent: the mode of the distribution has to match the actions of other firms.

We numerically show that there may exist a pure-strategy equilibrium in our model. This
equilibrium exists only when firms are sufficiently risk averse and the amount of uncertainty\(^3\) is sufficiently high. The result of having a pure-strategy equilibrium is important as mixed-strategy equilibria seem less relevant in the field of industrial organization: it does not seem reasonable to assume that firms always choose their actions randomly from a specific distribution. Therefore, the model we propose in this chapter makes it possible to apply price-quantity competition more widely as a framework for analyzing various market phenomena such as mergers or cartels and for policy analysis.

We analyze with numerical methods how the equilibrium depends on the degree of risk aversion and the amount of uncertainty in the model. In the pure-strategy equilibrium, aggregate production exceeds market demand for low degrees of risk aversion and firms are rationed. As firms become more risk averse, they decrease their production level and they will not satisfy the demand for their good eventually therefore consumers will be rationed. Our model shows that firms react differently to price uncertainty than to output uncertainty\(^4\): the equilibrium price is typically increasing while the production level is decreasing in the amount of price uncertainty but both the equilibrium price and production level decrease as the amount of output uncertainty increases. The reason for this difference is the following. Higher price uncertainty does not affect the profit variance directly but the price becomes a more efficient instrument for increasing the expected profit. In contrast, higher output uncertainty directly affects profit variance through the residual demand so firms have an incentive to reduce their price in order to decrease the chance of operating on the residual demand function. Our model can explain a seemingly anti-competitive behavior (both firms increase their price and decrease their production level) without collusion between firms: an increase in the amount of price uncertainty has exactly the aforementioned effect in equilibrium. Our analysis shows that a small degree of risk aversion is welfare enhancing and that the welfare is higher in equilibrium than the expected welfare in the mixed-strategy equilibrium of the standard model.

The chapter is organized as follows. First we review the literature on which the chapter

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\(^3\)We measure the amount of uncertainty as the variance of the distribution of conjectures.

\(^4\)Price/output uncertainty is the variance of the distribution of price/output conjectures.
builds in Section 4.2. Then we describe the standard model of price-quantity competition in Section 4.3. The model with strategic uncertainty and risk aversion is presented in Section 4.4. In Section 4.5 we numerically characterize the symmetric pure-strategy equilibrium and we analyze which parameter combinations lead to an equilibrium. Section 4.6 analyzes how the equilibrium depends on the degree of risk aversion and on the amount of uncertainty. Section 4.7 concludes. Derivations are presented in Appendices 4.A, 4.B and 4.C.

4.2 Related literature

The idea of simultaneous price-quantity competition was mentioned by Shubik (1955) first. However, the author does not investigate the equilibria of this model as other models are in the main focus of the paper. Levitan and Shubik (1978) analyze a duopoly in which firms produce a homogeneous good. The demand depends on the price linearly, production is costless but there is a fixed unit cost for disposing unsold products. The authors show that there exists no pure-strategy Nash equilibrium and they derive a mixed-strategy Nash equilibrium. Maskin (1986) analyzes the market of a homogeneous good and he considers two versions of price-quantity competition: production in advance (i.e. prices and quantities are set at the same time) and production to order (i.e. prices are set first and firms decide on production only after observing each other’s price). He proves the existence of a mixed-strategy Nash equilibrium under general demand and cost conditions. In his PhD thesis, Gertner (1986) analyzes a duopoly market of a homogeneous good with symmetric firms and increasing, constant and decreasing marginal costs. He shows that there is no pure-strategy Nash equilibrium in any of these cases but a mixed-strategy Nash equilibrium exists. He derives the unique mixed-strategy Nash equilibrium for a linear demand function and constant and equal marginal costs. This equilibrium has the feature that firms draw a price from a certain distribution and then both firms choose the production level that equals to the market demand at the price it drew. Consequently, one firm will serve the whole market while the other firm will not sell anything. Firms have zero expected
profit in equilibrium. McCulloch (2011) characterizes the mixed-strategy Nash equilibrium numerically for the case of an asymmetric duopoly with increasing marginal costs. He uses a fine grid for both prices and quantities. His findings support those of Gertner (1986): firms charge a (relatively) high price with a high probability and some lower prices with low probabilities. This result can be interpreted as firms often charging a high regular price and a lower sale price every now and then.

Even though there does not exist a pure-strategy equilibrium in the standard model, modified versions may lead to pure-strategy equilibria. In fact, Roy Chowdhury (2008) proposes a variation of the model that may lead to a pure-strategy Nash equilibrium. He analyzes a price-quantity model of a homogeneous good with discrete pricing over a grid and convex production costs. He shows that for a fixed grid size, there exists a unique Nash equilibrium if the number of firms is high enough. On the other hand, for a fixed number of firms, there is no pure-strategy equilibrium when the grid size is sufficiently small. In the model of this chapter we consider alternative assumptions under which the existence of pure-strategy equilibria is established for smaller number of firms and continuous action spaces too.

The way we model the conjectures of firms and the corresponding equilibrium concept are related to the random belief equilibrium introduced by Friedman and Mezzetti (2005). In their model, players hold beliefs regarding the other players’ actions and there are two equilibrium conditions. The first one is that players maximize their payoffs subject to their beliefs, and the second one is that beliefs are consistent with the other players’ actions in the sense that the expected choice of every firm coincides with the center of the belief distribution (i.e. the mode or the mean of the distribution is correctly specified). The same two conditions characterize the equilibrium in our model. Larue and Yapo (2000) and Andersson et al. (2012) take a similar approach. Their players hold a subjective belief about the action of the other players and they maximize their payoff subject to these beliefs. Larue and Yapo (2000) require beliefs to be consistent with the action of other players in equilibrium: the belief distribution is centered around the equilibrium action of the other player. Andersson et al. (2012) do not make consistency re-
quirements for beliefs but it is not necessary as they consider the limiting case when the amount of uncertainty goes to zero.

Models of price-quantity competition with differentiated goods are also characterized by the non-existence of pure-strategy Nash equilibria. With differentiated goods, one needs to model the spillover demand among the goods, that is the additional demand for a good when the supply of another good cannot satisfy the demand of that good. Friedman (1988) uses general demand, spillover demand and cost functions and considers three versions of price-quantity competition. He shows that there exists no pure-strategy Nash equilibrium when prices and production levels are set simultaneously. However, when production levels are set first and firms decide on prices only after observing the actual outputs, a pure-strategy Nash equilibrium exists when spillover effects are not too strong. Furthermore, when prices are set first, then there always exists a pure-strategy subgame-perfect Nash equilibrium. For further results on price-quantity competition with differentiated goods see Benassy (1986), Judd (1996) and Khan and Peeters (2011), for example.

After reviewing the literature, let us now discuss the standard model of simultaneous price-quantity setting.

### 4.3 Price-quantity competition

We consider the market for a homogeneous good that is produced by two firms. The firms engage in price-quantity competition and they both set prices and production levels simultaneously. Production levels correspond to actual production. That is, they are not simply capacity constraints in the sense that production must be implemented at the chosen level, firms may not supply less.

The market demand depends linearly on the price of the good. It is given by

\[
D(p) = \max \{a - bp, 0\},
\]

(4.1)
where $a$ and $b$ are positive parameters and $p$ is the price.

Since firms make their decisions simultaneously, a firm may end up with unsold products. Therefore, we have to distinguish production levels from sales. Sales depend on prices and production levels of both firms. The sales of firm $i$ are given by

$$s_i(p_i, q_i; p_j, q_j) = \begin{cases} 
\min \{q_i, D(p_i)\} & \text{if } p_i < p_j \\
\min \left\{ q_i, \frac{q_i}{q_i + q_j} D(p_i) \right\} & \text{if } p_i = p_j \\
\min \{q_i, r_i\} & \text{if } p_i > p_j
\end{cases}, \quad (4.2)$$

where $p_i$ and $q_i$ denote the price and production level of firm $i$ whereas subscript $j$ refers to firm $j$. Variable $r_i$ is the residual demand of firm $i$: $r_i = \max \{D(p_i) - s_j, 0\}$.\(^5\) Thus, we apply the efficient rationing rule\(^6\) in the model.

Formula (4.2) shows that the firm with the lowest price sells all its products, provided that its production level does not exceed the market demand at the price the firm chose. When firms charge the same price, they sell all their products if they do not serve the whole market together (i.e. $q_i + q_j \leq D(p)$ where $p$ is the price chosen by both firms). If, however, aggregate production exceeds the market demand, then the firms serve the whole market and we assume that sales are proportional to production levels.\(^7\) Finally, the firm with the highest price operates on its residual demand. If its residual demand exceeds its production level, then the firm will sell all its products. However, the firm will sell only a part of its products if the residual demand is smaller. Moreover, when the residual demand is 0, then the firm will not sell anything.

The profit of firm $i$ is given by

$$\pi_i = p_i s_i(p_i, q_i; p_j, q_j) - cq_i, \quad (4.3)$$

where $c$ is the marginal cost of production. Note that we assume that the marginal cost is

\(^{5}\)Note that we use $s_j$ in the definition of $s_i$. This is not problematic since $s_j$ affects the sales of firm $i$ only when $p_i > p_j$ and in this case $s_j = \min \{q_j, D(p_j)\}$.

\(^{6}\)See Tirole (1988) for more details about this rationing rule.

\(^{7}\)We will see later that this assumption does not affect the behavior of firms as the event that they charge the same price has probability 0 in our model.
constant and equal to $c$ for both firms.

There exists no pure-strategy Nash equilibrium in this setting when firms maximize their profit. To see this, consider an arbitrary situation where both prices are higher than the marginal cost. The firm with the lowest price has an incentive to increase its production level until it serves the whole market (it may have an incentive to change its price too). Thus, the firm with the lowest price must serve the whole market in any equilibrium. If it does so, the other firm undercut the price and it serves the whole market. This undercutting may continue until prices are equal to the marginal cost. However, there exists no equilibrium with prices equal to the marginal cost. If aggregate production exceeds the market demand, then both firms have an incentive to decrease their production level. But if aggregate production is lower than or equal to the market demand, then both firms have an incentive to charge a higher price and operate on their residual demand since they make zero profit otherwise.\(^8\)

The intuition behind the non-existence of a pure-strategy Nash equilibrium is that it is never optimal for a firm to choose the same price as the other firm: either serving the whole market at a slightly lower price or operating on the residual demand is more profitable than choosing the same price. Thus, the best response functions do not cross each other. The crucial condition is that firms can undercut each other (or choose to operate on the residual demand) with certainty. This, however, may not be the case if we introduce uncertainty in the model. In fact, we will see that there may exist a pure-strategy equilibrium in this case.

4.4 A model with strategic uncertainty and risk aversion

We introduce strategic uncertainty in the model through the beliefs about the actions of the other firm. Suppose that firms have a point forecast for the actions of the other firm but they are uncertain about the accuracy of these forecasts and they take this uncertainty into account when deciding on their own price and production level. Formally, the beliefs of firm $i$ about the price

\(^8\)If one of the firms does not produce anything when prices are equal to the marginal cost, then the other firm is better off by choosing the monopoly price and production level.
and production level of firm $j$ are given by

\[ p_j^b \sim N(p_j^f, \sigma_p^2), \]
\[ q_j^b \sim N(q_j^f, \sigma_q^2), \]

where $p_j^f$ and $q_j^f$ denote the point forecasts. Parameters $\sigma_p$ and $\sigma_q$ determine the perceived accuracy of the forecasts. Firm $i$ considers its forecasts perfectly accurate for $\sigma_p = \sigma_q = 0$. The higher the values of $\sigma_p$ and $\sigma_q$ are, the more inaccurate the forecasts are considered. We refer to $\sigma_p$ and $\sigma_q$ as the amount of price and output uncertainty. Thus, the beliefs $p_j^b$ and $q_j^b$ are independent normally distributed with mean $p_j^f$ and $q_j^f$ and variance $\sigma_p^2$ and $\sigma_q^2$, respectively.\(^9\)

The beliefs about the price and production level of firm $j$ generate profit conjectures in the following way: $\pi_i^c = p_i s_i(p_i, q_i, p_j^b, q_j^b) - c q_i$.\(^10\) We introduce risk aversion in the model by assuming that firms have mean-variance preferences. That is, they simultaneously solve the following constrained optimization problem:

\[
\max_{p_i, q_i} E(\pi_i^c) - \alpha \text{Var}(\pi_i^c) \tag{4.4}
\]
\[
s.t. \quad q_i \leq D(p_i),
\]

where $\alpha \geq 0$ measures the degree of risk aversion. For $\alpha = 0$ the firms are risk neutral and they maximize their expected profit only. The higher $\alpha$ is, the more disutility the variance gives, thus the more risk averse the firms are.\(^11\) Note that if $q_i > D(p_i)$, then firm $i$ will have some unsold products with certainty. Thus, the firm is always better-off by producing $q_i = D(p_i)$. Therefore, we can disregard the constraint $q_i \leq D(p_i)$ as it will always be satisfied.

---

\(^9\)Due to being normally distributed, beliefs involve negative as well as unreasonably high values. In order to analyze the importance of these extreme cases, we ran simulations using truncated normal distribution as well. This way we could make sure that $p_j^b$ takes values from the interval $[c, a]$ and $q_j^b$ from the interval $[0, a - bc]$. There was no visible change in the equilibria, thus extreme realizations do not have a substantial effect.

\(^10\)To keep the notation simple, we use $p_j^b$ and $q_j^b$ to denote both the random variable and a possible realization of the random variable.

\(^11\)Note that risk aversion in itself cannot lead to a pure-strategy Nash equilibrium since firms can still undercut each other’s price or operate on the residual demand with certainty.
Having discussed the model, we now turn to analyzing its equilibria. Variables $p_1^*, q_1^*$, $p_2^*$, and $q_2^*$ constitute an equilibrium if the following two conditions are satisfied:

1. $(p_t^*, q_t^*) \in \arg \max_{p_t, q_t} E(\pi_t^c) - \alpha \text{Var}(\pi_t^c)$ for $i = 1, 2$.

2. $p_t^f = p_t^*$ and $q_t^f = q_t^*$ for $i = 1, 2$.

The first condition means that actions are optimal given the conjectures. The second condition is a consistency requirement for the conjectures: it implies that conjectures are centered around the true values in equilibrium. Thus, in equilibrium the actions of the firms are optimal and their forecasts are correct.

This equilibrium concept is related to the random belief equilibrium (RBE), introduced by Friedman and Mezzetti (2005). They find empirical support for this concept in experimental data. Moreover, they compare RBE with the quantal response equilibrium (QRE) but the results are mixed: in some games RBE fits the subjects’ behavior better while QRE performs better in others. The authors conjecture that RBE fits the data better in non-zero sum games that have a unique completely mixed equilibrium. These conditions hold for the simultaneous price-quantity competition with linear demand function and constant and equal marginal costs.

In the remaining part of the chapter we focus on the symmetric equilibria of the model. In the next section we will see that there exists a unique symmetric pure-strategy equilibrium in this market, provided that firms are sufficiently risk averse and/or the amount of uncertainty is sufficiently high.

### 4.5 Symmetric pure-strategy equilibrium

The conditions that characterize the symmetric pure-strategy equilibria of the model are derived in Appendix 4.A. As these conditions are quite long and complex, we report them only in the Appendix, see equations (4.19)-(4.26). For simplifying the notation, let the first-order
conditions with respect to \( p_i \) and \( q_i \), evaluated at the fixed point \((p, q)\), be given by

\[
F_p(p, q) = 0, \tag{4.5}
\]

\[
F_q(p, q) = 0. \tag{4.6}
\]

This system of equations cannot be solved analytically: functions of \( p \) and \( q \) appear in the argument of the cumulative distribution function and the probability density function of the standard normal distribution in both equations. Therefore, we use numerical methods to find a solution and then we investigate whether it corresponds to the global maximum of the objective function of the firms. If it does, we can conclude to have found an equilibrium in pure strategies.

First we illustrate the existence of a symmetric pure-strategy equilibrium for a certain parameter specification and then we will analyze the set of parameters for which an equilibrium in pure strategies exists. We use the following parameter values in the calculations: \( a = 10, b = 1, c = 2, \alpha = 1 \) and \( \sigma_p = \sigma_q = 0.5 \). We numerically solve (4.5) and (4.6) by minimizing \( F_p^2 + F_q^2 \) with respect to \( p \) and \( q \).\(^\text{12}\) The resulting values are \( p^* \approx 3.87 \) and \( q^* \approx 2.94 \). The minimized value of the objective function is \( 1.36 \cdot 10^{-17} \). This value is very close to zero, which suggests that we have found a solution.

We cannot determine with this method whether there exist other points that satisfy the first order conditions. In order to answer this question, we numerically calculate the \((p, q)\) pairs that solve (4.5) and (4.6) separately and we study the number of \((p, q)\) combinations that satisfy both first-order conditions at the same time. We consider values for \( p \) from a fine grid in the interval \([c, \frac{a}{b}]\) and for each value of \( p \) we numerically calculate the value of \( q \) that satisfies (4.5) and (4.6), respectively. The upper left panel of Figure 4.1 shows the curves that consist of the points that satisfy the given first-order condition.\(^\text{13}\)

The figure shows that these curves cross each other at exactly one point. This point cor-

\(^\text{12}\)We use the \texttt{fminsearch} function in MATLAB for the minimization.

\(^\text{13}\)The curve for \( F_p \) does not look smooth for the following reason. It can be shown that \( F_p = 0 \) for any price when \( q = 0 \); and for some values of \( p \) the numerical procedure finds \( q = 0 \) instead of the positive solution for \( q \).
Figure 4.1: The \((p, q)\) pairs that satisfy the two first-order conditions separately (upper left panel) and the objective function of firm \(i\) with the price and production level of firm \(j\) fixed at \((p^*, q^*)\) (other panels). Parameter values: \(a = 10\), \(b = 1\), \(c = 2\), \(\alpha = 1\) and \(\sigma_p = \sigma_q = 0.5\).

responds to the previously calculated \((p^*, q^*)\). Thus, there exist a unique pair \((p, q)\) that may constitute an equilibrium in pure strategies. In order to conclude that this point is indeed an equilibrium, we need to examine if it corresponds to the global maximum of the objective function of a firm, keeping the price and production level of the other firm fixed at \((p^*, q^*)\). In other words, we need to check if choosing the same price and production level as the other firm is a best response. The upper right panel and the lower panels of Figure 4.1 depict the objective function of firm \(i\) for \(p_j = p^*\) and \(q_j = q^*\). We can observe that \((p^*, q^*)\) corresponds to the global maximum. The above analysis confirms that there exists a unique symmetric equilibrium in pure strategies for the parameter specification we considered.

