Bounded rationality and learning in market competition

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Chapter 4

Price-Quantity Competition under Strategic Uncertainty

4.1 Introduction

There are two traditional ways of modeling competition between firms producing homogeneous commodities. Cournot (1838) introduces quantity competition in which firms set the quantity of the good and the price adjusts such that the market clears. In contrast, Bertrand (1883) suggests a model in which price is the strategic variable and quantities clear the market. These two models serve as the basic framework in the literature of market competition. However, both models have their drawbacks. Under quantity competition, a market clearing mechanism is required to reach the price for which the demand equals aggregate production. In the basic model of price competition, Bertrand assumed that firms can produce any amount of the good and the output is realized immediately. Firms, however, might not be able to or might not want to[1] serve the whole market at a given price. One way to address this issue is to introduce capacity constraints

This chapter is based on Kopányi (2013b).

[1]This can occur with convex cost functions: undercutting the price of other firms may not be profitable due to the large increase in production costs. See Dastidar (1995), for example.

in the model: firms simultaneously choose capacity levels first and they decide about the price only after observing the capacity levels. See Edgeworth (1925), Kreps and Scheinkman (1983), Gelman and Salop (1983) and Davidson and Deneckere (1986), for example. These models, however, do not take into account that production takes time: firms typically need to produce the good in advance, without knowing the exact demand they will face. Taking these considerations into account, a reasonable alternative of modeling competition is to treat both prices and quantities as strategic variables to be set simultaneously.[2] An inconvenient characteristic of simultaneous price-quantity setting is that there does not exist a Nash equilibrium in pure strategies (see Levitan and Shubik, 1978 and Maskin, 1986). Roy Chowdhury (2008), however, proposes a variation of the model that may lead to a pure-strategy Nash equilibrium. In his model firms can choose their price from discrete values only.

In this chapter we consider alternative assumptions under which a pure-strategy equilibrium may exist in a market where firms simultaneously set both the price and the production level of the good. We consider the market for a homogeneous good that is produced by two firms. Firms are risk averse and they have mean-variance preferences. They hold probabilistic conjectures about the actions of the other firm and they choose the optimal actions given their conjectures. Conjectures are, however, not entirely in line with the actual action of the other firm. Thus, in this chapter we investigate the effect of weakening the consistency requirement on beliefs. Under perfect rationality the actual actions of other firms confirm the beliefs about their actions: the conjectured action is realized in case of pure-strategy equilibria while realized actions are in line with the conjectured probability distribution in case of mixed-strategy equilibria. In our model we require beliefs to be *weakly* consistent: the mode of the distribution has to match the actions of other firms.

We numerically show that there may exist a pure-strategy equilibrium in our model. This

[2] Judd (1996) provides an additional argument for price-quantity competition. He argues that the theoretical predictions of simpler models in which firms set price or quantity only, depend crucially on whether the strategic variables are strategic substitutes or strategic complements. Therefore, excluding either strategic variable from the analysis can have a substantial effect on the results so it is better to treat them together.

equilibrium exists only when firms are sufficiently risk averse and the amount of uncertainty[3] is sufficiently high. The result of having a pure-strategy equilibrium is important as mixed-strategy equilibria seem less relevant in the field of industrial organization: it does not seem reasonable to assume that firms always choose their actions randomly from a specific distribution. Therefore, the model we propose in this chapter makes it possible to apply price-quantity competition more widely as a framework for analyzing various market phenomena such as mergers or cartels and for policy analysis.

We analyze with numerical methods how the equilibrium depends on the degree of risk aversion and the amount of uncertainty in the model. In the pure-strategy equilibrium, aggregate production exceeds market demand for low degrees of risk aversion and firms are rationed. As firms become more risk averse, they decrease their production level and they will not satisfy the demand for their good eventually therefore consumers will be rationed. Our model shows that firms react differently to price uncertainty than to output uncertainty[4]: the equilibrium price is typically increasing while the production level is decreasing in the amount of price uncertainty but both the equilibrium price and production level decrease as the amount of output uncertainty increases. The reason for this difference is the following. Higher price uncertainty does not affect the profit variance directly but the price becomes a more efficient instrument for increasing the expected profit. In contrast, higher output uncertainty directly affects profit variance through the residual demand so firms have an incentive to reduce their price in order to decrease the chance of operating on the residual demand function. Our model can explain a seemingly anti-competitive behavior (both firms increase their price and decrease their production level) without collusion between firms: an increase in the amount of price uncertainty has exactly the aforementioned effect in equilibrium. Our analysis shows that a small degree of risk aversion is welfare enhancing and that the welfare is higher in equilibrium than the expected welfare in the mixed-strategy equilibrium of the standard model.

The chapter is organized as follows. First we review the literature on which the chapter

[3]We measure the amount of uncertainty as the variance of the distribution of conjectures.

[4]Price/output uncertainty is the variance of the distribution of price/output conjectures.

builds in Section 4.2. Then we describe the standard model of price-quantity competition in Section 4.3. The model with strategic uncertainty and risk aversion is presented in Section 4.4. In Section 4.5 we numerically characterize the symmetric pure-strategy equilibrium and we analyze which parameter combinations lead to an equilibrium. Section 4.6 analyzes how the equilibrium depends on the degree of risk aversion and on the amount of uncertainty. Section 4.7 concludes. Derivations are presented in Appendices 4.A, 4.B and 4.C.

4.2 Related literature

The idea of simultaneous price-quantity competition was mentioned by Shubik (1955) first. However, the author does not investigate the equilibria of this model as other models are in the main focus of the paper. Levitan and Shubik (1978) analyze a duopoly in which firms produce a homogeneous good. The demand depends on the price linearly, production is costless but there is a fixed unit cost for disposing unsold products. The authors show that there exists no pure-strategy Nash equilibrium and they derive a mixed-strategy Nash equilibrium. Maskin (1986) analyzes the market of a homogeneous good and he considers two versions of price-quantity competition: production in advance (i.e. prices and quantities are set at the same time) and production to order (i.e. prices are set first and firms decide on production only after observing each other's price). He proves the existence of a mixed-strategy Nash equilibrium under general demand and cost conditions. In his PhD thesis, Gertner (1986) analyzes a duopoly market of a homogeneous good with symmetric firms and increasing, constant and decreasing marginal costs. He shows that there is no pure-strategy Nash equilibrium in any of these cases but a mixed-strategy Nash equilibrium exists. He derives the unique mixed-strategy Nash equilibrium for a linear demand function and constant and equal marginal costs. This equilibrium has the feature that firms draw a price from a certain distribution and then both firms choose the production level that equals to the market demand at the price it drew. Consequently, one firm will serve the whole market while the other firm will not sell anything. Firms have zero expected profit in equilibrium. McCulloch (2011) characterizes the mixed-strategy Nash equilibrium numerically for the case of an asymmetric duopoly with increasing marginal costs. He uses a fine grid for both prices and quantities. His findings support those of Gertner (1986): firms charge a (relatively) high price with a high probability and some lower prices with low probabilities. This result can be interpreted as firms often charging a high regular price and a lower sale price every now and then.

Even though there does not exist a pure-strategy equilibrium in the standard model, modified versions may lead to pure-strategy equilibia. In fact, Roy Chowdhury (2008) proposes a variation of the model that may lead to a pure-strategy Nash equilibrium. He analyzes a price-quantity model of a homogeneous good with discrete pricing over a grid and convex production costs. He shows that for a fixed grid size, there exists a unique Nash equilibrium if the number of firms is high enough. On the other hand, for a fixed number of firms, there is no pure-strategy equilibrium when the grid size is sufficiently small. In the model of this chapter we consider alternative assumptions under which the existence of pure-strategy equilibria is established for smaller number of firms and continuous action spaces too.

The way we model the conjectures of firms and the corresponding equilibrium concept are related to the random belief equilibrium introduced by Friedman and Mezzetti (2005). In their model, players hold beliefs regarding the other players' actions and there are two equilibrium conditions. The first one is that players maximize their payoffs subject to their beliefs, and the second one is that beliefs are consistent with the other players' actions in the sense that the expected choice of every firm coincides with the center of the belief distribution (i.e. the mode or the mean of the distribution is correctly specified). The same two conditions characterize the equilibrium in our model. Larue and Yapo (2000) and Andersson et al. (2012) take a similar approach. Their players hold a subjective belief about the action of the other players and they maximize their payoff subject to these beliefs. Larue and Yapo (2000) require beliefs to be consistent with the action of other players in equilibrium: the belief distribution is centered around the equilibrium action of the other player. Andersson et al. (2012) do not make consistency requirements for beliefs but it is not necessary as they consider the limiting case when the amount of uncertainty goes to zero.

Models of price-quantity competition with differentiated goods are also characterized by the non-existence of pure-strategy Nash equilibria. With differentiated goods, one needs to model the *spillover demand* among the goods, that is the additional demand for a good when the supply of another good cannot satisfy the demand of that good. Friedman (1988) uses general demand, spillover demand and cost functions and considers three versions of price-quantity competition. He shows that there exists no pure-strategy Nash equilibrium when prices and production levels are set simultaneously. However, when production levels are set first and firms decide on prices only after observing the actual outputs, a pure-strategy Nash equilibrium exists when spillover effects are not too strong. Furthermore, when prices are set first, then there always exists a pure-strategy subgame-perfect Nash equilibrium. For further results on price-quantity competition with differentiated goods see Benassy (1986), Judd (1996) and Khan and Peeters (2011), for example.

After reviewing the literature, let us now discuss the standard model of simultaneous price-quantity setting.

4.3 Price-quantity competition

We consider the market for a homogeneous good that is produced by two firms. The firms engage in price-quantity competition and they both set prices *and* production levels simultaneously. Production levels correspond to *actual* production. That is, they are not simply capacity constraints in the sense that production must be implemented at the chosen level, firms may not supply less.

The market demand depends linearly on the price of the good. It is given by

$$D(p) = \max\{a - bp, 0\}, \tag{4.1}$$

where a and b are positive parameters and p is the price.

Since firms make their decisions simultaneously, a firm may end up with unsold products. Therefore, we have to distinguish production levels from sales. Sales depend on prices and production levels of both firms. The sales of firm i are given by

$$s_i(p_i, q_i, p_j, q_j) = \begin{cases} \min\{q_i, D(p_i)\} & \text{if } p_i < p_j \\ \min\left\{q_i, \frac{q_i}{q_i+q_j}D(p_i)\right\} & \text{if } p_i = p_j \\ \min\{q_i, r_i\} & \text{if } p_i > p_j \end{cases}, \tag{4.2}$$

where p_i and q_i denote the price and production level of firm i whereas subscript j refers to firm j. Variable r_i is the residual demand of firm i: $r_i = \max\{D(p_i) - s_j, 0\}$.[5] Thus, we apply the efficient rationing rule[6] in the model.

Formula (4.2) shows that the firm with the lowest price sells all its products, provided that its production level does not exceed the market demand at the price the firm chose. When firms charge the same price, they sell all their products if they do not serve the whole market together (i.e. $q_i + q_j \leq D(p)$ where p is the price chosen by both firms). If, however, aggregate production exceeds the market demand, then the firms serve the whole market and we assume that sales are proportional to production levels.[7] Finally, the firm with the highest price operates on its residual demand. If its residual demand exceeds its production level, then the firm will sell all its products. However, the firm will sell only a part of its products if the residual demand is smaller. Moreover, when the residual demand is 0, then the firm will not sell anything.

The profit of firm i is given by

$$\pi_i = p_i s_i(p_i, q_i, p_j, q_j) - cq_i, \tag{4.3}$$

where c is the marginal cost of production. Note that we assume that the marginal cost is

[5] Note that we use s_j in the definition of s_i. This is not problematic since s_j affects the sales of firm i only when $p_i > p_j$ and in this case $s_j = \min\{q_j, D(p_j)\}$.

