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Bohr chaoticity of principal algebraic actions and Riesz product measures

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Abstract. For a continuous $\mathbb{N}^d$ or $\mathbb{Z}^d$ action on a compact space, we introduce the notion of Bohr chaoticity, which is an invariant of topological conjugacy and which is proved stronger than having positive entropy. We prove that all principal algebraic $\mathbb{Z}$ actions of positive entropy are Bohr chaotic. The same is proved for principal algebraic actions of $\mathbb{Z}^d$ with positive entropy under the condition of existence of summable homoclinic points.

Key words: Bohr chaoticity, entropy, principal actions, algebraic dynamical systems, Riesz products
2020 Mathematics Subject Classification: 37A30, 37A46 (Primary); 37A20, 43A05 (Secondary)

1. *Introduction*

Bohr chaoticity is a topological invariant introduced in [9] for topological dynamical systems. For defining this invariant, we recall that a sequence $\boldsymbol{w} = (w_n)_{n\geq 0} \in \ell^\infty(\mathbb{N}, \mathbb{C})$ is a *non-trivial weight sequence* if

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |w_n| > 0. \tag{1.1}$$

A non-trivial weight sequence $\boldsymbol{w}$ is *orthogonal* to a topological dynamical system (X, T) if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} w_n f(T^n x) = 0 \tag{1.2}$$

for every continuous function $f \in C(X)$ and every $x \in X$.

Definition 1.1. [9] A topological dynamical system (X, T) is *Bohr chaotic* if it is non-orthogonal to every non-trivial weight sequence $\boldsymbol{w} \in \ell^\infty(\mathbb{N}, \mathbb{C})$. In other words, (X, T) is Bohr chaotic if we can find, for every non-trivial weight sequence $\boldsymbol{w} \in \ell^\infty(\mathbb{N}, \mathbb{C})$, a continuous function $g \in C(X)$ and a point $x \in X$ such that

$$\limsup_{N\to\infty} \frac{1}{N} \left| \sum_{n=0}^{N-1} w_n g(T^n x) \right| > 0. \tag{1.3}$$

Bohr chaotic systems must have positive entropy: for example, almost all $(\frac{1}{2}, \frac{1}{2})$ Bernoulli sequences taking values -1 and 1 are orthogonal to every topological dynamical system (X, T) with zero entropy (see [5]). We list some further basic results on Bohr chaoticity, taken from [9]:

- any extension of a Bohr chaotic topological dynamical system is Bohr chaotic;
- if a topological dynamical system (X, T) has a non-empty, closed, T-invariant subset $Y \subset X$ such that $(Y, T|_Y)$ is Bohr chaotic, then (X, T) is Bohr chaotic;
- no uniquely ergodic dynamical system is Bohr chaotic (this is generalized by Tal [31] to systems having at most countably many ergodic measures);
- all affine toral endomorphisms of positive entropy are Bohr chaotic;
- all systems having an m-order horseshoe, $m \geq 1$, are Bohr chaotic. By an m-order horseshoe K of a system (X, T), we mean a T^m-invariant closed non-empty set $K \subset X$ such that the subsystem (K, T^m) is conjugate either to the one-sided shift $(\{0, 1\}^{\mathbb{N}}, \sigma)$ or to the two-sided shift $(\{0, 1\}^{\mathbb{Z}}, \sigma)$;
- all subshifts of finite type with positive entropy are Bohr chaotic;
- all piecewise monotone C^1 interval maps of positive entropy are Bohr chaotic. For example, the β-shifts with $\beta > 1$;
- every $C^{1+\delta}$ ($\delta > 0$) diffeomorphism of a compact smooth manifold admitting an ergodic non-atomic Borel probability invariant measure with non-zero Lyapunov exponents is Bohr chaotic.

The reason for the last two classes is that any such system admits a subsystem which is conjugate to a subshift of finite type of positive entropy [15, 32].

It is interesting to note that for the examples of Bohr chaotic systems constructed in [9], the sets of points $x \in X$ satisfying equation (1.3) are large in the sense that they are of full Hausdorff dimension. Actually, weighted ergodic averages on typical dynamical systems would be multifractal and a study on symbolic spaces is carried out in [7].

In the present paper, we extend the notion of Bohr chaoticity from $\mathbb{Z}$- to $\mathbb{Z}^d$-actions and prove that a large class of algebraic dynamical systems—the so-called principal algebraic actions—are Bohr chaotic, provided they have positive entropy.

By analogy with equation (1.1), we say that a complex sequence $\boldsymbol{w} = (w_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{N}^d} \in \ell^\infty(\mathbb{N}^d, \mathbb{C})$ is a *non-trivial weight* if

$$\limsup_{N\to\infty} \frac{1}{N^d} \sum_{\boldsymbol{n}\in[0,N-1]^d} |w_{\boldsymbol{n}}| > 0.$$

Consider a continuous $\mathbb{N}^d$- or $\mathbb{Z}^d$-action $\alpha\colon \boldsymbol{n} \mapsto \alpha^{\boldsymbol{n}}$ on some compact space X. As in equation (1.2), say that a (non-trivial) weight $(w_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{N}^d}$ is *orthogonal* to the dynamical system (X, α) if

$$\lim_{N\to\infty} \frac{1}{N^d} \sum_{\boldsymbol{n}\in[0,N-1]^d} w_{\boldsymbol{n}} g(\alpha^{\boldsymbol{n}} x) = 0 \tag{1.4}$$

for every continuous function $g \in C(X)$ and every point $x \in X$.

Definition 1.2. If α is a continuous $\mathbb{N}^d$- or $\mathbb{Z}^d$-action on a compact space X, we call (X, α) *Bohr chaotic* if it is not orthogonal to any non-trivial weight, that is to say, if for any non-trivial weight $\boldsymbol{w} = (w_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{N}^d}$, there exist $g \in C(X)$ and $x \in X$ such that

$$\limsup_{N\to\infty} \frac{1}{N^d} \left| \sum_{\boldsymbol{n}\in[0,N-1]^d} w_{\boldsymbol{n}} g(\alpha^{\boldsymbol{n}} x) \right| > 0. \tag{1.5}$$

Note that if α is a continuous $\mathbb{N}^d$-action on X and if $(\bar{X}, \bar{\alpha})$ is the natural extension of (X, α) to a continuous $\mathbb{Z}^d$-action $\bar{\alpha}$ on a compact space $\bar{X}$, then $(\bar{X}, \bar{\alpha})$ is Bohr chaotic if and only if the same is true for (X, α). Conversely, if a continuous $\mathbb{Z}^d$-action is Bohr chaotic, it is obviously also Bohr chaotic as an $\mathbb{N}^d$-action. In view of this last property, we focus our attention in much of this paper on Bohr chaoticity of $\mathbb{Z}^d$-actions, referring to $\mathbb{N}^d$-actions only where necessary (like in Proposition 3.1 or Example 3.3).

As in the one-dimensional case, one can easily verify the following properties of continuous $\mathbb{Z}^d$-actions (X, α):

(i) if X has a closed, α-invariant subset Y such that $(Y, \alpha|_Y)$ is Bohr chaotic, then (X, α) is Bohr chaotic;

(ii) if (X, α) has a Bohr chaotic factor (Y, β) (that is, if (Y, β) is a Bohr chaotic $\mathbb{Z}^d$-action and there exists a continuous, surjective, equivariant map $\phi\colon X \to Y$), then (X, α) is Bohr chaotic.

In particular, Bohr chaoticity is an invariant of topological conjugacy.

Our main results will be proved by using Riesz product measures borrowed from harmonic analysis and the main technical tool is the notion of m-goodness.

The paper is organized as follows. In §2, we present algebraic $\mathbb{Z}^d$-actions and their basic properties, state our main results on Bohr chaoticity of principal algebraic $\mathbb{Z}^d$-actions (Theorems 2.1 and 2.3), and prove that Bohr chaotic algebraic $\mathbb{Z}^d$-actions have to have completely positive entropy (Example 3.2). In §3, we show that zero-entropy $\mathbb{Z}^d$-actions are not Bohr chaotic. Our main tool, Riesz products, is presented in §4 where lacunarity of polynomials is discussed. In §5, we prove that any principal algebraic $\mathbb{Z}^d$-action defined by a so-called *m-good* polynomial is Bohr chaotic (Theorem 5.1). Section 6 is devoted to the proof of m-goodness for all irreducible polynomials $f \in R_1$ with positive Mahler measure, and Theorem 2.1 (for $d = 1$) is proved there. Theorem 2.3 (for $d \geq 2$) is proved in §7, where we prove a gap theorem (Theorem 7.3) for irreducible polynomials which admit summable homoclinic points. This gap theorem is of independent interest. In §8, we speculate briefly on the necessity of our atorality assumptions for our main results and give some examples of Bohr chaotic principal actions arising from *toral* polynomials.

2. *Algebraic $\mathbb{Z}^d$-actions*

In this section, we present the principal algebraic $\mathbb{Z}^d$-actions which are our main objects of study and then state our main results (Theorems 2.1 and 2.3).

An *algebraic $\mathbb{Z}^d$-action* is an action of $\mathbb{Z}^d$ by (continuous) automorphisms of a compact metrizable abelian group. Algebraic $\mathbb{Z}^d$-actions provide a useful source of examples of continuous $\mathbb{Z}^d$-actions with a wide range of properties, both with zero and with positive entropy, and with or without Bohr chaoticity.

We are interested in a particular family of algebraic $\mathbb{Z}^d$-actions, the so-called *cyclic* actions. Denote by σ the shift-action of $\mathbb{Z}^d$ on $\mathbb{T}^{\mathbb{Z}^d}$ given by

$$\sigma^{\boldsymbol{m}}(x)_{\boldsymbol{n}} = x_{\boldsymbol{n}+\boldsymbol{m}} \tag{2.1}$$

for every $x = (x_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{Z}^d} \in \mathbb{T}^{\mathbb{Z}^d}$. A *cyclic algebraic $\mathbb{Z}^d$-action* is a pair (X, α_X), where $X \subset \mathbb{T}^{\mathbb{Z}^d}$ is a closed, shift-invariant subgroup and $\alpha_X = \sigma|_X$ is the restriction to X of the shift-action σ in equation (2.1).

To describe these actions in more detail, we denote by $R_d = \mathbb{Z}[z_1^{\pm 1}, \ldots, z_d^{\pm 1}]$ the ring of Laurent polynomials in the variables $z_1, \ldots, z_d$ with coefficients in $\mathbb{Z}$. Every $f \in R_d$ will be written as $f = \sum_{\boldsymbol{n}\in\mathbb{Z}^d} f_{\boldsymbol{n}} \boldsymbol{z}^{\boldsymbol{n}}$ with $f_{\boldsymbol{n}} \in \mathbb{Z}$ and $\boldsymbol{z}^{\boldsymbol{n}} = z_1^{n_1} \cdots z_d^{n_d}$ for every $\boldsymbol{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d$. The set $\mathsf{supp}(f) = \{\boldsymbol{n} \in \mathbb{Z}^d \mid f_{\boldsymbol{n}} \neq 0\}$ will be called the *support* of f, and we set $\|f\|_1 = \sum_{\boldsymbol{n}\in\mathbb{Z}^d} |f_{\boldsymbol{n}}|$ and $\|f\|_\infty = \max_{\boldsymbol{n}\in\mathbb{Z}^d} |f_{\boldsymbol{n}}|$. Following standard terminology, we call a non-zero element $f \in R_d$ *primitive* if the greatest common divisor $\gcd(\{f_{\boldsymbol{n}} \mid \boldsymbol{n} \in \mathbb{Z}^d\})$ of its coefficients is equal to 1, and *irreducible* if it is not a product of two non-units in R_d. For example, $2\boldsymbol{z}^{\boldsymbol{n}}$ is irreducible, but not primitive, while $2(1 + \boldsymbol{z}^{\boldsymbol{n}})$ is neither primitive nor irreducible, for every $\boldsymbol{n} \in \mathbb{Z}^d$.

Every non-zero $f = \sum_{\boldsymbol{n}\in\mathbb{Z}^d} f_{\boldsymbol{n}} \boldsymbol{z}^{\boldsymbol{n}} \in R_d$ defines a surjective group homomorphism $f(\sigma) = \sum_{\boldsymbol{m}\in\mathbb{Z}^d} f_{\boldsymbol{m}} \sigma^{\boldsymbol{m}} \colon \mathbb{T}^{\mathbb{Z}^d} \to \mathbb{T}^{\mathbb{Z}^d}$. Consider the closed, shift-invariant subgroup

$$X_f = \left\{ x \in \mathbb{T}^{\mathbb{Z}^d} \,\middle|\, \sum_{\boldsymbol{m}\in\mathbb{Z}^d} x_{\boldsymbol{n}+\boldsymbol{m}} f_{\boldsymbol{m}} = 0 \pmod 1 \text{ for all } \boldsymbol{n} \in \mathbb{Z}^d \right\} = \ker(f(\sigma)) \subset \mathbb{T}^{\mathbb{Z}^d}, \tag{2.2}$$

and denote by

$$\alpha_f = \sigma|_{X_f} \tag{2.3}$$

the restriction to X_f of the shift-action σ on $\mathbb{T}^{\mathbb{Z}^d}$. The dynamical system (X_f, α_f) is called the *principal algebraic action* corresponding to $f \in R_d$.