We cannot conclude from the previous analysis that a pair \((p, q)\) that satisfies the first-order
conditions, will always be an equilibrium. In fact, for certain combinations of $\alpha$, $\sigma_p$ and $\sigma_q$, the solution of the first-order conditions does not correspond to the global maximum of the objective function of a firm, keeping the price and production level of the other firm fixed at the value in the solution. For some parameter combinations the solution is a local but not the global maximum while it can be a saddle point for other parameters. Therefore, it is essential to investigate which parameter combinations lead to an equilibrium in pure strategies. The left panel of Figure 4.2 shows the $(\sigma_p, \sigma_q)$ combinations that lead to a symmetric equilibrium in pure strategies for $\alpha = 1$. The curve gives the boundary of the region of existence in the $(\sigma_p, \sigma_q)$ plane: the solution of the first-order conditions constitutes an equilibrium for the $(\sigma_p, \sigma_q)$ combinations that lie to the right of the curve. The right panel of Figure 4.2 illustrates how the existence region changes as $\alpha$ varies. For obtaining this figure, we consider a grid for $\sigma_p$ and $\sigma_q$ for a given value of $\alpha$, and for each parameter combination $(\alpha, \sigma_p, \sigma_q)$ we calculate the point that solves the first-order conditions (4.5)-(4.6). Then we compare the value of the objective function of a firm at this point with the global maximum (with the price and production level of the other firm fixed at the values in the solution). When these two values coincide, the solution corresponds to an equilibrium.\(^{14}\) Finally, for each $(\sigma_p, \alpha)$ pair we consider the minimal value of

\(^{14}\)For finding the global maximum, we evaluate the objective function on a grid with 1000 values from the
σ_q for which the equilibrium exists.\textsuperscript{15} This leads to the curves depicted in Figure 4.2.

The figure shows that there exists no pure-strategy equilibrium when both the price and the output uncertainty are small.\textsuperscript{16} This result is in line with the standard model with risk neutrality and no uncertainty: there exists no equilibrium in pure strategies in the standard model and the model we consider is close to the standard model for very small values of σ_p, σ_q and α. The figure also shows that the existence region expands in the degree of risk aversion and in the amount of price and output uncertainty. That is, the more risk averse the firms are or the more uncertainty they face, the more parameter combinations will lead to an equilibrium in pure strategies. Also note that price uncertainty is essential for having a pure-strategy equilibrium: the equilibrium region does not contain points for which σ_p = 0. On the other hand, numerical calculations show that neither output uncertainty nor risk aversion is crucial for existence: there are equilibria for α = 0 and for σ_q too.

In the previous analysis we used an arbitrary parameter specification for the demand function. In order to check the robustness of our results, we considered other parameter values as well, including higher and lower slopes for the market demand function. The results are robust: there exists a unique pair (p, q) that solves the two first-order conditions and the structure of the region of existence is the same. Having established the existence of a symmetric equilibrium in pure strategies, we now turn to the properties of this equilibrium.

\subsection*{4.6 Comparative statics}

The equilibrium depends on several parameters. Among these parameters, the degree of risk aversion α and the amount of uncertainty σ_p and σ_q are of particular interest. In this section we discuss how these parameters affect the equilibrium.

\textsuperscript{15} Preliminary simulations showed that an equilibrium may exist when σ_q is large enough, given σ_p and α. The analysis in Section 4.6 also confirms this.

\textsuperscript{16} There might exist an equilibrium when α is sufficiently high. However, it might not be reasonable to assume very high values of α since they correspond to extreme degrees of risk aversion.
### 4.6.1 The effect of prices and production levels on the objective function

Before investigating the effect of a parameter change on the equilibrium, it is worthwhile to analyze how a marginal increase in a price or a production level (keeping everything else fixed) affects the objective function of firms in equilibrium. This will be useful for understanding the intuition behind the results of the comparative statics analyses. Table 4.1 summarizes the marginal effect of the variables on the expected profit and the profit variance of firm $i$ in equilibrium. A $+$ ($-$) sign means that a marginal increase in the variable in the first row has a positive (negative) effect on the variable in the first column. For example, a marginal increase in $p_j$ increases $E(\pi_i^*)$ and decreases $Var(\pi_i^*)$ in equilibrium.\(^\text{17}\) We derive these effects in Appendix 4.B.

To understand the intuition behind these effects, note that there are four possibilities concerning the (conjectured) sales of firm $i$. If firm $i$ has a lower price than the conjectured price of firm $j$, then firm $i$ can sell all its products. If firm $i$ has a larger price, then there are three cases. When the conjectured production level of firm $j$ is low enough, such that the residual demand of firm $i$ exceeds $q_i$, firm $i$ can sell its whole production. For intermediate values of $q_j$, firm $i$ operates on its positive residual demand and sells strictly less than its production level. Finally, firm $i$ does not sell anything when the conjectured production level of firm $j$ is high enough, such that the residual demand of firm $i$ becomes 0. For simplicity, we refer to the cases when firm $i$ sells all its products as good cases and we call a case bad when the firm has unsold products.

\(^\text{17}\)More precisely, we should write $p_j^f$ and $q_j^f$ instead of $p_j$ and $q_j$ as we analyze the effect of a shift in the distribution of conjectures. However, in equilibrium $p_j = p_j^f$ and $q_j = q_j^f$ therefore we drop superscript $f$ for simplifying the notation.

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<td>$Var(\pi_i^*)$</td>
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Table 4.1: The marginal effect of prices and production levels on the expected profit and profit variance of firm $i$ in equilibrium.
When $p_i$ increases marginally, the price of firm $i$ will be larger than the conjectured price of firm $j$ with a higher probability. This has an increasing effect on the profit variance since there is more uncertainty about the sales and thus about the profit of firm $i$ when the firm operates on its residual demand: the uncertain production level of firm $j$ matters only if firm $i$ has the higher price. Another effect of an increase in $p_i$ is that the profit of firm $i$ increases when it has positive sales. Note that the profit increases by $q_i$ in the good cases while it increases less in the bad cases (by $r_i$ or 0). Thus, the profit difference between good and bad cases increases and this also has an increasing effect on the profit variance.

Since firms try to find the balance between the expected profit and the profit variance, in equilibrium it must hold that an increase in one of the decision variables of firm $i$ has the same marginal effect on its expected profit and profit variance (up to a factor $\alpha$). Consequently, the expected profit of firm $i$ should increase when $p_i$ increases.

Keeping everything else fixed, a marginal increase in the production level of firm $i$ increases the profit in the good cases. In contrast, the profit of firm $i$ decreases in the bad cases since the firm will have more unsold products. Therefore, the profit difference between good and bad states increases, resulting in a larger variance. Furthermore, when $q_i$ increases, the bad cases will occur with a higher chance since firm $i$ is less likely to sell all its products. Since the bad cases lead to more uncertainty, this further increases the variance. Similar arguments as for $p_i$ show that the expected profit of firm $i$ should increase.

An increase in $p_j$ is favorable for firm $i$ since it will have a lower price with a higher chance. This leads to a higher expected profit since the case with the highest profit occurs more often. Furthermore, the profit variance decreases since the most uncertain case (when firm $i$ operates on its positive residual demand) occurs with a lower chance.

When firm $j$ increases its production level, then the residual demand of firm $i$ will decrease. Thus, firm $i$ can sell less products in the bad cases, leading to a lower expected profit. Moreover, the most uncertain case occurs with a higher chance since firm $i$ is less likely to sell all its products when it has the higher price. This increases the profit variance.
Figure 4.3: The equilibrium price $p^*$ (upper panel), production level $q^*$ (horizontal middle panel) and production to demand ratio $PD = \frac{2q^*}{a-bp^*}$ (lower panel) as a function of $\alpha$, for $\sigma_p = 0.5, 0.75$ and 1. The left panel corresponds to $\sigma_q = 0.25$, the vertical middle panel to $\sigma_q = 0.5$ and the right panel to $\sigma_q = 1$. Other parameter values: $a = 10$, $b = 1$ and $c = 2$.

Having discussed the effect of prices and production levels, we can now turn to analyzing the effects of the model parameters on the equilibrium.

4.6.2 The effect of risk aversion

When firms become more risk averse, that is as $\alpha$ increases, they have an extra incentive for reducing the variance. As Table 4.1 shows, this can be achieved by decreasing the price or the production level. Figure 4.3 shows the equilibrium price (upper panel), production level (horizontal middle panel) and production to demand ratio $PD = \frac{2q^*}{a-bp^*}$ (lower panel) as a function of $\alpha$, for different values of $\sigma_p$ and $\sigma_q$. The vertical panels correspond to different
values of $\sigma_q$ and the lines on each plot correspond to different values of $\sigma_p$. The figure shows that an increase in $\alpha$ has typically a positive effect on the equilibrium price. Only when both $\sigma_p$ and $\sigma_q$ are high, we can observe a slight decrease in the equilibrium price for higher values of $\alpha$. The equilibrium production level monotonically decreases in $\alpha$. Thus, as firms become more risk averse, they decrease their production level to reduce the profit variance and they charge a higher price to compensate for the lower expected profit. However, when both the price and output uncertainty are high and the firms are sufficiently risk averse, they use also their price to decrease the variance.

The production to demand ratio $PD = \frac{2q^*}{a - bp^*}$ compares the aggregate production level ($2q^*$) to the demand ($a - bp^*$) in equilibrium. When $PD < 100\%$, demand exceeds aggregate production, so firms do not serve the whole market. For $PD = 100\%$, aggregate production equals the demand: firms sell all their products and the demand is satisfied. For $PD > 100\%$ there is overproduction: the demand is satisfied but firms end up with some unsold products. The PD ratio shows a decreasing pattern: aggregate production decreases more than the market demand as firms become more risk averse. Note that the PD ratio is always less than 200\%, thus individual production levels are always strictly less than the market demand in equilibrium. For lower degrees of risk aversion there is overproduction and firms are rationed but as firms become more risk averse, there is underproduction and consumers are rationed. The figure shows that for any amount of uncertainty, there exists a degree of risk aversion for which aggregate production equals the market demand in equilibrium.

### 4.6.3 The effect of price uncertainty

A change in $\sigma_p$ affects the variance of price conjectures. However, this variance itself is not relevant for the firm in equilibrium. To see this, note that the exact value of the price of the

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Note that the different lines start at different values of $\alpha$. This is due to the fact that for a given $(\sigma_p, \sigma_q)$ combination the pure-strategy equilibrium exists only if $\alpha$ is sufficiently high, as Figure 4.2 shows.

Note that the figure also shows that an increase in $\sigma_p$ has a positive effect while an increase in $\sigma_q$ has a negative effect on $p^*$. We will investigate these effects separately later in this section.

In contrast, both firms produce up to the market demand at the price they chose in the mixed-strategy Nash equilibrium of the standard model.
other firm is irrelevant for the profit of firm \( i \), the only thing that matters is whether this price is smaller or larger than the price of firm \( i \). Since firms charge the same price in equilibrium, the conjectures of a firm are such that the probability of having the lower price is always 50\%, it does not change as \( \sigma_p \) varies. Using the first-order conditions of the firms’ problem, it can easily be shown that a change in \( \sigma_p \) does not affect either the expected profit or the profit variance in equilibrium.

The importance of \( \sigma_p \) is that it determines the marginal effect of \( p_i \) on the expected profit and the profit variance of firm \( i \) in equilibrium. To see this, consider the equilibrium for a fixed \( \sigma_p \). It follows from the optimization problem (4.4) of firm \( i \) that \( \frac{\partial E(\pi_i^e)}{\partial p_i} = \alpha \frac{\partial \text{Var}(\pi_i^e)}{\partial p_i} \) in equilibrium, thus the gain from a higher expected profit (for an increase in \( p_i \)) is exactly offset by the increase in the variance. Now suppose that firms are in equilibrium and \( \sigma_p \) has increased marginally. Note that the probability density function of \( p_{bj} \) becomes lower around 0. Consequently, when \( p_i \) increases marginally, the probability of firm \( i \) having the lower price will decrease less compared to the situation with the lower value of \( \sigma_p \). This means that as \( \sigma_p \) increases, less probability mass is shifted towards the region that gives a lower profit and a higher variance when \( p_i \) increases. Thus, for a marginal increase in \( p_i \), the expected profit will increase more whereas the variance will increase less compared to the original situation with the lower \( \sigma_p \). This essentially means that \( p_i \) becomes a more efficient instrument for increasing the expected profit and firm \( i \) has extra incentives to increase its price.

Figure 4.4 depicts the equilibrium price, production level and production to demand ratio as a function of \( \sigma_p \), for different values of \( \alpha \) and \( \sigma_q \). The different lines on the plots correspond to different values of \( \alpha \) whereas the vertical panels show the results for different values of \( \sigma_q \). The figure shows that the equilibrium price increases, the production level decreases while the production to demand ratio remains constant essentially as \( \sigma_p \) becomes larger. Thus, firms charge a higher price to increase their expected profit while they reduce their production level to offset the increase in the profit variance. The lower production level is also explained by the negative effect of price on demand.
Figure 4.4: The equilibrium price $p^*$ (upper panel), production level $q^*$ (horizontal middle panel) and production to demand ratio $PD = \frac{2q^*}{a-bp^*}$ as a function of $\sigma_p$, for $\alpha = 0.1, 0.25, 0.5$ and 1. The left panel corresponds to $\sigma_q = 0.25$, the vertical middle panel to $\sigma_q = 0.5$ and the right panel to $\sigma_q = 1$. Other parameter values: $a = 10$, $b = 1$ and $c = 2$.

Note that prices and quantities move in the same direction as for an increase in $\alpha$. The PD ratio, however, changes differently. This can be explained by the different objectives in the two cases. When firms become more risk averse, they have more incentives to reduce the variance, and they aim to serve a smaller share of the demand at a higher price as $\alpha$ increases. In contrast, firms focus more on increasing the expected profit in the current situation, and they want to serve a constant share of the demand at a higher price as $\sigma_p$ increases.

Based on a price increase accompanied by reduced production levels, one might think that firms engage in some sort of an anti-competitive behavior. A remarkable feature of our model is that this need not be the case: higher price uncertainty leads to higher prices and lower production levels in our model.
4.6.4 The effect of output uncertainty

In contrast to price uncertainty, higher output uncertainty leads to a higher profit variance. This is due to the fact that the conjectured production level of firm $j$ directly affects the profit of firm $i$ through the residual demand. And as the uncertainty regarding the residual demand increases, the profit variance also increases. A change in $\sigma_q$ affects the expected profit too but it is not clear by intuition whether it has a positive or a negative effect on it.\textsuperscript{21} We numerically evaluated the marginal effect of $\sigma_q$ on the expected profit and profit variance for all the parameter combinations we considered in this chapter. We found that the expected profit always decreases and that the marginal effect on the profit variance is larger in absolute value than the marginal effect on the expected profit. Since the marginal effect on the profit variance dominates, firms face too much risk compared to the equilibrium before the parameter change. Therefore, their main objective should be to reduce the variance (and possibly offset the corresponding negative effect on the expected profit).

Figure 4.5 shows the equilibrium price, production level and production to demand ratio as a function of $\sigma_q$, for different values of $\alpha$ and $\sigma_p$. The different lines correspond to different values of $\alpha$ while the vertical panels correspond to different values of $\sigma_p$. Output uncertainty has a clear negative effect on the equilibrium price. The effect on the production level is, however, ambiguous. For low values of $\sigma_q$ the production level increases, whereas it decreases for high values of $\sigma_q$. The PD ratio typically decreases, except for the case $\sigma_p$ is high and $\alpha$ is low.

The results show that firms charge a lower price for reducing the variance. This is intuitive since firms have an incentive to avoid ending up on the residual demand because the residual demand becomes more uncertain than for a lower value of $\sigma_q$. When $\sigma_q$ is low, the profit variance is lower, and firms increase their production levels to offset the decrease in the expected profit, caused by the lower price. However, when $\sigma_q$ is large, firms face a higher profit variance and therefore they decrease their production level to reduce the variance. Thus, as output uncertainty

\textsuperscript{21}As $\sigma_q$ increases, extreme realizations of the production level of firm $j$ occur with higher probability. While extremely low realizations are favorable for firm $i$, extremely high realizations lead to low residual demands and low profits. It is ambiguous whether the total effect on the expected profit is positive or negative.
increases, firms use mainly their price to reduce the variance. The use of production level
depends on the relation between expected profit and profit variance. When the variance is
higher and thus relatively more important than the expected profit, then firms decrease their
production levels to further reduce the variance. Otherwise they use their production level to
increase their expected profit.

4.6.5 Welfare analysis

Next we investigate the welfare effect of changes in the degree of risk aversion and in the
amount of uncertainty. It is not easy to find a proper welfare measure in this model as the
profit variance in the objective function of firms is measured in a different unit than consumer
surplus and (expected) profits. Therefore we consider two alternative measures. First, we apply a non-standard measure and define welfare as the sum of consumer surplus and the value of the objective function of firms. This measure takes into account that firms care about the variance of their profit but the unit of the measure is unclear. Second, we apply the standard definition of welfare, being the sum of consumer surplus and profits. This measures welfare in monetary terms but it disregards the exact form of the preferences of firms. It can be interpreted as the evaluation of the market outcome from a rational point of view as if firms had standard preferences.

