Neutrino transport in core-collapse supernovae
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Chapter 1

Neutrino transport

In Sections 1.2-1.4 of this chapter, the two-moment equations of neutrino transport are presented. These equations are the basis of all subsequent chapters. The link between two-moment transport (TMT) and flux limited diffusion (FLD), a related transport scheme, is discussed in Section 1.5. But first some essentials of radiation transport theory, from which both TMT and FLD derive, are summarised, including a short review of how neutrino transport combines with the matter hydrodynamics into a set of equations used for core collapse calculations.

Neutrino transport is equivalent to radiation transport in general, because he “left hand side” of the Boltzmann equation is the same for any kind of radiation. For an introduction to radiation transport theory, one may turn to standard literature, like Mihalas & Mihalas (1984), Pomraning (1973), or Duderstadt & Martin (1979). Neutrino and photon transport differ in the quantum statistics: neutrinos are fermions, while photons are bosons. In what follows, this difference is apparent in just one expression, Eq.(1.19), the neutrino equilibrium distribution, being of the Fermi-Dirac type. Another difference is that neutrinos carry a chemical potential $\mu_\nu$, whereas photons have chemical potential equal to zero.

In the following, the symbols $c$, $h$, and $k$ stand, in line with commonplace notation, for the speed of light in vacuum, the Planck and the Boltzmann constant, respectively.

1.1 Transport equation

Neutrinos in core collapse are described by a distribution function $\mathcal{F}(x, p)$, defined such that

$$dN = \frac{1}{h^3} d^3x \, d^3p \, \mathcal{F}(x, p)$$

(1.1)

is the number of neutrinos in a (space-like) phase space volume $d^3x \, d^3p$. The distribution function $\mathcal{F}(x, p)$ depends on space-time coordinates $x = x^\alpha = (ct, \chi)$ and neutrino four-momentum $p = p^\alpha = (\omega/c)(1, \hat{\Omega})$, with $\omega = p^0c$ the neutrino energy, and $\hat{\Omega} = p/p^0$ the propagation direction. It is assumed here that neutrinos can be treated as massless particles, i.e., $p^0 = |p|$, and $\hat{\Omega}$ is a unit vector.

The evolution of $\mathcal{F}(x, p)$ is given by the Boltzmann transport equation, which has the form\(^1\):

$$\frac{1}{h^3} p^\alpha \mathcal{F}_{,\alpha} = C[\mathcal{F}] \quad .$$

(1.2)

The left hand side is the streaming term, the right hand side the collision kernel containing the interaction between matter and neutrinos. The neutrino-matter interactions incorporated in transport computations in this thesis are summarised in Table A.1 in Appendix A, where also expressions for the collision kernels are given. The transport equation is written in covariant

\(^1\)Strictly speaking, “Boltzmann equation” applies to systems in which the transported particles are also the collision targets, while the term “transport equation”, appropriate for neutrino transport, refers to the motion of particles through a medium. Both denominations will be used in the text.
Neutrino transport

form, derived first by Lindquist (1966) to account for special relativistic effects of fluid motion or general relativistic gravitational redshift. But even for classical transport, which is considered here, covariant notation is a practical approach when the streaming term is to be written in curvilinear coordinates. In a global inertial system the transport equation has the classical form

$$\frac{1}{c} \partial_t \mathcal{F} + \hat{\Omega} \cdot \nabla \mathcal{F} = B[\mathcal{F}]$$

(1.3)

with $B = \frac{b}{c} \omega C$. Neutrinos in this thesis are all living in spherical symmetry, with a notion of space via the radial coordinate $r$ and the cosine of the polar angle $\mu = \hat{\Omega} \cdot \hat{r}$. Then $\mathcal{F} = \mathcal{F}(r, t, \omega, \mu)$, and Eq.(1.3) takes the form

$$\frac{1}{c} \partial_t \mathcal{F} + \mu \partial_r \mathcal{F} + \frac{1 - \mu^2}{r} \partial_\mu \mathcal{F} = B[\mathcal{F}]$$

(1.4)

This is the basic equation for calculations of neutrino transport in subsequent chapters.

1.1.1 Radiation hydrodynamics

Although the dynamics of the material fluid itself is not the subject matter of the subsequent chapters, it is important to get a feel of how neutrino transport combines with the hydrodynamics through the exchange of energy, momentum, and lepton number between matter and neutrinos, given by Eqs.(1.10-1.11).