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Formally we can extend this definition of a principal action (X_f, α_f) to include the case $f = 0$, the zero polynomial in R_d. In this case, the definitions in equations (2.2) and (2.3) reduce to $X_f = \mathbb{T}^{\mathbb{Z}^d}$ and $\alpha_f = \sigma$, that is, (X_f, α_f) is simply the shift action of $\mathbb{Z}^d$ on $\mathbb{T}^{\mathbb{Z}^d}$.

For every cyclic algebraic action (X, α_X) with $X \subsetneq \mathbb{T}^{\mathbb{Z}^d}$, the set

$$I_X = \{f \in R_d \mid X \subset X_f\} \tag{2.4}$$

is an ideal in R_d (which is, of course, finitely generated since the ring R_d is Noetherian) and $X = \bigcap_{f\in I_X} X_f$. Conversely, if $I \subset R_d$ is an ideal, generated by $\{f^{(1)}, \ldots, f^{(r)}\}$, say, we denote by (X_I, α_I) the cyclic $\mathbb{Z}^d$-action defined by

$$X_I = \bigcap_{f\in I} X_f = \bigcap_{i=1}^{r} X_{f^{(i)}} \subset \mathbb{T}^{\mathbb{Z}^d} \quad \text{and} \quad \alpha_I = \sigma|_{X_I}, \tag{2.5}$$

and write λ_I for the normalized Haar measure of X_I. If the ideal $I \subset R_d$ is principal, $I = (f)$, say, we write (X_f, α_f) instead of $(X_{(f)}, \alpha_{(f)})$ and denote by λ_f the normalized Haar measure on X_f.

2.1. *Mahler measure.* The topological entropy $h_{\mathrm{top}}(\alpha_f)$ of a principal algebraic action (X_f, α_f), $f \in R_d \setminus \{0\}$, coincides with its measure-theoretic entropy $h_{\lambda_f}(\alpha_f)$ and is given by the (logarithmic) *Mahler measure* of its defining polynomial f:

$$h_{\mathrm{top}}(\alpha_f) = \mathsf{m}(f) := \int_0^1 \cdots \int_0^1 \log |f(e^{2\pi i t_1}, \ldots, e^{2\pi i t_d})| \, dt_1 \cdots dt_d. \tag{2.6}$$

For polynomials in a single variable (that is, for $f \in R_1$), Mahler measure can be computed using Jensen's formula: let $f(z) = f_0 + f_1 z + \cdots + f_k z^k$ with $f_0 f_k \neq 0$ and (complex) roots $\lambda_1, \ldots, \lambda_k$. Then

$$\mathsf{m}(f) = \log |f_k| + \sum_{j:\, |\lambda_j|>1} \log |\lambda_j| \tag{2.7}$$

(cf. [21, p. 597], [29, equation (16.2)], or [33]). The Kronecker lemma [17] states that if a polynomial $f \in R_1$ is irreducible, monic, and all its roots have absolute value at most 1, then f is *cyclotomic*, that is, for some integer n,

$$f(z) = \Phi_n(z) := \prod_{\substack{1\le \ell \le n\\ \gcd(\ell,n)=1}} (z - e^{2\pi i \ell/n}). \tag{2.8}$$

Since Mahler measure is additive in the sense that $\mathsf{m}(f \cdot g) = \mathsf{m}(f) + \mathsf{m}(g)$ for all $f, g \in R_d \setminus \{0\}$, any $f \in R_1 \setminus \{0\}$ with $\mathsf{m}(f) = 0$ must be a product of cyclotomic polynomials.

A similar statement is true for multivariate polynomials as well: if $f \in R_d \setminus \{0\}$, then $\mathsf{m}(f) = 0$ if and only if f is a product of so-called *generalized cyclotomic polynomials*

$$f(z) = \pm z^{\boldsymbol{n}_0} \Phi_{m_1}(z^{\boldsymbol{n}_1}) \cdots \Phi_{m_r}(z^{\boldsymbol{n}_r})$$

for some integers $m_1, \ldots, m_r$ and $\boldsymbol{n}_0, \boldsymbol{n}_1, \ldots, \boldsymbol{n}_r \in \mathbb{Z}^d$ ([2], [29, Theorem 19.5], [30]).

We recall the following properties of cyclic algebraic $\mathbb{Z}^d$-action (X_I, α_I) (cf. [29, Ch. 6]).

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- The normalized Haar measure λ_{X_I} of X_I is shift-invariant.
- If $I \subset R_d$ is non-zero and principal, $I = (f)$, say, the topological entropy of (X_f, α_f) is given by the Mahler measure in equation (2.6) of f; if $I \subset R_d$ contains at least two non-zero elements f, g which are relatively prime to each other (that is, without a non-trivial common factor), then $h_{\text{top}}(X_I, \alpha_I) = 0$.
- If $d > 1$, every principal $\mathbb{Z}^d$-action (X_f, α_f) is ergodic (with respect to λ_f); if $d = 1$, a principal $\mathbb{Z}$-action (X_f, α_f) is ergodic if and only if f has no cyclotomic divisor.
- For every non-zero and irreducible $f \in R_d$, the following conditions are equivalent:
  - λ_f is mixing under (X_f, α_f);
  - $h_{\text{top}}(X_f, \alpha_f) > 0$.

2.2. *Main results.* Our main results are the following theorems which will be proved in §§6 and 7.

THEOREM 2.1. *Suppose $f \in R_1 \setminus \{0\}$ with $\mathsf{m}(f) > 0$. Then the principal algebraic $\mathbb{Z}$-action (X_f, α_f) is Bohr chaotic.*

For the higher dimensional case, we need an extra condition.

Definition 2.2. [20] A non-zero Laurent polynomial $f \in R_d$ is *atoral* if it is not a unit in R_d and its *unitary variety*

$$\mathsf{U}(f) = \{(t_1, \ldots, t_d) \in \mathbb{T}^d \mid f(e^{2\pi i t_1}, \ldots, e^{2\pi i t_d}) = 0\}$$

of f has dimension $\leq d-2$. This includes the possibility that $\mathsf{U}(f) = \varnothing$, which is equivalent to *expansivity* of the $\mathbb{Z}^d$-action α_f. If $\mathsf{U}(f)$ has dimension $d-1$, f is called *toral*.

With this definition, the following is true.

THEOREM 2.3. *Suppose that $d \geq 2$ and that $f \in R_d$ is atoral. Then $h_{top}(X_f, \alpha_f) > 0$ and (X_f, α_f) is Bohr chaotic.*

We illustrate these definitions with a few examples.

Example 2.4. (Toral automorphisms) We start with a special case: let $f = f_0 + \cdots + f_k z^k \in R_1$ with $k \geq 1$ and $f_k = |f_0| = 1$. Then the principal $\mathbb{Z}$-action (X_f, α_f) is conjugate to the toral automorphism $(\mathbb{T}^k, A_f)$, where

$$A_f = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 & 1 \\ -f_0 & -f_1 & -f_2 & \ldots & -f_2 & -f_{k-2} & -f_{k-1} \end{pmatrix} \in \mathrm{GL}(k, \mathbb{Z}) \tag{2.9}$$

is the *companion matrix* of f. The map $\phi \colon X_f \to \mathbb{T}^k$, defined by

$$\phi(x) = \begin{pmatrix} x_0 \\ \vdots \\ x_{k-1} \end{pmatrix}$$

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for every $x = (x_n)_{n\in\mathbb{Z}}$, implements this conjugacy. We conclude that (X_f, α_f) and thus $(\mathbb{T}^k, A_f)$ is Bohr chaotic if and only if $\mathsf{m}(f) > 0$ (cf. equation (2.6)).

Consider now an irreducible toral automorphism $T_A\colon \mathbb{T}^k \to \mathbb{T}^k$ defined by a matrix $A \in GL_k(\mathbb{Z})$. Then the characteristic polynomial $f(z)$ of A is irreducible, and the principal algebraic $\mathbb{Z}$-action $(\mathbb{T}^k, A_f) \cong (X_f, \alpha_f)$ in equation (2.9) is a finite-to-one factor of $(\mathbb{T}^d, T_A)$. Hence, if $(\mathbb{T}^k, A_f)$ is Bohr chaotic (which is the case if and only if $(m)(f) > 0$), then $(\mathbb{T}^k, T_A)$ is also Bohr chaotic (as an extension).

If a toral automorphism $T_A\colon \mathbb{T}^k \to \mathbb{T}^k$ with $A \in GL_k(\mathbb{Z})$ is reducible (that is, has a proper invariant subtorus $V \subsetneq \mathbb{T}^k$), then the characteristic polynomial g of $T_A|_V$ will be a proper factor of f. If g' is one of the irreducible factors of g (and hence of f), then the system $(\mathbb{T}^{\deg(g')}, T_{A_{g'}})$ will be Bohr chaotic if and only if $\mathsf{m}(g') > 0$, in which case both $(V, T_A|_V)$ and $(\mathbb{T}^d, T_A)$ will be Bohr chaotic. By varying V over the A-invariant irreducible subtori of $\mathbb{T}^k$, we see that $(\mathbb{T}^k, T_A)$ is Bohr chaotic if and only if $h_{\mathrm{top}}(T_A) = \mathsf{m}(f) > 0$.

Example 2.5. (Constant polynomials) Suppose that $f = p \in \mathbb{N}$, $p > 1$, viewed as a constant polynomial in R_d. Then the principal algebraic action (X_f, α_f) arising from this polynomial is the shift-action in equation (2.1) on $X_p := \{0, \ldots, p - 1/p\}^{\mathbb{Z}^d}$, which is certainly Bohr chaotic. If $p = 1$ (or, more generally, if f is a unit in R_d), then X_f reduces to a single point and the $\mathbb{Z}^d$-action α_f becomes trivial. By default, (X_f, α_f) is not Bohr chaotic.

Example 2.6. (The zero polynomial) So far, we have always assumed that the polynomial $f \in R_d$ defining a principal algebraic action (X_f, α_f) is non-zero. If we deviate from this assumption and set $f = 0$ (the zero polynomial in R_d), then equation (2.2) reduces to $X_f = \mathbb{T}^{\mathbb{Z}^d}$, and α_f becomes the shift action σ of $\mathbb{Z}^d$ on $\mathbb{T}^{\mathbb{Z}^d}$. For every integer $p > 1$, $\mathbb{T}^{\mathbb{Z}^d}$ contains the closed, shift invariant subset $X_p := \{0, \ldots, p - 1/p\}^{\mathbb{Z}^d}$ in Example 2.5. Since (X_p, σ) is Bohr periodic, the same is true for $(X_f, \alpha_f) = (\mathbb{T}^{\mathbb{Z}^d}, \sigma)$.

Remark 2.7. To prove our Theorems 2.1 and 2.3, we may assume without loss of generality that the polynomial f in either of these theorems is primitive and irreducible.

Indeed, if f is not primitive, then $f = pg$ for some primitive polynomial $g \in R_d$, and (X_f, α_f) has the subsystem $(X_p, \alpha_p = \sigma|_{X_p})$ appearing in Example 2.5. Since (X_p, α_p) is Bohr chaotic, the same holds for (X_f, α_f).

Similarly, if $f \in R_d$ is reducible and $\mathsf{m}(f) > 0$, then at least one of the irreducible factors g of f has positive Mahler measure $\mathsf{m}(g) > 0$. If (X_g, α_g) is Bohr chaotic, the Bohr chaoticity of (X_f, α_f) follows immediately from Bohr chaoticity of the subsystem (X_g, α_g). For $d \geq 2$ in Theorem 2.3, we also note that atorality of a polynomial $f \in R_d$ is inherited by all its irreducible factors.

3. $\mathbb{Z}^d$ *actions of zero entropy*

Before going on to study our Bohr chaotic principal algebraic $\mathbb{Z}^d$-actions, we would like to justify that any continuous $\mathbb{N}^d$- or $\mathbb{Z}^d$-action with zero topological entropy is not Bohr chaotic (Proposition 3.1). This is an immediate consequence of the disjointness completely positive entropy systems and zero entropy systems. Also, we would like to point out that principal $\mathbb{Z}$-actions with zero topological entropy are disjoint from the Möbius function (Proposition 3.5).

3.1. *Zero entropy $\mathbb{Z}^d$-actions are not Bohr chaotic.* Consider a measure-preserving $\mathbb{N}^d$- or $\mathbb{Z}^d$-action γ on a Lebesgue space (Ω, μ), where Ω is a compact space equipped with its Borel field. We say that the measure-theoretic system (Ω, μ, γ) has *completely positive entropy* if any non-trivial factor of (Ω, μ, γ) has positive entropy. Bernoulli systems have complete positive entropy. For $d = 1$, the following result is folklore; for $d \geq 1$, we include a proof for completeness, based on a disjointness theorem due to Glasner, Thouvenot, and Weiss [**12**, Theorem 1].