[6] See Tirole (1988) for more details about this rationing rule.

[7] We will see later that this assumption does not affect the behavior of firms as the event that they charge the same price has probability 0 in our model.

constant and equal to c for both firms.

There exists no pure-strategy Nash equilibrium in this setting when firms maximize their profit. To see this, consider an arbitrary situation where both prices are higher than the marginal cost. The firm with the lowest price has an incentive to increase its production level until it serves the whole market (it may have an incentive to change its price too). Thus, the firm with the lowest price must serve the whole market in any equilibrium. If it does so, the other firm undercuts the price and it serves the whole market. This undercutting may continue until prices are equal to the marginal cost. However, there exists no equilibrium with prices equal to the marginal cost. If aggregate production exceeds the market demand, then both firms have an incentive to decrease their production level. But if aggregate production is lower than or equal to the market demand, then both firms have an incentive to charge a higher price and operate on their residual demand since they make zero profit otherwise.[8]

The intuition behind the non-existence of a pure-strategy Nash equilibrium is that it is never optimal for a firm to choose the same price as the other firm: either serving the whole market at a slightly lower price or operating on the residual demand is more profitable than choosing the same price. Thus, the best response functions do not cross each other. The crucial condition is that firms can undercut each other (or choose to operate on the residual demand) with *certainty*. This, however, may not be the case if we introduce uncertainty in the model. In fact, we will see that there may exist a pure-strategy equilibrium in this case.

4.4 A model with strategic uncertainty and risk aversion

We introduce strategic uncertainty in the model through the beliefs about the actions of the other firm. Suppose that firms have a point forecast for the actions of the other firm but they are uncertain about the accuracy of these forecasts and they take this uncertainty into account when deciding on their own price and production level. Formally, the beliefs of firm i about the price

[8] If one of the firms does not produce anything when prices are equal to the marginal cost, then the other firm is better off by choosing the monopoly price and production level.

and production level of firm j are given by

$$p_j^b \sim N(p_j^f, \sigma_p^2),$$
$$q_j^b \sim N(q_j^f, \sigma_q^2),$$

where p_j^f and q_j^f denote the point forecasts. Parameters σ_p and σ_q determine the perceived accuracy of the forecasts. Firm i considers its forecasts perfectly accurate for $\sigma_p = \sigma_q = 0$. The higher the values of σ_p and σ_q are, the more inaccurate the forecasts are considered. We refer to σ_p and σ_q as the amount of price and output uncertainty. Thus, the beliefs p_j^b and q_j^b are independent normally distributed with mean p_j^f and q_j^f and variance σ_p^2 and σ_q^2, respectively.[9]

The beliefs about the price and production level of firm j generate profit conjectures in the following way: $\pi_i^c = p_i s_i(p_i, q_i, p_j^b, q_j^b) - cq_i$.[10] We introduce risk aversion in the model by assuming that firms have mean-variance preferences. That is, they simultaneously solve the following constrained optimization problem:

$$\max_{p_i, q_i} E(\pi_i^c) - \alpha Var(\pi_i^c) \tag{4.4}$$
$$\text{s.t. } q_i \leq D(p_i),$$

where $\alpha \geq 0$ measures the degree of risk aversion. For $\alpha = 0$ the firms are risk neutral and they maximize their expected profit only. The higher α is, the more disutility the variance gives, thus the more risk averse the firms are.[11] Note that if $q_i > D(p_i)$, then firm i will have some unsold products with certainty. Thus, the firm is always better-off by producing $q_i = D(p_i)$. Therefore, we can disregard the constraint $q_i \leq D(p_i)$ as it will always be satisfied.

[9] Due to being normally distributed, beliefs involve negative as well as unreasonably high values. In order to analyze the importance of these extreme cases, we ran simulations using *truncated* normal distribution as well. This way we could make sure that p_j^b takes values from the interval $[c, \frac{a}{b}]$ and q_j^b from the interval $[0, a - bc]$. There was no visible change in the equilibria, thus extreme realizations do not have a substantial effect.

[10] To keep the notation simple, we use p_j^b and q_j^b to denote both the random variable and a possible realization of the random variable.

[11] Note that risk aversion in itself cannot lead to a pure-strategy Nash equilibrium since firms can still undercut each other's price or operate on the residual demand with certainty.

Having discussed the model, we now turn to analyzing its equilibria. Variables p_1^*, q_1^*, p_2^* and q_2^* constitute an equilibrium if the following two conditions are satisfied:

1. $(p_i^*, q_i^*) \in \arg\max_{p_i,q_i} E(\pi_i^c) - \alpha Var(\pi_i^c)$ for $i = 1, 2$.

2. $p_i^f = p_i^*$ and $q_i^f = q_i^*$ for $i = 1, 2$.

The first condition means that actions are optimal given the conjectures. The second condition is a consistency requirement for the conjectures: it implies that conjectures are centered around the true values in equilibrium. Thus, in equilibrium the actions of the firms are optimal and their forecasts are correct.

This equilibrium concept is related to the random belief equilibrium (RBE), introduced by Friedman and Mezzetti (2005). They find empirical support for this concept in experimental data. Moreover, they compare RBE with the quantal response equilibrium (QRE) but the results are mixed: in some games RBE fits the subjects' behavior better while QRE performs better in others. The authors conjecture that RBE fits the data better in non-zero sum games that have a unique completely mixed equilibrium. These conditions hold for the simultaneous price-quantity competition with linear demand function and constant and equal marginal costs.

In the remaining part of the chapter we focus on the symmetric equilibria of the model. In the next section we will see that there exists a unique symmetric pure-strategy equilibrium in this market, provided that firms are sufficiently risk averse and/or the amount of uncertainty is sufficiently high.

4.5 Symmetric pure-strategy equilibrium

The conditions that characterize the symmetric pure-strategy equilibria of the model are derived in Appendix 4.A. As these conditions are quite long and complex, we report them only in the Appendix, see equations (4.19)-(4.26). For simplifying the notation, let the first-order conditions with respect to p_i and q_i, evaluated at the fixed point (p, q), be given by

$$F_p(p, q) = 0, \tag{4.5}$$

$$F_q(p, q) = 0. \tag{4.6}$$

This system of equations cannot be solved analytically: functions of p and q appear in the argument of the cumulative distribution function and the probability density function of the standard normal distribution in both equations. Therefore, we use numerical methods to find a solution and then we investigate whether it corresponds to the global maximum of the objective function of the firms. If it does, we can conclude to have found an equilibrium in pure strategies.

First we illustrate the existence of a symmetric pure-strategy equilibrium for a certain parameter specification and then we will analyze the set of parameters for which an equilibrium in pure strategies exists. We use the following parameter values in the calculations: $a = 10$, $b = 1$, $c = 2$, $\alpha = 1$ and $\sigma_p = \sigma_q = 0.5$. We numerically solve (4.5) and (4.6) by minimizing $F_p^2 + F_q^2$ with respect to p and q.[12] The resulting values are $p^* \approx 3.87$ and $q^* \approx 2.94$. The minimized value of the objective function is $1.36 \cdot 10^{-17}$. This value is very close to zero, which suggests that we have found a solution.

We cannot determine with this method whether there exist other points that satisfy the first order conditions. In order to answer this question, we numerically calculate the (p, q) pairs that solve (4.5) and (4.6) separately and we study the number of (p, q) combinations that satisfy both first-order conditions at the same time. We consider values for p from a fine grid in the interval $\left[c, \frac{a}{b}\right]$ and for each value of p we numerically calculate the value of q that satisfies (4.5) and (4.6), respectively. The upper left panel of Figure 4.1 shows the curves that consist of the points that satisfy the given first-order condition.[13]

The figure shows that these curves cross each other at exactly one point. This point cor-

[12] We use the `fminsearch` function in MATLAB for the minimization.

[13] The curve for F_p does not look smooth for the following reason. It can be shown that $F_p = 0$ for any price when $q = 0$; and for some values of p the numerical procedure finds $q = 0$ instead of the positive solution for q.

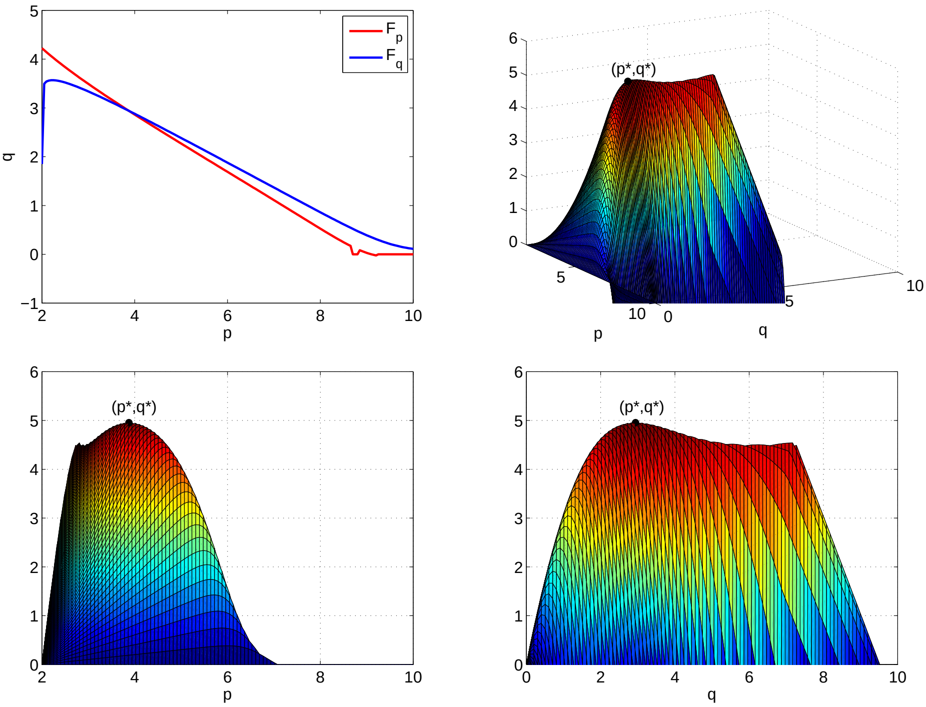

Figure 4.1: The (p, q) pairs that satisfy the two first-order conditions separately (upper left panel) and the objective function of firm i with the price and production level of firm j fixed at (p^*, q^*) (other panels). Parameter values: $a = 10$, $b = 1$, $c = 2$, $\alpha = 1$ and $\sigma_p = \sigma_q = 0.5$.

responds to the previously calculated (p^*, q^*). Thus, there exist a unique pair (p, q) that may constitute an equilibrium in pure strategies. In order to conclude that this point is indeed an equilibrium, we need to examine if it corresponds to the *global* maximum of the objective function of a firm, keeping the price and production level of the other firm fixed at (p^*, q^*). In other words, we need to check if choosing the same price and production level as the other firm is a best response. The upper right panel and the lower panels of Figure 4.1 depict the objective function of firm i for $p_j = p^*$ and $q_j = q^*$. We can observe that (p^*, q^*) corresponds to the global maximum. The above analysis confirms that there exists a unique symmetric equilibrium in pure strategies for the parameter specification we considered.