The equations describing the evolution of a neutrino-radiating hydrodynamical system derive from conservation laws expressing the conservation of baryon number, lepton number, and energy-momentum, of matter ($m$) and radiation ($\nu$) as a whole. Conservation of these quantities is stated by demanding that the four-divergence of the elementary flows, i.e., the baryon four-flow $N_B^\alpha$, lepton four-flow $N_L^\alpha$, and energy-momentum flow $T^{\alpha\beta}$, vanishes:

$$N_B^\alpha = 0$$ (baryon number conservation) ,

(1.5)

$$N_L^\alpha = 0$$ (lepton number conservation) ,

(1.6)

$$T^{\alpha\beta} = 0$$ (energy-momentum conservation) .

(1.7)

These equations can be interpreted as fluid equations that describe the evolution of the matter component, with the neutrino component entering as extra "sources". In radiation hydrodynamics, they are to be solved together with the transport equation (1.2) for the neutrino component. See Baron et al. (1989) for explicit expressions of the fluid equations in a general relativistic metric (spherical symmetry).

For a non-radiating (all neutrino flows zero) gas, in the classical limit, Eq.(1.5) becomes the mass-continuity equation (after multiplication with a mass-unit), Eq.(1.6) is redundant, and Eq.(1.7) represents the energy equation ($\alpha = 0$) and the Navier-Stokes equations ($\alpha = 1, 2, 3$). Mihalas & Mihalas (1984, Ch.4) demonstrate how the basic fluid equations are retrieved from the covariant derivatives in this case.

The lepton flow and energy-momentum flow of the neutrino component are given by (de Groot, van Weert & van Leeuwen, 1980)

$$N_{Lm}^\alpha(x) = \frac{c}{h^2} \int \frac{d^3p}{p^0} p^\alpha \mathcal{F}(x, p)$$

(1.8)

$$T_{\nu}^{\alpha\beta}(x) = \frac{c}{h^2} \int \frac{d^3p}{p^0} p^\alpha p^\beta \mathcal{F}(x, p)$$

(1.9)
1.2 Angular moments

Taking the four-divergence of these, and using Eq.(1.2), we have

\[ I(x) \equiv \left[ N_{\nu}^\alpha \right]_{\alpha \beta} = c \int \frac{d^3p}{p^0} C[F] \ , \quad (1.10) \]

\[ Q^\alpha (x) \equiv T_{\nu}^{\alpha \beta} = c \int \frac{d^3p}{p^0} p^\beta C[F] \ , \quad (1.11) \]

which define the lepton and energy-momentum transfer rates. These quantities enter as sources in the fluid equations, as can be seen by writing Eqs.(1.6-1.7) explicitly as

\[ N_{\nu}^\alpha = -I \ , \quad (1.12) \]

\[ T_{\nu}^{\alpha \beta} = -Q^\alpha \ . \quad (1.13) \]

The transfer rates are essential quantities in supernova theory, for they determine the amount of deleptonisation during the infall phase, the lepton and energy losses when the hydrodynamical shock breaks through the neutrino-sphere, how much energy matter may absorb during later phases in the hot outer atmosphere of the cooling nascent neutron star, etc.

For spherical symmetry, we write

\[ SI \equiv I/n_b = \frac{4\pi c}{n_b(hc)^3} \int_0^\infty d\omega \omega^2 \frac{1}{2} \int_{-1}^1 d\mu B[F] \ , \quad (1.14) \]

\[ SQ \equiv cQ^0/n_b = \frac{4\pi c}{n_b(hc)^3} \int_0^\infty d\omega \omega^3 \frac{1}{2} \int_{-1}^1 d\mu B[F] \ , \quad (1.15) \]

\[ SA \equiv Q^1/n_b = \frac{4\pi}{n_b(hc)^3} \int_0^\infty d\omega \omega^3 \frac{1}{2} \int_{-1}^1 d\mu B[F] \ , \quad (1.16) \]

normalized with respect to the local baryon density $n_b$. The lepton transfer rate is in units $[SI] = s^{-1}$, the energy transfer rate $[SQ] = \text{erg} \ s^{-1}$, and the momentum transfer rate $[SA] = \text{erg cm}^{-1}$.

1.2 Angular moments

In core collapse calculations the numerical solution of the transport equation (1.4) may be too much to ask for. The computational effort is simply too large to give enough results in, say, a thesis project time. In practically all core collapse calculations (see references in the Introduction), approximate transport schemes are used, with flux limited diffusion being the most fashionable. Two-moment neutrino transport, closely related to FLD, was first developed and applied by Dgan & Cernohorsky (1991), and used in the work of Cernohorsky & van Weert (1992). The methods are designed to describe transport phenomena in systems with a high degree of nonuniformity, typically a medium with an opaque interior where radiation is diffusive, surrounded by an extended atmosphere where the matter is eventually totally transparent and radiation is free streaming.