We have seen in Figure 4.3 that $p^*$ typically increases whereas $q^*$ decreases as firms become more risk-averse. These changes have a clear negative effect on consumers. Figure 4.6 shows the total profit of the two firms, the total objective function value and the standard welfare measure as a function of $\alpha$. The plots show that the profit of firms increases for small values of $\alpha$ but then it becomes decreasing. The reason for the initial increase in profits is that firms have unsold products for low degrees of risk aversion therefore their losses decrease as they produce less. This positive effect disappears when there is no more excess supply and profits start to decrease. The objective function value of firms shows a similar pattern. The decrease in the objective function value begins already for lower values of $\alpha$ since an increase in $\alpha$ gives a higher weight to profit variance, and this is not incorporated in the standard profit definition. The standard welfare measure also shows a similar pattern: welfare increases for low degrees for risk aversion whereas it decreases for high degrees of risk aversion. Thus, a small amount of risk aversion is welfare enhancing in our model. For comparison, welfare in the social optimum under the given demand and cost structure is $W^* = \frac{1}{2} \left( \frac{a}{b} - c \right) \left( a - bc \right) = 32$ while the expected welfare in the mixed-strategy Nash equilibrium in the standard model is $E(W) \approx 22.44$ (see the derivation in Appendix 4.C). Thus, the welfare loss in our model is at most 10% compared to the social optimum for the different parameter values we consider and our model results in a higher welfare than the standard model. The reason for this is that overproduction is much
higher in the standard model than in our model. The non-standard measure of risk aversion, which involves the objective function value of firms, always decreases.

We analyze the welfare effects of a change in the amount of uncertainty as well. We do not report the corresponding plots as the welfare effects are almost always monotonic. As price uncertainty increases, consumers are worse-off since firms charge a higher price and they produce less. Firms, on the other hand, are better-off both in terms of profits and in terms of their true objective function value. This is in line with our previous observation that an increase in price uncertainty is actually favorable for firms. Also, as Figure 4.4 shows, firms produce slightly less and they sell their good at a relatively higher price and this increases their

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22 Remember that firms produce up to the market demand at the price they drew in the mixed-strategy Nash equilibrium of the standard model, therefore the firm with the higher price does not sell any of its produced quantity.
(expected) profit. The overall welfare effect is, however, negative under both welfare measures.

As output uncertainty increases, consumer surplus increases: firms charge a lower price whereas their production level either slightly decreases or even increases. Firms, however, are always worse-off both in terms of profits and objective function value. The non-standard welfare measure is always decreasing in $\sigma_q$. The standard welfare measure is decreasing for low values of $\sigma_p$. For higher values of $\sigma_p$, it is first increasing but then it becomes decreasing again. Thus, even though output uncertainty is favorable for consumers, its overall welfare effect is ambiguous.

4.7 Discussion and concluding remarks

In this chapter we have deviated from the assumption of rationality by weakening the consistency requirement on the beliefs of firms. We have introduced strategic uncertainty and risk aversion in the standard model of price-quantity competition and we have numerically shown that there exists a symmetric equilibrium in pure strategies when uncertainty is sufficiently high or firms are sufficiently risk-averse. Thus, incorporating bounded rationality in models may not only alter equilibria in the sense that the equilibrium outcome is slightly different than in the standard model, but it may create an equilibrium even when the standard model does not have equilibria (in pure strategies). The importance of having a pure-strategy equilibrium is that there does not exist a Nash equilibrium in pure strategies in the standard model with risk neutral, profit-maximizing firms. Therefore, this modified version of price-quantity competition can be used more widely as a market structure for analyzing various market phenomena and for policy analysis.

Strategic uncertainty is introduced through the conjectures of firms: firms have a point forecast for the actions of the other firm but they take into account that these forecasts might not be accurate. This generates probabilistic conjectures. Risk aversion is incorporated in the model with firms having mean-variance preferences. First we have derived the first-order conditions of
the optimization problem of firms, and then we have numerically found a symmetric solution. There exists a unique solution, however, it does not necessarily lead to the global maximum of the objective function of firms. Additional analysis shows that when firms are sufficiently risk averse or the amount of uncertainty is sufficiently high, then the solution to the first-order conditions is the global maximum, consequently, it gives a symmetric equilibrium. We have numerically characterized the parameter region for which the equilibrium exists. In equilibrium, each firm produces strictly less than the market demand at the equilibrium price. Aggregate production, however, may exceed the market demand, depending on the parameters. Our analysis shows that aggregate production exceeds the market demand for low degrees of risk aversion while firms do not serve together the whole market when they are too risk averse. For any amount of uncertainty, there exists a degree of risk aversion such that demand equals supply in equilibrium, provided that the equilibrium exists.

We have analyzed how the equilibrium depends on important parameters of the model such as the degree of risk aversion and the amount of price and output uncertainty. The results show that as firms become more risk averse, they produce less to decrease the profit variance and they sell their products at a higher price to offset the negative effect on the expected profit. The effect of price uncertainty is similar: the equilibrium price increases and the production level decreases as price uncertainty increases. The reason for this is that price uncertainty affects the marginal effect of price: price becomes a more efficient instrument for increasing the expected profit. Firms react differently to output uncertainty than to price uncertainty: the equilibrium price always decreases, while the production level typically decreases as output uncertainty increases. The reason behind this difference is that price uncertainty is favorable for firms (the expected profit can be increased more efficiently with the price) while output uncertainty is not: it directly increases the profit variance through the residual demand function so firms try to avoid operating on the residual demand by charging a lower price. The welfare analysis shows that a small degree of risk aversion is welfare enhancing and that our model results in a higher welfare than the expected welfare in the mixed-strategy Nash equilibrium of the standard
model. For investigating the robustness of our results, we performed the previous analysis for
different demand parameters as well. We observed qualitatively the same effects as before.

Some of our results are in line with experimental findings. Cracau and Franz (2012) conduct
an experiment on simultaneous price-quantity setting with linear demand and constant and equal
marginal costs. They found that subjects did not play according to the mixed-strategy Nash
equilibrium: the average price was higher while the average production was lower than the
equilibrium prediction. Moreover, subjects did not always choose the production level that
corresponds to the market demand at the chosen price. This latter finding holds for our model
as well since the PD ratio is always smaller than 200%. Another similarity is that Cracau and
Franz (2012) found typically overproduction in the market: this occurs in our model for low
degrees of risk aversion. An important difference, however, is that subjects typically did not
converge to a fixed point whereas our model leads to a unique equilibrium. However, if we
consider parameter values for which there is no pure-strategy equilibrium, a dynamic version
of our model might be in line with the latter experimental result as prices and production levels
cannot settle down to a single value.

Our analysis can be extended in several ways. The predictions of the model about the effects
of a change in price or output uncertainty could be tested experimentally. The method outlined
in this chapter can be used to analyze different market models\textsuperscript{23} or asymmetric situations too.
Firms could have different marginal costs or different degree of risk aversion, for example.
They may also face different amount of uncertainty. The analysis of asymmetric situations is
left for future work. Another important extension is to turn the model into a dynamic one. This
can be done by specifying how forecasts for the price and production level of the other firm are
formed. For example, firms could use adaptive updating or estimations using observed data as
in Chapters 2 and 3.

\textsuperscript{23}We briefly analyzed the model with the same kind of strategic uncertainty and risk aversion in Bertrand and
Cournot competition as well. Preliminary analysis shows that the following relation holds for the equilibrium
production levels across the different market models: $q^C < q^P < q^B$. The opposite relation holds for prices.
Moreover, the equilibrium price in the Bertrand model exceeds the marginal cost of production.
Appendix 4.A The first-order conditions for the symmetric pure-strategy equilibrium

The first-order conditions of optimization problem (4.4)

The objective function

The expected profit of firm $i$ can be expressed as

$$E(\pi_i^e) = p_i E(s_i) - cq_i.$$  \hfill (4.7)

The variance of the profit is $Var(\pi_i^e) = Var(p_i s_i - cq_i) = p_i^2 Var(s_i)$, which leads to

$$Var(\pi_i^e) = p_i^2 \left[ E(s_i^2) - E(s_i)^2 \right].$$ \hfill (4.8)

Then the objective function of firm $i$ can be written as

$$p_i E(s_i) [1 + \alpha p_i E(s_i)] - cq_i - \alpha p_i^2 E(s_i^2).$$ \hfill (4.9)

Note that both $E(s_i)$ and $E(s_i^2)$ depend on $p_i$, $q_i$, $p_j$ and $q_j$.

First-order conditions

Firm $i$ maximizes (4.9) with respect to $p_i$ and $q_i$. The first-order condition with respect to $p_i$ reads as

$$E(s_i) + p_i \frac{\partial E(s_i)}{\partial p_i} + 2 \alpha p_i E(s_i)^2 + 2 \alpha p_i^2 E(s_i) \frac{\partial E(s_i)}{\partial p_i} - 2 \alpha p_i E(s_i^2) - \alpha p_i^2 \frac{\partial E(s_i^2)}{\partial p_i} = 0.$$  

This expression simplifies to

$$
(1 + 2 \alpha p_i E(s_i)) \left( E(s_i) + p_i \frac{\partial E(s_i)}{\partial p_i} \right) - \alpha p_i \left( 2 E(s_i^2) + p_i \frac{\partial E(s_i^2)}{\partial p_i} \right) = 0. \hfill (4.10)
$$
The first-order condition with respect to $q_i$ is given by

$$p_i \frac{\partial E(s_i)}{\partial q_i} + 2 \alpha p_i^2 E(s_i) \frac{\partial E(s_i)}{\partial q_i} - c - \alpha p_i^2 \frac{\partial E(s_i^2)}{\partial q_i} = 0,$$

which simplifies to

$$p_i \frac{\partial E(s_i)}{\partial q_i} (1 + 2 \alpha p_i E(s_i)) - c - \alpha p_i^2 \frac{\partial E(s_i^2)}{\partial q_i} = 0. \quad (4.11)$$

For further characterizing the solution, we need to give the formula for $E(s_i)$, $E(s_i^2)$ and for the partial derivatives of these terms with respect to $p_i$ and $q_i$. We derive these expressions in the next paragraphs. As beliefs are normally distributed, we can represent them in the following form:

$$p_{bj} = p_{fj} + \sigma p \varepsilon_{j,p},$$
$$q_{bj} = q_{fj} + \sigma q \varepsilon_{j,q},$$
where $\varepsilon_{j,p}$ and $\varepsilon_{j,q}$ are independent standard normal random variables. We use these representations throughout the appendices.

**Expected sales $E(s_i)$**

There are three possible cases concerning the value of $s_i$:

- $s_i = q_i$: Firm $i$ sells up to his production level $q_i$ either if it has the lower price or if it has the higher price and its residual demand exceeds its production level. The first condition is that $p_i < p_j^b$, or equivalently $\frac{p_i - p_j}{\sigma_p} < \varepsilon_{j,p}$. The second condition is that $p_i > p_j^b$ and $a - bp_i - q_j^b \geq q_i$, or equivalently $\frac{p_i - p_j}{\sigma_p} > \varepsilon_{j,p} + \frac{a - bp_i - q_j}{\sigma_q} \geq \varepsilon_{j,q}$. For simplifying notation, let $A = \frac{1}{\sigma_q} (a - bp_i - q_i - q_j)$ such that the latter condition reads as $A \geq \varepsilon_{j,q}$.

- $s_i = a - bp_i - q_j^b$: Firm $i$ sells up to his (positive) residual demand if it charges the higher price and its residual demand is positive. This leads to the conditions $p_i > p_j^b$ and $0 \leq a - bp_i - q_j^b < q_i$, or equivalently $\varepsilon_{j,p} < \frac{p_i - p_j}{\sigma_p}$ and $B \geq \varepsilon_{j,q} \geq A$, where $B = \frac{1}{\sigma_q} (a - bp_i - q_j)$.

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24 In the classification below we do not consider the case when both firms charge the same price as it has a measure 0 and thus does not affect the optimization problem of the firms.

25 Here we implicitly assume that $q_i \leq D(p_i)$.
• $s_i = 0$: Firm $i$ does not sell anything when it has the higher price and its residual demand at price $p_i$ is negative. This gives $p_i > p^b_j$ and $a - b p_i < q^b_j$, or equivalently $\varepsilon_{j,p} < \frac{p_i - p_j}{\sigma_p}$ and $B < \varepsilon_{j,q}$.

Therefore, expected sales can be calculated as

$$E(s_i) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} s_i \phi(x_q) \phi(x_p) \, dx_q \, dx_p = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} q_i \phi(x_q) \phi(x_p) \, dx_q \, dx_p$$

$$+ \int_{-\infty}^{\infty} \int_{-\infty}^{B} q_i \phi(x_q) \phi(x_p) \, dx_q \, dx_p + \int_{-\infty}^{B} \int_{A}^{\infty} (a - b p_i - q_j - \sigma_q x_q) \phi(x_q) \phi(x_p) \, dx_q \, dx_p$$

$$= q_i \left[ 1 - \Phi \left( \frac{p_i - p_j}{\sigma_p} \right) \right] + q_i \Phi \left( \frac{p_i - p_j}{\sigma_p} \right) \Phi(A)$$

$$+ \Phi \left( \frac{p_i - p_j}{\sigma_p} \right) \left\{ (a - b p_i - q_j) \left[ \Phi(B) - \Phi(A) \right] + \sigma_q \left[ \phi(B) - \phi(A) \right] \right\}$$

$$= q_i - q_i \Phi \left( \frac{p_i - p_j}{\sigma_p} \right) \left[ 1 - \Phi(A) \right]$$

$$+ \Phi \left( \frac{p_i - p_j}{\sigma_p} \right) \sigma_q \left\{ B \left[ \Phi(B) - \Phi(A) \right] + \phi(B) - \phi(A) \right\}. \quad (4.12)$$

For deriving the third term with the integral, we used the property that $\phi'(x) = -x \phi(x)$:

$$\int_{A}^{B} x_q \phi(x_q) \, dx_q = - \int_{A}^{B} \phi'(x_q) \, dx_q = [-\phi(x_q)]_{A}^{B} = \phi(A) - \phi(B).$$

**Expected squared sales $E(s^2_i)$**

For deriving $E(s^2_i)$ we can use the same steps as for deriving $E(s_i)$. We just need to replace $s_i$ with $s^2_i$ in the integral. Thus,

$$E(s^2_i) = q^2_i - q^2_i \Phi \left( \frac{p_i - p_j}{\sigma_p} \right) \left[ 1 - \Phi(A) \right] + \Phi \left( \frac{p_i - p_j}{\sigma_p} \right) M, \quad (4.13)$$

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where

\[ M = \int_{A}^{B} (a - bp_i - q_j - \sigma_q x_q)^2 \phi(x_q) \, dx_q \]

\[ = \int_{A}^{B} (a - bp_i - q_j)^2 \phi(x_q) \, dx_q - \int_{A}^{B} 2(a - bp_i - q_j)\sigma_q x_q \phi(x_q) \, dx_q + \int_{A}^{B} \sigma_q^2 x_q^2 \phi(x_q) \, dx_q \]

\[ = (a - bp_i - q_j)^2 [\Phi(B) - \Phi(A)] + 2\sigma_q (a - bp_i - q_j) [\phi(B) - \phi(A)] \]

\[ + \sigma_q^2 [A\phi(A) - B\phi(B) + \Phi(B) - \Phi(A)] , \]

which simplifies to

\[ M = \sigma_q^2 \left\{ (B^2 + 1) [\Phi(B) - \Phi(A)] + B [\phi(B) - \phi(A)] - \frac{q_i}{\sigma_q} \phi(A) \right\} . \]  \hspace{1cm} (4.14)

We used integration by parts for deriving the formula for \( M \):

\[ \int_{A}^{B} x_q^2 \phi(x_q) \, dx_q = - \int_{A}^{B} x_q (-x_q \phi(x_q)) \, dx_q = - \int_{A}^{B} x_q \phi'(x_q) \, dx_q \]

\[ = - \left[ x_q \phi(x_q) \right]_{A}^{B} - \int_{A}^{B} \phi(x_q) \, dx_q \]

\[ = A\phi(A) - B\phi(B) + \Phi(B) - \Phi(A) . \]

**Marginal effect of own price on expected sales:** \[ \frac{\partial E(s_i)}{\partial p_i} \]
\[ \frac{\partial E(s_i)}{\partial p_i} = - \frac{q_i}{\sigma_p} \phi \left( \frac{p_i - p_j}{\sigma_p} \right) \left[ 1 - \Phi(A) \right] - \frac{q_i}{\sigma_p} \phi \left( \frac{p_i - p_j}{\sigma_p} \right) \phi(A) \frac{b}{\sigma_p} \]

\[ + \frac{\sigma_q}{\sigma_p} \phi \left( \frac{p_i - p_j}{\sigma_p} \right) \left\{ B \left[ \Phi(B) - \Phi(A) \right] + \phi(B) - \phi(A) \right\} \]

\[ + \sigma_q \Phi \left( \frac{p_i - p_j}{\sigma_p} \right) \left\{ \left( - \frac{b}{\sigma_q} \right) \left[ \Phi(B) - \Phi(A) \right] + B \left[ - \phi(B) \frac{b}{\sigma_q} + \phi(A) \frac{b}{\sigma_q} \right] \right\} \]

which simplifies to

\[ \frac{\partial E(s_i)}{\partial p_i} = - \frac{q_i}{\sigma_p} \phi \left( \frac{p_i - p_j}{\sigma_p} \right) \left[ 1 - \Phi(A) \right] \]

\[ + \frac{\sigma_q}{\sigma_p} \phi \left( \frac{p_i - p_j}{\sigma_p} \right) \left\{ B \left[ \Phi(B) - \Phi(A) \right] + \phi(B) - \phi(A) \right\} \]

\[ - \frac{b}{\sigma_q} \Phi \left( \frac{p_i - p_j}{\sigma_p} \right) \left[ \Phi(B) - \Phi(A) \right] \cdot \tag{4.15} \]

**Marginal effect of own production on expected sales:** \( \frac{\partial E(s_i)}{\partial q_i} \)

\[ \frac{\partial E(s_i)}{\partial q_i} = 1 - \Phi \left( \frac{p_i - p_j}{\sigma_p} \right) \left[ 1 - \Phi(A) \right] - \frac{1}{\sigma_q} q_i \phi \left( \frac{p_i - p_j}{\sigma_p} \right) \phi(A) \]

\[ + \Phi \left( \frac{p_i - p_j}{\sigma_p} \right) \sigma_q \left\{ B \phi(A) \frac{1}{\sigma_q} - A \phi(A) \frac{1}{\sigma_q} \right\} \]

which simplifies to

\[ \frac{\partial E(s_i)}{\partial q_i} = 1 - \Phi \left( \frac{p_i - p_j}{\sigma_p} \right) \left[ 1 - \Phi(A) \right] \cdot \tag{4.16} \]

