The expansion of the neutrino-electron scattering rate in a Legendre series of the scattering angle is extended to include quadratic terms. This extension provides a considerable improvement of the 'fit' to the scattering rate. On the other hand, the effect of the quadratic terms on the neutrino transport during the infall phase of a Type II supernova is found to be negligible. This is partly due to the specific state of the matter background and the shape of the neutrino spectra which suppress the phase space where the quadratic Legendre approximant could be significant. Furthermore, the intrinsic structure of the Boltzmann equation causes the suppression of (nearly) coherent scattering in the forward direction where the scattering rate deviates most from a linear approximation.

5.1 Introduction

Neutrino-electron scattering (NES) plays an important role in the infall phase of a Type II supernova explosion (Bowers & Wilson 1982b; Brue"enn 1985; Myra et al. 1987; Brue"enn 1988; Mezzacappa & Brue"enn 1993c). The transport of neutrinos is governed by the Boltzmann-equation in which NES enters through the collision kernel (see also Eq.(4.1))

$$B_{\nu}[\mathcal{F}_\nu] = \int d^3p_{\nu'} \left[ R^{\text{in}}(p_{\nu}, p_{\nu'}) \mathcal{F}_{\nu'} \left(1 - \mathcal{F}_{\nu}\right) - R^{\text{out}}(p_{\nu}, p_{\nu'}) \mathcal{F}_{\nu} \left(1 - \mathcal{F}_{\nu'}\right) \right]$$

(5.1)

In this equation $\mathcal{F}_{\nu}(x, p)$ is the neutrino distribution function, which depends on spacetime coordinates $x = (t, \mathbf{x})$ and the neutrino four-momentum $p_{\nu} = (\omega, \mathbf{\hat{\Omega}})$. The primed distribution function has the following meaning: $\mathcal{F}_{\nu'} = \mathcal{F}_{\nu}(x, p_{\nu'})$. The scattering rates $R^{\text{in/out}}_{\nu}(\omega, \omega', \mathbf{\hat{\Omega}} \cdot \mathbf{\hat{\Omega}}')$ also depend on the matter temperature $T$ and electron degeneracy $\xi_e = \mu_e/T$, but this has not been explicitly indicated in equation (5.1).

In most practical situations where neutrino transport is computed in a supernova setting, the Boltzmann equation is not solved directly, but some approximate method is used. In the previous chapter, the role of NES was investigated in the two-moment transport (TMT) approach. In TMT, as in flux limited diffusion (FLD), angular moments of the Boltzmann equation are taken and then solved for the angular moments of the distribution function; for a description of TM and FLD theory, see Chapter 1. This approach requires that the scattering rates are expanded in a power series of the cosine of the scattering angle, $\cos \theta = \mathbf{\hat{\Omega}} \cdot \mathbf{\hat{\Omega}}'$, in order to write the collision kernel in terms of the angular moments of the distribution function.

For other scattering processes playing a role in gravitational collapse (like iso-energetic scattering on nucleons and nuclei), a Legendre expansion up to first order is sufficient, because
Legendre expansion of NES

5.2 Legendre expansion of NES

The $N$th order Legendre approximation of the NES scattering rate $R^{out}$ is written as follows:

$$R^{(N)}_{out}(\omega, \omega', \cos \theta) = \frac{1}{4\pi} \sum_{l=0}^{N} (2l + 1) \Phi_l^{out}(\omega, \omega') P_l(\cos \theta) ,$$

(5.2)

where the $P_l(\cos \theta)$ are Legendre polynomials. A similar approximation can be written for $R^{in}$. The expansion coefficients are obtained by evaluating

$$\Phi_l^{out}(\omega, \omega') = 2\pi \int_{-1}^{1} d(\cos \theta) P_l(\cos \theta) R^{out}(\omega, \omega', \cos \theta) ,$$