PROPOSITION 3.1. *Suppose that (Ω, μ, γ) has completely positive entropy, $\omega \in \Omega$ is a μ-generic point, and $\phi \in C(\Omega)$ is a continuous function having zero mean. Then $(\phi(\gamma^{\boldsymbol{n}}\omega))_{\boldsymbol{n}\in\mathbb{N}^d}$ is orthogonal to every zero entropy $\mathbb{N}^d$- or $\mathbb{Z}^d$-action (X, α). That is to say, for every $f \in C(X)$ and every $x \in X$, we have*

$$\lim_{N\to\infty} \frac{1}{N^d} \sum_{\boldsymbol{n}\in[0,N-1]^d} \phi(\gamma^{\boldsymbol{n}}\omega) f(\alpha^{\boldsymbol{n}}x) = 0. \tag{3.1}$$

In particular, continuous $\mathbb{N}^d$- or $\mathbb{Z}^d$-actions with zero topological entropy are not Bohr chaotic.

Proof. Suppose that for some f and some x, there exists a sequence (N_j) tending to infinity such that

$$\ell := \lim_{j\to\infty} \frac{1}{N_j^d} \sum_{\boldsymbol{n}\in[0,N_j-1]^d} \phi(\gamma^{\boldsymbol{n}}\omega) f(\alpha^{\boldsymbol{n}}x) \neq 0.$$

We can assume that along this sequence (N_j), the following weak limits of measures exist:

$$\lambda := \lim_{j\to\infty} \frac{1}{N_j^d} \sum_{\boldsymbol{n}\in[0,N_j-1]^d} \delta_{\gamma^{-\boldsymbol{n}}\omega} \times \delta_{\alpha^{-\boldsymbol{n}}x}, \quad \nu := \lim_{j\to\infty} \frac{1}{N_j^d} \sum_{\boldsymbol{n}\in[0,N_j-1]^d} \delta_{\alpha^{-\boldsymbol{n}}x},$$

where δ_ω and δ_x denote the point masses at the points ω and x, respectively. Clearly, the measure λ is $\gamma\times\alpha$-invariant, and the projection of λ on X is equal to ν. Since ω is μ-generic, the projection of λ onto Ω is equal to μ. In other words, λ is a joining of μ and ν, where ν has zero entropy. Since systems of completely positive entropy are disjoint from systems of zero entropy by [**12**, Theorem 1], we obtain that $\lambda = \mu\times\nu$. Thus, by the definition of λ and the hypothesis that $\mathbb{E}_\mu\phi = 0$, we get that

$$\ell = \mathbb{E}_\lambda(\phi \otimes f) = \mathbb{E}_\mu\phi \cdot \mathbb{E}_\nu f = 0,$$

which is a contradiction. □

COROLLARY 3.2. *Let (X, α) be an algebraic $\mathbb{Z}^d$-action which does not have completely positive entropy (with respect to the Haar measure λ_X). Then (X, α) is not Bohr chaotic.*

Proof. If (X, α) does not have completely positive entropy, then [29, Theorem 20.8] implies that there exists a non-trivial closed, α-invariant subgroup $Y \subset X$ such that the $\mathbb{Z}^d$-action $\alpha_{X/Y}$ induced by α on X/Y has zero entropy. Condition (ii) at the beginning of §3, combined with Proposition 3.1, shows that (X, α) cannot be Bohr chaotic. □

Example 3.3. (Furstenberg's example) Let $d = 2$ and let $I = (2 - z_1, 3 - z_2) \subset R_2$. Then $X_I = \{x \in \mathbb{T}^{\mathbb{Z}^2} \mid \sigma^{(1,0)}x = 2x,\ \sigma^{(0,1)}x = 3x\}$, so that $x_{k,l} = 2^k 3^l x_{(0,0)}$ for every $x \in X_I$ and $(k, l) \in \mathbb{Z}^2$. Since $f^{(2)} = 2 - z_1$ and $f^{(3)} = 3 - z_2$ are irreducible and relatively prime to each other, I is a prime ideal, and hence $h_{\text{top}}(X_I, \alpha_I) = 0$ [29, Proposition 17.5].

If γ is a continuous $\mathbb{Z}^2$-action on a compact space Ω, μ is a probability measure on Ω with completely positive entropy under γ, $\omega \in \Omega$ is a μ-generic point, and $\phi \in C(\Omega)$ has mean zero, Proposition 3.1 shows that

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{(m,n)\in[0,N-1]^2} \phi(\gamma^{(m,n)}\omega)h(2^m 3^n t) = 0$$

for every $h \in C(\mathbb{T})$ and $t \in \mathbb{T}$.

In [11], Furstenberg's example was defined as the $\mathbb{N}^2$-action α on $X = \mathbb{T}$ given by

$$\alpha^{(m,n)}t = 2^m 3^n t \ (\text{mod } 1)$$

for every $(m, n) \in \mathbb{N}^2$ and $t \in \mathbb{T}$.

We set $\Omega = \mathbb{T}^{\mathbb{N}^d}$, write the coordinates of every $\omega = (\omega_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{N}^d} \in \Omega$ in the form $\omega_{\boldsymbol{n}} = (\omega_{\boldsymbol{n}}^{(1)}, \ldots, \omega_{\boldsymbol{n}}^{(d)})$, and denote by γ the one-sided shift-action of $\mathbb{N}^d$ on Ω (cf. equation (2.1)). According to Franklin [10], for Lebesgue-almost everywhere (a.e.) $(\beta_1, \ldots, \beta_d)$ with $\beta_1 > 1, \ldots, \beta_d > 1$, the point $\beta = (\beta_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{N}^d} \in \Omega$ with $\beta_{\boldsymbol{n}} = (\beta_1^{n_1} \ (\text{mod } 1), \ldots, \beta_d^{n_d} \ (\text{mod } 1))$ for every $\boldsymbol{n} \in \mathbb{N}^d$ is Lebesgue-generic for γ on Ω. If $\phi \colon \Omega \to \mathbb{C}$ is the map defined by

$$\phi(\omega) = e^{2\pi i(\omega_{\boldsymbol{0}}^{(1)} + \cdots + \omega_{\boldsymbol{0}}^{(d)})},$$

then

$$\phi(\gamma^{\boldsymbol{n}}\beta) = e^{2\pi i(\beta_1^{n_1} + \cdots + \beta_d^{n_d})}$$

for every $\boldsymbol{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d$. By Proposition 3.1, the sequence $(\phi(\gamma^{\boldsymbol{n}}\beta))_{\boldsymbol{n}\in\mathbb{N}^d}$ is almost surely orthogonal to all systems of zero entropy. Since Furstenberg's example $(\mathbb{T}, \alpha)$ described in the preceding paragraph has zero entropy, we obtain the following corollary of Proposition 3.1.

COROLLARY 3.4. *For almost all (β_1, β_2) with $\beta_1 > 1$ and $\beta_2 > 1$,*

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{0\le m,n<N} e^{2\pi i(\beta_1^m + \beta_2^n)} f(2^m 3^n t) = 0$$

for every continuous function $f \in C(\mathbb{T})$ and every $t \in \mathbb{T}$.

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3.2. *Möbius disjointness and principal actions.* We have just shown that zero entropy $\mathbb{Z}^d$-actions are not Bohr chaotic. In fact, for principal actions, the result can be strengthened. We recall that a topological dynamical system (X, T) is *Möbius disjoint* if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mu(k) f(T^k x) = 0 \quad \text{for every } f \in C(X) \text{ and every } x \in X. \tag{3.2}$$

PROPOSITION 3.5. *A zero entropy principal $\mathbb{Z}$-algebraic action (X_f, α_f), $f \in R_1$, is Möbius disjoint.*

Proof. Since (X_f, α_f) has zero entropy, that is, $\mathsf{m}(f) = 0$, the Kronecker lemma implies that f has the form

$$f(z) = \pm z^{m_0} \Phi_{n_1}(z^{m_1}) \cdots \Phi_{n_k}(z^{m_k}), \tag{3.3}$$

where $m_0 \in \mathbb{Z}$, $n_j, m_j \in \mathbb{N}$, $j = 1, \ldots, k$, and Φ_n is the nth cyclotomic polynomial defined in equation (2.8). One immediately concludes from equation (3.3) that

$$f(z) = a_0 + a_1 z + \cdots + a_N z^N \quad \text{with } |a_0| = |a_N| = 1,$$

and hence (X_f, α_f) is topologically conjugate to the toral automorphism $(\mathbb{T}^N, T_A)$, where $T_A : \mathbb{T}^N \to \mathbb{T}^N$ is a linear automorphism with the matrix $A = A_f$—the companion matrix of f, see Example 2.4. However, toral automorphisms with zero entropy are known to be Möbius disjoint [23, Theorem 1.1]. In fact, toral automorphisms, and more generally affine maps of compact abelian groups, are the primary examples motivating Sarnak's (still unproven) conjecture that all topological dynamical systems with zero entropy are Möbius disjoint [28]. □

4. *Riesz product measures on X_f*

In this section, we start on the proofs of Theorems 2.1 and 2.3. As explained in Remark 2.7, we assume from now on—and without loss in generality—that the polynomial $f \in R_d$ defining our principal action (X_f, α_f) is primitive, irreducible, and has positive Mahler measure.

For the proofs of these theorems, we shall use a class of measures called Riesz products. First, we will recall the general construction of Riesz product measures on arbitrary compact abelian groups. Second, we will construct Riesz products on X_f based on lacunary polynomials in the dual group $\widehat{X}_f \subset R_d$.

4.1. *Riesz product measures.* Let X be a compact abelian group with dual group $\widehat{X}$.

Definition 4.1. [13] An infinite sequence of distinct characters $\Lambda = (\gamma_n)_{n\in\mathbb{N}} = \{\gamma_0, \gamma_1, \ldots\} \subset \widehat{X}$ is said to be *dissociate* if for every $k \geq 1$ and every k-tuple $(n_1, n_2, \ldots, n_k) \in \mathbb{N}^k$ of distinct non-negative integers, the equality

$$\gamma_{n_1}^{\varepsilon_1} \gamma_{n_2}^{\varepsilon_2} \cdots \gamma_{n_k}^{\varepsilon_k} = 1$$

with $\varepsilon_j \in \{-2, -1, 0, 1, 2\}$ for every $j = 1, \ldots, k$, implies that

$$\gamma_{n_1}^{\varepsilon_1} = \gamma_{n_2}^{\varepsilon_2} = \cdots = \gamma_{n_k}^{\varepsilon_k} = 1.$$

Equivalently, Λ is dissociate if any character in $\widehat{X}$ can be represented in at most one way as a finite product $\gamma_{n_1}^{\varepsilon_1}\gamma_{n_2}^{\varepsilon_2}\cdots\gamma_{n_k}^{\varepsilon_k}$ of elements of Λ, where all n_j are distinct and $\varepsilon_j \in \{-1, 0, 1\}$.

Using dissociate sequences of characters, Hewitt and Zuckermann [13] proposed a construction of interesting probability measures—the so-called Riesz products, generalizing Riesz products on $\mathbb{T}$ constructed by Riesz [27] in 1918. More precisely, denote by λ_X the Haar measure on X. Suppose that:

(i) $\Lambda = (\gamma_n)_{n\geq 0}$ is a dissociate sequence of characters in $\widehat{X}$;

(ii) $a = (a_n)_{n\geq 0}$ is a sequence of complex numbers such that $|a_n| \leq 1$ for all n.

For any $N \geq 0$, denote by $\mu_a^{(N)}$ the measure on X which is absolutely continuous with respect to λ_X with density

$$\frac{d\mu_a^{(N)}}{d\lambda_X}(x) = \prod_{n=0}^{N}(1 + \mathrm{Re}\, a_n\gamma_n(x)).$$

It is not very difficult to show that the sequence of measures $(\mu_a^{(N)})_{N\geq 0}$ converges weakly; the limiting measure $\mu_a = \lim_N \mu_a^{(N)}$ is called the *Riesz product*, and we denote it as

$$\mu_a = \prod_{n=0}^{\infty}(1 + \mathrm{Re}\, a_n\gamma_n(x)). \tag{4.1}$$

The Riesz product μ_a is absolutely continuous with respect to the Haar measure λ_X if and only if $\sum_n |a_n|^2 < \infty$, and it is singular to the Haar measure λ_X if and only if $\sum_n |a_n|^2 = \infty$ (see [25, 34]). We will omit dependence of μ_a on the sequence Λ, since Λ will usually be fixed.

Since

$$1 + \mathrm{Re}\, a_n\gamma_n(x) = 1 + \frac{a_n}{2}\gamma_n(x) + \frac{\overline{a}_n}{2}\gamma_n^{-1}(x),$$

the Riesz product μ_a, associated to the sequences Λ and a, can be characterized by the Fourier coefficients $\widehat{\mu}_a(\gamma) = \int \overline{\gamma}(x)\, d\mu_a(x)$, $\gamma \in \widehat{X}$, as follows.