We cannot conclude from the previous analysis that a pair (p, q) that satisfies the first-order

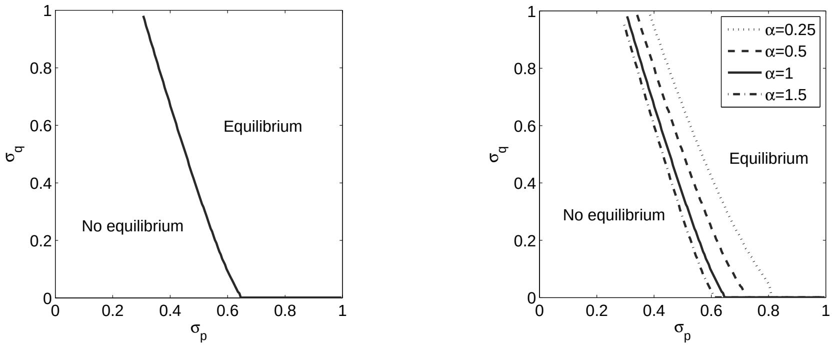

Figure 4.2: The parameter regions in the $\sigma_p - \sigma_q$ plane for which pure-strategy equilibria exist. Left panel: $\alpha = 1$. Right panel: $\alpha = 0.25, 0.5, 1$ and 1.5. There exists a symmetric equilibrium in pure strategies for the (σ_p, σ_q) combinations that lie to the right from a certain curve. Other parameters: $a = 10$, $b = 1$ and $c = 2$.

conditions, will always be an equilibrium. In fact, for certain combinations of α, σ_p and σ_q, the solution of the first-order conditions does not correspond to the global maximum of the objective function of a firm, keeping the price and production level of the other firm fixed at the value in the solution. For some parameter combinations the solution is a local but not the global maximum while it can be a saddle point for other parameters. Therefore, it is essential to investigate which parameter combinations lead to an equilibrium in pure strategies. The left panel of Figure 4.2 shows the (σ_p, σ_q) combinations that lead to a symmetric equilibrium in pure strategies for $\alpha = 1$. The curve gives the boundary of the region of existence in the (σ_p, σ_q) plane: the solution of the first-order conditions constitutes an equilibrium for the (σ_p, σ_q) combinations that lie to the right of the curve. The right panel of Figure 4.2 illustrates how the existence region changes as α varies. For obtaining this figure, we consider a grid for σ_p and σ_q for a given value of α, and for each parameter combination $(\alpha, \sigma_p, \sigma_q)$ we calculate the point that solves the first-order conditions (4.5)-(4.6). Then we compare the value of the objective function of a firm at this point with the global maximum (with the price and production level of the other firm fixed at the values in the solution). When these two values coincide, the solution corresponds to an equilibrium.[14] Finally, for each (σ_p, α) pair we consider the *minimal* value of

[14]For finding the global maximum, we evaluate the objective function on a grid with 1000 values from the

σ_q for which the equilibrium exists.[15] This leads to the curves depicted in Figure 4.2.

The figure shows that there exists no pure-strategy equilibrium when both the price and the output uncertainty are small.[16] This result is in line with the standard model with risk neutrality and no uncertainty: there exists no equilibrium in pure strategies in the standard model and the model we consider is close to the standard model for very small values of σ_p, σ_q and α. The figure also shows that the existence region expands in the degree of risk aversion and in the amount of price and output uncertainty. That is, the more risk averse the firms are or the more uncertainty they face, the more parameter combinations will lead to an equilibrium in pure strategies. Also note that price uncertainty is essential for having a pure-strategy equilibrium: the equilibrium region does not contain points for which $\sigma_p = 0$. On the other hand, numerical calculations show that neither output uncertainty nor risk aversion is crucial for existence: there are equilibria for $\alpha = 0$ and for σ_q too.

In the previous analysis we used an arbitrary parameter specification for the demand function. In order to check the robustness of our results, we considered other parameter values as well, including higher and lower slopes for the market demand function. The results are robust: there exists a unique pair (p, q) that solves the two first-order conditions and the structure of the region of existence is the same. Having established the existence of a symmetric equilibrium in pure strategies, we now turn to the properties of this equilibrium.

4.6 Comparative statics

The equilibrium depends on several parameters. Among these parameters, the degree of risk aversion α and the amount of uncertainty σ_p and σ_q are of particular interest. In this section we discuss how these parameters affect the equilibrium.

interval $(c, \frac{a}{b})$ for p_i and 1000 values from the interval $(0, a - bp_i]$ for q_i. This way we can get an *approximate* value of the global optimum. Then we conclude that the solution is an equilibrium if the value of the objective function at that point is larger than or equal to the approximate value of the global optimum.

[15]Preliminary simulations showed that an equilibrium may exist when σ_q is large enough, given σ_p and α. The analysis in Section 4.6 also confirms this.

[16]There might exist an equilibrium when α is sufficiently high. However, it might not be reasonable to assume very high values of α since they correspond to extreme degrees of risk aversion.

	p_i	q_i	p_j	q_j
$E(\pi_i^c)$	$+$	$+$	$+$	$-$
$Var(\pi_i^c)$	$+$	$+$	$-$	$+$

Table 4.1: The marginal effect of prices and production levels on the expected profit and profit variance of firm i in equilibrium.

4.6.1 The effect of prices and production levels on the objective function

Before investigating the effect of a parameter change on the equilibrium, it is worthwhile to analyze how a marginal increase in a price or a production level (keeping everything else fixed) affects the objective function of firms in equilibrium. This will be useful for understanding the intuition behind the results of the comparative statics analyses. Table 4.1 summarizes the marginal effect of the variables on the expected profit and the profit variance of firm i in equilibrium. A $+$ $(-)$ sign means that a marginal increase in the variable in the first row has a positive (negative) effect on the variable in the first column. For example, a marginal increase in p_j increases $E(\pi_i^c)$ and decreases $Var(\pi_i^c)$ in equilibrium.[17] We derive these effects in Appendix 4.B.

To understand the intuition behind these effects, note that there are four possibilities concerning the (conjectured) sales of firm i. If firm i has a lower price than the conjectured price of firm j, then firm i can sell all its products. If firm i has a larger price, then there are three cases. When the conjectured production level of firm j is low enough, such that the residual demand of firm i exceeds q_i, firm i can sell its whole production. For intermediate values of q_j^b, firm i operates on its positive residual demand and sells strictly less than its production level. Finally, firm i does not sell anything when the conjectured production level of firm j is high enough, such that the residual demand of firm i becomes 0. For simplicity, we refer to the cases when firm i sells all its products as good cases and we call a case bad when the firm has unsold products.

[17] More precisely, we should write p_j^f and q_j^f instead of p_j and q_j as we analyze the effect of a shift in the distribution of conjectures. However, in equilibrium $p_j = p_j^f$ and $q_j = q_j^f$ therefore we drop superscript f for simplifying the notation.

When p_i increases marginally, the price of firm i will be larger than the conjectured price of firm j with a higher probability. This has an increasing effect on the profit variance since there is more uncertainty about the sales and thus about the profit of firm i when the firm operates on its residual demand: the uncertain production level of firm j matters only if firm i has the higher price. Another effect of an increase in p_i is that the profit of firm i increases when it has positive sales. Note that the profit increases by q_i in the good cases while it increases less in the bad cases (by r_i or 0). Thus, the profit difference between good and bad cases increases and this also has an increasing effect on the profit variance.

Since firms try to find the balance between the expected profit and the profit variance, in equilibrium it must hold that an increase in one of the decision variables of firm i has the same marginal effect on its expected profit and profit variance (up to a factor α). Consequently, the expected profit of firm i should increase when p_i increases.

Keeping everything else fixed, a marginal increase in the production level of firm i increases the profit in the good cases. In contrast, the profit of firm i decreases in the bad cases since the firm will have more unsold products. Therefore, the profit difference between good and bad states increases, resulting in a larger variance. Furthermore, when q_i increases, the bad cases will occur with a higher chance since firm i is less likely to sell all its products. Since the bad cases lead to more uncertainty, this further increases the variance. Similar arguments as for p_i show that the expected profit of firm i should increase.

An increase in p_j is favorable for firm i since it will have a lower price with a higher chance. This leads to a higher expected profit since the case with the highest profit occurs more often. Furthermore, the profit variance decreases since the most uncertain case (when firm i operates on its positive residual demand) occurs with a lower chance.

When firm j increases its production level, then the residual demand of firm i will decrease. Thus, firm i can sell less products in the bad cases, leading to a lower expected profit. Moreover, the most uncertain case occurs with a higher chance since firm i is less likely to sell all its products when it has the higher price. This increases the profit variance.

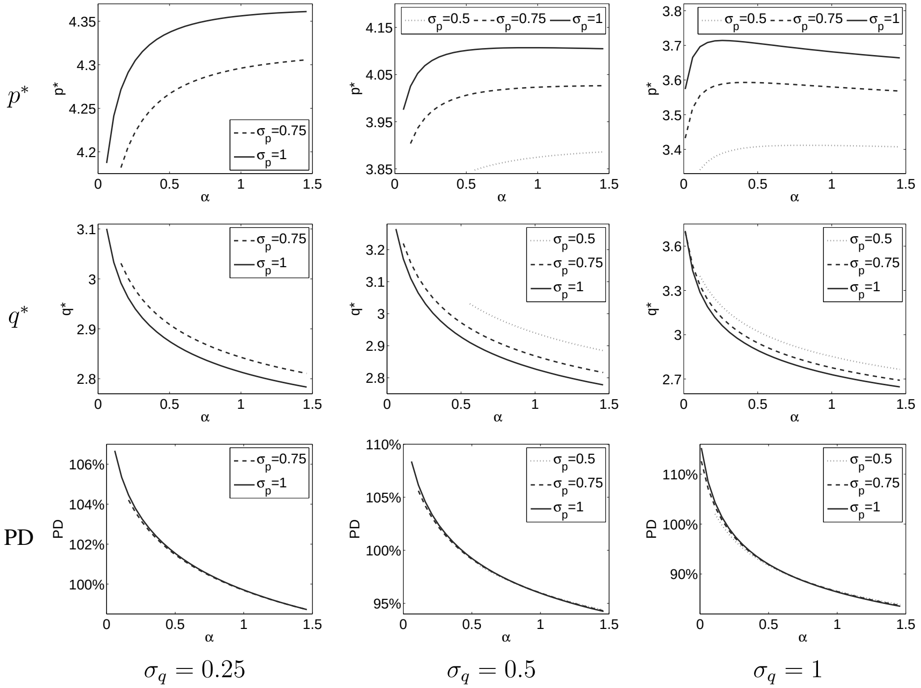

Figure 4.3: The equilibrium price p^* (upper panel), production level q^* (horizontal middle panel) and production to demand ratio $PD = \frac{2q^*}{a-bp^*}$ (lower panel) as a function of α, for $\sigma_p = 0.5, 0.75$ and 1. The left panel corresponds to $\sigma_q = 0.25$, the vertical middle panel to $\sigma_q = 0.5$ and the right panel to $\sigma_q = 1$. Other parameter values: $a = 10$, $b = 1$ and $c = 2$.

Having discussed the effect of prices and production levels, we can now turn to analyzing the effects of the model parameters on the equilibrium.

4.6.2 The effect of risk aversion

When firms become more risk averse, that is as α increases, they have an extra incentive for reducing the variance. As Table 4.1 shows, this can be achieved by decreasing the price or the production level. Figure 4.3 shows the equilibrium price (upper panel), production level (horizontal middle panel) and production to demand ratio $PD = \frac{2q^*}{a-bp^*}$ (lower panel) as a function of α, for different values of σ_p and σ_q. The vertical panels correspond to different values of σ_q and the lines on each plot correspond to different values of σ_p.[18] The figure shows that an increase in α has typically a positive effect on the equilibrium price. Only when both σ_p and σ_q are high, we can observe a slight decrease in the equilibrium price for higher values of α.[19] The equilibrium production level monotonically decreases in α. Thus, as firms become more risk averse, they decrease their production level to reduce the profit variance and they charge a higher price to compensate for the lower expected profit. However, when both the price and output uncertainty are high and the firms are sufficiently risk averse, they use also their price to decrease the variance.