(5.3)

which, following Yueh & Buchler (1977), leads to expressions of the kind

$$\Phi_l^{out}(\omega, \omega') = \frac{C}{\omega_0^2 \omega^2} \int_{\max(0, \omega' - \omega)}^{\infty} dE F_\epsilon(E) [1 - F_\epsilon(E + \omega - \omega')] H_l(\omega, \omega', E)$$

(5.4)

Here, $F_\epsilon(E)$ is the distribution function of electrons with energy $E$, see Eq.(4.6). The dimensional constant $C = 2 (2\pi)^{-3} G_F^2/(\hbar c)^6 = 5.5590 \times 10^{-14}$ MeV$^{-5} \text{ cm}^{-1}$, with the Fermi-constant $G_F/(\hbar c)^3 = 1.16639 \times 10^{-11}$ MeV$^{-2}$. Then $R^{(N)}_{out}$ has dimensions MeV$^{-3} \text{ cm}^{-1}$.

The functions $H_l(\omega, \omega', E)$ for $l = 0$ and $l = 1$ were derived by Yueh & Buchler$^1$ (1977). To go one step beyond this we computed the quadratic term $H_2(\omega, \omega', E)$; an expression is given in Appendix 5.A. The integrals in Eq.(5.4) over electron energy have to be done numerically when the electrons are arbitrarily degenerate. We used a 30-point Gauss-Legendre integration combined with a trailing 10-point Gauss-Laguerre integration.

Next, we compare the angular dependence of $R_{out}^{\epsilon}(\omega, \omega', \cos \theta)$ with the two Legendre approximants $R^{(1)}_{out}$ and $R^{(2)}_{out}$. Figure 5.1 shows the scattering rate $R_{out}^{\epsilon}(\omega, \omega', \cos \theta)$ as a function of the scattering angle at a fixed incoming neutrino energy $\omega$ for several scattered energies $\omega'$. The electron temperature and degeneracy, $T = 1.6$ MeV and $\xi_e = 17$ respectively, were chosen to represent the matter at infall (before bounce) in a typical supernova collapse. Down-scattering is the dominant process and therefore we depict only $\omega' < \omega$. The incoming energy $\omega = 30$ MeV is taken slightly above the Fermi-energy of the electrons. One sees from Fig.5.1 that the functional form of the scattering rate varies considerably: $R_{out}(\cos \theta)$ is an almost linear function of $\cos \theta$ for $\omega' = 5$ MeV, whereas it becomes very nonlinear when $\omega'$ approaches $\omega$. Figures 5.2a,b each show a selected curve from Fig.5.1 and their Legendre approximants $R^{(1)}_{out}$ (dashed curves) and $R^{(2)}_{out}$ (dotted curves). In Fig.5.2a the quadratic approximation evidently provides a better ‘fit’ to the scattering rate than the linear approximation. The fit becomes better as the transferred energy $\omega - \omega'$ increases. In Fig.5.2b, $R_{out}^{\epsilon}(\cos \theta)$ is more complicated, and

$^1$A few corrections are given by Bruenn (1985, below equation C.50)
5.2 Legendre expansion of NES

Figure 5.1: The scattering rate $R_{\text{out}}(\omega, \omega', \cos \theta)$ (in units $C = 5.559 \times 10^{-14} \text{ MeV}^{-3} \text{ cm}^{-1}$), at fixed neutrino energy $\omega = 30 \text{ MeV}$ and six different $\omega'$ energies: 1, 5, 10, 15, 25 and 28 MeV. The neutrino energies $\omega'$ are indicated in the figure. The electrons are taken at a temperature $T = 1.6 \text{ MeV}$ and degeneracy $\xi_e = \mu_e/T = 17$ (dimensionless).