**Marginal effect of own price on expected squared sales:** \( \frac{\partial E(s_i^2)}{\partial p_i} \)
\[
\frac{\partial E(s_i^2)}{\partial p_i} = -\frac{q_i^2}{\sigma_p} \phi \left( \frac{p_i - p_j}{\sigma_p} \right) [1 - \Phi(A)] - q_i^2 \Phi \left( \frac{p_i - p_j}{\sigma_p} \right) \phi(A) \frac{b}{\sigma_q} \\
+ \frac{\sigma_q^2}{\sigma_p} \phi \left( \frac{p_i - p_j}{\sigma_p} \right) \left\{ (B^2 + 1) [\Phi(B) - \Phi(A)] + B [\phi(B) - \phi(A)] - \frac{q_i}{\sigma_q} \phi(A) \right\} \\
+ \sigma_q^2 \Phi \left( \frac{p_i - p_j}{\sigma_p} \right) \left\{ -2B \frac{b}{\sigma_q} [\Phi(B) - \Phi(A)] + (B^2 + 1) \left[ -\phi(B) \frac{b}{\sigma_q} + \phi(A) \frac{b}{\sigma_q} \right] \\
- \frac{b}{\sigma_q} [\phi(B) - \phi(A)] + B \left[ B \phi(B) \frac{b}{\sigma_q} - A \phi(A) \frac{b}{\sigma_q} \right] \\
- \frac{q_i}{\sigma_q} \phi(A) \frac{b}{\sigma_q} \right\},
\]

which simplifies to

\[
\frac{\partial E(s_i^2)}{\partial p_i} = -\frac{q_i^2}{\sigma_p} \phi \left( \frac{p_i - p_j}{\sigma_p} \right) [1 - \Phi(A)] \\
+ \frac{\sigma_q^2}{\sigma_p} \phi \left( \frac{p_i - p_j}{\sigma_p} \right) \left\{ (B^2 + 1) [\Phi(B) - \Phi(A)] + B [\phi(B) - \phi(A)] - \frac{q_i}{\sigma_q} \phi(A) \right\} \\
- 2\sigma_q \Phi \left( \frac{p_i - p_j}{\sigma_p} \right) b \left\{ B [\Phi(B) - \Phi(A)] + \phi(B) - \phi(A) \right\}. \tag{4.17}
\]

**Marginal effect of own production on expected squared sales:**

\[
\frac{\partial E(s_i^2)}{\partial q_i} = 2q_i - 2q_i \Phi \left( \frac{p_i - p_j}{\sigma_p} \right) [1 - \Phi(A)] - q_i^2 \Phi \left( \frac{p_i - p_j}{\sigma_p} \right) \phi(A) \frac{1}{\sigma_q} \\
+ \sigma_q^2 \Phi \left( \frac{p_i - p_j}{\sigma_p} \right) \left\{ (B^2 + 1) \phi(A) \frac{1}{\sigma_q} - \frac{1}{\sigma_q} AB \phi(A) - \frac{1}{\sigma_q} \phi(A) - \frac{q_i}{\sigma_q} A \phi(A) \frac{1}{\sigma_q} \right\},
\]

which simplifies to

\[
\frac{\partial E(s_i^2)}{\partial q_i} = 2q_i - 2q_i \Phi \left( \frac{p_i - p_j}{\sigma_p} \right) [1 - \Phi(A)]. \tag{4.18}
\]
Thus, the first-order conditions of optimization problem (4.4) are characterized by equations (4.10)-(4.18).

**Symmetric pure-strategy equilibria**

For deriving the conditions that characterize the symmetric pure-strategy equilibria, we need to substitute \( p_i = p_j = p \) and \( q_i = q_j = q \) in equations (4.10)-(4.18). This yields the following equations:

\[
\left( 1 + 2\alpha p E(s_i) \right) \left( E(s_i) + p \frac{\partial E(s_i)}{\partial p_i} \right) - \alpha p \left( 2 E(s_i^2) + p \frac{\partial E(s_i^2)}{\partial p_i} \right) = 0, \tag{4.19}
\]

\[
p \frac{\partial E(s_i)}{\partial q_i} \left( 1 + 2\alpha p E(s_i) \right) - c - \alpha p^2 \frac{\partial E(s_i^2)}{\partial q_i} = 0, \tag{4.20}
\]

\[
E(s_i) = 0.5q \left[ 1 + \Phi(A) \right] + 0.5\sigma_q \left\{ B \left[ \Phi(B) - \Phi(A) \right] + \phi(B) - \phi(A) \right\}, \tag{4.21}
\]

\[
E(s_i^2) = 0.5q^2 \left[ 1 + \Phi(A) \right] + 0.5\sigma_q^2 \left\{ (B^2 + 1) \left[ \Phi(B) - \Phi(A) \right] + B \left[ \phi(B) - \phi(A) \right] - \frac{q}{\sigma_q} \phi(A) \right\}, \tag{4.22}
\]
(4.23)

\[ \frac{\partial E(s_i)}{\partial p_i} \bigg|_{(p,q)} = - \frac{1}{\sqrt{2\pi} \sigma_p} \frac{q}{\sigma_p} [1 - \Phi(A)] + \frac{1}{\sqrt{2\pi} \sigma_p} \{ B [\Phi(B) - \Phi(A)] + \phi(B) - \phi(A) \} - 0.5b [\Phi(B) - \Phi(A)], \]

(4.24)

\[ \frac{\partial E(s_i)}{\partial q_i} \bigg|_{(p,q)} = 0.5 [1 + \Phi(A)], \]

(4.25)

\[ \frac{\partial E(s_i^2)}{\partial p_i} \bigg|_{(p,q)} = - \frac{1}{\sqrt{2\pi} \sigma_p} \frac{q^2}{\sigma_p} [1 - \Phi(A)] + \frac{1}{\sqrt{2\pi} \sigma_p} \{ (B^2 + 1) [\Phi(B) - \Phi(A)] + B [\phi(B) - \phi(A)] - \frac{q^2}{\sigma_q} \phi(A) \} - b\sigma_q \{ B [\Phi(B) - \Phi(A)] + \phi(B) - \phi(A) \}, \]

(4.26)

\[ \frac{\partial E(s_i^2)}{\partial q_i} \bigg|_{(p,q)} = q [1 + \Phi(A)], \]

where \( A = \frac{1}{\sigma_p} (a - bp - 2q) \) and \( B = \frac{1}{\sigma_p} (a - bp - q). \)
Appendix 4.B  The marginal effect of prices and production levels in equilibrium

Marginal effect of own production level on expected profit: $\frac{\partial E(\pi^e_i)}{\partial q_i}$

Using (4.7), the marginal effect of $q_i$ on the expected profit is $p_i \frac{\partial E(s_i)}{\partial q_i} - c$. From (4.20) we know that

$$p \frac{\partial E(s_i)}{\partial q_i} \bigg|_{(p,q)} - c = -2\alpha p^2 E(s_i) \bigg|_{(p,q)} \frac{\partial E(s_i)}{\partial q_i} \bigg|_{(p,q)} + \alpha p^2 \frac{\partial E(s_i^2)}{\partial q_i} \bigg|_{(p,q)}$$

in equilibrium, so

$$\frac{\partial E(\pi^e_i)}{\partial q_i} \bigg|_{(p,q)} = \alpha p^2 \left( \frac{\partial E(s_i^2)}{\partial q_i} \bigg|_{(p,q)} - 2 E(s_i) \bigg|_{(p,q)} \frac{\partial E(s_i)}{\partial q_i} \bigg|_{(p,q)} \right).$$

Comparing (4.24) and (4.26), it can be seen that $\frac{\partial E(s_i^2)}{\partial q_i} \bigg|_{(p,q)} = 2q \frac{\partial E(s_i)}{\partial q_i} \bigg|_{(p,q)}$, so the marginal effect of $q_i$ on $E(\pi^e_i)$ reduces to

$$\frac{\partial E(\pi^e_i)}{\partial q_i} \bigg|_{(p,q)} = 2\alpha p^2 \frac{\partial E(s_i)}{\partial q_i} \bigg|_{(p,q)} \left( q - E(s_i) \bigg|_{(p,q)} \right).$$

It is easy to see from (4.24) that $\frac{\partial E(s_i)}{\partial q_i} \bigg|_{(p,q)} > 0$. The term $q - E(s_i) \bigg|_{(p,q)}$ is obviously positive since $q_i$ is the maximal value of $s_i$, therefore $q_i > E(s_i)$. Consequently,

$$\frac{\partial E(\pi^e_i)}{\partial q_i} \bigg|_{(p,q)} > 0.$$

Marginal effect of own production level on profit variance: $\frac{\partial \text{Var}(\pi^e_i)}{\partial q_i}$

It follows from the first-order conditions of optimization problem (4.4) that

$$\frac{\partial E(\pi^e_i)}{\partial q_i} = \alpha \frac{\partial \text{Var}(\pi^e_i)}{\partial q_i}$$

in equilibrium. We have shown that $\frac{\partial E(\pi^e_i)}{\partial q_i} > 0$ in equilibrium, thus $\frac{\partial \text{Var}(\pi^e_i)}{\partial q_i} \bigg|_{(p,q)} > 0$ must also hold.
Marginal effect of other production level on expected profit: \( \frac{\partial E(s_i)}{\partial q_j} \)

It follows from (4.12) that

\[
\frac{\partial E(s_i)}{\partial q_j} = q_i \Phi \left( \frac{p_i - p_j}{\sigma_p} \right) \phi(A) \left( -\frac{1}{\sigma_q} \right) + \Phi \left( \frac{p_i - p_j}{\sigma_p} \right) \sigma_q \left\{ \left( -\frac{1}{\sigma_q} \right) \left[ \Phi(B) - \Phi(A) \right] + B \left[ \phi(B) \left( -\frac{1}{\sigma_q} \right) - \phi(A) \left( -\frac{1}{\sigma_q} \right) \right] \right. \\
- B \phi(B) \left( -\frac{1}{\sigma_q} \right) + A \phi(A) \left( -\frac{1}{\sigma_q} \right) \right\},
\]

which simplifies to \( \frac{\partial E(s_i)}{\partial q_j} = -\Phi \left( \frac{p_i - p_j}{\sigma_p} \right) \left[ \Phi(B) - \Phi(A) \right]. \) Since \( A < B \) and \( \Phi(x) \) is an increasing function, \( \frac{\partial E(s_i)}{\partial q_j} < 0. \) Thus, the marginal effect of \( q_j \) on \( E(\pi_i^c) \) is also negative: \( \frac{\partial E(\pi_i^c)}{\partial q_j} = p_i \frac{\partial E(s_i)}{\partial q_j} < 0 \) since \( p_i > 0. \)

Marginal effect of other production level on profit variance: \( \frac{\partial \text{Var}(\pi_i^c)}{\partial q_j} \)

Using (4.13) and (4.14), the marginal effect of \( q_j \) on \( E(s_i^2) \) is

\[
\frac{\partial E(s_i^2)}{\partial q_j} = q_i^2 \Phi \left( \frac{p_i - p_j}{\sigma_p} \right) \phi(A) \left( -\frac{1}{\sigma_q} \right) + \Phi \left( \frac{p_i - p_j}{\sigma_p} \right) \sigma_q^2 \left\{ 2B \left( -\frac{1}{\sigma_q} \right) [\Phi(B) - \Phi(A)] \right. \\
+ (B^2 + 1) \left[ \phi(B) \left( -\frac{1}{\sigma_q} \right) - \phi(A) \left( -\frac{1}{\sigma_q} \right) \right] \right. \\
+ \left( -\frac{1}{\sigma_q} \right) [\phi(B) - \phi(A)] \\
+ B \left[ -B \phi(B) \left( -\frac{1}{\sigma_q} \right) + A \phi(A) \left( -\frac{1}{\sigma_q} \right) \right] \\
+ \frac{q_i}{\sigma_q} A \phi(A) \left( -\frac{1}{\sigma_q} \right) \left\},
\]
from which

\[
\frac{\partial E(s_i^2)}{\partial q_j} = -\frac{q_i^2}{\sigma_q} \left( \frac{p_i - p_j}{\sigma_p} \right) \phi(A) \\
- \Phi \left( \frac{p_i - p_j}{\sigma_p} \right) \sigma_q \left\{ 2B \left[ \Phi(B) - \Phi(A) \right] + (B^2 + 1) \left[ \phi(B) - \phi(A) \right] \right\} \\
- \Phi \left( \frac{p_i - p_j}{\sigma_p} \right) \sigma_q \left\{ \phi(B) - \phi(A) - B^2 \phi(B) + AB \phi(A) + \frac{q_i}{\sigma_q} A \phi(A) \right\},
\]

which simplifies to

\[
\frac{\partial E(s_i^2)}{\partial q_j} = -2\Phi \left( \frac{p_i - p_j}{\sigma_p} \right) \sigma_q \left\{ B \left[ \Phi(B) - \Phi(A) \right] + \phi(B) - \phi(A) \right\}.
\]

This expression is negative since \( \Phi \left( \frac{p_i - p_j}{\sigma_p} \right) \sigma_q \left\{ B \left[ \Phi(B) - \Phi(A) \right] + \phi(B) - \phi(A) \right\} \) is the contribution to the expected sales of the case \( 0 < r_i < q_i \) (see formula (4.12)), which must be positive.

Using (4.8), the marginal effect of \( q_j \) on \( \text{Var}(\pi_i^c) \) is given by

\[
\frac{\partial \text{Var}(\pi_i^c)}{\partial q_j} = p_i^2 \left[ \frac{\partial E(s_i^2)}{\partial q_j} - 2E(s_i) \frac{\partial E(s_i)}{\partial q_j} \right].
\]

The sign of this term is ambiguous since both derivatives are negative and \( E(s_i) > 0 \). In order to determine the sign of this expression, we need to know the exact value of \( p^* \) and \( q^* \). Therefore we evaluated \( \frac{\partial \text{Var}(\pi_i^c)}{\partial q_j} \bigg|_{(p,q)} \) numerically for all parameter combinations that we considered in this chapter. All calculations show that \( \frac{\partial \text{Var}(\pi_i^c)}{\partial q_j} \bigg|_{(p,q)} \) is positive, that is an increase in \( q_j \) increases the profit variance of firm \( i \) in equilibrium.

**Marginal effect of other price on expected profit:** \( \frac{\partial E(\pi_i^c)}{\partial p_j} \)

Using (4.12), the expected sales of firm \( i \) can be expressed in the following form: \( E(s_i) = \)
\[ q_i + \Phi \left( \frac{p_i - p_j}{\sigma_p} \right) X_1, \]

where

\[ X_1 = -q_i [1 - \Phi(A)] + \sigma_q \left\{ B [\Phi(B) - \Phi(A)] + \phi(B) - \phi(A) \right\}. \] (4.27)

The expected sales is smaller than \( q_i \) (since \( q_i \) is the maximal value of \( s_i \)), thus \( X_1 < 0 \) must hold. Furthermore, \( X_1 \) is independent of \( p_j \). It is easy to see that \( \Phi \left( \frac{p_i - p_j}{\sigma_p} \right) \) is decreasing in \( p_j \), therefore \( \frac{\partial E(s_i)}{\partial p_j} > 0 \). This implies that the marginal effect of \( p_j \) on \( E(\pi_i) \) is positive:

\[ \frac{\partial E(\pi_i)}{\partial p_j} = p_i \frac{\partial E(s_i)}{\partial p_j} > 0 \text{ since } p_i > 0. \]

**Marginal effect of other price on profit variance:** \( \frac{\partial \text{Var}(\pi_i)}{\partial p_j} \)

Using (4.13) and (4.14), the expected squared sales of firm \( i \) can be expressed as \( E(s_i^2) = q_i^2 + \Phi \left( \frac{p_i - p_j}{\sigma_p} \right) X_2 \) with

\[ X_2 = -q_i^2 [1 - \Phi(A)] + \sigma_q^2 \left\{ (B^2 + 1) [\Phi(B) - \Phi(A)] + B [\Phi(B) - \Phi(A)] - \frac{q}{\sigma_q} \phi(A) \right\}. \] (4.28)

Since \( q_i^2 \) is the maximal value of \( s_i^2 \), \( E(s_i^2) \) must be smaller than \( q_i^2 \) and consequently \( X_2 < 0 \). Therefore, \( \frac{\partial E(s_i^2)}{\partial p_j} > 0 \) since \( X_2 \) is independent of \( p_j \) and \( \Phi \left( \frac{p_i - p_j}{\sigma_p} \right) \) is decreasing in \( p_j \).

From (4.8) the marginal effect of \( p_j \) on \( \text{Var}(\pi_i) \) is given by

\[ \frac{\partial \text{Var}(\pi_i)}{\partial p_j} = p_i^2 \left[ \frac{\partial E(s_i^2)}{\partial p_j} - 2E(s_i) \frac{\partial E(s_i)}{\partial p_j} \right]. \]

The sign of this term is ambiguous since both derivatives are positive and \( E(s_i) > 0 \). In order to determine the sign of this expression, we need to know the exact value of \( p^* \) and \( q^* \). Therefore we evaluated \( \frac{\partial \text{Var}(\pi_i)}{\partial p_j} \bigg|_{(p,q)} \) numerically for all parameter combinations that we considered in this chapter. All calculations show that \( \frac{\partial \text{Var}(\pi_i)}{\partial p_j} \bigg|_{(p,q)} \) is negative, that is an increase in \( p_j \) decreases the profit variance of firm \( i \) in equilibrium.

**Marginal effect of own price on expected profit:** \( \frac{\partial E(\pi_i^2)}{\partial p_i} \)

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Combining (4.23) with (4.27), it can be seen that
\[
\frac{\partial E(s_i)}{\partial p_i} \bigg|_{(p,q)} = \frac{1}{\sqrt{2\pi}\sigma_p} X_1 - 0.5b [\Phi(B) - \Phi(A)].
\]
This expression is negative since \(X_1 < 0\) and \(\Phi(B) > \Phi(A)\).

Using (4.7), the marginal effect of \(p_i\) on \(E(\pi_i^c)\) in equilibrium is given by
\[
\frac{\partial E(\pi_i^c)}{\partial p_i} \bigg|_{(p,q)} = E(s_i) \bigg|_{(p,q)} + p \frac{\partial E(s_i)}{\partial p_i} \bigg|_{(p,q)}.
\]
The sign of this expression is ambiguous since the first term is positive while the second one is negative.