(a) For any finite set of distinct characters $\{\gamma_{n_1}, \gamma_{n_2}, \ldots, \gamma_{n_k}\} \subset \Lambda$ and any $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k) \in \{-1, 0, 1\}^k$,

$$\widehat{\mu}_a(\gamma_{n_1}^{\varepsilon_1}\gamma_{n_2}^{\varepsilon_2}\cdots\gamma_{n_k}^{\varepsilon_k}) = a_{n_1}^{(\varepsilon_1)}a_{n_2}^{(\varepsilon_2)}\cdots a_{n_k}^{(\varepsilon_k)}, \tag{4.2}$$

where $a_n^{(\varepsilon)} = (a_n/2)$, 0, or $\overline{a}_n/2$, depending on whether $\varepsilon = 1, 0$, or -1.

(b) For any character $\gamma \in \widehat{X}$ not of the form $\gamma_{n_1}^{\varepsilon_1}\gamma_{n_2}^{\varepsilon_2}\cdots\gamma_{n_k}^{\varepsilon_k}$ with $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k \in \{-1, 0, 1\}$ as in case (a) above, one has

$$\widehat{\mu}_a(\gamma) = 0. \tag{4.3}$$

For any two Riesz products μ_a and μ_b, it is proved in [25] that μ_a and μ_b are mutually singular if $\sum |a_n - b_n|^2 = \infty$, and mutually equivalent if $\sum |a_n - b_n|^2 < \infty$ and $\sup_n |a_n| < 1$. For any Riesz product μ_a, it is proved in [6] that the orthogonal series $\sum c_n(\gamma_n(x) - a_n/2)$ (with $c_n \in \mathbb{C}$) converges μ_a-a.e. if and only if $\sum |c_n|^2 < \infty$. Such convergence results will be useful to us in the proofs of Theorems 2.1 and 2.3. Riesz products on $\mathbb{T}$ and some generalized Riesz products appear as spectral measures of some dynamical systems (see [**1**, **18**, **26**]). Riesz products are tools in harmonic analysis (see [**14**, **16**, **34**]).

4.2. *The dual group* $\widehat{X}_f$. Before constructing Riesz products on X_f, let us describe the dual group of X_f (cf. [**19**, **29**]). Every Laurent polynomial with integer coefficients

$$h(z) = \sum_{\boldsymbol{m}\in\mathbb{Z}^d} h_{\boldsymbol{m}} z^{\boldsymbol{m}} \in R_d$$

defines a character $\gamma^{(h)} \in \widehat{\mathbb{T}^{\mathbb{Z}^d}}$, given by

$$\gamma^{(h)}(x) := e^{2\pi i\langle h,x\rangle},$$

where

$$\langle h, x\rangle = \sum_{\boldsymbol{m}\in\mathbb{Z}} h_{\boldsymbol{m}} x_{\boldsymbol{m}}$$

for every $x \in \mathbb{T}^{\mathbb{Z}^d}$. Conversely, every character of $\mathbb{T}^{\mathbb{Z}^d}$ is of the form $\gamma = \gamma^{(h)}$ for some $h \in R_d$, so that we may identify $\widehat{\mathbb{T}^{\mathbb{Z}^d}}$ with R_d. Note, however, that the group operation in R_d is addition, whereas in $\widehat{\mathbb{T}^{\mathbb{Z}^d}}$, it is multiplication:

$$\gamma^{(h+h')} = \gamma^{(h)}\gamma^{(h')}$$

for all $h, h' \in R_d$.

Since X_f is a subgroup of $\mathbb{T}^{\mathbb{Z}^d}$, every character $\gamma^{(h)} \in \widehat{\mathbb{T}^{\mathbb{Z}^d}}$, $h \in R_d$, restricts to a character $\tilde{\gamma}^{(h)} \in \widehat{X}_f$. From the definition of X_f in equation (2.2), it is clear that, for any two polynomials $h, h' \in R_d$, $\tilde{\gamma}^{(h)} = \tilde{\gamma}^{(h')}$ if and only if $h - h'$ is a multiple of f. This allows us to identify the dual group $\widehat{X}_f$ with $R_d/(f)$, where $(f) = R_d \cdot f$ is the principal ideal in R_d generated by f:

$$\widehat{X}_f = R_d/(f).$$

More generally, if $I \subset R_d$ is an ideal and X_I is given by equation (2.5), then

$$\widehat{X}_I = R_d/I.$$

4.3. *Lacunary polynomials.* For the construction of Riesz product measures on X_f, we have to take a closer look at dissociate families $\Lambda \subset \widehat{X}_f$ in the sense of Definition 4.1.

Definition 4.2. Given an integer $m \in \mathbb{N}$, we say that a primitive irreducible polynomial $f \in R_d$ is *m-good* if the following conditions hold.

(C1) The collection of characters

$$\{\tilde{\gamma}^{(z^{m\boldsymbol{n}})} \mid \boldsymbol{n} \in \mathbb{N}^d\} \subset \widehat{X}_f$$

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is dissociate. Explicitly, this means that any non-zero polynomial of the form $g(z^m)$ where

$$g(z) = \sum_{\boldsymbol{n} \in \mathbb{Z}^d} \varepsilon_{\boldsymbol{n}} z^{\boldsymbol{n}}$$

with $\varepsilon_{\boldsymbol{n}} \in \{-2, -1, 0, 1, 2\}$ is not divisible by f.

(C2) For any $\boldsymbol{k} \in \mathbb{N}^d / m\mathbb{N}^d$, any two points $\boldsymbol{n} \neq \boldsymbol{n}'$ in $\mathbb{Z}^d$, and any non-zero polynomial of the form $g(z) := \sum_{\boldsymbol{n} \in \mathbb{Z}^d} \varepsilon_{\boldsymbol{n}} z^{\boldsymbol{n}}$ with $\varepsilon_{\boldsymbol{n}} \in \{-1, 0, 1\}$, the polynomial

$$z^{m\boldsymbol{n}+\boldsymbol{k}} - z^{m\boldsymbol{n}'+\boldsymbol{k}} + g(z^m)$$

is not divisible by f.

For a given principal algebraic action (X_f, α_f), where f is m-good, Riesz product measures μ_a can be constructed using the countable dissociate collection of characters $\Lambda = \{\tilde{\gamma}^{(z^{m\boldsymbol{n}})} \mid \boldsymbol{n} \in \mathbb{N}^d\}$ (cf. condition (C1)). Condition (C2) ensures that any shifted family of characters $\Lambda_{\boldsymbol{k}} = \{\tilde{\gamma}^{(z^{m\boldsymbol{n}+\boldsymbol{k}})} \mid \boldsymbol{n} \in \mathbb{N}^d\}$ (with $\boldsymbol{k} \in [0, m-1]^d \setminus \{\boldsymbol{0}\}$ being fixed) is a μ_a-orthogonal system, as a direct consequence of equation (4.2) applied with $k = 2$—a useful property which will help us control the behavior of weighted ergodic averages. As we will see, the coefficient sequence a will be chosen depending on the non-trivial weight sequence $\boldsymbol{w}$.

5. (X_f, α_f) *is Bohr chaotic when f is m-good*

The following theorem will allow us to reduce the proof of Bohr chaoticity of (X_f, α_f) to checking the m-goodness of the polynomial f.

THEOREM 5.1. *If a primitive irreducible polynomial $f \in R_d$ with positive Mahler measure is m-good, that is, if the conditions* (C1) *and* (C2) *hold for some positive integer m, then (X_f, α_f) is Bohr chaotic.*

We begin with a simple auxiliary lemma.

LEMMA 5.2. *Let α be a continuous $\mathbb{Z}^d$-action on a compact metrizable space X, and let $\boldsymbol{w} = (w_{\boldsymbol{n}})_{\boldsymbol{n} \in \mathbb{N}^d}$ be a non-trivial weight. Then (X, α) is not disjoint from $\boldsymbol{w} = (w_{\boldsymbol{n}})$ if and only if for any $\boldsymbol{k} \in \mathbb{N}^d$, (X, α) is not disjoint from the weight $\widetilde{\boldsymbol{w}} = (\widetilde{w}_{\boldsymbol{n}})$ defined by $\widetilde{w}_{\boldsymbol{n}} = w_{\boldsymbol{n}+\boldsymbol{k}}$ for all $\boldsymbol{n} \in \mathbb{N}^d$.*

Proof. Introduce the following notation: for a continuous function ϕ on X, let

$$S_N^{\boldsymbol{w}} \phi(x) = \sum_{\boldsymbol{n} \in [0, N-1]^d} w_{\boldsymbol{n}} \phi(\alpha^{\boldsymbol{n}} x).$$

For any $\boldsymbol{k} \in \mathbb{N}^d$ and for any $x \in X$, one has

$$\begin{aligned} |S_{N+\|\boldsymbol{k}\|_\infty}^{\boldsymbol{w}} \phi(x) - S_N^{\widetilde{\boldsymbol{w}}} \phi(\alpha^{\boldsymbol{k}} x)| &= \bigg| \sum_{\boldsymbol{n} \in [0, N+\|\boldsymbol{k}\|_\infty - 1]^d} w_{\boldsymbol{n}} \phi(\alpha^{\boldsymbol{n}} x) - \sum_{\boldsymbol{n} \in [0, N-1]^d} w_{\boldsymbol{n}+\boldsymbol{k}} \phi(\alpha^{\boldsymbol{n}+\boldsymbol{k}} x) \bigg| \\ &\le \|\phi\|_\infty \cdot |[0, N + \|\boldsymbol{k}\|_\infty - 1]^d \triangle (\boldsymbol{k} + [0, N-1]^d)|. \end{aligned}$$

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To finish the proof, it suffices to notice that the cardinality of the symmetric difference is of order $O(N^{d-1})$. $\square$

Proof of Theorem 5.1. Fix $m \in \mathbb{N}$ such that conditions (C1) and (C2) hold. Assume that $\boldsymbol{w}$ is a non-trivial weight (cf. equation (1.5)). Then for some $\boldsymbol{k} \in [0, \ldots, m-1]^d$, one has

$$\limsup_{N\to\infty} \frac{1}{N^d} \sum_{\boldsymbol{n} | m\boldsymbol{n}+\boldsymbol{k} \in [0,N-1]^d} |w_{m\boldsymbol{n}+\boldsymbol{k}}| > 0. \tag{5.1}$$

Without loss of generality, we can assume $\boldsymbol{k} = \boldsymbol{0}$. Otherwise, consider the shifted weight $\widetilde{\boldsymbol{w}} = (\widetilde{w}_{\boldsymbol{n}})$ with $\widetilde{w}_{\boldsymbol{n}} = w_{\boldsymbol{n}+\boldsymbol{k}}$. By Lemma 5.2, (X_f, α_f) is not disjoint from $\boldsymbol{w}$ if and only if (X_f, α_f) is not disjoint from $\widetilde{\boldsymbol{w}}$. Thus, it is sufficient to consider the weight $\widetilde{\boldsymbol{w}}$ for which we can assume that equation (5.1) holds with $\boldsymbol{k} = 0$. In the following, we consider an arbitrary such weight.

Step 1. Choice of the function ϕ and the point x. We are going to show that equation (1.5) holds for $\phi(x) := e^{2\pi i x_0} = e^{2\pi i \langle 1, x\rangle}$ and for almost all $x \in X_f$ with respect to an appropriately chosen Riesz product measure. Note that for all $\boldsymbol{n} \in \mathbb{N}^d$,

$$\phi(\alpha_f^{\boldsymbol{n}} x) = e^{2\pi i x_{\boldsymbol{n}}} = e^{2\pi i \langle z^{\boldsymbol{n}}, x\rangle} = \gamma^{(z^{\boldsymbol{n}})}(x).$$

Step 2. Choice of the measure. By condition (C1), the collection of characters

$$\Lambda := \{\gamma_{\boldsymbol{n}} = \gamma^{(z^{m\boldsymbol{n}})} \mid \boldsymbol{n} \in \mathbb{N}^d\}$$

is dissociate. Consider now the following collection of coefficients:

$$a := \{a_{\boldsymbol{n}} = e^{-i \arg w_{m\boldsymbol{n}}} \mid \boldsymbol{n} \in \mathbb{N}^d\}.$$

Since $|a_{\boldsymbol{n}}| = 1$ for all $\boldsymbol{n}$, the Riesz product μ_a in equation (4.1) is well defined.

Step 3. Orthonormality. For each $\boldsymbol{k} \in [0, m-1]^d \setminus \{\boldsymbol{0}\}$, consider the following collection of functions:

$$\mathcal{F}_{\boldsymbol{k}} := \{\gamma^{(z^{m\boldsymbol{n}+\boldsymbol{k}})}(x) = \phi \circ \alpha_f^{m\boldsymbol{n}+\boldsymbol{k}}(x) \mid \boldsymbol{n} \in \mathbb{N}^d\}.$$

We claim that for each $\boldsymbol{k} \in [0, m-1]^d \setminus \{\boldsymbol{0}\}$, $\mathcal{F}_{\boldsymbol{k}}$ is orthonormal in $L^2(X_f, \mu_a)$. Indeed, for each $\boldsymbol{n} \neq \boldsymbol{n}'$, condition (C2) means that the character corresponding to the polynomial $z^{m\boldsymbol{n}+\boldsymbol{k}} - z^{m\boldsymbol{n}'+\boldsymbol{k}}$:

$$\gamma^{(z^{m\boldsymbol{n}+\boldsymbol{k}} - z^{m\boldsymbol{n}'+\boldsymbol{k}})}(x) = \gamma^{(z^{m\boldsymbol{n}+\boldsymbol{k}})}(x)\overline{\gamma^{(z^{m\boldsymbol{n}'+\boldsymbol{k}})}(x)},$$

cannot be expressed as a product of characters in Λ, and hence using equation (4.3) for the Fourier coefficients of Riesz products, one gets that

$$\int_{X_f} \gamma^{(z^{m\boldsymbol{n}+\boldsymbol{k}})}(x)\overline{\gamma^{(z^{m\boldsymbol{n}'+\boldsymbol{k}})}(x)}\, d\mu_a(x) = \widehat{\mu_a}(\overline{\gamma^{(z^{m\boldsymbol{n}+\boldsymbol{k}} - z^{m\boldsymbol{n}'+\boldsymbol{k}})}}) = 0.$$

Since $|\gamma^{(z^{m\boldsymbol{n}+\boldsymbol{k}})}(x)|^2 = 1$ for all x, the orthonormality of $\mathcal{F}_{\boldsymbol{k}}$ is thus proved.