The production to demand ratio $PD = \frac{2q^*}{a-bp^*}$ compares the aggregate production level $(2q^*)$ to the demand $(a - bp^*)$ in equilibrium. When $PD < 100\%$, demand exceeds aggregate production, so firms do not serve the whole market. For $PD = 100\%$, aggregate production equals the demand: firms sell all their products and the demand is satisfied. For $PD > 100\%$ there is overproduction: the demand is satisfied but firms end up with some unsold products. The PD ratio shows a decreasing pattern: aggregate production decreases more than the market demand as firms become more risk averse. Note that the PD ratio is always less than 200%, thus individual production levels are always strictly less than the market demand in equilibrium.[20] For lower degrees of risk aversion there is overproduction and firms are rationed but as firms become more risk averse, there is underproduction and consumers are rationed. The figure shows that for any amount of uncertainty, there exists a degree of risk aversion for which aggregate production equals the market demand in equilibrium.

4.6.3 The effect of price uncertainty

A change in σ_p affects the variance of price conjectures. However, this variance itself is not relevant for the firm in equilibrium. To see this, note that the *exact value* of the price of the

[18] Note that the different lines start at different values of α. This is due to the fact that for a given (σ_p, σ_q) combination the pure-strategy equilibrium exists only if α is sufficiently high, as Figure 4.2 shows.

[19] Note that the figure also shows that an increase in σ_p has a positive effect while an increase in σ_q has a negative effect on p^*. We will investigate these effects separately later in this section.

[20] In contrast, both firms produce up to the market demand at the price they chose in the mixed-strategy Nash equilibrium of the standard model.

other firm is irrelevant for the profit of firm i, the only thing that matters is whether this price is smaller or larger than the price of firm i. Since firms charge the same price in equilibrium, the conjectures of a firm are such that the probability of having the lower price is always 50%, it does not change as σ_p varies. Using the first-order conditions of the firms' problem, it can easily be shown that a change in σ_p does not affect either the expected profit or the profit variance *in equilibrium*.

The importance of σ_p is that it determines the marginal effect of p_i on the expected profit and the profit variance of firm i in equilibrium. To see this, consider the equilibrium for a fixed σ_p. It follows from the optimization problem (4.4) of firm i that $\frac{\partial E(\pi_i^c)}{\partial p_i} = \alpha \frac{\partial Var(\pi_i^c)}{\partial p_i}$ in equilibrium, thus the gain from a higher expected profit (for an increase in p_i) is exactly offset by the increase in the variance. Now suppose that firms are in equilibrium and σ_p has increased marginally. Note that the probability density function of p_j^b becomes lower around 0. Consequently, when p_i increases marginally, the probability of firm i having the lower price will decrease less compared to the situation with the lower value of σ_p. This means that as σ_p increases, less probability mass is shifted towards the region that gives a lower profit and a higher variance when p_i increases. Thus, for a marginal increase in p_i, the expected profit will increase more whereas the variance will increase less compared to the original situation with the lower σ_p. This essentially means that p_i becomes a more efficient instrument for increasing the expected profit and firm i has extra incentives to increase its price.

Figure 4.4 depicts the equilibrium price, production level and production to demand ratio as a function of σ_p, for different values of α and σ_q. The different lines on the plots correspond to different values of α whereas the vertical panels show the results for different values of σ_q. The figure shows that the equilibrium price increases, the production level decreases while the production to demand ratio remains constant essentially as σ_p becomes larger. Thus, firms charge a higher price to increase their expected profit while they reduce their production level to offset the increase in the profit variance. The lower production level is also explained by the negative effect of price on demand.

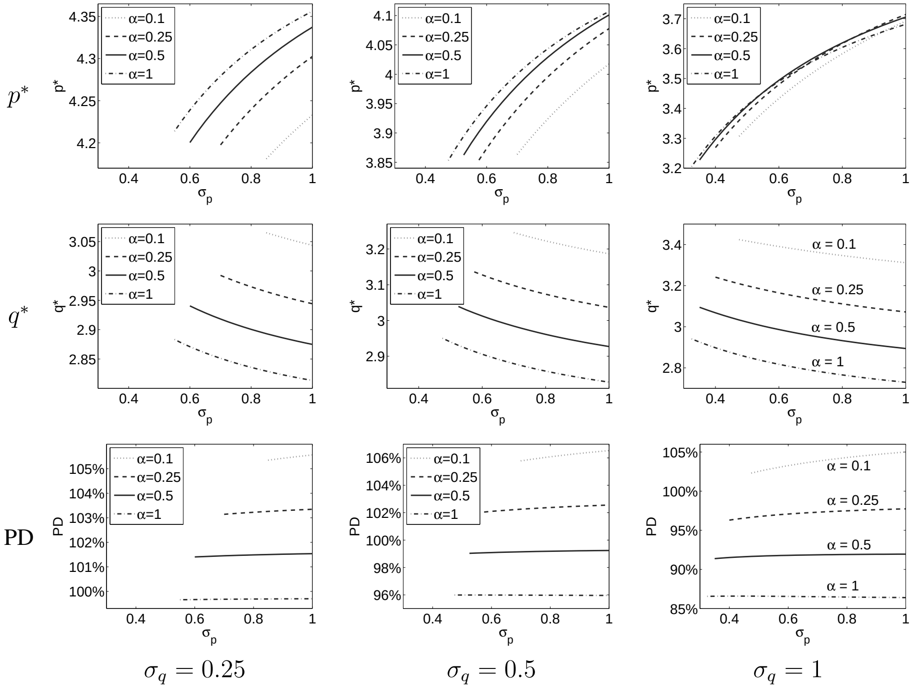

Figure 4.4: The equilibrium price p^* (upper panel), production level q^* (horizontal middle panel) and production to demand ratio $PD = \frac{2q^*}{a-bp^*}$ as a function of σ_p, for $\alpha = 0.1, 0.25, 0.5$ and 1. The left panel corresponds to $\sigma_q = 0.25$, the vertical middle panel to $\sigma_q = 0.5$ and the right panel to $\sigma_q = 1$. Other parameter values: $a = 10$, $b = 1$ and $c = 2$.

Note that prices and quantities move in the same direction as for an increase in α. The PD ratio, however, changes differently. This can be explained by the different objectives in the two cases. When firms become more risk averse, they have more incentives to reduce the variance, and they aim to serve a smaller share of the demand at a higher price as α increases. In contrast, firms focus more on increasing the expected profit in the current situation, and they want to serve a constant share of the demand at a higher price as σ_p increases.

Based on a price increase accompanied by reduced production levels, one might think that firms engage in some sort of an anti-competitive behavior. A remarkable feature of our model is that this need not be the case: higher price uncertainty leads to higher prices and lower production levels in our model.

4.6.4 The effect of output uncertainty

In contrast to price uncertainty, higher output uncertainty leads to a higher profit variance. This is due to the fact that the conjectured production level of firm j directly affects the profit of firm i through the residual demand. And as the uncertainty regarding the residual demand increases, the profit variance also increases. A change in σ_q affects the expected profit too but it is not clear by intuition whether it has a positive or a negative effect on it.[21] We numerically evaluated the marginal effect of σ_q on the expected profit and profit variance for all the parameter combinations we considered in this chapter. We found that the expected profit always decreases and that the marginal effect on the profit variance is larger in absolute value than the marginal effect on the expected profit. Since the marginal effect on the profit variance dominates, firms face too much risk compared to the equilibrium before the parameter change. Therefore, their main objective should be to reduce the variance (and possibly offset the corresponding negative effect on the expected profit).

Figure 4.5 shows the equilibrium price, production level and production to demand ratio as a function of σ_q, for different values of α and σ_p. The different lines correspond to different values of α while the vertical panels correspond to different values of σ_p. Output uncertainty has a clear negative effect on the equilibrium price. The effect on the production level is, however, ambiguous. For low values of σ_q the production level increases, whereas it decreases for high values of σ_q. The PD ratio typically decreases, except for the case σ_p is high and α is low.

The results show that firms charge a lower price for reducing the variance. This is intuitive since firms have an incentive to avoid ending up on the residual demand because the residual demand becomes more uncertain than for a lower value of σ_q. When σ_q is low, the profit variance is lower, and firms increase their production levels to offset the decrease in the expected profit, caused by the lower price. However, when σ_q is large, firms face a higher profit variance and therefore they decrease their production level to reduce the variance. Thus, as output uncertainty

[21] As σ_q increases, extreme realizations of the production level of firm j occur with higher probability. While extremely low realizations are favorable for firm i, extremely high realizations lead to low residual demands and low profits. It is ambiguous whether the total effect on the expected profit is positive or negative.

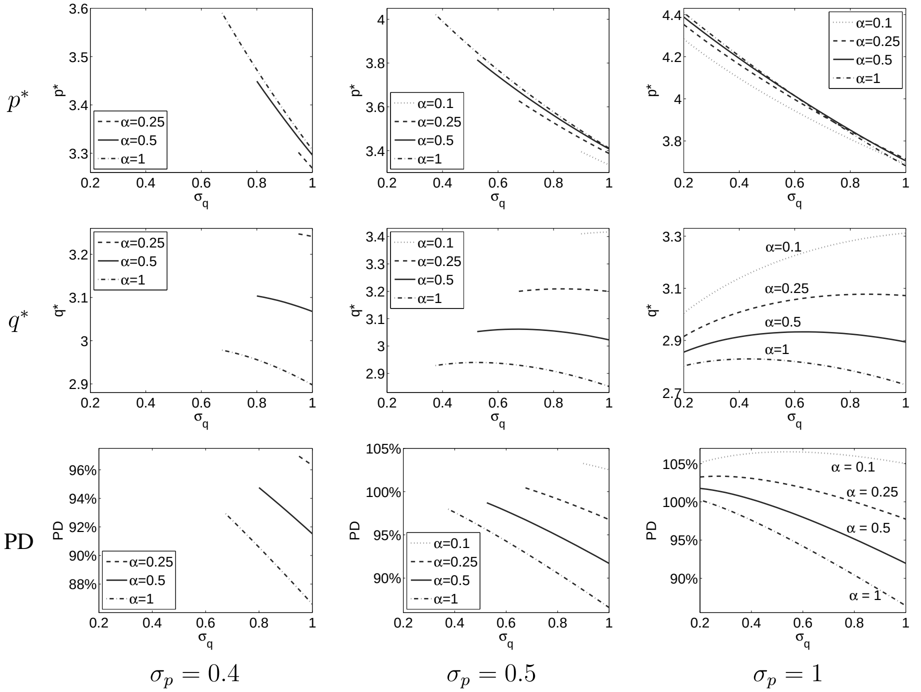

Figure 4.5: The equilibrium price p^* (upper panel), production level q^* (horizontal middle panel) and production to demand ratio $PD = \frac{2q^*}{a-bp^*}$ as a function of σ_q, for $\alpha = 0.1, 0.25, 0.5$ and 1. The left panel corresponds to $\sigma_p = 0.4$, the vertical middle panel to $\sigma_p = 0.5$ and the right panel to $\sigma_p = 1$. Other parameter values: $a = 10$, $b = 1$ and $c = 2$.

increases, firms use mainly their price to reduce the variance. The use of production level depends on the relation between expected profit and profit variance. When the variance is higher and thus relatively more important than the expected profit, then firms decrease their production levels to further reduce the variance. Otherwise they use their production level to increase their expected profit.

4.6.5 Welfare analysis

Next we investigate the welfare effect of changes in the degree of risk aversion and in the amount of uncertainty. It is not easy to find a proper welfare measure in this model as the profit variance in the objective function of firms is measured in a different unit than consumer surplus and (expected) profits. Therefore we consider two alternative measures. First, we apply a non-standard measure and define welfare as the sum of consumer surplus and the value of the objective function of firms. This measure takes into account that firms care about the variance of their profit but the unit of the measure is unclear. Second, we apply the standard definition of welfare, being the sum of consumer surplus and profits. This measures welfare in monetary terms but it disregards the exact form of the preferences of firms. It can be interpreted as the evaluation of the market outcome from a rational point of view as if firms had standard preferences.