We evaluated \(\frac{\partial E(\pi_i^c)}{\partial p_i} \bigg|_{(p,q)}\) numerically for all parameter combinations that we considered in this chapter. All calculations show that \(\frac{\partial E(\pi_i^c)}{\partial p_i} \bigg|_{(p,q)}\) is positive, that is an increase in \(p_i\) increases the expected profit of firm \(i\) in equilibrium.

**Marginal effect of own price on profit variance:** \(\frac{\partial \text{Var}(\pi_i^c)}{\partial p_i} \bigg|_{(p,q)}\)

We can show that \(\frac{\partial E(s_i^2)}{\partial p_i} \bigg|_{(p,q)}\) is negative. The sum of the first two terms in (4.25) equals \(\frac{1}{\sqrt{2\pi}\sigma_p} X_2\) and this is negative since \(X_2 < 0\). The last term is also negative since
\[
\{ B [\Phi(B) - \Phi(A)] + \phi(B) - \phi(A) \} > 0,
\]
as this is the contribution to the expected sales of the case \(0 < r_i < q_i\) (see formula (4.12)), which must be positive.

The marginal effect of \(p_i\) on \(\text{Var}(\pi_i^c)\) is given by
\[
\frac{\partial \text{Var}(\pi_i^c)}{\partial p_i} = 2p_i \left[ E(s_i^2) - (E(s_i))^2 \right] + p_i^2 \left[ \frac{\partial E(s_i^2)}{\partial p_i} - 2E(s_i) \frac{\partial E(s_i)}{\partial p_i} \right].
\]
Note that the first term is positive since the term in the brackets is the variance of \(s_i\). The sign of the second term is, however, ambiguous: \(\frac{\partial E(s_i^2)}{\partial p_i} < 0\) while \(-2E(s_i)\frac{\partial E(s_i)}{\partial p_i} > 0\) since \(\frac{\partial E(s_i)}{\partial p_i} \bigg|_{(p,q)} < 0\).

We evaluated \(\frac{\partial \text{Var}(\pi_i^c)}{\partial p_i} \bigg|_{(p,q)}\) numerically for all parameter combinations that we considered in this chapter. All calculations show that \(\frac{\partial \text{Var}(\pi_i^c)}{\partial p_i} \bigg|_{(p,q)}\) is positive, that is an increase in \(p_i\) increases the profit variance of firm \(i\) in equilibrium.

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Appendix 4.C  Expected welfare in the standard model

As Gertner (1986) shows, firms draw their prices from the distribution function

\[
F(x) = \begin{cases} 
0 & \text{if } x < c \\
1 - \frac{x}{a} & \text{if } x \in \left[c, \frac{a}{b}\right] \\
1 & \text{if } x > \frac{a}{b}
\end{cases},
\]

and they produce up to the market demand at the price they drew.\(^{26}\) Thus, the firm with the lower price serves the whole market and the other firm cannot sell any of the goods it produced.

Let \(p = \min\{p_1, p_2\}\) and \(\overline{p} = \max\{p_1, p_2\}\). Then consumer surplus is given by \(CS = \frac{1}{2b}(a - b\overline{p})^2\). The profit of the firm with the lower price is \(\pi = (p - c)(a - b\overline{p})\). The firm with the higher price does not sell anything, therefore its profit is \(\pi = -c(a - b\overline{p})\). Adding up these expressions and simplifying the result gives \(W = -\frac{b}{2}\overline{p}^2 + bc\overline{p} + a\left(\frac{1}{2b}a - c\right) - c(a - b\overline{p})\). The expected welfare is therefore

\[
E(W) = -\frac{b}{2}E(p^2) + bcE(p) + a\left(\frac{1}{2b}a - c\right) - c[a - bE(\overline{p})].
\]  (4.29)

To evaluate the expected welfare, we need to derive the formula for \(E(p)\), \(E(p^2)\) and \(E(\overline{p})\). First we derive the distribution function of \(\overline{p}\).\(^{27}\) \(P(p > x) = P(p_1 > x)P(p_2 > x) = [1 - F(x)]^2\), therefore the distribution function is \(G(x) = P(p \leq x) = 1 - [1 - F(x)]^2\), with the corresponding probability density function \(g(x) = 2(1 - F(x))f(x)\). Using that \(f(x) = \frac{c}{x^2}\), \(g(x)\) simplifies to \(g(x) = 2\frac{c^2}{x^3}\). Then \(E(p)\) can be calculated as follows.

\[
E(p) = \int_c^{a/b} 2\frac{c^2}{x^2} dx + P(p = \frac{a}{b}) \cdot \frac{a}{b} = 2c^2 \left[-\frac{1}{x}\right]_c^{\frac{a}{b}} + \left(\frac{bc}{a}\right)^2 \cdot \frac{a}{b} = c \left(2 - \frac{bc}{a}\right). \]  (4.30)

\(^{26}\)Note that the distribution function is discontinuous at \(x = \frac{a}{b}\).

\(^{27}\)To simplify notation, we focus on the non-constant part of the distribution function, i.e. when \(x \in \left[c, \frac{a}{b}\right]\).
Similarly, $E(p^2)$ is given by

$$E(p^2) = \int_c^{a/b} 2c^2 \frac{x}{c} \, dx + P\left(p = \frac{a}{b}\right) \left(\frac{a}{b}\right)^2 = 2c^2 \left[\ln x\right]_c^{a/b} + \left(\frac{bc}{a}\right)^2 \left(\frac{a}{b}\right)^2 = c^2 \left(1 + 2 \ln \frac{bc}{a}\right).$$

(4.31)

Next we derive the distribution function of $p$. $P(p < x) = P(p_1 < x)P(p_2 < x) = F(x)^2$, therefore the distribution function is $H(x) = F(x)^2$, with the corresponding probability density function $h(x) = 2F(x)f(x) = 2\left(1 - \frac{c}{x}\right) \frac{c}{x^2}$. Then

$$E(\bar{p}) = \int_c^{a/b} 2 \left(1 - \frac{c}{x}\right) \frac{c}{x} \, dx + P\left(\bar{p} = \frac{a}{b}\right) \frac{a}{b} = 2c \left[\ln x + \frac{c}{x}\right]_c^{a/b} + \left(2 - \frac{bc}{a}\right) \frac{bc}{a} \frac{a}{b}$$

$$= 2c \left[\ln \frac{a}{bc} + \frac{bc}{a} - 1\right] + \left(2 - \frac{bc}{a}\right) c = c \left(2 \ln \frac{a}{bc} + \frac{bc}{a}\right).$$

(4.32)

For deriving this result we used that $P\left(\bar{p} = \frac{a}{b}\right) = \left(2 - \frac{bc}{a}\right) \frac{bc}{a}$, which follows from $P\left(\bar{p} < \frac{a}{b}\right) = F^2\left(\frac{a}{b}\right) = (1 - \frac{bc}{a})^2$.

Combining equations (4.29)-(4.32), we get a formula for the expected welfare as the function of $a, b$ and $c$. For the parameter values we consider in this chapter, the expected welfare is approximately 22.4378 (with $E(p) = 3.6$ and $E(\bar{p}) \approx 6.8378$).
Chapter 5

Endogenous Information Disclosure in Experimental Oligopolies

5.1 Introduction

Information sharing about prices or production levels can be efficiency enhancing. When firms can observe prices or production levels of their competitors, then they know all the relevant variables that affect their demand and therefore they may learn uncertain demand conditions better. In contrast, when not all the information that affects demand is available, firms can only work with a misspecified model and this can lead to a welfare loss.\(^1\) However, information sharing may have anti-competitive effects as well. Competition authorities are concerned that information sharing can facilitate collusion: when firms observe past prices or quantities, then it is possible to identify the firm(s) that broke a cartel agreement, which makes cartels more sustainable.\(^2\) Therefore competition authorities raise concerns about the dissemination of firm-

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\(^1\) See Bischi et al. (2004) for example.

specific data typically but not about aggregate values.³

The effect of the type of available information on the market outcome has been analyzed by means of laboratory experiments. Experimental results show that the view of competition authorities is not necessarily valid. Huck et al. (1999) and Huck et al. (2000) find that additional information about rivals’ actions and profits leads to a more competitive outcome. On the other hand, Offerman et al. (2002) find more collusion when firms receive information about individual production levels but not about profits. When, however, profit information is available, the outcome is more competitive. Dufwenberg and Gneezy (2002) find that receiving information about all the bids leads to more collusive outcomes in a first-price sealed-bid auction.

This paper contributes to the debate on how the publication of aggregate or individual data affects competitiveness. We conduct a laboratory experiment in which we vary the information available to subjects about their competitors’ past actions. Previous experiments on this question lack an important characteristic of markets, namely that sharing information is often a firm’s own decision, therefore the information structure is endogenous. The choice of sharing information gives firms an important tool: the possibility for signaling. By voluntarily sharing information, firms can signal their willingness to collude with their competitors. When, however, the information structure is exogenously given, firms automatically receive information, removing important strategic considerations behind information sharing. This may lead to biased conclusions about the effect of information. Therefore we analyze in this paper how endogenous information sharing affects the market outcome. In particular, we address the following questions. Does the market outcome become more collusive as firms receive more detailed information about their competitors? Does it matter for the market outcome whether information sharing is compulsory or voluntary? Do firms use information sharing as a signaling device? Can it be desirable to make it compulsory for firms to share information (in order to remove signaling possibilities)?

We use the same market structure as Offerman et al. (2002). We consider the market of a

³For more details about regulation and examples for cases see Kühn and Vives (1995), Section 8.3 in Buccirossi (2008) and OECD (2010).
homogeneous good where firms compete in quantities. The demand function is nonlinear in the aggregate production and firms face increasing marginal costs. Subjects play the role of firms in the market and each market consists of three firms. There are four treatments in the experiment, each one consists of three parts. In each part, subjects act on the same market for 30 rounds, then a new market is formed. The market structure remains the same across parts, only the composition of groups and the available information changes. In the first two parts subjects either receive information about the other subjects’ choices or not, while in part 3 information sharing is voluntary.\textsuperscript{4} Treatments differ in the kind of information subjects receive (whenever they receive information). Subjects are informed either about aggregate production levels or about individual quantities, but not about profits.

The experimental results confirm that the voluntary nature of information sharing can have important consequences for the market outcome. The average individual production is significantly lower when subjects share information compared to when they decide not to share information. Furthermore, when subjects decide not to share information, production levels are significantly higher than when information is not available by default. Voluntary information sharing can enhance coordination as well. Individual production levels are closer to each other when information is shared voluntarily and subjects receive aggregate information. Concerning the type of information subjects receive, we find that the average total production is lower when firms receive individual information but this difference is not significant. We could observe more attempts for collusion under individual information though. This supports the view of competition authorities that firm-specific data has anti-competitive effects. This suggests that the concern about disseminating individual data is justified.

The effect of different information about competitors’ actions has been analyzed in the experimental literature. Huck et al. (1999) conduct an experiment on a homogeneous Cournot market with four firms. One dimension of their experimental design is the amount of information subjects receive about their competitors: they either receive no information or they are

\textsuperscript{4}We vary the order of the parts with and without information to control for the order effect. This ordering gives one of the treatment dimensions.
informed about the quantity choice of their competitors as well as the corresponding profits. The results show that more information about competitors leads to a more competitive market outcome.

Huck et al. (2000) analyze the effect of data aggregation on the market outcome. They consider the market of a differentiated good where firms compete either in prices or in quantities. Subjects are informed either about the aggregate action of other firms or they receive detailed information about firm-specific prices or quantities as well as about profits. The authors find that providing disaggregated information about actions and profits yield a more competitive outcome in case of quantity competition whereas there is no significant difference in competitiveness when firms compete in prices.

Offerman et al. (2002) consider a homogeneous Cournot oligopoly with three firms and they vary the available information about competitors. In treatment \( Q \) subjects receive information about aggregate production only, while they are informed about individual production levels as well in treatment \( Q_q \). Finally, individual profits are reported too in treatment \( Q_q \pi \). The results show that subjects produce less when they receive additional information about firm-specific production levels, moreover, there is evidence for collusion in some cases. On the other hand, the market tends to be more competitive when profit information becomes available but there is evidence for collusive behavior as well.

Our paper differs from the previous literature in important aspects. Most importantly, we impose an endogenous information structure in which subjects can decide whether they want to share information with others or not. This gives the possibility for subjects to signal their willingness to cooperate and this has important consequences for the market outcome. Second, we do not give information about profits. The reason for this is that in practice information sharing is implemented by trade associations and they collect information about prices or quantities but not about profits typically. From price or quantity information it is not necessarily possible to draw conclusion about profits since firms may not know the production technology of their competitors or the agreement they have with suppliers. Not providing profit information
explains the difference between our conclusions and those of Huck et al. (1999) and Huck et al. (2000). They do not investigate the case when subjects receive information about quantities or prices only. Our results are in line with those of Offerman et al. (2002). We also do not observe a significant difference between total production under aggregate and individual information and we also find more attempts for collusion when subjects receive individual information.

The paper is organized as follows. The market structure and the benchmark outcomes are discussed in Section 5.2. Then we present the experimental design in Section 5.3. Section 5.4 summarizes our hypotheses, then we report the experimental results in Section 5.5. Section 5.6 concludes. Instructions of the experiment are presented in Appendix 5.A.

5.2 Market and information structures

We use the same market structure as Offerman et al. (2002). We consider the market for a homogeneous good that is produced by 3 firms. Firms compete in quantities and the inverse demand function is given by

\[ P(Q) = 45 - \sqrt{3Q}, \]

where \( Q = \sum_{i=1}^{3} q_i \) is the total production of the three firms. Firms face the same cost function, the production costs of each firm \( i \) are given by

\[ C_i(q_i) = \frac{q_i^3}{4}, \]

where \( q_i \) is the production level of firm \( i \). The number of firms, the inverse demand and the cost functions are common knowledge.

We consider this setup for the following reasons. Note that we need at least 3 firms in the market in order to have a difference between information about aggregate and individual production levels from the perspective of a given firm. However, Huck et al. (2004) do not observe any collusion when there are more than 3 firms on a Cournot market. Therefore we
choose to have 3 firms so that collusion would be feasible in the experiment. We use a nonlinear setup as the Cournot adjustment process (i.e. naive best-response dynamics) does not converge under linear demand and cost conditions for more than 2 firms (see Theocharis, 1960). Finally, the setup also facilitates comparisons with Offerman et al. (2002).

There are three benchmark outcomes for this market: the Nash equilibrium (N), collusion (C) and the Walrasian equilibrium (W). Table 5.1 summarizes the individual and total quantities and the individual profits in these outcomes.\(^5\)

We consider a situation where firms compete on the market for a finite and known number of periods. When firms are perfectly rational, the Nash equilibrium of the stage game is the unique subgame-perfect equilibrium of the repeated game. Collusion cannot be sustained with rational players since the stage game is finitely repeated with a known end period. There is, however, ample empirical evidence that the assumption of perfect rationality and the standard equilibrium prediction does not necessarily describe actual behavior well.\(^6\) Finding the Nash equilibrium and coordinating on it are complex tasks and subjects might not have sufficient cognitive and computational abilities to do so. In this situation, subjects may use different decision rules that have lower deliberation costs than calculating the Nash equilibrium. Examples for such rules are different versions of imitation. Vega-Redondo (1997) proposes a rule where firms imitate the action of the firm that made the highest profit in the previous round. Under this rule quantities converge to the Walrasian outcome. Offerman et al. (2002) propose an alternative rule where firms imitate the action of a so-called exemplary firm. This firm is the one whose action would have resulted in the highest total profit if each firm had chosen the same action. This imitation

\(^5\)See Offerman et al. (2002) for the general formulas.

\(^6\)See Conlisk (1996) for an overview of such results.

<table>
<thead>
<tr>
<th></th>
<th>(q_i)</th>
<th>(Q)</th>
<th>(\pi_i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>N</td>
<td>81</td>
<td>243</td>
<td>729</td>
</tr>
<tr>
<td>C</td>
<td>56.25</td>
<td>168.75</td>
<td>843.75</td>
</tr>
<tr>
<td>W</td>
<td>100</td>
<td>300</td>
<td>500</td>
</tr>
</tbody>
</table>

rule leads to the collusive outcome.\footnote{These imitation rules lead to the Walrasian and collusive outcomes respectively, only when each firm uses the same rule. It is, however, unlikely that subjects use the same imitation rule in an experiment. In fact, subjects do not necessarily use imitation rules at all. Bosch-Domènech and Vriend (2003) investigate whether subjects use imitation more often as the environment is perceived to be more complicated. They find that the more complicated the market environment seems, the less frequently the subjects imitate each other.}

In order to investigate how the market outcome depends on the type of available information and on firms’ information sharing decisions, we conduct a laboratory experiment. In the next section we discuss the experimental design we use.

### 5.3 Experimental design and procedures

The experiment was conducted in the CREED laboratory of the University of Amsterdam in June 2014. In total, 180 subjects participated in 4 treatments in 14 sessions. None of the subjects participated more than once. Participants were mainly undergraduate students from different fields. Each session lasted about 2 hours, and participants earned on average 25.5 euros. Earnings were paid privately in cash at the end of the experiment. The experiment was computerized, and programmed in php. Participants read the instructions at their own pace from the computer screens, and questions were answered privately. After reading the instructions subjects had to answer control questions in order to ensure they understood the situation they faced in the experiment.

Each session consisted of 3 parts. In each part 3 participants formed a market. This market composition was fixed for the whole part but it changed between different parts: subjects were rematched in their matching group of 6. Subjects were informed that there are 3 parts and that they will not play in the exact same market again but we did not inform them about the size of the matching group. Each part consisted of 30 rounds. The number of rounds was known to the participants. During the experiment, subjects earned points in each part. At the beginning of each part they received a starting capital of 6000 points. At the end of the experiment one part was randomly chosen by rolling a die, and all participants’ earnings from that part were converted to euros. Participants received 1 euro for each 1100 points they earned in the given
In every round subjects had to decide simultaneously how much to produce. They could choose integer production levels between 40 and 125. The parts differed only in the information participants received about other firms’ production decisions. The demand and cost structure was the same through the whole experiment, as given in Section 5.2 and this was commonly known. In parts 1 and 2, either no information about others’ production was provided, or aggregate / individual production details were provided. The order of these two parts differed across treatments to control for the possible order effects. In part 3 subjects decided not only about their production but they also decided at the same time whether to share information about their production with the other two subjects in their group. If they shared information, the other two firms in the market received information about this firm’s production regardless of their own information sharing decision.\textsuperscript{8} Subjects received feedback after each round. This feedback contained their own production, the market price, their own revenue, cost and profit. Additionally, they received information about other firms’ production in the full information part, and in part 3 if applicable.