Figure 5.2: Selected curves from Fig.5.1 with $\omega' = 15 \text{ MeV}$ (left panel) and $\omega' = 28 \text{ MeV}$ (right). Shown are $R_{\text{out}}$ (solid line), the linear Legendre approximant (dashed line) and the quadratic Legendre approximant (dotted line).

in terms of a good fit, neither of the two approximants seems to be the better one. The peak of $R_{\text{out}}$ near $\cos \theta = 1$ in Fig.5.2b becomes more pronounced as $\omega'$ approaches $\omega$ and eventually (not shown in our figure) becomes a delta-peak in the limit of coherent scattering, $\omega' = \omega$. The picture sketched above is found for a wide range of the neutrino energies $\omega$ and $\omega'$, matter temperature and electron degeneracy.
5.3 Quadratic NES in two-moment equations

How NES is incorporated in two-moment neutrino transport was explained in Section 4.2.2 of the previous chapter. Below we simply list the modifications of relevant formulae when the quadratic term \( \Phi_2(\omega, \omega') \) is taken into account.

**Boltzmann equation**

When the Legendre expansion, Eq.(5.2) is substituted in Eq.(5.1), the scattering kernel can be written as:

\[
B_{\text{NES}} = \kappa^0 - \mathcal{F}_\nu \kappa^\rho + \sum_i \Omega_i \hat{\kappa}_i^f - \mathcal{F}_\nu \sum_i \Omega_i \kappa^\rho_i + \sum_{i,j} \Omega_i \Omega_j \hat{\kappa}_{ij}^p - \mathcal{F}_\nu \sum_{i,j} \Omega_i \Omega_j \kappa^\rho_{ij},
\]

which is equation Eq.(4.13) with additional terms involving \( \hat{\kappa}^p \) and \( \kappa^\rho \):

\[
\hat{\kappa}_{ij}^p = 5 \int d\omega' \omega'^2 e(\omega') \frac{1}{2} [3p_{ij}(\omega') - \delta_{ij}] \Phi_2^p(\omega, \omega') \]

\[
\kappa_{ij}^\rho = 5 \int d\omega' \omega'^2 e(\omega') \frac{1}{2} [3p_{ij}(\omega') - \delta_{ij}] \left[ \Phi_2^{\rho \text{in}}(\omega, \omega') - \Phi_2^{\rho \text{out}}(\omega, \omega') \right]
\]

**Moments of the Boltzmann equation**

The second order NES contribution to the moment equations (compare with equations (4.19-4.20)) is:

\[
S^e_{\text{NES}} = \int_{4\pi} \frac{d\Omega}{4\pi} B_{\text{NES}} = \kappa^0 - e \kappa^e - e \sum_i f_i \kappa_i^f - e \sum_{ij} p_{ij} \kappa_{ij}^p,
\]

\[
S^f_{\text{NES},i} = \int_{4\pi} \frac{d\Omega}{4\pi} \Omega_i B_{\text{NES}} = -e f_i \kappa^e + 1/3 \hat{\kappa}_i^f - e \sum_j p_{ij} \kappa_j^f - e \sum_{jk} q_{ijk} \kappa_{jk}^p.
\]

The third angular moment of \( \mathcal{F}_\nu \) is defined as

\[
q_{ijk} = \frac{1}{4\pi} \int_{4\pi} d\Omega \Omega_i \Omega_j \Omega_k \mathcal{F}_\nu
\]

Note that \( \hat{\kappa}^p \) is absent in both equations, due to the fact that Trace(\( \hat{\kappa}^p \))=0 and that the odd angular integrations of \( \hat{\kappa}^p \) vanish.