The two approximate schemes, FLD and TMT, are based on reducing the dimensionality of the problem by considering the first three angular moments

\[ \{ E_\omega, F_\omega/c, \bar{F}_\omega \} \equiv \int_{4\pi} d\Omega \{ 1, \Omega, \Omega \otimes \Omega \} F(\omega, \Omega) \quad (1.17) \]
of the distribution function, and solving for these moments, rather than for the full angular dependence of $F(\omega, \Omega)$. The subscripts indicate that the angular moments are spectral quantities, depending on the neutrino energy. To illustrate the procedure, we consider a particular collision kernel in the transport equation:

$$1 \frac{1}{c} \frac{\partial}{\partial t} F + \Omega \cdot \nabla F = \kappa_a (F^0 - F) + \kappa_s \left( \frac{E_\omega}{4\pi} - F \right) + \kappa_s \frac{F_\omega \cdot \Omega}{4\pi c} \quad (1.18)$$

(see Appendix A). Absorption and emission are contained in the first term on the r.h.s., with the absorption opacity $\kappa_a(\omega)$. The coefficients $\kappa_a(\omega)$ and $\kappa_s(\omega)$ account for iso-energetic scattering, composed of an isotropic ($\kappa_i$) and an anisotropic ($\kappa_a$) part. To avoid unnecessarily complicating the formulae, Compton-scattering ($\nu e \rightarrow \nu'e'$) has been excluded from the collision kernel. The equilibrium neutrino distribution is

$$F^0(\omega) = \frac{1}{e^{(\omega - \mu_\nu)/kT} + 1}, \quad (1.19)$$

with $\mu_\nu$ the neutrino equilibrium chemical potential, given by

$$\mu_\nu = \mu_e - (\mu_n - \mu_p). \quad (1.20)$$

The electron, proton, and neutron chemical potentials $\mu_e$, $\mu_p$, and $\mu_n$ follow from the equation of state.

The first two angular moments of the transport equation are obtained by taking the angular averages $\int_{4\pi} d\Omega$ and $\int_{4\pi} d\Omega \Omega$ of Eq.(1.18), and this gives

$$\frac{1}{c} \frac{\partial}{\partial t} E_\omega + \nabla \cdot F_\omega = c \kappa_s (B_\omega - E_\omega) \quad (1.21)$$

$$\frac{1}{c^2} \frac{\partial}{\partial t} F_\omega + \nabla \cdot \tilde{F}_\omega = -\kappa F_\omega / c, \quad (1.22)$$

where $\kappa = \kappa_a + \kappa_s - \frac{1}{2} \kappa_s$ and $B_\omega = 4\pi F^0$. Equations (1.21) and (1.22) will be referred to as the spectral energy balance and spectral momentum balance equations, where “spectral” means that everything depends on the neutrino energy $\omega$.

### 1.3 Two-moment transport

In TMT, the assumption is made that the angular moments in Eq.(1.17) suffice as a description of the radiation field, for which the evolution is prescribed by the two moment equations, Eqs.(1.21) and (1.22) above. Because the problem contains more variables than equations, an additional relation $\tilde{F}_\omega(E_\omega, F_\omega)$ must be supplied. This relation is usually established in terms of the ratios

$$\tilde{f} = F_\omega / (c E_\omega) \quad \text{and} \quad \tilde{\rho} = \tilde{F}_\omega / E_\omega, \quad (1.23)$$

via a prescribed “closure” of the form

$$\tilde{\rho} = \tilde{\rho}(\tilde{f}). \quad (1.24)$$

For systems with local axial symmetry, there is a preferred direction $\hat{a}$ (which is the radial direction in our case of a spherical coordinate system), and consequently, the radiation field
1.4 Flux limited diffusion

has a simple angular dependence: \( F(\hat{n}) = F(\mu) \), with \( \mu = \hat{n} \cdot \hat{n} \). This allows to make the decompositions

\[
\bar{f} = f \hat{n} \quad \text{and} \quad \bar{p} = \frac{1}{2}(1-p)\bar{I} + \frac{1}{2}(3p-1)\hat{n}\hat{n}
\]

with

\[
\{f(\omega), p(\omega)\} = \frac{1}{2E_\omega} \int_{-1}^{+1} d\mu \{\mu, \mu^2\} F(\omega, \mu).
\]