The second treatment dimension was the type of information subjects received about others’ production. We had two treatments with aggregate information and two with individual information. In case of aggregate information subjects were informed about the sum of the others’ production and they could not recover individual production levels from this information. In part 3, subjects could observe how many subjects shared information in their market and the aggregate output of the others who shared information. In case of individual information, individual production levels were shown and subjects could identify which firm produced a given amount.\textsuperscript{9} In part 3, firms could see which firms shared information, and the exact production level of those firms who shared. This $2 \times 2$ design leads to 4 different treatments, which are

\textsuperscript{8}That is, we applied a non-exclusionary disclosure rule. See Vives (1990) for a discussion about the effect of different disclosure rules on information sharing incentives.

\textsuperscript{9}Subjects were distinguished as Firm A, Firm B and Firm C in the market. This firm ID was fixed in a given part.
summarized in Table 5.2.\textsuperscript{10} Subjects were informed about the kind of information they would receive and about the decision(s) they need to make in the different parts at the beginning of the experiment.

During the decision making, subjects could use a profit calculator which was built in the screen. Here subjects could enter a hypothetical own production and a hypothetical total production of others, and the calculator gave the corresponding price and profit. Participants could use the profit calculator as often as they wanted.

In each part, a history screen was always available for every subject. This screen contained information about past production in their own market. In the part with no information subjects could see their own production, price and profit for every round.\textsuperscript{11} In the part with full information they additionally saw either the total output (in case of aggregate information) or the production levels of the other two subjects by firm ID (in case of individual information). In part 3, the additional information was their own information sharing decision and either the number of other firms who shared information and their total production (in case of aggregate information), or the individual production of the other firms by firm ID, showing “n.a.” if a firm did not share information (in case of individual information). An example of the history screen for part 3 and instructions for treatment I-NF can be found in Appendix 5.A.

\begin{table}[h]
\centering
\begin{tabular}{|l|c|c|c|}
\hline
Treatment & Type of info & Sequence of parts 1 and 2 & \# of matching groups \\
\hline
A-NF & aggregate & No info - Full info & 8 \\
A-FN & aggregate & Full info - No info & 8 \\
I-NF & individual & No info - Full info & 8 \\
I-FN & individual & Full info - No info & 6 \\
\hline
\end{tabular}
\caption{Overview of the treatments}
\end{table}

\textsuperscript{10}We wanted to have 8 matching groups in treatment I-FN too but we had to cancel some sessions due to low show up.

\textsuperscript{11}Even though subjects were not informed about total productions, they could infer this information using the inverse demand function or the profit calculator. Based on the usage of the profit calculator, subjects did not perform such calculations.
5.4 Hypotheses

Subjects receive different kinds of information about each other’s choices across treatments and parts and this may affect their behavior. In this section we discuss how subjects may behave under the different information structures and we summarize our hypotheses.

When subjects do not receive any information about their competitors, we expect to observe substantial differences in individual production levels as subjects choose their quantities by trial and error. Even though the aggregate production can be calculated using the demand function and thus subjects could play the best response in principle, we do not expect them to perform such calculations.

When subjects receive aggregate information about their competitors, then we expect to observe smaller differences in individual production levels for two reasons. First, when aggregate production is directly observable, it becomes easier to find the best response, which drives the outcome towards the Nash equilibrium. Second, even if subjects do not calculate the best response, observing what the other subjects chose may drive individual quantities closer to each other through imitating the average, for example. Since playing the best response leads to the Nash equilibrium, we expect production levels to be distributed around the Nash equilibrium quantity. Subjects might also try to collude as they can monitor their competitors to some extent.

When individual information is available, we expect to observe more collusion. This is in line with the theoretical considerations that firms can monitor each other’s behavior better. Moreover, as Offerman et al. (2002) show, the “imitate the exemplary firm” rule leads to the collusive outcome and subjects have enough information to use this rule.

When the information structure becomes endogenous, subjects can signal their willingness to cooperate. This is due to the fact that sharing information unilaterally gives an informational advantage to the competitors but it is not directly beneficial for the firm that shares information. The rationale of sharing information is that subjects may induce their competitors to share

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12 The exemplary firm is the one whose action would have resulted in the highest total profit if each firm had chosen the same action.
their production level as well. Once cooperation is established in terms of information sharing, subjects might be able to collude in production levels easier. Similarly, when subjects decide not to share information with each other, this signals their willingness to compete and then it is hard to imagine that they will collude in their production choice. Thus, we expect a difference in the behavior of subjects that share information and those who do not share information.

Based on the above considerations, we formulate the following hypotheses about the results. The first set of hypotheses compares the *average total quantities under the aggregate and the individual information* treatments. When subjects receive no information about their competitors’ production levels, it does not matter whether they are in the aggregate or in the individual information treatment, the average total quantities are the same:

**Hypothesis 1.A.** Average total production is the same under the individual and the aggregate information treatments when subjects receive no information about their competitors: $\bar{Q}_{I,NI} = \bar{Q}_{A,NI}$.

When, however, subjects do receive information about their competitors, the average total production is lower under individual information. Thus:

**Hypothesis 1.B.** Average total production is lower under the individual information treatment when subjects have full information about their competitors: $\bar{Q}_{I,FI} < \bar{Q}_{A,FI}$.

Under voluntary information sharing, we expect to observe some subjects that decide to share information with each other. Then, in line with Hypothesis 1.B, we expect to observe lower average total production under individual information.

**Hypothesis 1.C.** Average total production is lower under the individual information treatment when subjects voluntarily share information with each other: $\bar{Q}_{I,VS} < \bar{Q}_{A,VS}$.

The second set of hypotheses concerns the *effect of voluntary information sharing on total output*. As information sharing may work as a signaling device, we expect to observe more attempts for collusion, and consequently lower average production, under voluntary information
sharing when subjects share information than under full information. The reason for this is that there might be subjects who would not collude by default but they cooperate with their competitors after receiving information from them.

**Hypothesis 2.A.** Average total production is lower under voluntary information sharing when subjects share information with each other than under full information: \( \bar{Q}_{VS|\text{share}} < \bar{Q}_{FI} \).\(^{13}\)

A similar reasoning holds when subjects decide not to share information with each other. If a subject would be willing to cooperate with his competitors but he is on a market where others decide not to share their production choice, then the subject in question may more easily give up attempts for cooperation compared to the case when information about production levels is not available by default. This leads to higher average total production under voluntary sharing when subjects do not share information with each other than under no information.

**Hypothesis 2.B.** Average total production is higher under voluntary information sharing when subjects do not share information with each other than under no information: \( \bar{Q}_{VS|\text{no share}} > \bar{Q}_{NI} \).\(^{14}\)

Finally, as subjects may show their willingness to cooperate by sharing information, we expect to observe lower average production level under voluntary sharing when subjects share information with each other compared to the case when they do not share information.

**Hypothesis 2.C.** Under voluntary information sharing, average total production is lower when subjects share information with each other than when they do not share information: \( \bar{Q}_{VS|\text{share}} < \bar{Q}_{VS|\text{no share}} \).

In the last set of hypotheses we discuss the effect of voluntary information sharing on coordination. We call a group coordinating when individual productions in a given round on the

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\(^{13}\)For calculating \( \bar{Q}_{VS|\text{share}} \), we consider the total production on markets where at least 2 subjects share information in a given round and we calculate the average of these total productions over markets and rounds.

\(^{14}\)For calculating \( \bar{Q}_{VS|\text{no share}} \), we consider the total production on markets where at most 1 subject shares information in a given round and we calculate the average of these total productions over markets and rounds.
same market are close to each other.\footnote{Thus, coordination means that subjects choose similar production levels, not that they are coordinating on one of the benchmark outcomes. We specify this definition in the next section.} Then we measure the amount of coordination as the share of coordinating groups. Since voluntary information sharing may enhance collusion, we expect to observe a higher amount of coordination under voluntary information sharing when subjects share information than under full information:

**Hypothesis 3.A.** Coordination is higher under voluntary information sharing when subjects share information with each other than under full information: $C_{VS|\text{share}} > C_{FI}$.\footnote{$C$ denotes the measure of coordination. For calculating $C_{VS|\text{share}}$, we consider a one-round lag in information sharing: we consider the markets where at least 2 subjects share information in a given round and we examine whether individual production levels are close to each other in the following round.}

We do not expect to observe a difference in the amount of coordination when subjects decide not to share information with each other and when information is not available by default as subjects cannot observe each other’s choice in either case.

**Hypothesis 3.B.** The amount of coordination is the same under voluntary information sharing when subjects do not share information with each other as under no information: $C_{VS|\text{no share}} = C_{NI}$.\footnote{For calculating $C_{VS|\text{no share}}$, we consider again a one-round lag in information sharing: we consider the markets where at most 1 subject shares information in a given round and we examine whether individual production levels are close to each other in the following round.}

Finally, we expect to observe a higher amount of coordination when subjects choose to share information with each other than when they decide not to share information since observing what others choose helps coordination.

**Hypothesis 3.C.** Under voluntary information sharing, coordination is higher when subjects share information with each other than when they do not share information: $C_{VS|\text{share}} > C_{VS|\text{no share}}$.

In the next section we investigate whether the experimental results confirm our hypotheses.
5.5 Results

In this section we report the experimental findings. First we checked whether the different order of the parts with no information and full information had an effect on subjects’ behavior. To do so, we tested with the Mann-Whitney ranksum test whether subjects behaved differently in part 3 if they faced No information or Full information first. We did not find significant differences in individual production, in total output and in the information sharing decision.\footnote{The test was performed on matching group level. We compared part 3 behavior for A-NF and A-FN, and for I-NF and I-FN. For each test the p-value is at least 0.529.} We also compared individual production levels and total output for parts No information and Full information for A-NF and A-FN, and for I-NF and I-FN. Here we did not find any significant difference either.\footnote{The tests were performed on matching group level again. The p-values are between 0.302 and 0.796. We also plotted the average production over time, and the plots are very similar to each other.} Because of these findings we conclude that there is no order effect in our experiment. Subjects behaved in the same way when they were facing the part with no information first as when they were facing the part with full information first. Thus we merge our data on one treatment dimension, and analyze them together. This means that we end up with two treatments: Aggregate (A) and Individual (I) with 16 and 14 matching groups, respectively.

The remainder of this section is organized as follows. In Section 5.5.1 we compare the total outputs in the different parts of the two treatments. In Section 5.5.2 we focus on the effect of voluntary information sharing on the subjects’ production choice and on total output. Then in Section 5.5.3 we investigate how groups coordinate under different information structures. Finally, in Section 5.5.4 we analyze the factors that influence the information sharing and production choice by means of panel regressions.

5.5.1 Output decisions

Figure 5.1 shows the average output over time for each treatment. We can see that there is no substantial difference in output across parts and across treatments. The average output seems to be lower under Individual information, however these differences are not significant. Further-
Fig. 5.1: Average total output over time in the Aggregate and in the Individual treatments

Table 5.3: Average total output across treatments and parts, and the corresponding test results

<table>
<thead>
<tr>
<th>Treatment</th>
<th>NI</th>
<th>FI</th>
<th>VS</th>
<th>NI vs. FI</th>
<th>NI vs. VS</th>
<th>FI vs. VS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aggregate</td>
<td>234.12 (27.28)</td>
<td>232.43 (29.35)</td>
<td>235.38 (21.91)</td>
<td>1.00</td>
<td>0.68</td>
<td>0.76</td>
</tr>
<tr>
<td>Individual</td>
<td>231.59 (27.67)</td>
<td>232.46 (31.31)</td>
<td>230.97 (31.25)</td>
<td>0.83</td>
<td>0.98</td>
<td>0.59</td>
</tr>
<tr>
<td>Agg. vs. Ind.</td>
<td>0.59</td>
<td>0.43</td>
<td>0.20</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Notes: The numbers in brackets are standard deviations of the output. p-values are according to two-sided rank sum test with $n_A = 16$ and $n_I = 14$ for the treatment differences, and Wilcoxon-test for the differences between parts.

more, we can observe an end-game effect after round 27. The output before that is quite stable, subjects quickly learn how the market works. Non-parametric tests (presented in Table 5.3) also confirm the similarities across treatments and parts; there is no significant difference between the output levels in different treatments or parts. Based on these tests we reject Hypotheses 1.B and 1.C, but we accept Hypothesis 1.A.

Although we cannot find any significant difference in average total productions across treatments, Figure 5.2 suggests that there might be differences across treatments and parts with respect to the distribution of total output. Interestingly there is a significant difference in distributions between treatments not only in the case of voluntary sharing but also in the case of No information (p=0.000 for both cases by the two-sample Kolmogorov-Smirnov test). In the case of No information total production is close to the Nash equilibrium value more often in the Aggregate treatment than in the Individual treatment. In the case of Full information there
is no difference between treatments (p=0.18). The parts with Full information can be compared to the $Q$ and $Qq$ treatments in Offerman et al. (2002). Their distribution in treatment $Q$ is very similar to ours in A-FI. The shape of the distributions in their $Qq$ and in our I-FI is also similar but we observe a smaller peak close to the collusive outcome. In our experiment, subjects try to collude more often in the case of Voluntary sharing.

The distributions are also significantly different within treatments across parts. As we can see from the figures (and from the standard deviations in Table 5.3), the dispersion of the total output is not higher in NI than in other parts. This suggests that subjects could not coordinate better when they received information about each other’s choices. The figure also shows that the total output was hardly ever around the Walrasian outcome of $Q_W = 300$. Attempts for collusion were observed a bit more often, especially in the case of VS in the Individual treatment.\footnote{However, the collusive outcome was hardly ever reached. The spike in case of I-VS is around 180.}

We will return to this issue in Section 5.5.2. Our findings can be summarized as follows.

**Result 1.** Production levels tend to be lower under individual information than under aggregate information but there are no significant differences in average total outputs across treatments and parts. When subjects receive individual information, we can observe collusive behavior

\footnote{In case of the Aggregate (Individual) treatment, the difference between NI and FI is significant at 10% (5%) level, between NI and VS at 5% (5%) level, and between FI and VS at 1% (10%) level.}
Table 5.4: Percentage of outcomes in the $\pm 5$ neighborhood of the Nash equilibrium quantity $Q_N$ under voluntary information sharing.

Table 5.4 summarizes how often the total output was in the neighborhood of the Nash outcome, and among these cases how often individual production levels were close to the individual Nash quantity.\(^{22}\) The Nash outcome was always reached less often in the Individual treatment than in the Aggregate treatment. Furthermore, we can see that in the Individual treatment subjects are more often in the $\pm 5$ range of the individual Nash quantity in case of Full information and VS. This suggests that subjects may coordinate better in that treatment when they receive information.

### 5.5.2 Consequences of information sharing on output decisions

Now we will focus on voluntary information sharing, and investigate how the endogeneity of the information structure affects production choice. Figure 5.3 presents the average frequency of information sharing in the Aggregate and in the Individual treatments. As we can see from the graph, subjects share information more often in the Individual treatment than in the Aggregate treatment, with an average information sharing of 60.2% in the Aggregate and 67.8% in the Individual case. These values are not significantly different from each other using the ranksum test on matching group levels ($p=0.38$).

Table 5.5 shows the average individual production levels in parts NI, FI, and VS. We separated the average production in VS by the observations in which subjects shared information,

\(^{22}\)We also checked in how many cases chose all 3 firms close to the individual Nash quantity, but the results are not substantially different from the ones reported in the table. We have only 74 cases (about 16%) where all three individuals chose around the individual Nash quantity, but the total quantity is not in the $\pm 5$ range of the Nash output.
Figure 5.3: Average frequency of information sharing over time

Table 5.5: Average individual and total production and their test results

<table>
<thead>
<tr>
<th>Treatment</th>
<th>NI</th>
<th>FI</th>
<th>VS\textsubscript{n}</th>
<th>VS\textsubscript{s}</th>
<th>VS\textsubscript{s} vs. VS\textsubscript{n}</th>
<th>NI vs. VS\textsubscript{n}</th>
<th>FI vs. VS\textsubscript{s}</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aggregate</td>
<td>q</td>
<td>78.04</td>
<td>77.48</td>
<td>81.93</td>
<td>76.48</td>
<td>0.00***</td>
<td>0.01**</td>
</tr>
<tr>
<td>Q</td>
<td>234.12</td>
<td>232.43</td>
<td>241.62</td>
<td>232.97</td>
<td>0.03**</td>
<td>0.04**</td>
<td>1.00</td>
</tr>
<tr>
<td>Individual</td>
<td>q</td>
<td>77.20</td>
<td>77.49</td>
<td>84.53</td>
<td>74.74</td>
<td>0.00***</td>
<td>0.02**</td>
</tr>
<tr>
<td>Q</td>
<td>231.59</td>
<td>232.46</td>
<td>244.87</td>
<td>226.39</td>
<td>0.01***</td>
<td>0.06*</td>
<td>0.30</td>
</tr>
<tr>
<td>Agg. vs. Ind.</td>
<td>q</td>
<td>0.84</td>
<td>0.77</td>
<td>0.43</td>
<td>0.34</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Q</td>
<td>0.59</td>
<td>0.43</td>
<td>0.60</td>
<td>0.34</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Notes: ***: significant at 1%-level, **: significant at 5% level, *: significant at 10% level according to two-sided ranksum test with \( n_\text{A} = 16 \) and \( n_\text{I} = 14 \) (and \( n_{\text{I-\text{VS}}|n} = 13 \)) for the treatment differences, and Wilcoxon-test for the differences between parts. The averages in the table are based on the matching group averages.

and those in which they did not share. Although there are still no significant differences between treatments, now we can see a clear difference in behavior across different information structures. Subjects indeed choose different production levels in part VS when they voluntarily decide to share information than when they intentionally decide not to share it. Those who share information choose significantly lower production levels than those who do not share information. Furthermore, subjects who do not share information in part 3 choose significantly higher production level than the average production level in case of NI. If subjects share information in part 3, their average production is lower than the average production in FI (though this difference is not significant). These results are in line with our hypotheses about the effect of voluntary sharing on production choice. We can conclude that subjects self-select to sharing / not sharing information if they have an intention to collude / compete with the others. We
also checked the relationship between the own production and the information sharing decision in part 3 with simple correlation: in both treatments this relationship is significantly negative.\textsuperscript{23} Although these results are well in line with our hypotheses about the market outcome, we cannot accept or reject them yet since groups might be heterogeneous with respect to information sharing behavior, therefore we cannot draw conclusions about group behavior based on individual behavior only.