In spherical symmetry, the NES-part of right hand side of the moment equations reduces to

\[
S^e_{\text{NES}} = \kappa^0 - e \kappa^e - ef \kappa^f - \frac{1}{2} e(3p - 1) \kappa^\rho,
\]

\[
S^f_{\text{NES}} = \frac{1}{3} \hat{\kappa} - ef \kappa^e - ep \kappa^f - \frac{1}{2} e(3q - f) \kappa^\rho,
\]

in which

\[
f \equiv f_r, \quad p \equiv p_{rr}, \quad q \equiv q_{rrr}, \quad \hat{\kappa} \equiv \hat{\kappa}_r, \quad \kappa^f \equiv \kappa^f_r, \quad \kappa^\rho \equiv \kappa^\rho_{rr},
\]

where \( r \) is the radial component.
5.4 Neutrino transport with quadratic NES

In transport calculations, the quadratic terms may be expected to play a role in those regions of the iron core, where, first of all, NES is important compared to other interactions, and secondly, where the neutrino radiation field deviates from isotropy, i.e., where $p \neq \frac{1}{3}$. In contrast, the quadratic terms will be of no consequence whenever $p = \frac{1}{3}$, because this kills $\kappa^p$ via Eq.(5.7) and also with the prefactor $(3p - 1)$ in Eq.(5.11). Figure 5.3 shows the flux ratio $f$ and the Eddington factor $p$ in the iron core for several neutrino energies, from neutrino transport without the quadratic term, as calculated in Chapter 4. Radiation is diffusive and isotropic where $f \approx 0$ and $p \approx 1/3$. Moving outwards in radius, radiation decouples from the matter, and approaches free streaming ($f \to 1$, $p \to 1$). The figures clearly show how the more energetic neutrinos...
decouple at lower densities than less energetic neutrinos. Comparing these figures with the energy transfer rate $\Sigma_{\text{uw}}$ in Fig.4.4, it is observed that at high energies, the neutrino radiation is practically isotropic ($p \approx \frac{1}{2}$) in the region where NES is effective. But at low energies, the radiation is quite anisotropic ($f > 0.2, p > \frac{1}{2}$) in the NES scattering zone. Which of the two energy regimes is more important will depend on what part of the $(\omega, \omega')$ space is relevant. This in turn depends on the distribution function $\mathcal{F}_\nu$, because the NES collision kernel is a quadratic functional of $\mathcal{F}_\nu$, and therefore a full transport calculation, with the quadratic terms included, has to be performed.

We performed numerical transport computations on the same stationary background model that we used in the previous chapter (model M1). This model is a $1.17 M_\odot$ iron core during infall, with a central density $\rho_c = 4.1 \times 10^{12}$ g cm$^{-3}$, temperature $T_c = 2.3$ MeV and electron degeneracy $\xi_{e,c} = 24$. The maximum infall velocity is $v = -1.34 \times 10^4$ km s$^{-1}$ at masspoint $M = 0.91 M_\odot (\rho = 2.0 \times 10^{10}$ g cm$^{-3}$).

Computing stationary state neutrino transport with the quadratic term included we find, compared with first order calculations, no significant change in any relevant quantity e.g. neutrino fraction, flux, or transfer rates of lepton number, energy or momentum to the stellar matter. The NES coefficients $\kappa^a$ are all smaller than $\kappa^f$ by a factor of order $10^6$, and in combination with $f, p$ and $q$, their contribution to the moment equations is even less. This result is obtained for many different closure relations. Thus the fact that for MEC calculations we used Minerbo’s closure on the third angular moment $q$ is immaterial.

5.5 Discussion

Mezzacappa & Bruenn (1993abc) were able to solve the Boltzmann equation, even with NES included (Mezzacappa & Bruenn 1993c). In their treatment the NES rates need no expansion into a Legendre series and are retained exactly. For comparison they also computed the results with a (spectral) FLD approximation which involved a linear expansion of NES. They found that differences between the Boltzmann and the FLD transport were somewhat larger for several relevant neutrino quantities (such as the neutrino density and luminosity) when NES was included in the transport. They concluded however that those effects did not result from truncating the neutrino-electron scattering kernels at $R^{(1)}_{\text{in/out}}$ but rather from the intrinsic difference between the two transport methods. Because the distribution function $\mathcal{F}_\nu$ at $\omega = 5$ MeV was found to be fairly isotropic ($f < 0.2$) at matter densities above $\rho \approx 5 \times 10^{11}$ g cm$^{-3}$, they expected higher moments of the NES kernel not to contribute much during infall.