The closure,

\[
p = p(f)
\]

is known as the variable Eddington factor. The functional form of \( p(f) \) must be such, that it accurately mimics the behaviour of radiation in two limiting cases. In the diffusion limit, radiation is isotropic \((\partial F/\partial \mu = 0)\), and we have \( f = 0 \), and \( p = 1/3 \):

\[
\lim_{f \to 0} p(f) = \frac{1}{3}
\]

The free streaming limit is characterised by radiation flowing in a single direction \((F \propto \delta^{(2)}(\hat{n} \cdot \hat{n}))\), and we have

\[
\lim_{f \to 1} p(f) = 1
\]

An additional constraint (Levermore 1984),

\[
f^2 \leq p \leq 1
\]

follows from the fact that \( f \) and \( p \) are normalised averages of a distribution.

In spherical symmetry, the two-moment equations, Eqs (1.21,1.22) become

\[
\frac{1}{c} \partial_t e + \frac{1}{r^2} \partial_r (r^2 ef) = \kappa (b-e) ,
\]

\[
\frac{1}{c} \partial_t (ef) + \frac{1}{r^2} \partial_r (r^2 ep) + \frac{(p-1)e}{r} = -\kappa ef ,
\]

where we have introduced the occupation density \( e(\omega) \equiv E_\omega/(4\pi) \), and \( b \equiv B_\omega/(4\pi) = F^0 \). These equations, with a suitable prescription \( p(f) \) to close the set, are used in the following chapters in neutrino transport calculations.

1.4 Flux limited diffusion

Flux limited diffusion deals with only the spectral energy balance equation (1.21), with a closure at the lowest level, \( F_\omega = F_\omega(E_\omega) \). The spectral momentum balance equation is not entirely ignored, but serves as a guide in deriving a closure. For example, with the Eddington approximation, \( P_\omega = \frac{1}{3} E_\omega \bar{I} \), and assuming \( \partial_t F_\omega = 0 \), Eq.(1.22) gives

\[
F_\omega = -\frac{c}{3} \nabla E_\omega / \kappa.
\]
With this diffusion expression for the flux substituted into Eq. (1.21), a diffusion equation for \( E_\omega(r, t, \omega) \) is obtained. For atmospheric problems, however, in regions where \( \kappa \to 0 \), this expression does not obey the physical requirement that the spectral flux \( F_\omega \) cannot exceed the spectral energy density \( E_\omega \) times \( c \) (stated also in Eq. (1.30)). Flux limited diffusion theories (Levermore 1979, Levermore & Pomraning 1981, Levermore 1984) modify Eq. (1.33) to

\[
F_\omega = cE_\omega \lambda(R) R
\]  

(1.34)

with

\[
R = -\frac{\nabla E_\omega}{\kappa_{\text{eff}} E_\omega},
\]

(1.35)

and \( \kappa_{\text{eff}} \) defined below in Eq. (1.39). Here \( \lambda(R) \) is the "flux limiter", whose main property is to tend to zero as \( |\nabla \ln E_\omega| / \kappa_{\text{eff}} \) grows indefinitely, in such a way that \( |F_\omega| \leq E_\omega c \) remains satisfied:

\[
\lim_{R \to \infty} R \lambda(R) = 1 \quad .
\]

(1.36)

In the diffusion limit, reached when the effective mean free path \( 1/\kappa_{\text{eff}} \) is much shorter than the local scale height \( 1/|\nabla \ln E_\omega| \), the flux limiter must tend to one third:

\[
\lim_{R \to 0} \lambda(R) = 1/3 \quad .
\]

(1.37)

Levermore (1984) showed how in general a variable Eddington factor \( p(f) \) can be used to construct a flux limiter \( \lambda(R) \). His basic idea was to assume that spatial and temporal derivatives of \( f(x, t) \) and \( \tilde{p}(x, t) \) are small compared with derivatives of \( E_\omega(x, t) \). These derivatives vanish in the diffusion and free streaming limit, so the assumption is that they are small in the intermediate regime where matter and radiation decouple. Using the energy balance equation (1.21) to eliminate the \( \partial_t E_\omega \) term, Levermore wrote the spectral momentum balance equation (1.22) as

\[
(\tilde{p} - pf) \cdot \nabla E_\omega + E_\omega (\partial f + \nabla \cdot \tilde{p} - f \nabla \cdot f) = -\kappa_{\text{eff}} E_\omega f \quad ,
\]

(1.38)

with the effective opacity

\[
\kappa_{\text{eff}} = \kappa_s B_\omega / E_\omega + \kappa_s - 1/3 \kappa_s \quad .
\]