As a next step, we categorized groups as \textit{sharing} group if there were 2 or 3 firms sharing information in a given round and \textit{non-sharing} group otherwise. Note that with this definition a group might be a sharing group in some rounds while non-sharing in other rounds. Besides individual production, Table 5.5 also shows the average total output for the different information structure and the test results for differences across parts and treatments. We can see from the table that our results from the individual production still hold in this case (though the p-values are a bit higher for the sign-rank tests). So based on these test results, we reject Hypothesis 2.A but we accept Hypotheses 2.B and 2.C, which leads to our next result:

Result 2. \textit{Information sharing works as a signaling device: subjects produce significantly less when they share information voluntarily compared to the case when they decide not to share information. Furthermore, if they decide not to share information, the market is significantly more competitive than if information is not available by default. However, there is no significant difference in the market outcome if firms receive information exogenously or they decide to share information voluntarily.}

Since we have seen that sharing groups choose significantly lower production than non-sharing groups in VS, we investigated whether sharing groups can explain the small peak close to the collusive outcome in the Individual treatment in Figure 5.2. The analysis shows that there were 4 sharing groups which shared information most of the times with each other (30, 29, 25 and 21 times out of 30), and chose low production levels (with an average total production below

\textsuperscript{23}The correlation coefficient is -0.1855 for the Aggregate treatment and -0.2026 for the Individual treatment, with p=0.000 for both cases.
Figure 5.4: Average share of coordinating groups over time in the Aggregate and in the Individual treatments

200). Mainly the production of these four groups led to the smaller spike near the collusive outcome. Even though these groups were very successful in collusion, they only constitute less than one third of all the experimental markets. So letting firms decide about sharing firm-specific data can indeed result in collusion, but because firms are heterogeneous with respect to information sharing, this might not actually happen.

### 5.5.3 Coordination

To further analyze what causes the difference in the distributions of total output, we defined coordinating and non-coordinating groups by the following rule: a group is *coordinating* in a given round if the maximal absolute difference between the individual production levels and the average production level in the market is at most 5. Using this definition, Figure 5.4 shows the average share of coordinating groups over time in the different parts and treatments. Interestingly there is no substantial difference between parts in the Aggregate treatment, so more information does not seem to help coordination here. In contrast, more information clearly helps in the Individual treatment, irrespective of whether information is given exogenously or endogenously. Groups are able to coordinate more often under Full information and VS than under No information, and non-parametric tests, presented in Table 5.6 confirm that these dif-
Table 5.6: Average share of coordinating groups across different treatments and different parts, and the corresponding test results

| Treatment | NI vs. FI | NI vs. VS | FI vs. VS | VS|s vs. VS|n | VS|s vs. NI | VS|s vs. FI |
|-----------|-----------|-----------|-----------|-----------------|-----------------|-----------------|-----------------|-----------------|
| Aggregate | 0.37      | 0.19      | 0.00***   | 0.44            | 0.92            | 0.01***         |                   |
| Individual| 0.00***   | 0.00***   | 0.95      | 0.00***         | 0.67            | 0.03**          |                   |

Notes: ***, significant at 1%-level, **: significant at 5% level according to two-sided ranksum test with \( n_A = 16 \) and \( n_I = 14 \) (and \( n_{I-VS} = 12 \)) for the treatment differences, and Wilcoxon-test for the differences between parts. The averages in the table are based on the matching group averages.

So far we have focused on the *amount* of coordination. Next we analyze the *production levels* of groups that coordinated in a given round. Although in the Aggregate treatment coordination improved only between FI and VS, the distribution of the total output for the coordinating cases shows exactly the opposite result. Figure 5.5 presents these distributions. Even though the distributions look alike in the Aggregate treatment, that is, they all have a big spike close to the Nash outcome, the Kolmogorov-Smirnov test shows that NI and FI, and NI and VS are significantly different from each other (with p-values 0.002 and 0.056, respectively), but there is no difference between FI and VS (p=0.274). In the Individual treatment the difference between parts are more striking: the highest spike close to the Nash equilibrium production level is under FI, and the lowest is under VS. On the other hand, the highest spike near the collusive outcome...
Figure 5.5: Frequencies of total output for the coordinating groups in the Aggregate and in the Individual treatments. For each $Q$ the plots show the percentage of outcomes that lie in the $\pm 5$ neighborhood of a given $Q$.

is in VS, and there is no spike here in NI at all. These differences are all highly significant with $p$-values between 0.000 and 0.007. The distributions are also different across treatments for given part (all $p$-values are 0.000 here).\footnote{We have also checked whether the average output for coordinators is different across treatments and parts, but we only found one weakly significant difference: in the Aggregate treatment the total output for coordinating groups was significantly higher than the total output in FI ($233.09 > 224.67$ with $p=0.07$). Note that much more groups managed to coordinate in VS than in FI, and it seems that they were indeed good at coordinating on the Nash outcome (see also Figure 5.5).}

If we compare the distribution of coordinating groups to the total distributions in Figure 5.2, we can see that in the Aggregate treatment total output is less dispersed, coordinating groups more often reach an outcome around the Nash equilibrium than the whole population. On the other hand, distributions in the Individual treatment change differently across parts. It is common that all distributions for the coordinating groups have more spikes now (compared to the grand total where the most cases lie around the Nash equilibrium outcome), but the size of these peaks are different. If subjects get individual information about others’ production, they tend to coordinate more often on a lower total output, they try to be more collusive. However, if information sharing is not voluntary, the mode is still around the Nash outcome. This analysis leads to the following result.
Result 3. The voluntary nature of information sharing improves coordination in case of aggregate information but not with individual information. Coordinating markets still tend to reach the Nash outcome under aggregate information, whereas there are more observations for collusive behavior in the Individual treatment.

We also investigate whether coordination and information sharing are side-by-side by checking the correlation between being a coordinating and being a sharing group. For both treatments the correlation is significantly positive, though for the Aggregate treatment it is weakly significant (correlation is 0.06 with p=0.072). For the Individual treatment the correlation coefficient is much higher, 0.39 with p=0.000. Information sharing should help coordination. This is confirmed by the fact that coordinating groups share information more often in the preceding round than non-coordinating groups: in the Individual treatment in 90.86% of the cases the group was a sharing group in the preceding round if they were a coordinating group in a given round, whereas this percentage is only 55.21% for the non-coordinating groups. In the Aggregate treatment this difference is not that substantial: 65.48% vs. 58.39% for coordinating and non-coordinating groups. This is not surprising since sharing information in the Aggregate treatment does not reveal individual productions, thus it does not facilitate coordination to the same extent.

5.5.4 Factors influencing behavior

Finally we investigate what drives information sharing and production decisions. First we analyze how subjects decide whether to share information. As we have seen in the previous section, there is a significant and negative correlation between information sharing and production in a given round. Although correlation does not mean causation, we can say that production decisions have an effect on information sharing decisions, since firms make profit by choosing a production level, and if they want to collude to get higher profit, they can signal this by sharing

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25 The correlation was calculated based on coordination in the given round and information sharing type of the previous round. We decided to take the first lag of the information sharing type because the information firms receive about the others are more likely to have an effect on the decision in the next round.
### Table 5.7: Regression results for information sharing as dependent variable

<table>
<thead>
<tr>
<th>Dependent variable: Information sharing</th>
<th>Model I</th>
<th>Model II</th>
</tr>
</thead>
<tbody>
<tr>
<td>own production</td>
<td>-0.05 (0.01)***</td>
<td>-0.05 (0.01)***</td>
</tr>
<tr>
<td>1(^{st}) lag of infosharing</td>
<td>1.23 (0.12)***</td>
<td>1.22 (0.12)***</td>
</tr>
<tr>
<td>2(^{nd}) lag of infosharing</td>
<td>0.38 (0.13)***</td>
<td>0.34 (0.13)***</td>
</tr>
<tr>
<td>3(^{rd}) lag of infosharing</td>
<td>0.42 (0.12)***</td>
<td>0.43 (0.12)***</td>
</tr>
<tr>
<td>1(^{st}) lag of others sharing</td>
<td>0.62 (0.15)***</td>
<td></td>
</tr>
<tr>
<td>1(^{st}) lag of 1 other firm sharing</td>
<td>-</td>
<td>0.55 (0.16)***</td>
</tr>
<tr>
<td>1(^{st}) lag of 2 other firms sharing</td>
<td>-</td>
<td>1.52 (0.37)***</td>
</tr>
<tr>
<td>treatment * 1(^{st}) lag of 2 other firms sharing</td>
<td>-</td>
<td>-0.88 (0.37)**</td>
</tr>
</tbody>
</table>

Notes: ***: significant at the 1% level, **: significant at the 5% level. Std errors are in parentheses after the coefficients. Panels are the individuals. Infosharing is 1 if the subject shares information in the given round, 0 otherwise. Others sharing is 1 if there is at least one other firm sharing information in the given round. 1 (2) other firm(s) sharing is 1 if there is exactly 1 (2) other firm sharing information in the given round, 0 otherwise. Treatment*2 other firms sharing is an interaction between treatment dummy (0=Aggregate, 1=Individual) and the dummy for 2 firms sharing.

Information about their production level.

Table 5.7 shows the estimation outputs of the fixed effect panel logit regressions. The table shows that the own production has a significantly negative effect on the likelihood of sharing information. This is in line with the signaling feature of information sharing. Information sharing decisions in the previous 3 rounds have a significant and positive effect on present information sharing decisions. If subjects previously shared information, they are more likely to share information again. Additionally, if at least one firm shared information in the previous round, subjects are significantly more likely to share information (see Model I).\(^{26}\) In Model II we can also see that subjects react differently on the number of firms that share information. If two firms share information, this has a larger effect on the information sharing likelihood than if only one firm shares information. However, in the Individual treatment the effect of two other firms sharing is smaller than in the Aggregate treatment. This might be due to the fact that sharing information is much more costly in the individual treatment in the sense that competitors receive more detailed information about the firm’s behavior. Reciprocity plays a smaller role here, since it was already taken into account when one other firm shares information. There is no treatment-difference in either model in the effect of the other variables.

\(^{26}\)Further lags of these dummies are not significant.
As Table 5.7 shows, the information sharing decision not only depends on others’ information sharing decision, but the initial own decision also gives priming to the behavior. The intention for collusion is also present in the first round: the correlation between information sharing and own production is -0.33 which is significantly negative at 1% level (p=0.000). However, this is not necessarily the only source of the first round information sharing decision. Subjects have already gained experience with no information and with full information in parts 1 and 2, and they know their earnings in both parts. The difference in earnings under full information and under no information positively correlates with the information sharing decision in the first round: the higher this difference is (that is, the more they earned under FI compared to NI), the more likely the subject shares information. However, this relationship is only weakly significant (the correlation coefficient is 0.13 with p=0.09).

Finally, we investigate how subjects decide about their production in different parts. First of all, based on the usage of the profit calculator, subjects did not seem to calculate best responses. They used the profit calculator in total 6511 times during the $3 \cdot 30$ rounds (with 501 times in the very first round in part 1).\footnote{Thus, a subject used the profit calculator approximately 2 times in every 5 rounds on average.} If information was available about others’ decision, subjects used this information in only about 50\% of the cases when the profit calculator was used. Subjects are considered to use the information about others’ decision if they entered a number that was within the $\pm 5$ range of the previously observed value.\footnote{We considered the $\pm 5$ range as subjects may take into account that the others make a different choice in the current round. That is, subject may not focus on the naive best response.} These together suggest that subjects did not calculate best response against others’ decision.

Table 5.8 shows which variables affect production choices in different parts. VS is divided into three parts according to the number of other firms sharing information. In all 5 cases we have, price has a significantly negative effect on production. This result is counter-intuitive at first sight because if the price is higher, production should be increased to make more revenue, and profit. However, subjects might expect a decrease in the price, which leads them to choose a lower production. Unfortunately we cannot test this hypothesis with our data. In VS, when

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27 Thus, a subject used the profit calculator approximately 2 times in every 5 rounds on average.
28 We considered the $\pm 5$ range as subjects may take into account that the others make a different choice in the current round. That is, subject may not focus on the naive best response.
at least one firm shares information, subjects also react on the price changes. After a price increase, production significantly increases which is in line with our expectations. In contrast to the price, the profit has a significantly positive effect on production (except for the case in VS where 2 others share information). Furthermore, if the profit and the quantity change in the same direction, then subjects significantly increase their production (except for the same VS case again with 2 others sharing).\textsuperscript{29} This result is consistent with gradient learning where production is adjusted based on previous experience on profit and production changes (as we assumed in Chapter 2). If full information was available for the subjects, own production is (weakly) significantly decreasing in others’ production choices. This result is also in line with theory since quantities are strategic substitutes in this market. However, if only one other subject shared information, this information does not significantly affect production choice.

We also investigate whether there is a difference in behavior between treatments by including interaction terms in the model. Table 5.8 shows only the final models, we do not include the insignificant interaction terms. As we can see from the table, in case of No information there is no difference in behavior across treatments which is what we expect, since there is no difference in the information subjects receive. This is also the same for VS without sharing subjects in the group. In these two cases subjects can base their decision on the same information from the previous round, and the corresponding parameter estimates are very similar to each other. Note however, that in case of VS, the constant term is much larger than in case of NI which suggests that subjects not only choose higher production if they intentionally do not share information compared to the case when they cannot share, but they also produce more if the others intentionally do not share information. On the other hand, in case of Full information and VS with 1 other sharing the direction of the profit and production change has a significantly lower effect in the Individual treatment. In fact, in VS this effect in the Individual treatment is not significant any more, and it is only weakly significant in the Full information case (as the effect of the others’ quantity is also only weakly significant in the Individual treatment). Furthermore, the

\textsuperscript{29}To illustrate this result: if a subject increased his quantity compared to the previous round and he earned a higher profit, then it makes sense to increase the production to increase the profit again.
### Table 5.8: Fixed effect linear panel regression results for individual production as dependent variable

<table>
<thead>
<tr>
<th>Dependent variable:</th>
<th>No info</th>
<th>Full info</th>
<th>0 others share</th>
<th>1 other shares</th>
<th>2 others share</th>
</tr>
</thead>
<tbody>
<tr>
<td>Individual production</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>L. price</td>
<td>-5.01 (0.50)**</td>
<td>-6.36 (0.58)**</td>
<td>-5.56 (1.06)**</td>
<td>-4.81 (0.60)**</td>
<td>-6.20 (1.03)**</td>
</tr>
<tr>
<td>L. price change</td>
<td>0.43 (0.35)</td>
<td>0.37 (0.23)</td>
<td>0.14 (0.22)</td>
<td>0.72 (0.30)**</td>
<td>0.72 (0.34)**</td>
</tr>
<tr>
<td>L. profit</td>
<td>0.06 (0.01)**</td>
<td>0.04 (0.01)**</td>
<td>0.04 (0.02)*</td>
<td>0.05 (0.01)**</td>
<td>0.03 (0.02)</td>
</tr>
<tr>
<td>L. $\pi/q$ change</td>
<td>1.68 (0.25)**</td>
<td>1.92 (0.48)**</td>
<td>1.51 (0.49)**</td>
<td>1.45 (0.65)**</td>
<td>0.49 (0.33)</td>
</tr>
<tr>
<td>L. others’ (seen) quantity</td>
<td>-</td>
<td>-0.20 (0.05)**</td>
<td>-</td>
<td>0.04 (0.04)</td>
<td>-0.17 (0.10)*</td>
</tr>
<tr>
<td>tr * L. $\pi/q$</td>
<td>-</td>
<td>-1.15 (0.59)*</td>
<td>-</td>
<td>-1.70 (0.92)*</td>
<td>-</td>
</tr>
<tr>
<td>tr * L. others’ q</td>
<td>-</td>
<td>0.12 (0.02)**</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>tr * L. price change</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-1.15 (0.45)**</td>
<td>-</td>
</tr>
<tr>
<td>constant</td>
<td>124.93 (6.06)**</td>
<td>192.98 (19.08)**</td>
<td>150.14 (16.14)**</td>
<td>129.60 (11.14)**</td>
<td>199.70 (35.31)***</td>
</tr>
</tbody>
</table>

Notes: ***, **: significant at the 1% level, **: significant at the 5% level, *: significant at the 10% level. Std errors are in parentheses after the coefficients. Panels are the individuals. In VS the sample is unbalanced. The variables are lagged. Changes are calculated by taking the first difference of the variables. $\pi/q$ change is -1 if profit and quantity changes in opposite direction, 1 if they change in the same direction, and 0 if one of them is unchanged. Others’ (seen) quantity is the total quantity the subject can see about others’ production. In VS it depends on the others’ info-sharing decision. Finally, the treatment effects (shown by tr) are calculated by the interaction between treatment dummy (0 for Aggregate, and 1 for Individual) and other variables. Insignificant treatment effects are not included in the final models, therefore they are also not reported here.
price change in case of VS with 2 others sharing is also insignificant for the Individual treatment. Here subjects seem to care a bit more about others’ quantity than about price change or profit change. Note that FI and VS with 2 sharing firms have the same available information from the previous round, but subjects react on these pieces of information differently. The variables are less significant in case of VS which might be due to the fact that here information provision is endogenous, which cannot be captured by this model. Previously we have already seen that others’ information sharing decision affects subjects’ information sharing decision which highly correlates with the own production decision.