Yueh & Buchler (1977), already suspected that a linear approximation of the NES scattering rate would suffice, but this was based on transport calculations using a matter background in the later phases of shock propagation, with a central density of $\rho_c = 10^{14}$ g cm$^{-3}$. At that time the effects of NES no longer vitally determine the iron core evolution.

By deriving an analytic expression for the quadratic Legendre term we were able to show explicitly that in TMT and FLD neutrino transport it is sufficient to retain only the linear terms of the NES Legendre expansion, at least during infall.

This result must be due to the specific state of the neutrino fluid in this matter setting: the radiation is on average isotropic where NES is effective, and apparently the anisotropy at low energies has no weight in the problem. In Fig.5.3, the region of isotropy, where the Eddington factor $p$ is close to $\frac{1}{3}$, extends somewhat further out than the diffusive region (where $f \propto \partial e/\partial r \ll 1$). As discussed earlier, the second order expansion occurs in the spectral...
energy balance equation, Eq.(5.11), as the product \((3p(\omega) - 1) \kappa^p\). The opacity \(\kappa^p\) is itself a functional of \((3p(\omega') - 1)\). So, effectively, the deviation from isotropy enters quadratically in the energy balance equation, which greatly suppresses the second order term of the Legendre expansion. In the outer atmospheric regions \(p \to 1.0\), but there the \(\nu\epsilon\) scattering rate is too low to establish any change in the neutrino fluid.

Still, the qualitative discussion above only partly explains the fact that \(\kappa^p\) is so small in comparison with the other NES opacities. There are two ways to make \(\kappa^p\) small: the first, \(p \approx \frac{1}{3}\), has just been discussed. The second is by favouring \(\nu\epsilon\) scatterings with a large energy loss \(\Delta \omega = \omega - \omega'\). Figure 5.1 shows that the scattering rate \(R^\text{out}\) is nearly linear for \(\omega \ll \omega'\). If the solution of neutrino transport favours mainly large energy losses, then \(R^\text{out}(\omega, \omega', \cos \theta)\) is mostly linear in \(\cos \theta\) and \(\kappa^p\) is small even for \(p \neq \frac{1}{3}\). To check this, we consider the average energy lost per scattering, defined as

\[
\langle \Delta \omega \rangle = \frac{\int d^3p_\nu \int d^3p_\nu' \omega \, R^\text{out}(\omega, \omega', \cos \theta) \, \mathcal{F}_\nu(1 - \mathcal{F}_\nu)}{\int d^3p_\nu \int d^3p_\nu' \, R^\text{out}(\omega, \omega', \cos \theta) \, \mathcal{F}_\nu(1 - \mathcal{F}_\nu)} ,
\]

and compare it with the average energy \(\langle \omega \rangle\) per neutrino, given by

\[
\langle \omega \rangle = \frac{\int d^3p_\nu \omega \, \mathcal{F}_\nu}{\int d^3p_\nu \, \mathcal{F}_\nu} .
\]

Figure 5.4 displays these two quantities, where \(\langle \Delta \omega \rangle\) was calculated up to first order in the scattering rate. When we have \(\langle \Delta \omega \rangle \approx \langle \omega \rangle\), a significant amount of energy is exchanged per average \(\nu\epsilon\) collision. The figure shows that this is the case in the entire collapsing core. For this reason, it must be concluded that the rates \(R^\text{in\,out}\) are nearly linear in the cosine of the scattering angle.