We have seen in Figure 4.3 that p^* typically increases whereas q^* decreases as firms become more risk-averse. These changes have a clear negative effect on consumers. Figure 4.6 shows the total profit of the two firms, the total objective function value and the standard welfare measure as a function of α. The plots show that the profit of firms increases for small values of α but then it becomes decreasing. The reason for the initial increase in profits is that firms have unsold products for low degrees of risk aversion therefore their losses decrease as they produce less. This positive effect disappears when there is no more excess supply and profits start to decrease. The objective function value of firms shows a similar pattern. The decrease in the objective function value begins already for lower values of α since an increase in α gives a higher weight to profit variance, and this is not incorporated in the standard profit definition. The standard welfare measure also shows a similar pattern: welfare increases for low degrees for risk aversion whereas it decreases for high degrees of risk aversion. Thus, a small amount of risk aversion is welfare enhancing in our model. For comparison, welfare in the social optimum under the given demand and cost structure is $W^* = \frac{1}{2}\left(\frac{a}{b} - c\right)(a - bc) = 32$ while the expected welfare in the mixed-strategy Nash equilibrium in the standard model is $E(W) \approx 22.44$ (see the derivation in Appendix 4.C). Thus, the welfare loss in our model is at most 10% compared to the social optimum for the different parameter values we consider and our model results in a higher welfare than the standard model. The reason for this is that overproduction is much

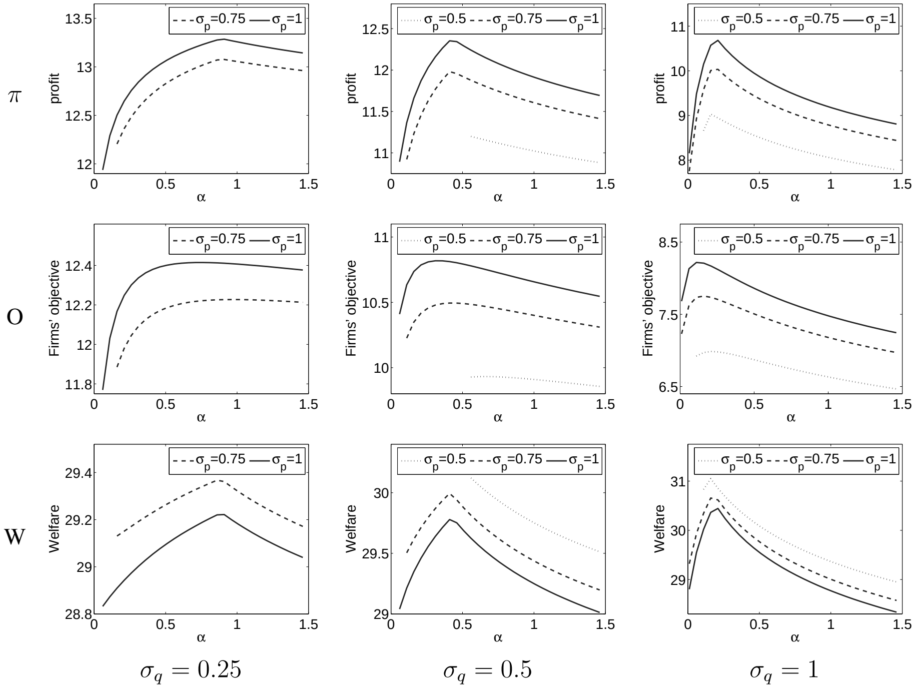

Figure 4.6: Profits (upper panel), firms' objective function value (horizontal middle panel) and welfare as a function of α, for $\sigma_p = 0.5, 0.75$ and 1. The left panel corresponds to $\sigma_q = 0.25$, the vertical middle panel to $\sigma_q = 0.5$ and the right panel to $\sigma_q = 1$. Other parameter values: $a = 10$, $b = 1$ and $c = 2$.

higher in the standard model than in our model.[22] The non-standard measure of risk aversion, which involves the objective function value of firms, always decreases.

We analyze the welfare effects of a change in the amount of uncertainty as well. We do not report the corresponding plots as the welfare effects are almost always monotonic. As price uncertainty increases, consumers are worse-off since firms charge a higher price and they produce less. Firms, on the other hand, are better-off both in terms of profits and in terms of their true objective function value. This is in line with our previous observation that an increase in price uncertainty is actually favorable for firms. Also, as Figure 4.4 shows, firms produce slightly less and they sell their good at a relatively higher price and this increases their

[22]Remember that firms produce up to the market demand at the price they drew in the mixed-strategy Nash equilibrium of the standard model, therefore the firm with the higher price does not sell any of its produced quantity.

(expected) profit. The overall welfare effect is, however, negative under both welfare measures.

As output uncertainty increases, consumer surplus increases: firms charge a lower price whereas their production level either slightly decreases or even increases. Firms, however, are always worse-off both in terms of profits and objective function value. The non-standard welfare measure is always decreasing in σ_q. The standard welfare measure is decreasing for low values of σ_p. For higher values of σ_p, it is first increasing but then it becomes decreasing again. Thus, even though output uncertainty is favorable for consumers, its overall welfare effect is ambiguous.

4.7 Discussion and concluding remarks

In this chapter we have deviated from the assumption of rationality by weakening the consistency requirement on the beliefs of firms. We have introduced strategic uncertainty and risk aversion in the standard model of price-quantity competition and we have numerically shown that there exists a symmetric equilibrium in pure strategies when uncertainty is sufficiently high or firms are sufficiently risk-averse. Thus, incorporating bounded rationality in models may not only alter equilibria in the sense that the equilibrium outcome is slightly different than in the standard model, but it may create an equilibrium even when the standard model does not have equilibria (in pure strategies). The importance of having a pure-strategy equilibrium is that there does not exist a Nash equilibrium in pure strategies in the standard model with risk neutral, profit-maximizing firms. Therefore, this modified version of price-quantity competition can be used more widely as a market structure for analyzing various market phenomena and for policy analysis.

Strategic uncertainty is introduced through the conjectures of firms: firms have a point forecast for the actions of the other firm but they take into account that these forecasts might not be accurate. This generates probabilistic conjectures. Risk aversion is incorporated in the model with firms having mean-variance preferences. First we have derived the first-order conditions of the optimization problem of firms, and then we have numerically found a symmetric solution. There exists a unique solution, however, it does not necessarily lead to the global maximum of the objective function of firms. Additional analysis shows that when firms are sufficiently risk averse or the amount of uncertainty is sufficiently high, then the solution to the first-order conditions is the global maximum, consequently, it gives a symmetric equilibrium. We have numerically characterized the parameter region for which the equilibrium exists. In equilibrium, each firm produces strictly less than the market demand at the equilibrium price. Aggregate production, however, may exceed the market demand, depending on the parameters. Our analysis shows that aggregate production exceeds the market demand for low degrees of risk aversion while firms do not serve together the whole market when they are too risk averse. For any amount of uncertainty, there exists a degree of risk aversion such that demand equals supply in equilibrium, provided that the equilibrium exists.

We have analyzed how the equilibrium depends on important parameters of the model such as the degree of risk aversion and the amount of price and output uncertainty. The results show that as firms become more risk averse, they produce less to decrease the profit variance and they sell their products at a higher price to offset the negative effect on the expected profit. The effect of price uncertainty is similar: the equilibrium price increases and the production level decreases as price uncertainty increases. The reason for this is that price uncertainty affects the marginal effect of price: price becomes a more efficient instrument for increasing the expected profit. Firms react differently to output uncertainty than to price uncertainty: the equilibrium price always decreases, while the production level typically decreases as output uncertainty increases. The reason behind this difference is that price uncertainty is favorable for firms (the expected profit can be increased more efficiently with the price) while output uncertainty is not: it directly increases the profit variance through the residual demand function so firms try to avoid operating on the residual demand by charging a lower price. The welfare analysis shows that a small degree of risk aversion is welfare enhancing and that our model results in a higher welfare than the expected welfare in the mixed-strategy Nash equilibrium of the standard model. For investigating the robustness of our results, we performed the previous analysis for different demand parameters as well. We observed qualitatively the same effects as before.

Some of our results are in line with experimental findings. Cracau and Franz (2012) conduct an experiment on simultaneous price-quantity setting with linear demand and constant and equal marginal costs. They found that subjects did not play according to the mixed-strategy Nash equilibrium: the average price was higher while the average production was lower than the equilibrium prediction. Moreover, subjects did not always choose the production level that corresponds to the market demand at the chosen price. This latter finding holds for our model as well since the PD ratio is always smaller than 200%. Another similarity is that Cracau and Franz (2012) found typically overproduction in the market: this occurs in our model for low degrees of risk aversion. An important difference, however, is that subjects typically did not converge to a fixed point whereas our model leads to a unique equilibrium. However, if we consider parameter values for which there is no pure-strategy equilibrium, a dynamic version of our model might be in line with the latter experimental result as prices and production levels cannot settle down to a single value.

Our analysis can be extended in several ways. The predictions of the model about the effects of a change in price or output uncertainty could be tested experimentally. The method outlined in this chapter can be used to analyze different market models[23] or asymmetric situations too. Firms could have different marginal costs or different degree of risk aversion, for example. They may also face different amount of uncertainty. The analysis of asymmetric situations is left for future work. Another important extension is to turn the model into a dynamic one. This can be done by specifying how forecasts for the price and production level of the other firm are formed. For example, firms could use adaptive updating or estimations using observed data as in Chapters 2 and 3.

[23]We briefly analyzed the model with the same kind of strategic uncertainty and risk aversion in Bertrand and Cournot competition as well. Preliminary analysis shows that the following relation holds for the equilibrium production levels across the different market models: $q^C < q^{PQ} < q^B$. The opposite relation holds for prices. Moreover, the equilibrium price in the Bertrand model exceeds the marginal cost of production.

Appendix 4.A The first-order conditions for the symmetric pure-strategy equilibrium

The first-order conditions of optimization problem (4.4)

The objective function

The expected profit of firm i can be expressed as

$$E(\pi_i^c) = p_i E(s_i) - cq_i. \tag{4.7}$$

The variance of the profit is $Var(\pi_i^c) = Var\left(p_i s_i - cq_i\right) = p_i^2 Var(s_i)$, which leads to

$$Var(\pi_i^c) = p_i^2\left[E(s_i^2) - E(s_i)^2\right]. \tag{4.8}$$

Then the objective function of firm i can be written as

$$p_i E(s_i)\left[1 + \alpha p_i E(s_i)\right] - cq_i - \alpha p_i^2 E(s_i^2). \tag{4.9}$$

Note that both $E(s_i)$ and $E(s_i^2)$ depend on p_i, q_i, p_j and q_j.

First-order conditions

Firm i maximizes (4.9) with respect to p_i and q_i. The first-order condition with respect to p_i reads as

$$E(s_i) + p_i\frac{\partial E(s_i)}{\partial p_i} + 2\alpha p_i E(s_i)^2 + 2\alpha p_i^2 E(s_i)\frac{\partial E(s_i)}{\partial p_i} - 2\alpha p_i E(s_i^2) - \alpha p_i^2\frac{\partial E(s_i^2)}{\partial p_i} = 0.$$

This expression simplifies to

$$(1 + 2\alpha p_i E(s_i))\left(E(s_i) + p_i\frac{\partial E(s_i)}{\partial p_i}\right) - \alpha p_i\left(2E(s_i^2) + p_i\frac{\partial E(s_i^2)}{\partial p_i}\right) = 0. \tag{4.10}$$

The first-order condition with respect to q_i is given by

$$p_i \frac{\partial E(s_i)}{\partial q_i} + 2\alpha p_i^2 E(s_i) \frac{\partial E(s_i)}{\partial q_i} - c - \alpha p_i^2 \frac{\partial E(s_i^2)}{\partial q_i} = 0,$$

which simplifies to

$$p_i \frac{\partial E(s_i)}{\partial q_i} \left(1 + 2\alpha p_i E(s_i)\right) - c - \alpha p_i^2 \frac{\partial E(s_i^2)}{\partial q_i} = 0. \tag{4.11}$$

For further characterizing the solution, we need to give the formula for $E(s_i)$, $E(s_i^2)$ and for the partial derivatives of these terms with respect to p_i and q_i. We derive these expressions in the next paragraphs. As beliefs are normally distributed, we can represent them in the following form: $p_j^b = p_j^f + \sigma_p \varepsilon_{j,p}$ and $q_j^b = q_j^f + \sigma_q \varepsilon_{j,q}$, where $\varepsilon_{j,p}$ and $\varepsilon_{j,q}$ are independent standard normal random variables. We use these representations throughout the appendices.