For $\boldsymbol{k} = 0$, we set

$$\mathcal{F}_{\boldsymbol{0}} := \left\{\gamma^{(z^{m\boldsymbol{n}})}(x) - \frac{a_{\boldsymbol{n}}}{2} \mid \boldsymbol{n} \in \mathbb{N}^d\right\}.$$

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Direct application of equations (4.2) and (4.3) immediately gives that the collection of functions $\mathcal{F}_{\mathbf{0}}$ is orthogonal in $L^2(X_f, \mu_a)$, and that

$$\int_{X_f} |\gamma^{(z^{m\boldsymbol{n}})}(x)|^2 \, d\mu_a(x) = 1 - \frac{|a_{\boldsymbol{n}}|^2}{4} = \frac{3}{4} \quad \text{for all } \boldsymbol{n} \in \mathbb{N}^d.$$

Step 4. Almost everywhere convergence. Write

$$S_N^{\boldsymbol{w}}\phi(x) = \sum_{\boldsymbol{n}\in[0,N-1]^d} w_{\boldsymbol{n}}\phi(\alpha_f^{\boldsymbol{n}}x) = \sum_{\boldsymbol{k}\in[0,m-1]^d} S_{N,\boldsymbol{k}}^{\boldsymbol{w}}\phi(x),$$

where

$$S_{N,\boldsymbol{k}}^{\boldsymbol{w}}\phi(x) := \sum_{\{\boldsymbol{n} \mid m\boldsymbol{n}+\boldsymbol{k}\in[0,N-1]^d\}} w_{m\boldsymbol{n}+\boldsymbol{k}}\phi(\alpha_f^{m\boldsymbol{n}+\boldsymbol{k}}x).$$

We claim that for any $\boldsymbol{k} \in [0, m-1]^d \setminus \{\mathbf{0}\}$, one has

$$\frac{1}{N^d} S_{N,\boldsymbol{k}}^{\boldsymbol{w}}\phi(x) \to 0 \quad \mu_a\text{–}a.e., \tag{5.2}$$

and for $\boldsymbol{k} = 0$, one has

$$\frac{1}{N^d}\bigg(S_{N,\mathbf{0}}^{\boldsymbol{w}}\phi(x) - \frac{1}{2}\sum_{\{\boldsymbol{n} \mid m\boldsymbol{n}\in[0,N-1]^d\}} |w_{\boldsymbol{n}}|\bigg) \to 0 \quad \mu_a\text{–}a.e. \tag{5.3}$$

Now we write

$$\frac{1}{N^d} S_N^{\boldsymbol{w}}\phi(x) = \frac{1}{N^d}\bigg(S_N^{\boldsymbol{w}}\phi(x) - \frac{1}{2}\sum_{\{\boldsymbol{n} \mid m\boldsymbol{n}\in[0,N-1]^d\}} |w_{\boldsymbol{n}}|\bigg) + \frac{1}{2N^d}\sum_{\{\boldsymbol{n} \mid m\boldsymbol{n}\in[0,N-1]^d\}} |w_{\boldsymbol{n}}|.$$

If equations (5.2) and (5.3) are indeed true, the first term in the brackets on the right-hand side converges to 0 for μ_a-almost all $x \in X_f$, and the second term does not converge to 0 by equation (5.1). Hence, we will be able to conclude that

$$\limsup_{N\to\infty} \frac{1}{N^d}|S_N^{\boldsymbol{w}}\phi(x)| > 0, \quad \mu_a\text{-a.e.},$$

and thus, that (X_f, α_f) is Bohr chaotic.

Finally, to establish equations (5.2) and (5.3), we will use the following multivariate generalization of the result of Davenport, Erdös, and LeVeque [3] due to Fan, Fan, and Qiu [8, Theorem 6.1]: suppose that $\{\xi_{\boldsymbol{\ell}} \mid \boldsymbol{\ell} \in \mathbb{N}^d\}$ is a collection of measurable complex valued uniformly bounded functions on a probability space $(\Omega, \mathbb{P})$ such that

$$\sum_{N=1}^{\infty} \frac{1}{N} \int_{\Omega} |Z_N|^2 \, d\mathbb{P} < \infty, \tag{5.4}$$

where

$$Z_N = \frac{1}{N^d} \sum_{\boldsymbol{\ell}\in[0,N-1]^d} \xi_{\boldsymbol{\ell}} \quad (N \geq 1).$$

Then $Z_N \to 0$ as $N \to \infty$ $\mathbb{P}$-a.e. on Ω.

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In particular, if $\{\xi_{\boldsymbol{\ell}} \mid \boldsymbol{\ell} \in \mathbb{N}^d\}$ are uniformly bounded and orthogonal in $L^2(\Omega, \mathbb{P})$, then

$$\frac{1}{N}\int_\Omega |Z_N|^2\, d\mathbb{P} = \frac{1}{N^{2d+1}} \sum_{\boldsymbol{\ell}\in[0,N-1]^d} \int_X |\xi_{\boldsymbol{\ell}}|^2\, d\mathbb{P} \le \frac{C}{N^{d+1}},$$

and hence equation (5.4) holds for any $d \ge 1$.

If we now apply this result to the orthogonal families of bounded functions

$$\mathcal{F}_{\boldsymbol{k}}^{\boldsymbol{w}} = \{w_{m\boldsymbol{n}+\boldsymbol{k}}\psi \circ \alpha^{m\boldsymbol{n}+\boldsymbol{k}}(x) \mid \boldsymbol{n} \in \mathbb{Z}_+^d\}, \quad \boldsymbol{k} \in [0, m-1]^d \setminus \{\boldsymbol{0}\},$$

and

$$\mathcal{F}_{\boldsymbol{0}}^{\boldsymbol{w}} = \left\{ w_{m\boldsymbol{n}}\left(\psi \circ \alpha^{m\boldsymbol{n}}(x) - \frac{a_{\boldsymbol{n}}}{2}\right) \mid \boldsymbol{n} \in \mathbb{N}^d \right\},$$

we obtain equations (5.2) and (5.3), and hence, we complete the proof. □

6. *Bohr chaoticity of (X_f, α_f): the case of $d = 1$*

In this section, we complete the proof of Theorem 2.1 in the case where $d = 1$: every principal algebraic $\mathbb{Z}$-action (X_f, α_f) with positive entropy is Bohr chaotic. Theorem 2.1 will follow from Remark 2.7, Theorem 5.1, and the following result.

THEOREM 6.1. *Every primitive irreducible polynomial $f \in R_1$ with $\mathsf{m}(f) > 0$ is m-good for some positive integer m.*

The proof of Theorem 6.1 consists of the following three lemmas.

LEMMA 6.2. (Preliminary lemma) *Let $f(z) = f_0 + f_1 z + \cdots + f_r z^r \in \mathbb{Z}[z]$ be an irreducible polynomial with $r \ge 1$ and $f_0 f_r \ne 0$. If $\mathsf{m}(f) > 0$, then at least one of the following statements is true:*

(1) *there exists a root of f in $\mathbb{C}$ which is not on the unit circle;*
(2) *there exists a prime $p \ge 2$ such that f admits a root λ in the algebraic closure $\overline{\mathbb{Q}}_p$ of the field of p-adic numbers $\mathbb{Q}_p$ such that $|\lambda|_p > 1$.*

Proof. Suppose $r = 1$, that is, $f(z) = f_0 + f_1 z$. If the only root of f lies on the unit circle, then necessarily $|f_0| = |f_1|$. The irreducibility of f implies that $|f_0| = |f_1| = 1$, and thus $\mathsf{m}(f) = 0$ by equation (2.7). Therefore, assuming $\mathsf{m}(f) > 0$ for irreducible f with $\deg(f) = 1$, we conclude that $|f_0| \ne |f_1|$, and hence condition (1) must hold.

Suppose $r \ge 2$ and the roots of f are all on the unit circle, that is, suppose that condition (1) does not hold. If $|f_r| = 1$, then Kronecker's theorem [17] implies that all roots of f are roots of unity, and equation (2.7) shows that $\mathsf{m}(f) = 0$, in contradiction to our hypothesis. However, if $|f_r| > 1$, then

$$f(z) = f_r\left(z^r + \frac{f_{r-1}}{f_r} z^{r-1} + \cdots + \frac{f_0}{f_r}\right),$$

and Vieta's formula implies that $|f_0/f_r| = |\prod_{\zeta\in\mathbb{C}: f(\zeta)=0} \zeta| = 1$, that is, that $|f_0| = |f_r|$. Since f is irreducible, f_j/f_r is not integer for some $1 \le j \le r-1$. Then there exists a rational prime p such that $|f_j/f_r|_p > 1$. Let λ_i, $1 \le i \le r$, be the roots of f in the algebraic

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closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, the p-adic rationals. By considering the jth elementary symmetric polynomials of the roots and once again applying Vieta's formulas, we get that

$$1 < \left|\frac{f_j}{f_r}\right|_p = \left|\sum_{1\le i_1<i_2<\cdots<i_j\le r} \lambda_{i_1}\cdots\lambda_{i_j}\right|_p \le (\max_{1\le i\le r}|\lambda_i|_p)^j.$$

Thus one has $|\lambda_i|_p > 1$ for some $i \in \{1, \ldots, r\}$. □

The following key lemma will be used to show that for sufficiently large m, the sequence of polynomials $\{z^{nm}\}_{n\ge 0}$ gives a dissociate sequence of characters of X_f.

LEMMA 6.3. (Condition (C1)) *Suppose that* $f = f_0 + f_1 z + \cdots + f_r z^r \in \mathbb{Z}[z]$ *has a root in* $\mathbb{C}$ *or in* $\overline{\mathbb{Q}}_p$ *(for some p) of absolute value larger than* 1. *Then for any sufficiently large m and any* $D \ge 0$*, the polynomials*

$$P(z) = \sum_{j=0}^{D} \varepsilon_j z^{mj} \quad \textit{with } \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_D \in \{-2, -1, 0, 1, 2\}, \tag{6.1}$$

are not divisible by f unless $\epsilon_0 = \epsilon_1 = \cdots = \epsilon_D = 0$.

Proof. First we consider the case that f has a root in $\mathbb{C}$ of modulus larger than 1. For any polynomial g, we introduce the notation

$$\rho_g = \max\{|z| \mid g(z) = 0\}.$$

Without loss of generality, we may assume that $\varepsilon_D \neq 0$ and consider the reduced polynomial

$$\widetilde{P}(z) = \sum_{j=0}^{D} \varepsilon_j z^j,$$

such that $P(z) = \widetilde{P}(z^m)$. Clearly, $\rho_{\widetilde{P}} = \rho_P^m$.

However, using the Cauchy bound on the roots of polynomials, one gets that

$$\rho_{\widetilde{P}} \le 1 + \max_{j=0,\ldots,D-1}\left|\frac{\varepsilon_j}{\varepsilon_D}\right| \le 3,$$

and hence $\rho_P \le 3^{1/m}$. Choose an integer $M \ge 1$ large enough such that $3^{1/M} < \rho_f$ (this is possible because $\rho_f > 1$). Thus for all $m \ge M$, we have $\rho_P < \rho_f$. However, if $P(z)$ was divisible by f, we would have $\rho_f \le \rho_P$, thus arriving to a contradiction.

If f has a root in $\overline{\mathbb{Q}}_p$ (for some prime p) of absolute value larger than 1, the same argument works with $|\cdot|$ replaced by $|\cdot|_p$. Indeed, suppose ζ is a root of f with $|\zeta|_p > 1$. If $f\,|\,P$, then ζ is also a root of P, so one has

$$|\zeta|_p^{mD} = |\zeta^{mD}|_p = \left|\sum_{j=0}^{D-1}\frac{\varepsilon_j}{\varepsilon_D}\zeta^{mj}\right|_p \le \max_{j=0,\ldots,D-1}\left|\frac{\varepsilon_j}{\varepsilon_D}\zeta^{mj}\right|_p \le \left(\max_{j=0,\ldots,D-1}\left|\frac{\varepsilon_j}{\varepsilon_D}\right|_p\right)|\zeta|_p^{m(D-1)}.$$

Thus arriving at a contradiction. □

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LEMMA 6.4. (Condition (C2)) *Suppose that $f(z) = f_r z^r + \cdots + f_1 z + f_0 \in \mathbb{Z}[z]$ has a root in $\mathbb{C}$ or in $\overline{\mathbb{Q}}_p$ (for some prime p) of absolute value larger than 1. Then for all sufficiently large integers m, any integer k with $1 \le k < m$, every $D \ge 0$, and all arbitrary $(D+1)$-tuples $\varepsilon = (\varepsilon_0, \ldots, \varepsilon_D)$ and $\delta = (\delta_0, \ldots, \delta_D)$ in $\{-1, 0, 1\}^{D+1}$, the polynomial*

$$Q(z) = \sum_{j=0}^{D} \varepsilon_j z^{mj} - \sum_{j=0}^{D} \delta_j z^{mj+k} \tag{6.2}$$

is not divisible by f unless $Q(z) \equiv 0$, that is, unless all ϵ_j and δ_j are equal to zero.