It remains to be investigated if the quadratic extension is important in a different matter setting and/or for other neutrino flavors.
5.A Quadratic expansion of the NES rate

An expression is given here for the second-order legendre coefficient $H_2(\omega, \omega')$ in Eq.(5.4. They are composed of three terms, of which the third is negligible for relativistic electrons;

$$H_1(\omega, \omega', E) = \alpha_1 H_1^1(\omega, \omega', E) + \alpha_2 H_1^2(\omega, \omega', E),$$

(A.1)

with $\alpha_1, \alpha_2$ listed in Eq.(4.5) for electron neutrinos, for other species, see Cernohorsky (1990). The quadratic corrections $H_2^{\text{II}}(\omega, \omega', E)$ are found by evaluating the integral in equation A.3 of Yueh & Buchler (1977), and we obtain:

$$\omega^2 \omega'^2 H_1^{\text{II}} = a_2^{\text{II}}(\omega, \omega') E + c_2^{\text{II}}(\omega, \omega') E^2$$

$$\omega^2 \omega'^2 H_1^{\text{II}} = a_2^{\text{II}}(\omega, \omega') E + c_2^{\text{II}}(\omega, \omega') E^2$$

$$= Y_1^2(\omega, \omega', E) + \Theta(\omega' - \omega) \Gamma_2^2(\omega, \omega', E)$$

(A.2)

with

$$a_2^{\text{II}}(\omega, \omega') = \frac{16}{35} \omega^9 - \frac{36}{35} \omega^8 \omega' + \frac{64}{105} \omega^7 \omega'^2) \Theta(\omega' - \omega)$$

$$+ \frac{12}{35} \omega^4 \omega'^5 - \frac{8}{7} \omega^3 \omega'^6 + \frac{128}{35} \omega^2 \omega'^7 - \frac{24}{7} \omega \omega'^8 + \frac{8}{7} \omega'^9) \Theta(\omega - \omega')$$

(A.4)

$$b_2^{\text{II}}(\omega, \omega') = b_2^{\text{I}}(\omega, \omega)$$

(A.5)

$$c_2^{\text{II}}(\omega, \omega') = c_2^{\text{I}}(\omega, \omega)$$

(A.9)

and

$$Y_1^2(\omega, \omega', E) = \frac{5}{4} E^9 + \frac{24}{7} E^8(2\omega - \omega') + \frac{16}{35} E^7(12\omega^2 - 13\omega \omega' + 2\omega'^2)$$

$$+ \frac{12}{35} E^6(56 \omega^3 - 36 \omega^2 \omega' + 12 \omega \omega'^2) + \frac{12}{35} E^5(12 \omega^4 - 42 \omega^3 \omega' + \frac{72}{5} \omega^2 \omega'^2)$$

$$+ \frac{5}{4} E^4(-9 \omega^4 \omega' + 10 \omega^3 \omega'^2) + \frac{3}{4} E^3 \omega^4 \omega'^2$$

(A.10)

$$\Gamma_2^2(\omega, \omega', E) = \Gamma_2^2(-\omega', -\omega, E)$$

(A.11)

$$\Gamma_1^2(\omega, \omega', E) = -\frac{4}{105} E^2(\omega' - \omega)(72 \omega^6 + 9 \omega^5 \omega' + 16 \omega^4 \omega'^2$$

$$+ 16 \omega^3 \omega'^3 + 16 \omega^2 \omega'^4 + 9 \omega \omega'^5 + 72 \omega'^6)$$

$$+ \frac{4}{105} E(\omega' - \omega)^3(54 \omega^6 + 18 \omega^5 \omega' + 17 \omega^4 \omega'^2$$

$$+ 16 \omega^3 \omega'^3 + 15 \omega^2 \omega'^4 + 9 \omega \omega'^5)$$

$$- \frac{4}{105} (\omega' - \omega)^3(12 \omega^6 + 9 \omega^5 \omega')$$

$$+ 7 \omega^4 \omega'^2 + 6 \omega^3 \omega'^3 + 6 \omega^2 \omega'^4 + 30 \omega'^6)$$

(A.12)

$$\Gamma_2^2(\omega, \omega', E) = \Gamma_1^2(-\omega', -\omega, E)$$

(A.13)

In the equations above $\Theta$ is the unit step function.