Expected sales $E(s_i)$

There are three possible cases concerning the value of s_i.[24]

- $s_i = q_i$: Firm i sells up to his production level q_i either if it has the lower price[25] or if it has the higher price and its residual demand exceeds its production level. The first condition is that $p_i < p_j^b$, or equivalently $\frac{p_i - p_j}{\sigma_p} < \varepsilon_{j,p}$. The second condition is that $p_i > p_j^b$ and $a - bp_i - q_j^b \geq q_i$, or equivalently $\frac{p_i - p_j}{\sigma_p} > \varepsilon_{j,p}$ and $\frac{a - bp_i - q_i - q_j}{\sigma_q} \geq \varepsilon_{j,q}$. For simplifying notation, let $A = \frac{1}{\sigma_q}\left(a - bp_i - q_i - q_j\right)$ such that the latter condition reads as $A \geq \varepsilon_{j,q}$.

- $s_i = a - bp_i - q_j^b$: Firm i sells up to his (positive) residual demand if it charges the higher price and its residual demand is positive. This leads to the conditions $p_i > p_j^b$ and $0 \leq a - bp_i - q_j^b < q_i$, or equivalently $\varepsilon_{j,p} < \frac{p_i - p_j}{\sigma_p}$ and $B \geq \varepsilon_{j,q} \geq A$, where $B = \frac{1}{\sigma_q}\left(a - bp_i - q_j\right)$.

[24] In the classification below we do not consider the case when both firms charge the same price as it has a measure 0 and thus does not affect the optimization problem of the firms.

[25] Here we implicitly assume that $q_i \leq D(p_i)$.

- $s_i = 0$: Firm i does not sell anything when it has the higher price and its residual demand at price p_i is negative. This gives $p_i > p_j^b$ and $a - bp_i < q_j^b$, or equivalently $\varepsilon_{j,p} < \frac{p_i - p_j}{\sigma_p}$ and $B < \varepsilon_{j,q}$.

Therefore, expected sales can be calculated as

$$
\begin{aligned}
E(s_i) = &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s_i \phi(x_q)\phi(x_p)\ dx_q\ dx_p = \int_{\frac{p_i-p_j}{\sigma_p}}^{\infty}\int_{-\infty}^{\infty} q_i \phi(x_q)\phi(x_p)\ dx_q\ dx_p \\
&+ \int_{-\infty}^{\frac{p_i-p_j}{\sigma_p}}\int_{-\infty}^{A} q_i \phi(x_q)\phi(x_p)\ dx_q\ dx_p + \int_{-\infty}^{\frac{p_i-p_j}{\sigma_p}}\int_{A}^{B} (a - bp_i - q_j - \sigma_q x_q)\, \phi(x_q)\phi(x_p)\ dx_q\ dx_p \\
&= q_i\left[1 - \Phi\left(\frac{p_i - p_j}{\sigma_p}\right)\right] + q_i \Phi\left(\frac{p_i - p_j}{\sigma_p}\right)\Phi(A) \\
&+ \Phi\left(\frac{p_i - p_j}{\sigma_p}\right)\left\{(a - bp_i - q_j)\left[\Phi(B) - \Phi(A)\right] + \sigma_q\left[\phi(B) - \phi(A)\right]\right\} \\
&= q_i - q_i \Phi\left(\frac{p_i - p_j}{\sigma_p}\right)\left[1 - \Phi(A)\right] \\
&+ \Phi\left(\frac{p_i - p_j}{\sigma_p}\right)\sigma_q\left\{B\left[\Phi(B) - \Phi(A)\right] + \phi(B) - \phi(A)\right\}. \qquad (4.12)
\end{aligned}
$$

For deriving the third term with the integral, we used the property that $\phi'(x) = -x\phi(x)$:

$$
\int_A^B x_q \phi(x_q)\ dx_q = \int_A^B \left(-\phi'(x_q)\right)\ dx_q = \left[-\phi(x_q)\right]_A^B = \phi(A) - \phi(B).
$$

Expected squared sales $E(s_i^2)$

For deriving $E(s_i^2)$ we can use the same steps as for deriving $E(s_i)$. We just need to replace s_i with s_i^2 in the integral. Thus,

$$
E(s_i^2) = q_i^2 - q_i^2 \Phi\left(\frac{p_i - p_j}{\sigma_p}\right)\left[1 - \Phi(A)\right] + \Phi\left(\frac{p_i - p_j}{\sigma_p}\right) M, \qquad (4.13)
$$

where

$$
\begin{aligned}
M &= \int_A^B \left(a - bp_i - q_j - \sigma_q x_q\right)^2 \phi(x_q)\, dx_q \\
&= \int_A^B \left(a - bp_i - q_j\right)^2 \phi(x_q)\, dx_q - \int_A^B 2(a - bp_i - q_j)\sigma_q x_q \phi(x_q)\, dx_q + \int_A^B \sigma_q^2 x_q^2 \phi(x_q)\, dx_q \\
&= (a - bp_i - q_j)^2 \left[\Phi(B) - \Phi(A)\right] + 2\sigma_q (a - bp_i - q_j)\left[\phi(B) - \phi(A)\right] \\
&\quad + \sigma_q^2 \left[A\phi(A) - B\phi(B) + \Phi(B) - \Phi(A)\right],
\end{aligned}
$$

which simplifies to

$$
M = \sigma_q^2 \left\{ (B^2 + 1)\left[\Phi(B) - \Phi(A)\right] + B\left[\phi(B) - \phi(A)\right] - \frac{q_i}{\sigma_q}\phi(A) \right\}. \tag{4.14}
$$

We used integration by parts for deriving the formula for M :

$$
\begin{aligned}
\int_A^B x_q^2 \phi(x_q)\, dx_q &= - \int_A^B x_q \left(-x_q \phi(x_q)\right)\, dx_q = - \int_A^B x_q \phi'(x_q)\, dx_q \\
&= - \left(\left[x_q \phi(x_q)\right]_A^B - \int_A^B \phi(x_q)\, dx_q \right) = A\phi(A) - B\phi(B) + \Phi(B) - \Phi(A).
\end{aligned}
$$

Marginal effect of own price on expected sales: $\frac{\partial E(s_i)}{\partial p_i}$

$$\frac{\partial E(s_i)}{\partial p_i} = -\frac{q_i}{\sigma_p}\phi\left(\frac{p_i - p_j}{\sigma_p}\right)[1-\Phi(A)] - q_i\Phi\left(\frac{p_i - p_j}{\sigma_p}\right)\phi(A)\frac{b}{\sigma_q}$$
$$+\frac{\sigma_q}{\sigma_p}\phi\left(\frac{p_i - p_j}{\sigma_p}\right)\{B\left[\Phi(B)-\Phi(A)\right]+\phi(B)-\phi(A)\}$$
$$+\sigma_q\Phi\left(\frac{p_i - p_j}{\sigma_p}\right)\left\{\left(-\frac{b}{\sigma_q}\right)[\Phi(B)-\Phi(A)] + B\left[-\phi(B)\frac{b}{\sigma_q}+\phi(A)\frac{b}{\sigma_q}\right]\right.$$
$$\left.+B\phi(B)\frac{b}{\sigma_q} - A\phi(A)\frac{b}{\sigma_q}\right\},$$

which simplifies to

$$\begin{aligned}\frac{\partial E(s_i)}{\partial p_i} =& -\frac{q_i}{\sigma_p}\phi\left(\frac{p_i - p_j}{\sigma_p}\right)[1-\Phi(A)] \\ &+\frac{\sigma_q}{\sigma_p}\phi\left(\frac{p_i - p_j}{\sigma_p}\right)\{B\left[\Phi(B)-\Phi(A)\right]+\phi(B)-\phi(A)\} \\ &- b\Phi\left(\frac{p_i - p_j}{\sigma_p}\right)[\Phi(B)-\Phi(A)].\end{aligned} \tag{4.15}$$

Marginal effect of own production on expected sales: $\frac{\partial E(s_i)}{\partial q_i}$

$$\frac{\partial E(s_i)}{\partial q_i} = 1-\Phi\left(\frac{p_i - p_j}{\sigma_p}\right)[1-\Phi(A)] - \frac{1}{\sigma_q}q_i\Phi\left(\frac{p_i - p_j}{\sigma_p}\right)\phi(A)$$
$$+\Phi\left(\frac{p_i - p_j}{\sigma_p}\right)\sigma_q\left\{B\phi(A)\frac{1}{\sigma_q} - A\phi(A)\frac{1}{\sigma_q}\right\},$$

which simplifies to

$$\frac{\partial E(s_i)}{\partial q_i} = 1-\Phi\left(\frac{p_i - p_j}{\sigma_p}\right)[1-\Phi(A)]. \tag{4.16}$$

Marginal effect of own price on expected squared sales: $\frac{\partial E(s_i^2)}{\partial p_i}$

$$
\begin{aligned}
\frac{\partial E(s_i^2)}{\partial p_i} = &-\frac{q_i^2}{\sigma_p}\phi\left(\frac{p_i-p_j}{\sigma_p}\right)[1-\Phi(A)] - q_i^2\Phi\left(\frac{p_i-p_j}{\sigma_p}\right)\phi(A)\frac{b}{\sigma_q} \\
&+\frac{\sigma_q^2}{\sigma_p}\phi\left(\frac{p_i-p_j}{\sigma_p}\right)\left\{(B^2+1)\left[\Phi(B)-\Phi(A)\right]+B\left[\phi(B)-\phi(A)\right]-\frac{q_i}{\sigma_q}\phi(A)\right\} \\
&+\sigma_q^2\Phi\left(\frac{p_i-p_j}{\sigma_p}\right)\left\{-2B\frac{b}{\sigma_q}\left[\Phi(B)-\Phi(A)\right]+(B^2+1)\left[-\phi(B)\frac{b}{\sigma_q}+\phi(A)\frac{b}{\sigma_q}\right]\right. \\
&\qquad -\frac{b}{\sigma_q}\left[\phi(B)-\phi(A)\right]+B\left[B\phi(B)\frac{b}{\sigma_q}-A\phi(A)\frac{b}{\sigma_q}\right] \\
&\qquad \left.-\frac{q_i}{\sigma_q}A\phi(A)\frac{b}{\sigma_q}\right\},
\end{aligned}
$$

which simplifies to

$$
\begin{aligned}
\frac{\partial E(s_i^2)}{\partial p_i} = &-\frac{q_i^2}{\sigma_p}\phi\left(\frac{p_i-p_j}{\sigma_p}\right)[1-\Phi(A)] \\
&+\frac{\sigma_q^2}{\sigma_p}\phi\left(\frac{p_i-p_j}{\sigma_p}\right)\left\{(B^2+1)\left[\Phi(B)-\Phi(A)\right]+B\left[\phi(B)-\phi(A)\right]-\frac{q_i}{\sigma_q}\phi(A)\right\} \\
&-2\sigma_q\Phi\left(\frac{p_i-p_j}{\sigma_p}\right)b\left\{B\left[\Phi(B)-\Phi(A)\right]+\phi(B)-\phi(A)\right\}. \qquad (4.17)
\end{aligned}
$$

Marginal effect of own production on expected squared sales: $\frac{\partial E(s_i^2)}{\partial q_i}$

$$
\begin{aligned}
\frac{\partial E(s_i^2)}{\partial q_i} = &\,2q_i - 2q_i\Phi\left(\frac{p_i-p_j}{\sigma_p}\right)[1-\Phi(A)] - q_i^2\Phi\left(\frac{p_i-p_j}{\sigma_p}\right)\phi(A)\frac{1}{\sigma_q} \\
&+\sigma_q^2\Phi\left(\frac{p_i-p_j}{\sigma_p}\right)\left\{(B^2+1)\phi(A)\frac{1}{\sigma_q}-\frac{1}{\sigma_q}AB\phi(A)-\frac{1}{\sigma_q}\phi(A)-\frac{q_i}{\sigma_q}A\phi(A)\frac{1}{\sigma_q}\right\},
\end{aligned}
$$

which simplifies to

$$
\frac{\partial E(s_i^2)}{\partial q_i} = 2q_i - 2q_i\Phi\left(\frac{p_i-p_j}{\sigma_p}\right)[1-\Phi(A)]\,. \qquad (4.18)
$$

Thus, the first-order conditions of optimization problem (4.4) are characterized by equations (4.10)-(4.18).