Proof. Assume Q is divisible by f. We treat the complex case first. Namely, assume $\zeta \in \mathbb{C}$ is such that $f(\zeta) = 0$ and $R := |\zeta| > 1$. Without loss of generality, we may assume that $|\varepsilon_D| + |\delta_D| > 0$. We distinguish two cases.

Case I. $\delta_D \neq 0$. If the polynomial $Q(z)$, defined by equation (6.2), is divisible by f, then $Q(\zeta) = 0$, in other words,

$$\delta_D \zeta^{mD+k} = \sum_{j=0}^{D} \varepsilon_j \zeta^{mj} - \sum_{j=0}^{D-1} \delta_j \zeta^{mj+k}. \tag{6.3}$$

It follows that

$$R^{mD+k} \le \sum_{j=0}^{D} R^{mj} + \sum_{j=0}^{D-1} R^{mj+k} = \frac{R^{(D+1)m} - 1}{R^m - 1} + \frac{R^{Dm} - 1}{R^m - 1} \cdot R^k,$$

and hence

$$R^k < \frac{R^m}{R^m - 1} + \frac{R^k}{R^m - 1}.$$

As $m \to \infty$, the right-hand side of this inequality converges to 1, but the left-hand side remains equal to $R^k > 1$. If m is large enough, our assumption that Q is divisible by f leads to a contradiction.

Case II. $\delta_D = 0$ *but* $\varepsilon_D \neq 0$. In this case, we have

$$\varepsilon_D \zeta^{mD} = -\sum_{j=0}^{D-1} \varepsilon_j \zeta^{jm} + \sum_{j=0}^{D-1} \delta_j \zeta^{jm+k}. \tag{6.4}$$

It follows that

$$1 < \frac{1}{R^m - 1} + \frac{R^k}{R^m - 1} = \frac{R^k + 1}{R^m - 1} \le \frac{R^{m-1} + 1}{R^m - 1}.$$

Since $R > 1$, the last inequality is violated for all sufficiently large m, and we again arrive at a contradiction with our assumption that Q is divisible by f.

In the p-adic case, the argument is simpler because of the non-archimedean triangle inequality $|\zeta + \xi|_p \le \max(|\zeta|_p, |\xi|_p)$. Indeed, from equation (6.3), we get that $|\zeta|_p^{mD+k} \le |\zeta|_p^{mD}$ (impossible), and from equation (6.4), we get that $|\zeta|_p^{mD} \le |\zeta|_p^{m(D-1)+k}$ (equally impossible). □

Proof of Theorem 2.1. By Remark 2.7, we may assume that the polynomial $f(z) = f_0 + \cdots + f_r z^r \in \mathbb{Z}[z]$ is primitive, irreducible, and has positive Mahler measure (cf. equation (2.7)). If f has a root in $\mathbb{C}$ of absolute value larger than 1, Lemmas 6.3 and 6.4 show

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that f is m-good for some $m \in \mathbb{N}$ (in fact, for all sufficiently large m). An application of Theorem 5.1 completes the proof of Theorem 2.1 in this case.

If all complex roots of f have absolute value ≤ 1, and if ζ is a root of f, we denote by $\mathbf{k} = \mathbb{Q}(\zeta)$ the algebraic number field generated by ζ and write $P_{\mathbf{k}}$ for the set of *places* (or *valuations*) of $\mathbf{k}$. If $|\zeta|_v$ is the absolute value of ζ at a place $v \in P_{\mathbf{k}}$, then our last assumption implies that $|\zeta|_v \leq 1$ for every infinite place $v \in P_{\mathbf{k}}$. Since the product formula for algebraic number fields shows that $\prod_{v \in P_{\mathbf{k}}} |\zeta|_v = 1$, we either have that $|\zeta|_v = 1$ for every $v \in P_{\mathbf{k}}$, or that there exists a *finite* place $v \in P_{\mathbf{k}}$ with $|\zeta|_v > 1$. In the first case, ζ is a root of unity, which makes f cyclotomic and $\mathsf{m}(f) = 0$, contrary to our hypothesis. In the second case, we can once again use the Lemmas 6.3 and 6.4 to show that f is m-good for all sufficiently large m, and Theorem 5.1 completes the proof of Theorem 2.1 as above. □

7. *Bohr chaoticity of (X_f, α_f): the case of $d \geq 2$*

This section is devoted to the proof of Theorem 2.3 for $d \geq 2$, which will again be based on Theorem 5.1. As before, we assume that the polynomial $f \in R_d$ is primitive and irreducible.

7.1. *Homoclinic points of atoral polynomials in R_d and the gap theorem.*

For every $t \in \mathbb{T}$, we set

$$\|t\| = \min_{q \in \mathbb{Z}} |t - q|.$$

Definition 7.1. A point $x \in X_f$ is *homoclinic* (or, more precisely, *homoclinic to* 0) if $\lim_{\mathbf{n} \to \infty} \|x_{\mathbf{n}}\| = 0$. A homoclinic point $x \in X_f$ is *summable* if $\sum_{\mathbf{n} \in \mathbb{Z}^d} \|x_{\mathbf{n}}\| < \infty$.

The existence of non-zero summable homoclinic points of (X_f, α_f) is equivalent to *atorality* of the polynomial f.

THEOREM 7.2. [20] *If $0 \neq f \in R_d$, the following conditions are equivalent:*

(1) *the principal algebraic action (X_f, α_f) has a non-zero summable homoclinic point;*
(2) *the Laurent polynomial f is atoral in the sense of Definition 2.2.*

For a principal algebraic $\mathbb{Z}^d$-action (X_f, α_f), the existence of summable homoclinic points has a number of important consequences (cf. [19]): it implies positivity of entropy and very strong specification properties of the action, and it guarantees the coincidence of entropy with the logarithmic growth rate of the number of periodic points of α_f (that is, of points in X_f with finite orbits under α_f—cf. [19, 20]). Somewhat surprisingly, it also plays a role in the *gap theorem* stated below, which will imply conditions (C1) and (C2) in Definition 4.2.

We remark in passing that some of these consequences of atorality also hold for toral polynomials, but with considerably harder proofs and/or weaker conclusions—cf. e.g., [4] or [20]). However, it is not known if specification or gap properties hold in the toral case.

To state the gap theorem referred to above, we consider, for any non-empty subset $\mathcal{S} \subset \mathbb{Z}^d$ and any integer $H \geq 1$, the set $\mathcal{P}(\mathcal{S}, H) \subset R_d$ of all Laurent polynomials with support in $\mathcal{S}$ and coefficients bounded in absolute value by H:

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$$\mathcal{P}(\mathcal{S}, H) = \{v \in R_d \mid \mathsf{supp}(v) \subseteq \mathcal{S} \quad \text{and} \quad \|v\|_\infty \leq H\}.$$

For every $\boldsymbol{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d$, we set $\|\boldsymbol{n}\| = \max\{|n_1|, \ldots, |n_d|\}$. Then the following is true.

THEOREM 7.3. (Gap theorem) *Suppose that $g \in R_d$ is primitive, irreducible, and atoral. For every $H \geq 1$, there exists an integer $m \geq 1$ with the following property: for every pair of sets $\mathcal{S}, \mathcal{S}' \subset \mathbb{Z}^d$ with distance*

$$d(\mathcal{S}, \mathcal{S}') := \min_{\boldsymbol{n} \in \mathcal{S}, \boldsymbol{n}' \in \mathcal{S}'} \|\boldsymbol{n} - \boldsymbol{n}'\| \geq m,$$

and for every $v = \sum_{\boldsymbol{n} \in \mathcal{S} \cup \mathcal{S}'} v_{\boldsymbol{n}} \boldsymbol{z}^{\boldsymbol{n}} \in \mathcal{P}(\mathcal{S} \cup \mathcal{S}', H)$ which is divisible by g, the restriction of v to $\mathcal{S}$,

$$v_{\mathcal{S}} = \sum_{\boldsymbol{n} \in \mathcal{S}} v_{\boldsymbol{n}} \boldsymbol{z}^{\boldsymbol{n}}, \tag{7.1}$$

is also divisible by g.

For the proof of Theorem 7.3, we consider the algebra $\ell^1(\mathbb{Z}^d, \mathbb{R})$ of all functions $v \colon \boldsymbol{n} \mapsto v_{\boldsymbol{n}}$ from $\mathbb{Z}^d$ to $\mathbb{R}$ with $\|v\|_1 = \sum_{\boldsymbol{n} \in \mathbb{Z}^d} |v_{\boldsymbol{n}}| < \infty$, furnished with its usual multiplication (or convolution) $(v, w) \mapsto v \cdot w$ and involution $w \mapsto w^*$, given by

$$(v \cdot w)_{\boldsymbol{n}} = \sum_{\boldsymbol{m} \in \mathbb{Z}^d} v_{\boldsymbol{m}} w_{\boldsymbol{n}-\boldsymbol{m}} = \sum_{\boldsymbol{m} \in \mathbb{Z}^d} v_{\boldsymbol{n}-\boldsymbol{m}} w_{\boldsymbol{m}}, \tag{7.2}$$

and

$$w^*_{\boldsymbol{m}} = w_{-\boldsymbol{m}} \tag{7.3}$$

for every $v, w \in \ell^1(\mathbb{Z}^d, \mathbb{R})$ and $\boldsymbol{m}, \boldsymbol{n} \in \mathbb{Z}^d$. If we denote by $\ell^1(\mathbb{Z}^d, \mathbb{Z}) \subset \ell^1(\mathbb{Z}^d, \mathbb{R})$ the set of all integer-valued elements of $\ell^1(\mathbb{Z}^d, \mathbb{R})$ and identify every $h = \sum_{\boldsymbol{n} \in \mathbb{Z}^d} h_{\boldsymbol{n}} \boldsymbol{z}^{\boldsymbol{n}} \in R_d$ with the element $(h_{\boldsymbol{n}})_{\boldsymbol{n} \in \mathbb{Z}^d} \in \ell^1(\mathbb{Z}^d, \mathbb{Z})$, we obtain an embedding

$$R_d = \ell^1(\mathbb{Z}^d, \mathbb{Z}) \subset \ell^1(\mathbb{Z}^d, \mathbb{R})$$

in which the multiplication $(h, h') \mapsto h \cdot h'$ of Laurent polynomials extends to the composition in equation (7.2) in $\ell^1(\mathbb{Z}^d, \mathbb{R})$. In fact, the multiplication $(v, w) \mapsto v \cdot w$ in equation (7.2) is also well defined for $w \in \ell^1(\mathbb{Z}^d, \mathbb{R})$ and $v \in \ell^\infty(\mathbb{Z}^d, \mathbb{R})$, the space of all bounded sequences $(v_{\boldsymbol{n}})_{\boldsymbol{n} \in \mathbb{Z}^d}$ in the supremum norm $\|v\|_\infty = \sup_{\boldsymbol{n} \in \mathbb{Z}^d} |v_{\boldsymbol{n}}|$, and

$$\|v \cdot w\|_\infty \leq \|v\|_\infty \|w\|_1$$

for all $w \in \ell^1(\mathbb{Z}^d, \mathbb{R})$ and $v \in \ell^\infty(\mathbb{Z}^d, \mathbb{R})$.

The shift action $\bar{\sigma}$ of $\mathbb{Z}^d$ on $\ell^\infty(\mathbb{Z}^d, \mathbb{R})$, defined exactly as in equation (2.1) by

$$(\bar{\sigma}^{\boldsymbol{m}} v)_{\boldsymbol{n}} = v_{\boldsymbol{m}+\boldsymbol{n}} \tag{7.4}$$

for every $\boldsymbol{m} \in \mathbb{Z}^d$ and $v \in \ell^\infty(\mathbb{Z}^d, \mathbb{R})$, extends to an action $w \mapsto w(\bar{\sigma})$ of $\ell^1(\mathbb{Z}^d, \mathbb{R})$ on $\ell^\infty(\mathbb{Z}^d, \mathbb{R})$ by bounded linear operators with

$$w(\bar{\sigma}) = \sum_{\boldsymbol{m} \in \mathbb{Z}^d} w_{\boldsymbol{m}} \bar{\sigma}^{\boldsymbol{m}} \colon \ell^\infty(\mathbb{Z}^d, \mathbb{R}) \to \ell^\infty(\mathbb{Z}^d, \mathbb{R})$$

for every $w \in \ell^1(\mathbb{Z}^d, \mathbb{R})$. Equation (7.4) implies that

$$(w(\bar{\sigma})v)_{\boldsymbol{n}} = \sum_{\boldsymbol{m}\in\mathbb{Z}^d} w_{\mathbf{m}}(\bar{\sigma}^{\boldsymbol{m}} v)_{\boldsymbol{n}} = \sum_{\boldsymbol{m}\in\mathbb{Z}^d} w_{\boldsymbol{m}} v_{\boldsymbol{m}+\boldsymbol{n}} = (w^* \cdot v)_{\boldsymbol{n}},$$

so that

$$w(\bar{\sigma})v = w^* \cdot v \tag{7.5}$$

for every $w \in \ell^1(\mathbb{Z}^d, \mathbb{R})$ and $v \in \ell^\infty(\mathbb{Z}^d, \mathbb{R})$ (cf. equation (7.3)).