(1.39)

Under the prevailing assumption, the second term in Eq. (1.38) can be dropped. Then, restricting \( f \) and \( \tilde{p} \) to be of the form given by Eq. (1.25) (this restriction is not necessary, but simplifies the discussion), Eq. (1.38) reduces to

\[
f = (p - f^2) R \quad ,
\]

(1.40)

or, equivalently an expression for the flux

\[
F_\omega = (p - f^2) cE_\omega R \quad .
\]

(1.41)

Comparing the last equation with Eq. (1.34), we have the identification

\[
\lambda(R) = p - f^2 \quad .
\]

(1.42)

This expression relates a given closure \( p(f) \) to a flux limiter \( \lambda(R) \) and vice versa.

An example may help to illustrate the procedure. Take as a closure: \( p(f) = \frac{1}{2} - \frac{1}{2} f + f^2 \). This closure has the properties Eq. (1.28) and (1.29). Substitution in Eq. (1.42) gives \( \lambda = \frac{1}{2} (1 - f) \), and using this in \( f = \lambda R \), we find Bowers & Wilson's "minimal" flux limiter (Bowers & Wilson, 1982a): \( \lambda(R) = 1/(3 + R) \), which has the required properties Eq. (1.36) and (1.37). Levermore (1984) gives a variety of closures and their corresponding flux limiter.
1.5 TMT versus FLD

From the last two sections it should be clear that two-moment transport and flux limited diffusion are closely related. By discarding several terms in the spectral momentum balance equation, one gets FLD from TMT by relating the variable Eddington factor used in TMT to a flux limiter for FLD. The last statement also implies that TMT is more accurate than FLD: in FLD it must be assumed that some terms can be discarded because they are not important. The assumption is, that the collection of terms,

\[
\left(\frac{1}{c} \partial_t f + \nabla \cdot \mathbf{p} - f \nabla \cdot \mathbf{f} \right) \equiv \delta ,
\]

is negligible in the spectral momentum balance equation, Eq.(1.38). It was shown by Cernohorsky & van den Horn (1990) that this assumption was explicitly violated in solutions they obtained with FLD. They quantified the "error" \( \delta \) in the a posteriori computed momentum balance. Janka (1991, 1992), from neutrino transport calculations using Monte Carlo techniques, demonstrated that \( \delta \) is even the dominant term that determines the evolution and the flow pattern of the neutrino flux in the nearly transparent outer layers of collapsed iron cores.

To correct for this error, Janka (1991) and Dgani & Janka (1992) proposed a subtle modification of the FLD scheme by introducing the concept of an "artificial opacity". Rewriting the spectral momentum balance equation, Eq.(1.38), slightly,

\[
(\mathbf{p} - \mathbf{f}) \cdot \nabla \ln E_\nu + \Xi_{\text{art}} f = - \kappa_{\text{eff}} f ,
\]

with the artificial opacity defined by

\[
\Xi_{\text{art}} f = \delta ,
\]

it follows that

\[
(\mathbf{p} - \mathbf{f}) \cdot \mathbf{R} = f ,
\]

but now with \( \mathbf{R} \) defined by

\[
\mathbf{R} = - \frac{\nabla \ln E_\nu}{\kappa_{\text{eff}} + \Xi_{\text{art}}} .
\]

The last equation illustrates how \( \Xi_{\text{art}} \), with dimension of an inverse mean free path, got its name. With a given flux limiter \( \lambda(R) \), using Eq.(1.47) instead of Eq.(1.35) in

\[
f = \lambda(R) \mathbf{R} ,
\]

an FLD scheme results that does not violate momentum balance, because the artificial opacity accounts for the error otherwise made in standard FLD where it is assumed that \( \Xi_{\text{art}} = 0 \).

Naturally, we can’t have our cake and eat it: with the introduction of artificial opacity we are back at the original moment equations, since Eq.(1.46), with Eq.(1.47), is really the spectral momentum balance equation rewritten. This was also noted by Dgani & Janka (1992), who presented FLD plus artificial opacity as a numerical alternative to TMT with a variable Eddington factor. It is pointed out however, that the fundamental set of equations is not changed. In that sense, the term "diffusion" is inappropriate for FLD plus artificial opacity, because we are no longer solving a diffusion equation. A diffusion equation is a parabolic partial differential equation. The two-moment equations are actually a hyperbolic set of equations, as will be discussed in detail in Chapter 3.

\[2\]It is commented here that in more than one dimension, in general \( \delta \) will not be parallel to \( f \) and a scalar artificial opacity cannot be defined then.