Symmetric pure-strategy equilibria

For deriving the conditions that characterize the symmetric pure-strategy equilibria, we need to substitute $p_i = p_j = p$ and $q_i = q_j = q$ in equations (4.10)-(4.18). This yields the following equations:

$$\left(1 + 2\alpha p\, E(s_i)\big|_{(p,q)}\right)\left(E(s_i)\big|_{(p,q)} + p\,\frac{\partial E(s_i)}{\partial p_i}\bigg|_{(p,q)}\right) \\ -\alpha p\left(2\, E(s_i^2)\big|_{(p,q)} + p\,\frac{\partial E(s_i^2)}{\partial p_i}\bigg|_{(p,q)}\right) = 0, \tag{4.19}$$

$$p\,\frac{\partial E(s_i)}{\partial q_i}\bigg|_{(p,q)}\left(1 + 2\alpha p\, E(s_i)\big|_{(p,q)}\right) - c - \alpha p^2\,\frac{\partial E(s_i^2)}{\partial q_i}\bigg|_{(p,q)} = 0, \tag{4.20}$$

$$E(s_i)\big|_{(p,q)} = 0.5q\left[1 + \Phi(A)\right] + 0.5\sigma_q\left\{B\left[\Phi(B) - \Phi(A)\right] + \phi(B) - \phi(A)\right\}, \tag{4.21}$$

$$E(s_i^2)\big|_{(p,q)} = 0.5q^2\left[1 + \Phi(A)\right] \\ + 0.5\sigma_q^2\left\{(B^2 + 1)\left[\Phi(B) - \Phi(A)\right] + B\left[\phi(B) - \phi(A)\right] - \frac{q}{\sigma_q}\phi(A)\right\}, \tag{4.22}$$

$$
\left.\frac{\partial E(s_i)}{\partial p_i}\right|_{(p,q)} = -\frac{1}{\sqrt{2\pi}}\frac{q}{\sigma_p}\left[1-\Phi(A)\right] + \frac{1}{\sqrt{2\pi}}\frac{\sigma_q}{\sigma_p}\left\{B\left[\Phi(B)-\Phi(A)\right]+\phi(B)-\phi(A)\right\} \tag{4.23}
$$

$$
-0.5b\left[\Phi(B)-\Phi(A)\right],
$$

$$
\left.\frac{\partial E(s_i)}{\partial q_i}\right|_{(p,q)} = 0.5\left[1+\Phi(A)\right], \tag{4.24}
$$

$$
\begin{aligned}
\left.\frac{\partial E(s_i^2)}{\partial p_i}\right|_{(p,q)} = & -\frac{1}{\sqrt{2\pi}}\frac{q^2}{\sigma_p}\left[1-\Phi(A)\right] \qquad (4.25)\\
& + \frac{1}{\sqrt{2\pi}}\frac{\sigma_q^2}{\sigma_p}\left\{(B^2+1)\left[\Phi(B)-\Phi(A)\right] + B\left[\phi(B)-\phi(A)\right] - \frac{q}{\sigma_q}\phi(A)\right\}\\
& - b\sigma_q\left\{B\left[\Phi(B)-\Phi(A)\right]+\phi(B)-\phi(A)\right\},
\end{aligned}
$$

$$
\left.\frac{\partial E(s_i^2)}{\partial q_i}\right|_{(p,q)} = q\left[1+\Phi(A)\right], \tag{4.26}
$$

where $A = \frac{1}{\sigma_q}(a-bp-2q)$ and $B = \frac{1}{\sigma_q}(a-bp-q)$.

Appendix 4.B The marginal effect of prices and production levels in equilibrium

Marginal effect of own production level on expected profit: $\frac{\partial E(\pi_i^c)}{\partial q_i}$

Using (4.7), the marginal effect of q_i on the expected profit is $p_i \frac{\partial E(s_i)}{\partial q_i} - c$. From (4.20) we know that

$$p \left. \frac{\partial E(s_i)}{\partial q_i} \right|_{(p,q)} - c = -2\alpha p^2 \left. E(s_i) \right|_{(p,q)} \left. \frac{\partial E(s_i)}{\partial q_i} \right|_{(p,q)} + \alpha p^2 \left. \frac{\partial E(s_i^2)}{\partial q_i} \right|_{(p,q)}$$

in equilibrium, so

$$\left. \frac{\partial E(\pi_i^c)}{\partial q_i} \right|_{(p,q)} = \alpha p^2 \left(\left. \frac{\partial E(s_i^2)}{\partial q_i} \right|_{(p,q)} - 2 \left. E(s_i) \right|_{(p,q)} \left. \frac{\partial E(s_i)}{\partial q_i} \right|_{(p,q)} \right).$$

Comparing (4.24) and (4.26), it can be seen that $\left. \frac{\partial E(s_i^2)}{\partial q_i} \right|_{(p,q)} = 2q \left. \frac{\partial E(s_i)}{\partial q_i} \right|_{(p,q)}$, so the marginal effect of q_i on $E(\pi_i^c)$ reduces to

$$\left. \frac{\partial E(\pi_i^c)}{\partial q_i} \right|_{(p,q)} = 2\alpha p^2 \left. \frac{\partial E(s_i)}{\partial q_i} \right|_{(p,q)} \left(q - \left. E(s_i) \right|_{(p,q)} \right).$$

It is easy to see from (4.24) that $\left. \frac{\partial E(s_i)}{\partial q_i} \right|_{(p,q)} > 0$. The term $q - \left. E(s_i) \right|_{(p,q)}$ is obviously positive since q_i is the maximal value of s_i, therefore $q_i > E(s_i)$. Consequently,

$$\left. \frac{\partial E(\pi_i^c)}{\partial q_i} \right|_{(p,q)} > 0.$$

Marginal effect of own production level on profit variance: $\frac{\partial Var(\pi_i^c)}{\partial q_i}$

It follows from the first-order conditions of optimization problem (4.4) that $\frac{\partial E(\pi_i^c)}{\partial q_i} = \alpha \frac{\partial Var(\pi_i^c)}{\partial q_i}$ in equilibrium. We have shown that $\frac{\partial E(\pi_i^c)}{\partial q_i} > 0$ in equilibrium, thus $\left. \frac{\partial Var(\pi_i^c)}{\partial q_i} \right|_{(p,q)} > 0$ must also hold.

Marginal effect of other production level on expected profit: $\frac{\partial E(\pi_i^c)}{\partial q_j}$

It follows from (4.12) that

$$
\begin{aligned}
\frac{\partial E(s_i)}{\partial q_j} &= q_i \Phi\left(\frac{p_i - p_j}{\sigma_p}\right)\phi(A)\left(-\frac{1}{\sigma_q}\right) \\
&\quad + \Phi\left(\frac{p_i - p_j}{\sigma_p}\right)\sigma_q \left\{\left(-\frac{1}{\sigma_q}\right)[\Phi(B) - \Phi(A)] + B\left[\phi(B)\left(-\frac{1}{\sigma_q}\right) - \phi(A)\left(-\frac{1}{\sigma_q}\right)\right]\right. \\
&\qquad \left. - B\phi(B)\left(-\frac{1}{\sigma_q}\right) + A\phi(A)\left(-\frac{1}{\sigma_q}\right)\right\},
\end{aligned}
$$

which simplifies to $\frac{\partial E(s_i)}{\partial q_j} = -\Phi\left(\frac{p_i - p_j}{\sigma_p}\right)[\Phi(B) - \Phi(A)]$. Since $A < B$ and $\Phi(x)$ is an increasing function, $\frac{\partial E(s_i)}{\partial q_j} < 0$. Thus, the marginal effect of q_j on $E(\pi_i^c)$ is also negative: $\frac{\partial E(\pi_i^c)}{\partial q_j} = p_i \frac{\partial E(s_i)}{\partial q_j} < 0$ since $p_i > 0$.

Marginal effect of other production level on profit variance: $\frac{\partial Var(\pi_i^c)}{\partial q_j}$

Using (4.13) and (4.14), the marginal effect of q_j on $E(s_i^2)$ is

$$
\begin{aligned}
\frac{\partial E(s_i^2)}{\partial q_j} &= q_i^2 \Phi\left(\frac{p_i - p_j}{\sigma_p}\right)\phi(A)\left(-\frac{1}{\sigma_q}\right) \\
&\quad + \Phi\left(\frac{p_i - p_j}{\sigma_p}\right)\sigma_q^2 \left\{2B\left(-\frac{1}{\sigma_q}\right)[\Phi(B) - \Phi(A)]\right. \\
&\qquad + (B^2 + 1)\left[\phi(B)\left(-\frac{1}{\sigma_q}\right) - \phi(A)\left(-\frac{1}{\sigma_q}\right)\right] \\
&\qquad + \left(-\frac{1}{\sigma_q}\right)[\phi(B) - \phi(A)] \\
&\qquad + B\left[-B\phi(B)\left(-\frac{1}{\sigma_q}\right) + A\phi(A)\left(-\frac{1}{\sigma_q}\right)\right] \\
&\qquad \left. + \frac{q_i}{\sigma_q} A\phi(A)\left(-\frac{1}{\sigma_q}\right)\right\},
\end{aligned}
$$

from which

$$\begin{aligned}\frac{\partial E(s_i^2)}{\partial q_j} = & -\frac{q_i^2}{\sigma_q}\Phi\left(\frac{p_i-p_j}{\sigma_p}\right)\phi(A) \\ & -\Phi\left(\frac{p_i-p_j}{\sigma_p}\right)\sigma_q\left\{2B\left[\Phi(B)-\Phi(A)\right]+(B^2+1)\left[\phi(B)-\phi(A)\right]\right\} \\ & -\Phi\left(\frac{p_i-p_j}{\sigma_p}\right)\sigma_q\left\{\phi(B)-\phi(A)-B^2\phi(B)+AB\phi(A)+\frac{q_i}{\sigma_q}A\phi(A)\right\},\end{aligned}$$

which simplifies to

$$\frac{\partial E(s_i^2)}{\partial q_j} = -2\Phi\left(\frac{p_i-p_j}{\sigma_p}\right)\sigma_q\left\{B\left[\Phi(B)-\Phi(A)\right]+\phi(B)-\phi(A)\right\}.$$

This expression is negative since $\Phi\left(\frac{p_i-p_j}{\sigma_p}\right)\sigma_q\left\{B\left[\Phi(B)-\Phi(A)\right]+\phi(B)-\phi(A)\right\}$ is the contribution to the expected sales of the case $0 < r_i < q_i$ (see formula (4.12)), which must be positive.