We define a surjective group homomorphism $\eta\colon \ell^\infty(\mathbb{Z}^d, \mathbb{R}) \to \mathbb{T}^{\mathbb{Z}^d}$ by setting

$$\eta(v)_{\boldsymbol{n}} = v_{\boldsymbol{n}} \pmod 1 \tag{7.6}$$

for every $v = (v_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{Z}^d}$ and $\boldsymbol{n} \in \mathbb{Z}^d$. Note that η is shift-equivariant in the sense that

$$\eta \circ \bar{\sigma}^{\boldsymbol{n}} = \sigma^{\boldsymbol{n}} \circ \eta$$

for every $\boldsymbol{n} \in \mathbb{Z}^d$; more generally, if $w \in R_d = \ell^1(\mathbb{Z}^d, \mathbb{Z})$, then

$$\eta \circ w(\bar{\sigma}) = w(\sigma)\eta. \tag{7.7}$$

For every $x \in \mathbb{T}^{\mathbb{Z}^d}$, there exists a unique point $x^\# \in (-\frac{1}{2}, \frac{1}{2}]^{\mathbb{Z}^d} \subset \ell^\infty(\mathbb{Z}^d, \mathbb{R})$, called the *lift* of x, such that

$$\eta(x^\#) = x. \tag{7.8}$$

Let $g = \sum_{\boldsymbol{n}\in\mathbb{Z}^d} g_{\boldsymbol{n}} z^{\boldsymbol{n}} \in R_d$ be the Laurent polynomial appearing in the statement of Theorem 7.3 and set

$$f = g^* = \sum_{\boldsymbol{n}\in\mathbb{Z}^d} g_{\boldsymbol{n}} z^{-\boldsymbol{n}}.$$

Since $U(f) = U(g)$, f is again atoral and has non-trivial summable homoclinic points by Theorem 7.2.

LEMMA 7.4. *For every $x \in \mathbb{T}^{\mathbb{Z}^d}$, the following is true:*

(1) *$x \in X_f$ if and only if $f(\bar{\sigma})x^\# \in \ell^\infty(\mathbb{Z}^d, \mathbb{Z})$, that is, $x^\# \cdot f^* \in \ell^\infty(\mathbb{Z}^d, \mathbb{Z})$ (cf. equation (7.5));*

(2) *x is a non-trivial summable homoclinic point of α_f if and only if $x^\# \in \ell^1(\mathbb{Z}^d, \mathbb{R})$, $h := x^\# \cdot f^* \in \ell^1(\mathbb{Z}^d, \mathbb{Z}) = R_d$, and h is not divisible by f^* in R_d.*

Proof. (1) Suppose that $x \in \mathbb{T}^{\mathbb{Z}^d}$. By equation (7.7), we have

$$\eta(f(\bar{\sigma})x^\#) = f(\sigma)\eta(x^\#) = f(\sigma)x.$$

So, $x \in X_f$, that is, $f(\sigma)x = 0$ if and only if $f(\bar{\sigma})x^\# \in \ell^\infty(\mathbb{Z}^d, \mathbb{Z})$.

(2) If x is a non-trivial summable homoclinic point of α_f, then $x^\# \in \ell^1(\mathbb{Z}^d, \mathbb{R})$, and part (1) of this proof implies that $h = x^\# \cdot f^* \in \ell^1(\mathbb{Z}^d, \mathbb{Z}) = R_d$. If h were divisible by f^*, that is, if $x^\# \cdot f^* = h \cdot f^*$ for some $h \in R_d$, then $(x^\# - h) \cdot f^* = 0$ and [22, Theorem 2.1] would imply that $x^\# = h$ and $x = \eta(x^\#) = \eta(h) = 0$. This violates our conditions on x. The converse is obvious. □

Proof of Theorem 7.3. Since $f = g^*$ is atoral, there exists a non-trivial summable homoclinic point $x \in X_f$. Let $x^\# \in \ell^1(\mathbb{Z}^d, \mathbb{R})$ be the lift of x (cf. equation (7.8)), and let $h = f(\bar{\sigma})x^\# = x^\# \cdot f^* \in R_d$ (cf. Lemma 7.4). Since $x^\# \in \ell^1(\mathbb{Z}^d, \mathbb{R})$, there exists an integer $R = R(x, f, H)$ such that

$$\sum_{\|\boldsymbol{n}\| \geq R} |x^\#_{\boldsymbol{n}}| < \frac{1}{2H\|f\|_1}.$$

For every non-empty subset $\mathcal{S} \subset \mathbb{Z}^d$, we set

$$B_R(\mathcal{S}) = \{\boldsymbol{n} \in \mathbb{Z}^d \mid d(\boldsymbol{n}, \mathcal{S}) = \min_{\boldsymbol{n}' \in \mathcal{S}} \|\boldsymbol{n} - \boldsymbol{n}'\| \leq R\}.$$

Let $\mathcal{S}, \mathcal{S}' \subset \mathbb{Z}^d$ be two subsets of $\mathbb{Z}^d$ with distance $d(\mathcal{S}, \mathcal{S}') \geq 3R$. Suppose that a Laurent polynomial $v \in \mathcal{P}(\mathcal{S} \cup \mathcal{S}', H)$ is divisible by f^*, that is, that $v = \phi \cdot f^*$ for some $\phi \in R_d$. Then:

(i) $v \cdot x^\# \in R_d$;

(ii) $\mathsf{supp}(v \cdot x^\#) \subset B_R(\mathcal{S}) \cup B_R(\mathcal{S}')$.

Indeed, item (i) follows from Lemma 7.4(2):

$$v \cdot x^\# = (\phi \cdot f^*) \cdot x^\# = \phi \cdot (f^* \cdot x^\#) = \phi \cdot h \in R_d,$$

because both ϕ and h belong to R_d. For item (ii), we note that every $\boldsymbol{n} \notin B_R(\mathcal{S}) \cup B_R(\mathcal{S}')$ satisfies that $d(\boldsymbol{n}, \mathcal{S} \cup \mathcal{S}') > R$. Then $v_{\boldsymbol{n}-\boldsymbol{m}} = 0$ for all $\boldsymbol{m}$ with $\|\boldsymbol{m}\| \leq R$, and hence

$$|(v \cdot x^\#)_{\boldsymbol{n}}| = \bigg|\sum_{\boldsymbol{m} \in \mathbb{Z}^d} x^\#_{\boldsymbol{m}} v_{\boldsymbol{n}-\boldsymbol{m}}\bigg| \leq \|v\|_\infty \sum_{\|\boldsymbol{m}\| > R} |x^\#_{\boldsymbol{m}}| < H \cdot \frac{1}{2H\|f\|_1} \leq \frac{1}{2}.$$

Since $(v \cdot x^\#)_{\boldsymbol{n}} \in \mathbb{Z}$ by item (i), it follows that $(v \cdot x^\#)_{\boldsymbol{n}} = 0$.

Let ψ be the restriction of $v \cdot x^\#$ to $B_R(\mathcal{S})$, and let $v_{\mathcal{S}}$ and $v_{\mathcal{S}'}$ be the restrictions of v to $\mathcal{S}$ and $\mathcal{S}'$, respectively. Then $\psi \in R_d$ by item (i), and we claim that

$$\|\psi - v_{\mathcal{S}} \cdot x^\#\|_\infty < \frac{1}{2\|f\|_1}, \tag{7.9}$$

that is, that

$$|\psi_{\boldsymbol{n}} - (v_{\mathcal{S}} \cdot x^\#)_{\boldsymbol{n}}| < \frac{1}{2\|f\|_1} \quad \text{for every } \boldsymbol{n} \in \mathbb{Z}^d. \tag{7.10}$$

Indeed, if $\boldsymbol{n} \in B_R(\mathcal{S})$, then $d(\boldsymbol{n}, B_R(\mathcal{S}')) \geq R$, and hence

$$|(v_{\mathcal{S}'} \cdot x^\#)_{\boldsymbol{n}}| = \bigg|\sum_{\boldsymbol{m} \in \mathcal{S}'} v_{\boldsymbol{m}} x^\#_{\boldsymbol{n}-\boldsymbol{m}}\bigg| \leq \|v\|_\infty \sum_{\|\boldsymbol{\ell}\| \geq R} |x^\#_{\boldsymbol{\ell}}| \leq H \cdot \frac{1}{2H\|f\|_1} = \frac{1}{2\|f\|_1}. \tag{7.11}$$

Since $\psi_{\boldsymbol{n}} = (v_{\mathcal{S}} \cdot x^\#)_{\boldsymbol{n}} + (v_{\mathcal{S}'} \cdot x^\#)_{\boldsymbol{n}}$, it follows that

$$|\psi_{\boldsymbol{n}} - (v_{\mathcal{S}} \cdot x^\#)_{\boldsymbol{n}}| = |(v_{\mathcal{S}'} \cdot x^\#)_{\boldsymbol{n}}| < \frac{1}{2\|f\|_1}$$

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by equation (7.11). However, if $\boldsymbol{n} \notin B_R(\mathcal{S})$, then

$$|\psi_{\boldsymbol{n}} - (v_{\mathcal{S}} \cdot x^{\#})_{\boldsymbol{n}}| = |(v_{\mathcal{S}} \cdot x^{\#})_{\boldsymbol{n}}| = \bigg|\sum_{\boldsymbol{m} \in \mathcal{S}} (v_{\boldsymbol{m}} x^{\#}_{\boldsymbol{n}-\boldsymbol{m}})\bigg| \leq H \cdot \frac{1}{2H\|f\|_1} = \frac{1}{2\|f\|_1}.$$

This proves equation (7.10) for every $\boldsymbol{n} \in \mathbb{Z}^d$.

Since both $v_{\mathcal{S}} \cdot x^{\#} \cdot f^* = v_{\mathcal{S}} \cdot h$ and ψ lie in R_d, we have that $(\psi - v_{\mathcal{S}} \cdot x^{\#}) \cdot f^* \in R_d$, but the smallness of the coordinates of $\psi - v_{\mathcal{S}} \cdot x^{\#}$ in equation (7.10) implies that $(\psi - v_{\mathcal{S}} \cdot x^{\#}) \cdot f^* = 0$. Thus, we have proved that $\psi \cdot f^* = v_{\mathcal{S}} \cdot x^{\#} \cdot f^* = v_{\mathcal{S}} \cdot h$, where h is not divisible by f^* (cf. Lemma 7.4(2)). As $g = f^*$ is irreducible, we have proved that $v_{\mathcal{S}}$ is divisible by g, as claimed in the statement of this theorem.

This completes the proof of Theorem 7.3 with $m \geq 3R$. □

7.2. *The conditions (C1) and (C2): divisibility by f of lacunary polynomials.* According to Theorem 5.1, to prove Theorem 2.3, it suffices to prove that any irreducible and atoral polynomial $f \in R_d$ is m-good for a sufficiently large $m \in \mathbb{N}$. Now we are going to prove this and finish the proof of Theorem 2.3.

Theorem 7.3 has an immediate corollary which implies that any atoral polynomial is m-good for sufficiently large m.

COROLLARY 7.5. *Suppose that $f \in R_d$ is irreducible and atoral, and that $|\mathrm{supp}(f)| > 1$. Then there exists, for every $H \geq 1$, an integer $m \geq 1$ with the following property: for any set $\mathcal{S} \in \mathbb{Z}^d$ which is m-separated in the sense that*

$$\|\boldsymbol{k} - \boldsymbol{n}\| \geq m \quad \text{for any pair } \boldsymbol{k}, \boldsymbol{n} \in \mathcal{S},\ \boldsymbol{k} \neq \boldsymbol{n},$$

no non-zero polynomial $g \in \mathcal{P}(\mathcal{S}, H)$ is divisible by f.

Proof. For $H \geq 1$ and f fixed, choose m as in the statement of Theorem 7.3 (that is, $m \geq 3R$ in the proof of that theorem). Consider an arbitrary m-separated set $\mathcal{S}$ and any non-trivial polynomial $v = \sum_{\boldsymbol{n} \in \mathcal{S}} v_{\boldsymbol{n}} z^{\boldsymbol{n}} \in \mathcal{P}(\mathcal{S}, H)$.