As a final test, we also ran separate regressions in case of Full info and VS with 2 others sharing for the Individual treatment in which we included the absolute difference of the two other firms’ production. By doing so, we can test whether getting disaggregated data has an effect on the production. In both cases the effect is insignificant, therefore we omit the regression outputs.

5.6 Conclusion

This chapter has stressed that the voluntary nature of sharing information with competitors has important consequences for market competitiveness. The reason for this is that information sharing works as a signaling device: firms can show their willingness to collude by sharing information unilaterally. On the other hand, not sharing information signals competitiveness.

We have conducted a laboratory experiment in which subjects act as firms in the market of a homogeneous good. Three subjects form a market, where they compete in quantities. When subjects receive feedback about their competitors’ actions, we vary the type of information they receive. Subjects can observe either the total output of their competitors (aggregate information) or the production level of the firms separately (individual information). Moreover, in one part of the experiment subjects can choose whether they want to share information with others or not, making the information structure endogenous. This kind of endogeneity has not been taken
into account in previous work analyzing the effect of aggregate and individual information on market competitiveness.

Our results show that both the voluntary nature of information sharing and the level of data aggregation can have important consequences for the market outcome. Subjects produce significantly less when they decide to share information with their competitors compared to the case when they intentionally withhold this information. This confirms that subjects use information sharing as a signaling device. When subjects decide not to share information, they produce significantly more compared to the situation when information sharing is not possible and subjects receive no information about competitors. On the other hand, there is no significant difference between average productions when subjects decide to share information and when information is available by default. Voluntary information sharing helps coordination as well compared to the case when subjects automatically get aggregate information about their competitors’ action: individual production levels tend to be closer to each other. However, coordination does not improve between Full information and Voluntary information sharing in case of individual information. Concerning further effects of aggregate and individual information, our results show that the average total output is lower when subjects receive individual information. This difference is, however, not significant. Thus, the market outcome does not become significantly less competitive under individual information but we could observe more attempts for collusion in the individual treatment. This is in line with the view of competition authorities regarding the anti-competitive nature of individual information. So publication of aggregate data does not lead to a higher level of collusion though it does not increase competitiveness either. Publication of individual data is a bit more dangerous, especially if firms voluntarily decide whether to share the information. Of course this depends on the distribution of the different types of firms in the market: the more collusive types are in the market, the more likely collusion is if firms can decide to share information. Thus, allowing firms to decide about sharing individual data might not be desirable.

In the paper we have focused on information sharing about actions. However, there is an-
other branch of the literature on information sharing in oligopolies. In this other branch firms face either demand or cost uncertainty, they receive an individual signal about some unknown parameter and they may share their signal with each other.\textsuperscript{30} Note that information sharing concerns different types of uncertainty in the two branches of the literature. In the branch that this chapter belongs to, firms are assumed to know the market characteristics and they learn about the behavior of their competitors through information sharing. In contrast, the demand or cost structure is not fully known in the other branch and information sharing helps firms learning the true market characteristics.\textsuperscript{31} It would be interesting to combine the two branches as both sources of uncertainty (market characteristics vs. competitors’ behavior) can be relevant in real markets. One possibility is to assume that firms do not know the demand function, for example, and they learn both demand conditions and their competitors’ behavior from market observations. When firms do not know the demand function and they cannot observe the actions of their competitors when learning about demand conditions, then they may reach a substantially different outcome than under known demand structure (see e.g. Brousseau and Kirman, 1992 and Chapters 2 and 3 of this thesis). This might result in welfare loss and this possible loss should be taken into account for analyzing the effects of information on the market outcome. Huck et al. (1999) investigate the effect of receiving additional information about market conditions and they find that more information about the market leads to less competitive outcomes. As we have seen, the voluntary nature of information sharing has important consequences for the market outcome, therefore it would be interesting to extend our current design with treatments where demand conditions are unknown.

\textsuperscript{30}See Raith (1996) for a general model that incorporates different market structures and sources of uncertainty. Kühn and Vives (1995) give a good overview of the results in this literature.

\textsuperscript{31}Moreover, Mailath (1989) and Jin (1994) point out another important difference between the two branches. While firms cannot affect their signal about the environment, they can strategically choose their price or production level to alter the beliefs of their competitors about market conditions.
Appendix 5.A  Instructions for Treatment I-NF

This section contains the instructions for treatment I-NF. The other instructions are similar and are available upon request.

Welcome to this experiment on decision making. Please read the following instructions carefully. If you have any questions, please raise your hand, and we will come to your table to answer your question in private. The experiment will consist of three parts. Each part consists of 30 rounds. Your overall earnings will be equal to your total earnings in one randomly chosen part. At the end of the experiment we will publicly roll a die to determine which part will be paid out. If the result of the roll is 1 or 2, then part 1 will be paid. If the result is 3 or 4, part 2 will be paid. If the result is 5 or 6, part 3 will be paid.

When everyone has finished reading the instructions and before the experiment starts, you will receive a handout with a summary of these instructions. At the start of each part, you will receive a starting capital of 6000 points. You will not have to pay back this starting capital. In addition, you will earn points in every round based on your decisions in combination with the decisions of other participants. At the end of the experiment, your earnings in points will be transferred into money. Each 100 points will be exchanged for 0.091 eurocent. This means that for each 1100 points you earn, you will receive 1 euro. Your earnings will be privately paid to you in cash.

In this experiment, you will be randomly assigned to a “market” consisting of 3 firms denoted by firm A, firm B and firm C. During each part your market will not change, you will play with the same two other participants, and your role will be fixed to be one of the three above-mentioned firms. However, after each part of the experiment the composition of your market
will change, and you will never be part of the exact same market again.

In the experiment you run one of the firms. You are interacting on the market with 2 other firms run by other participants. In every round you need to decide how much you want to produce. The price at which you sell the products depends on your production choice and on the production choice of the two other firms. Your earnings in each round will be equal to your profit from the production (which equals to your revenue minus your costs).

PAGE 2

MARKET CHARACTERISTICS

In the experiment the market is characterized by the same structure in all three parts. In each part you need to decide how much you want to produce. You can choose any integer production level between (and including) 40 and 125 units. The price will depend on your production choice and on the production choice of the two other firms. The higher the total production (your production plus the production of the other firms) is, the lower the price is. The price is determined by the total production, according to the following formula: $\text{price} = 45 - \sqrt{3} \sqrt{\text{total production}}$. The following graph illustrates how the price depends on the total production:

The price is the same for all firms in your market. Your total revenue is determined by the price
and your own production in the following way: revenue = price * own production.

Production is costly. Your production cost depends on your own production choice only: cost = (own production)*√(own production). The more you produce, the higher your production cost is. The production costs of the other firms depend on their own production level in the same way. The following graph illustrates how your total cost depends on your production:

Your profit in a given round is given by the revenue minus production cost:

Profit = \(45 - \sqrt{3 \cdot \sqrt{(total \ production)}}\) * (own production) – (own production) ∗ √(own production)

DECISIONS IN EACH ROUND

PART 1

At the beginning of each round you choose your production, without knowing how much the other firms produce. After all three firms made their production choice, the price and the pay-offs are determined. At the end of each round you are informed about your production choice, the price and your payoff (revenue, cost and profit) in the round.
PART 2

At the beginning of each round you choose your production, without knowing how much the other firms produce. After all three firms made their production choice, the price and the payoffs are determined. At the end of each round you are informed about your production choice, the price and your payoff (revenue, cost and profit) in the round. Additionally, all three firms are informed about the individual production levels of the other firms.

PART 3

At the beginning of each round you need to decide whether you want to inform the other firms about your production choice. They will receive this information only at the end of the round, after they made their own decision about informing others and production choice. Informing other firms is free, and both of your partners will receive the same information. Furthermore, you also choose your production, without knowing how much the other firms produce, and whether they decided to share information. After all three firms made their choices, the price and the payoffs are determined. At the end of each round you are informed about your production choice, the price and your payoff (revenue, cost and profit) in the round. Additionally, all three firms are also informed about the individual production levels of those firms that decided to inform others about their production choice.

PAGE 3

PROFIT CALCULATOR

When you are making your decision about production choice, you will see on the left-hand side of the screen a profit calculator. Here you can enter hypothetical production levels about
your own production and the other two firms’ total production, and you can calculate your profit with these production details. You can use the profit calculator as often as you want.

HISTORY OVERVIEW

On the lower part of the screen, a history screen will be provided. There, you can see the production details in your market for each round in the given part. One row contains information about one round. The history screen updates after every round. The observations are sorted descending by round, so you can find the most recent round always at the top. The history screen clears after every part.

The history screen is different in every part. In each part you will see your own production, the market price and your own profit. In addition, in part 2 you will see the individual production levels of the other firms by firm ID. In part 3, you will also see your information sharing decision, the individual production levels of those firms who shared. If a firm does not share information in certain rounds, you will see “n.a.” for that firm in the table. Below you can find an example for the history screen in part 3.

<table>
<thead>
<tr>
<th>Round</th>
<th>Information sharing</th>
<th>Own production</th>
<th>Production of firm B</th>
<th>Production of firm C</th>
<th>Price</th>
<th>Profit</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>yes</td>
<td>76</td>
<td>113</td>
<td>“n.a.”</td>
<td>18</td>
<td>705.45</td>
</tr>
<tr>
<td>2</td>
<td>yes</td>
<td>95</td>
<td>“n.a.”</td>
<td>75</td>
<td>19.72</td>
<td>947.45</td>
</tr>
<tr>
<td>1</td>
<td>no</td>
<td>84</td>
<td>70</td>
<td>101</td>
<td>17.34</td>
<td>888.68</td>
</tr>
</tbody>
</table>

On the next screens you will be requested to answer some control questions. Please answer these questions now.
Chapter 6

Summary

In this thesis we have investigated the consequences of incorporating bounded rationality and learning in different models of market competition. The relevance of the analysis is that empirical evidence shows that observed behavior is not always in line with the assumption of perfect rationality and that models of bounded rationality may describe behavior better in some situations. In the thesis we have focused on bounded rationality on the supply side of the market in order to complement the recent literature on behavioral industrial organization, which mainly focuses on deviations from standard rational models on the demand side.

There are many different ways to incorporate bounded rationality in market competition. We have focused on two aspects of rationality throughout the chapters of this thesis. Under perfect rationality, firms know the market environment (or they have correct beliefs about it) and their beliefs about the action of their competitors are consistent with the competitors’ actual choice. First we have relaxed the assumption that firms perfectly know the demand condition in the market and we have considered different learning methods that firms may apply. Second, we have weakened the consistency requirement on beliefs about the action of other firms, under a known demand structure. We have combined analytical investigation with numerical analysis to explore how the market outcome changes compared to the standard equilibrium prediction and what the possible welfare effects are. Moreover, we have investigated behavior by means of a laboratory experiment as well.
In Chapter 2 we have focused on the interaction between different learning methods. We have considered a situation where firms do not know the demand conditions in the market and they can use two different methods for finding the optimal action. We have shown that different methods can lead to different market outcomes, different methods can coexist in the market at the same time and the coexistence of different methods may affect the convergence properties of the methods. Thus, this chapter has stressed that it is important to take into account learning and heterogeneity in economic models.

In Chapter 3 we have further investigated the properties of least squares learning. We have considered a model where firms can observe all the relevant variables that affect the demand for their good and the functional form they use in the estimation is locally correct. We have analytically shown that some firms may end up in a suboptimal situation as firms do not learn the whole demand function correctly, only one part of it. These firms could increase their profit by charging a different price. Thus, this chapter has demonstrated that least squares learning can lead to a suboptimal outcome even when the estimation seems correct in the sense that the estimated function perfectly matches the observations.

In Chapter 4 we have analyzed the effects of weakening the consistency requirement on beliefs about the competitors’ actions. We have shown that this may induce existence of a pure-strategy equilibrium in a situation where such an equilibrium does not exist otherwise. We have modified the standard version of simultaneous price and quantity setting by introducing risk aversion and uncertainty in the beliefs of firms about the price and quantity choice of their competitor. We have numerically shown that the modified model may have an equilibrium in pure strategies. This chapter has illustrated that incorporating bounded rationality in a model can substantially change the equilibrium prediction compared to the corresponding rational model and that a small amount of bounded rationality may be welfare enhancing.

In Chapter 5 we have investigated by means of a laboratory experiment how information about the competitors’ production choices affects the corresponding market outcome. Subjects play the role of firms in the experiment and they receive information either about the total
production in the market or about firm-specific production levels. We have found that total output is typically lower when subjects receive firm-specific information. This supports the view of competition authorities that reporting detailed firm-specific information has anti-competitive effects. We have also found that voluntary information sharing serves as a signaling device: subjects show their willingness to collude by sharing information with each other. This chapter has shown that both the voluntary nature of information sharing and the level of data aggregation can have important consequences for the market outcome.

To summarize, this thesis has stressed that bounded rationality and learning should be taken into account in economic models. Models with bounded rationality and learning can lead to substantially different market outcomes than standard rational models and these outcomes may be closer to observed behavior.
Bibliography


Samenvatting (Summary in Dutch)

In dit proefschrift worden de consequenties van begrensde rationaliteit en het gebruik van leermethoden op verschillende soorten markten onderzocht. Empirisch onderzoek laat zien dat de aanname van volledige rationaliteit geobserveerd gedrag niet altijd goed kan beschrijven en dat modellen met begrensde rationaliteit sommige uitkomsten juist beter kunnen verklaren. Dit onderstreept het belang van dit onderwerp. We vullen de recente literatuur, die zich vooral richt op consumenten met begrensde rationaliteit, aan door modellen te beschouwen waar juist bedrijven niet volledig rationeel zijn.

Begrensde rationaliteit kan op verschillende manieren in marktcompetitie ingebouwd worden. In dit proefschrift focussen we op twee aspecten van rationaliteit. Onder volledige rationaliteit wordt er verondersteld dat bedrijven de marktomgeving volledig kennen (of dat hun inschatting van die marktomgeving niet systematisch fout is) en dat hun verwachtingen over de acties van hun concurrenten consistent zijn met de daadwerkelijk gekozen acties. Eerst laten we de aanname dat bedrijven de marktvraag volledig kennen los en beschouwen we verschillende leermethoden die de bedrijven kunnen gebruiken. Daarna laten we de eis dat, gegeven een bekende marktvraag, verwachtingen consistent met werkelijke acties moeten zijn, vallen. We gebruiken analytische en numerieke methoden om te onderzoeken hoe de marktuitkomst verandert in vergelijking tot de standaard evenwichtsvoorspelling en wat de bijbehorende welvaartsgevolgen zijn. Bovendien bestuderen we daadwerkelijk menselijk gedrag in dit soort marktomgevingen met behulp van een laboratoriumexperiment.

In Hoofdstuk 2 focussen we op de interactie tussen verschillende leermethoden. We be-
schouwen een situatie waarbij bedrijven de marktvraag niet kennen en ze twee verschillende methoden kunnen gebruiken om de optimale prijs te bepalen. We laten zien dat de methoden tot verschillende uitkomsten leiden, dat de verschillende methoden op de markt naast elkaar kunnen bestaan en dat deze coëxistentie invloed kan hebben op de dynamische eigenschappen van de methoden. Dit hoofdstuk toont daarmee aan dat het belangrijk is om met leren en heterogeneiteit rekening te houden in economische modellen.

In Hoofdstuk 3 onderzoeken we de eigenschappen van de zogenaamde kleinste-kwadratenleermethode verder. We beschouwen een model waar bedrijven alle relevante variabelen die invloed op de vraag voor hun product hebben kunnen observeren en ze in de regressie een lokaal goed gespecificeerde functievorm gebruiken. We tonen aan dat sommige bedrijven een suboptimale situatie kunnen bereiken omdat ze niet de hele vraagfunctie goed kunnen leren maar slechts één deel daarvan. Dit hoofdstuk demonstreert dus dat kleinste-kwadratenleren tot een suboptimale uitkomst kan leiden, ook als de schatting correct lijkt omdat de geschatte functie volledig overeenkomt met de waarnemingen.

In Hoofdstuk 4 analyseren we de gevolgen van het laten vallen van de eis dat verwachtingen volkomen consistent met de werkelijke acties van concurrenten moeten zijn. We laten zien dat er een evenwicht in zuivere strategieën kan bestaan in een marktmodel dat geen evenwicht in pure strategieën heeft onder de standaard-aannames. We passen de standaard versie van simultane prijs-hoeveelheidscompetitie op de volgende manier aan: we veronderstellen dat de bedrijven risicomijdend zijn en dat ze onzeker zijn over de prijs en hoeveelheidskeuze van hun concurrent. We laten met behulp van numerieke methoden zien dat het aangepaste model een evenwicht in zuivere strategieën kan hebben. Dit hoofdstuk illustreert dat de uitkomst van een model met begrensde rationaliteit wezenlijk kan veranderen in vergelijking tot het rationele model en dat een kleine afwijking van het rationele model bovendien tot een hogere welvaart kan leiden.

In Hoofdstuk 5 beschrijven we een laboratoriumexperiment dat is uitgevoerd om te onderzoeken hoe de informatie over de hoeveelheidskeuze van de concurrenten de marktuitkomst beïnvloedt. Deelnemers spelen de rol van bedrijven in het experiment en ze krijgen informatie

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hetzij over de totale geproduceerde hoeveelheid in de markt of over de individuele productieniveaus. Uit onze resultaten blijkt dat de totale productie typisch lager is als deelnemers bedrijfsspecifieke informatie krijgen. Dit is in overeenstemming met het inzicht van mededingingsautoriteiten dat bedrijfsspecifieke informatie een concurrentiebeperkend effect kan hebben. We vinden ook dat het vrijwillig delen van informatie als een signaal werkt: deelnemers kunnen aan elkaar laten zien dat ze bereid zijn om samen te werken door informatie met de anderen te delen. Dit hoofdstuk laat zien dat zowel het vrijwillige karakter van het delen van informatie, als het niveau van data-aggregatie belangrijke gevolgen voor de marktuitkomst kan hebben.

Samenvattend benadrukt dit proefschrift dat er in economische modellen rekening met begrensde rationaliteit en leren gehouden moet worden. Modellen met begrensde rationaliteit en leren kunnen tot wezenlijk andere uitkomsten leiden dan het standaard rationele model en deze alternatieve uitkomsten liggen mogelijk dichter bij werkelijk marktgedrag.
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