Using (4.8), the marginal effect of q_j on $Var(\pi_i^c)$ is given by

$$\frac{\partial Var(\pi_i^c)}{\partial q_j} = p_i^2\left[\frac{\partial E(s_i^2)}{\partial q_j} - 2E(s_i)\frac{\partial E(s_i)}{\partial q_j}\right].$$

The sign of this term is ambiguous since both derivatives are negative and $E(s_i) > 0$. In order to determine the sign of this expression, we need to know the exact value of p^* and q^*. Therefore we evaluated $\left.\frac{\partial Var(\pi_i^c)}{\partial q_j}\right|_{(p,q)}$ numerically for all parameter combinations that we considered in this chapter. All calculations show that $\left.\frac{\partial Var(\pi_i^c)}{\partial q_j}\right|_{(p,q)}$ is positive, that is an increase in q_j increases the profit variance of firm i in equilibrium.

Marginal effect of other price on expected profit: $\frac{\partial E(\pi_i^c)}{\partial p_j}$

Using (4.12), the expected sales of firm i can be expressed in the following form: $E(s_i) =$

$q_i + \Phi\left(\frac{p_i - p_j}{\sigma_p}\right) X_1$, where

$$X_1 = -q_i \left[1 - \Phi(A)\right] + \sigma_q \left\{B\left[\Phi(B) - \Phi(A)\right] + \phi(B) - \phi(A)\right\}. \tag{4.27}$$

The expected sales is smaller than q_i (since q_i is the maximal value of s_i), thus $X_1 < 0$ must hold. Furthermore, X_1 is independent of p_j. It is easy to see that $\Phi\left(\frac{p_i - p_j}{\sigma_p}\right)$ is decreasing in p_j, therefore $\frac{\partial E(s_i)}{\partial p_j} > 0$. This implies that the marginal effect of p_j on $E(\pi_i^c)$ is positive: $\frac{\partial E(\pi_i^c)}{\partial p_j} = p_i \frac{\partial E(s_i)}{\partial p_j} > 0$ since $p_i > 0$.

Marginal effect of other price on profit variance: $\frac{\partial Var(\pi_i^c)}{\partial p_j}$

Using (4.13) and (4.14), the expected squared sales of firm i can be expressed as $E(s_i^2) = q_i^2 + \Phi\left(\frac{p_i - p_j}{\sigma_p}\right) X_2$ with

$$X_2 = -q_i^2 \left[1 - \Phi(A)\right] + \sigma_q^2 \left\{(B^2 + 1)\left[\Phi(B) - \Phi(A)\right] + B\left[\Phi(B) - \Phi(A)\right] - \frac{q}{\sigma_q}\phi(A)\right\}. \tag{4.28}$$

Since q_i^2 is the maximal value of s_i^2, $E(s_i^2)$ must be smaller than q_i^2 and consequently $X_2 < 0$. Therefore, $\frac{\partial E(s_i^2)}{\partial p_j} > 0$ since X_2 is independent of p_j and $\Phi\left(\frac{p_i - p_j}{\sigma_p}\right)$ is decreasing in p_j.

From (4.8) the marginal effect of p_j on $Var(\pi_i^c)$ is given by

$$\frac{\partial Var(\pi_i^c)}{\partial p_j} = p_i^2 \left[\frac{\partial E(s_i^2)}{\partial p_j} - 2E(s_i)\frac{\partial E(s_i)}{\partial p_j}\right].$$

The sign of this term is ambiguous since both derivatives are positive and $E(s_i) > 0$. In order to determine the sign of this expression, we need to know the exact value of p^* and q^*. Therefore we evaluated $\left.\frac{\partial Var(\pi_i^c)}{\partial p_j}\right|_{(p,q)}$ numerically for all parameter combinations that we considered in this chapter. All calculations show that $\left.\frac{\partial Var(\pi_i^c)}{\partial p_j}\right|_{(p,q)}$ is negative, that is an increase in p_j decreases the profit variance of firm i in equilibrium.

Marginal effect of own price on expected profit: $\frac{\partial E(\pi_i^c)}{\partial p_i}$

Combining (4.23) with (4.27), it can be seen that $\left.\frac{\partial E(s_i)}{\partial p_i}\right|_{(p,q)} = \frac{1}{\sqrt{2\pi}\sigma_p} X_1 - 0.5b\left[\Phi(B) - \Phi(A)\right].$ This expression is negative since $X_1 < 0$ and $\Phi(B) > \Phi(A)$.

Using (4.7), the marginal effect of p_i on $E(\pi_i^c)$ in equilibrium is given by $\left.\frac{\partial E(\pi_i^c)}{\partial p_i}\right|_{(p,q)} = \left.E(s_i)\right|_{(p,q)} + p\left.\frac{\partial E(s_i)}{\partial p_i}\right|_{(p,q)}$. The sign of this expression is ambiguous since the first term is positive while the second one is negative.

We evaluated $\left.\frac{\partial E(\pi_i^c)}{\partial p_i}\right|_{(p,q)}$ numerically for all parameter combinations that we considered in this chapter. All calculations show that $\left.\frac{\partial E(\pi_i^c)}{\partial p_i}\right|_{(p,q)}$ is positive, that is an increase in p_i increases the expected profit of firm i in equilibrium.

Marginal effect of own price on profit variance: $\frac{\partial Var(\pi_i^c)}{\partial p_i}$

We can show that $\left.\frac{\partial E(s_i^2)}{\partial p_i}\right|_{(p,q)}$ is negative. The sum of the first two terms in (4.25) equals $\frac{1}{\sqrt{2\pi}\sigma_p} X_2$ and this is negative since $X_2 < 0$. The last term is also negative since

$$\left\{B\left[\Phi(B) - \Phi(A)\right] + \phi(B) - \phi(A)\right\} > 0,$$

as this is the contribution to the expected sales of the case $0 < r_i < q_i$ (see formula (4.12)), which must be positive.

The marginal effect of p_i on $Var(\pi_i^c)$ is given by

$$\frac{\partial Var(\pi_i^c)}{\partial p_i} = 2p_i\left[E(s_i^2) - (E(s_i))^2\right] + p_i^2\left[\frac{\partial E(s_i^2)}{\partial p_i} - 2E(s_i)\frac{\partial E(s_i)}{\partial p_i}\right].$$

Note that the first term is positive since the term in the brackets is the variance of s_i. The sign of the second term is, however, ambiguous: $\frac{\partial E(s_i^2)}{\partial p_i} < 0$ while $-2E(s_i)\frac{\partial E(s_i)}{\partial p_i} > 0$ since $\left.\frac{\partial E(s_i)}{\partial p_i}\right|_{(p,q)} < 0$.

We evaluated $\left.\frac{\partial Var(\pi_i^c)}{\partial p_i}\right|_{(p,q)}$ numerically for all parameter combinations that we considered in this chapter. All calculations show that $\left.\frac{\partial Var(\pi_i^c)}{\partial p_i}\right|_{(p,q)}$ is positive, that is an increase in p_i increases the profit variance of firm i in equilibrium.

Appendix 4.C Expected welfare in the standard model

As Gertner (1986) shows, firms draw their prices from the distribution function

$$F(x)=\begin{cases} 0 & \text{if } x<c \\ 1-\frac{c}{x} & \text{if } x\in\left[c,\frac{a}{b}\right] \\ 1 & \text{if } x>\frac{a}{b} \end{cases},$$

and they produce up to the market demand at the price they drew.[26] Thus, the firm with the lower price serves the whole market and the other firm cannot sell any of the goods it produced.

Let $\underline{p}=\min\{p_1,p_2\}$ and $\overline{p}=\max\{p_1,p_2\}$. Then consumer surplus is given by $CS=\frac{1}{2b}\left(a-b\underline{p}\right)^2$. The profit of the firm with the lower price is $\underline{\pi}=(\underline{p}-c)(a-b\underline{p})$. The firm with the higher price does not sell anything, therefore its profit is $\overline{\pi}=-c(a-b\overline{p})$. Adding up these expressions and simplifying the result gives $W=-\frac{b}{2}\underline{p}^2+bc\underline{p}+a\left(\frac{1}{2}\frac{a}{b}-c\right)-c(a-b\overline{p})$. The expected welfare is therefore

$$E(W)=-\frac{b}{2}E\left(\underline{p}^2\right)+bcE\left(\underline{p}\right)+a\left(\frac{1}{2}\frac{a}{b}-c\right)-c\left[a-bE\left(\overline{p}\right)\right]. \tag{4.29}$$

To evaluate the expected welfare, we need to derive the formula for $E\left(\underline{p}\right)$, $E\left(\underline{p}^2\right)$ and $E\left(\overline{p}\right)$. First we derive the distribution function of $\underline{p}$.[27] $P\left(\underline{p}>x\right)=P(p_1>x)P(p_2>x)=[1-F(x)]^2$, therefore the distribution function is $G(x)=P\left(\underline{p}\leq x\right)=1-[1-F(x)]^2$, with the corresponding probability density function $g(x)=2(1-F(x))f(x)$. Using that $f(x)=\frac{c}{x^2}$, $g(x)$ simplifies to $g(x)=2\frac{c^2}{x^3}$. Then $E\left(\underline{p}\right)$ can be calculated as follows.

$$E\left(\underline{p}\right)=\int_c^{a/b}2\frac{c^2}{x^2}\,dx+P\left(\underline{p}=\frac{a}{b}\right)\frac{a}{b}=2c^2\left[-\frac{1}{x}\right]_c^{\frac{a}{b}}+\left(\frac{bc}{a}\right)^2\frac{a}{b}=c\left(2-\frac{bc}{a}\right). \tag{4.30}$$

[26] Note that the distribution function is discontinuous at $x=\frac{a}{b}$.

[27] To simplify notation, we focus on the non-constant part of the distribution function, i.e. when $x\in\left[c,\frac{a}{b}\right]$.

Similarly, $E\left(\underline{p}^2\right)$ is given by

$$E\left(\underline{p}^2\right) = \int\limits_c^{a/b} 2\frac{c^2}{x}\,dx + P\left(\underline{p} = \frac{a}{b}\right)\left(\frac{a}{b}\right)^2 = 2c^2\left[\ln x\right]_c^{\frac{a}{b}} + \left(\frac{bc}{a}\right)^2\left(\frac{a}{b}\right)^2 = c^2\left(1 + 2\ln\frac{bc}{a}\right). \tag{4.31}$$

Next we derive the distribution function of $\overline{p}$. $P\left(\overline{p} < x\right) = P(p_1 < x)P(p_2 < x) = F(x)^2$, therefore the distribution function is $H(x) = F(x)^2$, with the corresponding probability density function $h(x) = 2F(x)f(x) = 2\left(1 - \frac{c}{x}\right)\frac{c}{x^2}$. Then

$$\begin{aligned} E\left(\overline{p}\right) &= \int\limits_c^{a/b} 2\left(1 - \frac{c}{x}\right)\frac{c}{x}\,dx + P\left(\overline{p} = \frac{a}{b}\right)\frac{a}{b} = 2c\left[\ln x + \frac{c}{x}\right]_c^{\frac{a}{b}} + \left(2 - \frac{bc}{a}\right)\frac{bc}{a}\frac{a}{b} \\ &= 2c\left(\ln\frac{a}{bc} + \frac{bc}{a} - 1\right) + \left(2 - \frac{bc}{a}\right)c = c\left(2\ln\frac{a}{bc} + \frac{bc}{a}\right). \end{aligned} \tag{4.32}$$

For deriving this result we used that $P\left(\overline{p} = \frac{a}{b}\right) = \left(2 - \frac{bc}{a}\right)\frac{bc}{a}$, which follows from $P\left(\overline{p} < \frac{a}{b}\right) = F^2\left(\frac{a}{b}\right) = \left(1 - \frac{bc}{a}\right)^2$.

Combining equations (4.29)-(4.32), we get a formula for the expected welfare as the function of a, b and c. For the parameter values we consider in this chapter, the expected welfare is approximately 22.4378 (with $E\left(\underline{p}\right) = 3.6$ and $E\left(\overline{p}\right) \approx 6.8378$).