If $|\mathrm{supp}(v)| = 1$, then v cannot be divisible by f, since $|\mathrm{supp}(f)| > 1$ by assumption. Assume therefore that $|\mathrm{supp}(v)| \geq 2$, and that v is divisible by f. Since for any $\boldsymbol{n} \in \mathrm{supp}(v)$, the sets

$$\mathcal{T} = \{\boldsymbol{n}\}, \quad \mathcal{T}' = \mathrm{supp}(v) \setminus \{\boldsymbol{n}\}$$

have distance at least m and hence, by Theorem 7.3, the restriction of v to $\mathcal{T}$, that is, $v_{\mathcal{T}} = v_{\boldsymbol{n}} z^{\boldsymbol{n}}$ must be divisible by f, which is impossible, v is not divisible by f. □

The condition that $|\mathrm{supp}(f)| > 1$ in Corollary 7.5 is obviously necessary: the polynomial $f = 2$ is obviously irreducible and atoral, and divides $2g$ for every $g \in R_d$ (irrespective of whether g is m-separated or not).

COROLLARY 7.6. *Suppose that $f \in R_d$ is irreducible and atoral, and that $|\mathrm{supp}(f)| > 1$. For all sufficiently large $m \geq 1$ and every $\boldsymbol{k} \in [0, m-1]^d \setminus \{\boldsymbol{0}\}$, no $v \in \mathcal{P}(m\mathbb{Z}^d \cup (m\mathbb{Z}^d + \boldsymbol{k}), 1)$ with $v \neq 0$ is divisible by f.*

Proof. Put $H = 1$ and let $m \geq 6R$, where R is the number appearing in the proof of Theorem 7.3. Suppose $v \in \mathcal{P}(m\mathbb{Z}^d \cup (m\mathbb{Z}^d + \boldsymbol{k}), 1)$ is a non-trivial polynomial divisible by f. Consider the decomposition $\mathsf{supp}(v) = \mathcal{S}_0 \sqcup \mathcal{S}_1$ where

$$\mathcal{S}_0 = \mathsf{supp}(v) \cap m\mathbb{Z}^d, \quad \mathcal{S}_1 = \mathsf{supp}(v) \cap (m\mathbb{Z}^d + \boldsymbol{k}).$$

Both sets $\mathcal{S}_0, \mathcal{S}_1$ are m-separated, as subsets of $m\mathbb{Z}^d$ and $m\mathbb{Z}^d + \boldsymbol{k}$, respectively.

We claim that for any $\boldsymbol{n} \in \mathcal{S}_0$, there exists $\boldsymbol{n}' = \boldsymbol{n}'(\boldsymbol{n}) \in \mathcal{S}_1$ such that $d(\boldsymbol{n}, \boldsymbol{n}') < 3R$. Otherwise, there exists $\boldsymbol{n} \in \mathcal{S}_0$ such that $d(\boldsymbol{n}, \mathcal{S}_1) \geq 3R$ so that $d(\boldsymbol{n}, \mathsf{supp}(v) \setminus \{\boldsymbol{n}\}) \geq 3R$. Then, by Theorem 7.3, the restriction of v to $\{\boldsymbol{n}\}$, that is, $\pm z^{\boldsymbol{n}}$, is divisible by f, which is impossible. Similarly, for any $\boldsymbol{n}' \in \mathcal{S}_1$, there exists $\boldsymbol{n} \in \mathcal{S}_0$ such that $d(\boldsymbol{n}, \boldsymbol{n}') < 3R$. Thus, the support of v is a union of distinct pairs:

$$\mathsf{supp}(v) = \bigcup_{\boldsymbol{n} \in \mathcal{S}_0} \{\boldsymbol{n}, \boldsymbol{n}'\},$$

where the distance within each pair is at most $3R$.

Given a pair $\{\boldsymbol{n}, \boldsymbol{n}'\}$, consider the decomposition of $\mathsf{supp}(v)$:

$$\mathcal{S} = \{\boldsymbol{n}, \boldsymbol{n}'\}, \quad \mathcal{S}' = \mathsf{supp}(v) \setminus \mathcal{S}.$$

The fact that $m \geq 6R$ implies $d(\mathcal{S}, \mathcal{S}') \geq 3R$. Indeed, $d(\boldsymbol{n}, \mathcal{S}') = d(\boldsymbol{n}, \boldsymbol{n}^*)$ for some $\boldsymbol{n}^* \in \mathcal{S}'$ and

$$\begin{cases} d(\boldsymbol{n}, \boldsymbol{n}^*) \geq m & \text{if } \boldsymbol{n}^* \in \mathcal{S}_0; \\ d(\boldsymbol{n}, \boldsymbol{n}^*) \geq d(\boldsymbol{n}', \boldsymbol{n}^*) - d(\boldsymbol{n}', \boldsymbol{n}) \geq m - 3R & \text{if } \boldsymbol{n}^* \in \mathcal{S}_1. \end{cases}$$

It follows that $d(\boldsymbol{n}, \mathcal{S}') > 3R$. Similarly, $d(\boldsymbol{n}', \mathcal{S}') > 3R$.

Applying Theorem 7.3 to $\mathcal{S}$ and $\mathcal{S}'$, we conclude that the restriction of v to $\mathcal{S} = \{\boldsymbol{n}, \boldsymbol{n}'\}$, that is,

$$v_{\mathcal{S}} = v_{\boldsymbol{n}} z^{\boldsymbol{n}} + v_{\boldsymbol{n}'} z^{\boldsymbol{n}'}, \quad v_{\boldsymbol{n}}, v_{\boldsymbol{n}'} \in \{-1, 1\},$$

must be divisible by f, which is impossible, since $v_{\mathcal{S}}$ is of the form

$$\pm z^{\boldsymbol{m}}(1 \pm z^{\boldsymbol{\ell}}), \quad \boldsymbol{m} \in \mathbb{Z}^d, \boldsymbol{\ell} \in \mathbb{N}^d,$$

and hence is a product of a unit $(\pm z^{\boldsymbol{m}})$ and a generalized cyclotomic polynomial $(1 \pm z^{\boldsymbol{\ell}})$, and thus must have zero Mahler measure $m(v_{\mathcal{S}}) = 0$. This implies that $\mathsf{m}(f) = 0$, in violation of Theorem 7.2. $\square$

Proof of Theorem 2.3. Since Remark 2.7 allows us to assume without loss in generality that the polynomial $f \in R_d$ is primitive and irreducible, the proof of Bohr chaoticity under the additional assumption of atorality of f is now complete.

Indeed, if $|\mathsf{supp}(f)| = 1$, atorality implies that we are in the situation of Example 2.5 with $p > 1$, so that (X_f, α_f) is Bohr chaotic. If $|\mathsf{supp}(f)| \geq 2$, Corollary 7.5 for $H = 2$ and Corollary 7.6 show that conditions (C1) and (C2) are satisfied. Therefore, Bohr chaoticity of (X_f, α_f) for irreducible atoral polynomials $f \in R_d$ follows from Theorem 5.1. $\square$

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8. *Concluding remarks: toral polynomials*

We have shown that a principal $\mathbb{Z}$-action is Bohr chaotic if and only if it has positive entropy. A principal $\mathbb{Z}^d$-action, $d > 1$, is shown to be Bohr chaotic if it has positive entropy and is atoral. We believe that the atorality assumption (equivalently, existence of a non-trivial summable homoclinic point) can be removed.

Irreducible toral polynomials come in two flavors: those for which X_f has no non-trivial homoclinic points (cf. e.g., [**19**, Example 7.1]), and those for which X_f has no summable homoclinic points, but uncountably many non-zero homoclinic points $v \in X_f$ with the property that $v^{\#} \cdot f^* \in f^* R_d$ (cf. e.g., [**19**, Example 7.3]; for notation, we refer to equation (7.8)). Unfortunately, none of these latter homoclinic points can be used in our proof of the gap theorem (Theorem 7.3), since the key Lemma 7.4 is not valid in this case.

Remarkably, for toral examples of the kind illustrated in [**19**, Example 7.1], we can still prove Bohr chaoticity by using Theorems 5.1 and 6.1.

THEOREM 8.1. *Let $g \in R_1$ be an irreducible polynomial with positive Mahler measure and with all roots of absolute value* 1*, and define $f \in R_d$ by $f(z_1, \ldots, z_d) = g(z_1)$. Then (X_f, α_f) is Bohr chaotic.*

Remark 8.2. There exist infinitely many distinct irreducible polynomials $g \in R_1$ with the properties required in Theorem 8.1. To see this, we follow a short note from math.stackexchange.com [**24**].

Let ϑ be a totally real algebraic number all of whose conjugates $\vartheta_1 = \vartheta, \vartheta_2, \ldots, \vartheta_r$ have absolute values strictly less than 2 (to find such a ϑ, take any totally real algebraic number and divide by a big integer). Assume also that ϑ is not itself an algebraic integer. Let β be a solution to the equation

$$\beta + 1/\beta = \vartheta.$$

All the conjugates of ϑ are, by assumption, real numbers in the interval $(-2, 2)$. This forces all the conjugates of β to be complex numbers of absolute value one—which is what we want. Moreover, β will not be a root of unity, since otherwise, ϑ would be an algebraic integer. Then β is a root of the polynomial

$$h(x) = \prod_{i=1}^{r} (x^2 - \vartheta_i x + 1).$$

Clearing denominators in h, one gets the desired polynomial g. It will be irreducible, because, by looking at infinite places, $[\mathbb{Q}(\beta) : \mathbb{Q}(\vartheta)] = 2$.

For example, $\vartheta = 1/2$ yields $g = 2z^2 - 1 + 2$, $\vartheta = -6/5$ yields $g = 5z^2 - 6z + 5$, $\vartheta = 1/\sqrt{2}$ yields $g = 2z^4 + 3z^2 + 2$, etc.

Proof of Theorem 8.1. Since g with $\mathsf{m}(g) > 0$ is m-good for some sufficiently large m by Theorem 6.1, $f(z_1, \ldots, z_d) = g(z_1)$ is also m-good, but now viewed as a polynomial in d-variables. Indeed, if $f(z) = g(z_1)$ divides a non-trivial polynomial h of the form

$$h(\boldsymbol{z}) = \sum_{\boldsymbol{n} \in \mathbb{Z}^d} \varepsilon_{\boldsymbol{n}} \boldsymbol{z}^{m\boldsymbol{n}}$$

with $\varepsilon_{\boldsymbol{n}} = \varepsilon_{(n_1,n_2,\ldots,n_d)} \in \{-2, -1, 0, 1, 2\}$, then by rewriting h as

$$\begin{aligned} h(\boldsymbol{z}) &= \sum_{(n_2,\ldots,n_d)\in\mathbb{Z}^{d-1}} \bigg(\sum_{n_1\in\mathbb{Z}} \varepsilon_{(n_1,n_2,\ldots,n_d)} z_1^{mn_1} \bigg) z_2^{mn_2} \cdots z_d^{mn_d} \\ &=: \sum_{(n_2,\ldots,n_d)\in\mathbb{Z}^{d-1}} h_{n_2,\ldots,n_d}(z_1^m) z_2^{mn_2} \cdots z_d^{mn_d}, \end{aligned}$$

we conclude that $g(z_1)$ must divide all polynomials $h_{n_2,\ldots,n_d}(z_1^m)$, some of which are non-zero. Since $g(z_1)$ is m-good, the resulting contradiction proves that condition (C1) is valid for f. For the proof of condition (C2), we can proceed similarly.

Theorem 8.1 now follows from Theorem 5.1. □

Theorem 8.1 obviously applies only to an extremely restricted class of toral polynomials. A more interesting example of an irreducible toral polynomial with positive entropy in [**19**, Example 7.3] is given by the polynomial

$$f(z_1, z_2) = 3 - z_1 - \frac{1}{z_1} - z_2 - \frac{1}{z_2}.$$

The unitary variety of f is a smooth real-analytic curve

$$\mathsf{U}(f) = \bigg\{ (e^{2\pi i s}, e^{2\pi i t}) : t = \pm\frac{1}{2\pi} \cos^{-1}\bigg(\frac{3}{2} - \cos 2\pi s\bigg), -\frac{1}{6} \le s \le \frac{1}{6} \bigg\}.$$

Moreover, $\mathsf{U}(f)$ is connected and has curvature bounded away from zero. As explained in [**19**, Example 7.3], one can use this fact to prove the existence of probability measures supported on $\mathsf{U}(f)$ whose Fourier transforms (as functions on $\widehat{\mathbb{S}^d} = \mathbb{Z}^d$) vanish at infinity. By translating this information back to the system (X_f, α_f), one obtains uncountably many homoclinic points $x \in X_f$ satisfying

$$|x_{\boldsymbol{n}}| \le \frac{C}{1 + \|\boldsymbol{n}\|^{\frac{1}{2}}} \quad \text{for all } \boldsymbol{n} \in \mathbb{Z}^2.$$

Unfortunately, none of these homoclinic points can be used in our proof of the gap theorem (Theorem 7.3), since they are not summable.

It would be interesting to see whether one can prove the gap theorem for $f = 3 - z_1 - (1/z_1) - z_2 - (1/z_2)$ directly, using some elementary methods, or establish Bohr chaoticity of X_f by some other means.

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