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Finite-temperature correlations in the Lieb-Liniger one-dimensional Bose gas

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We address the problem of calculating finite-temperature response functions of an experimentally relevant low-dimensional, strongly correlated system: the integrable one-dimensional Bose gas with a repulsive δ-function interaction (the Lieb-Liniger model). Focusing on the dynamical density-density function, we present a Bethe ansatz-based method allowing for its accurate evaluation in finite but large systems, over broad ranges of momenta, frequencies, temperatures, and interaction parameters which are difficult to access using other methods. This allows us to quantify the reshaping of the zero-temperature critical behavior by thermal fluctuations, in experimentally accessible regimes.

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I. INTRODUCTION

Important examples of strongly correlated systems occur in reduced dimensionality [1], in which the nonperturbative effects of interactions break any single-particle picture and can lead to quantum critical states. In particular, bosonic quantum gases confined to one-dimensional (1D) channels have recently been subjected to intense theoretical and experimental investigations [2]. On the theoretical side, the physical responses of these systems, despite much progress, are still insufficiently understood to allow for high-quality experimental phenomenology. At low temperatures, 1D gases benefit from a universal Luttinger liquid description [3] allowing for a phenomenology. At low temperatures, 1D gases benefit from a universal Luttinger liquid description [3] allowing for a phenomenology. At low temperatures, 1D gases benefit from a universal Luttinger liquid description [3] allowing for a phenomenology. At low temperatures, 1D gases benefit from a universal Luttinger liquid description [3] allowing for a phenomenology. At low temperatures, 1D gases benefit from a universal Luttinger liquid description [3] allowing for a phenomenology.

In experimental situations [5–10] thermal fluctuations cannot be discounted; since typical measurements (using, e.g., Bragg spectroscopy [8,11]) require response functions away from the low-energy, long-distance asymptotics of observable correlations [1]. Alternately, methods based on integrability have allowed for the computation of ground-state (zero-temperature) dynamical correlations at arbitrary energy for continuum gases with contact interactions [4]. In this paper we focus on the dynamical density-density response of the integrable Lieb-Liniger 1D Bose gas [19] at finite temperatures. We present a Bethe ansatz-based approach valid for interactions and temperatures covering physically interesting regimes. The nontrivial line shapes obtained give quantitative predictions for eventual matching with experimental data.

II. THE MODEL

The Hamiltonian of the 1D Bose gas (Lieb-Liniger model [19]) is (setting ℏ2 = 2m = 1)

\[ H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + 2c \sum_{i>j}^{N} \delta(x_i - x_j) - \mu N, \tag{1} \]

where \( x_i \) denotes the position of the \( i \)th atom and \( \mu \) is the chemical potential. The coupling \( c \) is related to the scattering length [20]. At finite temperatures, the gas is characterized by two parameters: the interaction strength \( \gamma = c/n \) and temperature \( T \), where \( n = N/L \) is the 1D density. Hereafter we set \( n = 1 \) and \( k_B T = 1 \). Hamiltonian (1) is exactly diagonalizable (in each sector of fixed particle number \( N \)) by the Bethe ansatz [19]. Imposing periodicity, eigenstates (labeled by quantum numbers \( \{I_j\}_{j=1}^{N} \)) are fully characterized by rapidities solving the Bethe equations [19]

\[ \lambda_j + \sum_{k=1}^{N} \phi(\lambda_j - \lambda_k) = \frac{2\pi}{L} I_j, \quad j = 1, \ldots, N. \tag{2} \]

Here \( \phi(\lambda) = 2 \arctan(\lambda/c) \) is the two-particle phase shift. The momentum and energy are

\[ P_j = \sum_{j=1}^{N} \lambda_j, \quad E_j = \sum_{j=1}^{N} \lambda_j^2. \tag{3} \]

The ground state is formed by a Fermi-sea-like configuration of quantum numbers [19]. Low-lying excitations can be classified in terms of particles and holes, with these following respective dispersion relations \( \omega_{\mu}(k) \) [19]. At finite temperatures, the equilibrium state is (similarly to a free fermionic gas) a "melted" Fermi sea with smoothly varying densities of the particles and holes [21].

We are interested in dynamical properties in equilibrium at finite temperatures. Although our method in principle applies to any few-point correlator, we focus on the experimentally relevant density-density function

\[ S_T(k,\omega) = \frac{2\pi}{L} \sum_{\lambda'} |\langle \lambda'|\hat{\rho}_T|\lambda\rangle|^2 \delta(\omega - E_{\lambda'} + E_{\rho_T}), \tag{4} \]

where \( |\rho_T\rangle \) is the thermal equilibrium state [21], and the density operator is \( \hat{\rho}(x) = \sum_{j=1}^{N} \delta(x - x_j) \). Its matrix element for any two eigenstates of the system and any value of the interaction parameter is known exactly [22] from the algebraic Bethe ansatz (see Ref. [23] and references therein).

The density-density correlation function at \( T = 0 \) is characterized by a singular behavior along the dispersion lines \( \omega_{\mu}(k) \) [24,25]. It vanishes below the lower dispersion \( \omega_{-}(k) \) and has a power-law singularity around \( \omega_{+}(k) \). At small momentum...
by a closest-matching state. Ref. [21]. Choosing a fixed

\begin{equation}
\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \omega S(k,\omega) = nk^2.
\end{equation}

Table I. The levels of saturation of the $f$-sum rule combined with the detailed balance relation [Eq. (6)] for the intermediate interaction strengths and two values of momentum.

\begin{tabular}{|l|l|l|}
\hline
\text{c} & \text{T} & n_{k_f} & n_{2k_f} \\
\hline
1 & $T = 1/4$ ($N = 100$) & 0.991 & 0.975 \\
& $T = 1/2$ ($N = 64$) & 0.991 & 0.979 \\
4 & $T = 1/2$ ($N = 80$) & 0.992 & 0.982 \\
& $T = 1$ ($N = 50$) & 0.987 & 0.982 \\
16 & $T = 1$ ($N = 100$) & 0.990 & 0.981 \\
& $T = 2$ ($N = 64$) & 0.997 & 0.989 \\
\hline
\end{tabular}

Repeating calculations for different system sizes then explicitly showed convergence to the thermodynamic limit.

III. RESULTS

A. Momentum space

The full $k$- and $\omega$-dependent density-density correlation function for various temperatures and interaction strengths is plotted in Fig. 1. Representative $f$-sum rule saturations are presented in Table I. The $\omega$ dependence of the correlation is shown in Fig. 2 where fixed momentum cuts (at $k = k_F$) are plotted. Figure 3 illustrates finite-size effects.

![Figure 1](image1.png)

**FIG. 1.** (Color online) The full dynamic correlator plotted for the intermediate values of interaction $c = 1, 4, 16$ and for zero and high temperature. As the temperature increases, the correlation becomes smeared but stays approximately within the same region in the $k$-$\omega$ plane. The exception is the small correlation region at low momentum and negative energy visible for $c = 16$.

![Figure 2](image2.png)

**FIG. 2.** (Color online) Fixed momentum cuts through the correlation function for $c = 1, 4, 16, 64$ and at increasing values of temperatures from the top to the bottom. A finite temperature drastically modifies the line shape of the correlation. The upper threshold singularities are washed out and the correlation becomes almost symmetric around its maximum. Results agree with the perturbative expansion in $1/c$ (dots) of Ref. [13]. The error bars, based on the $f$-sum rule (see Table I), are below the plot resolution (in the worst case they are around 1.5%; we include also the finite-size smoothening effects in this estimate).

![Figure 3](image3.png)
In the small momentum limit the dispersion relation of excitations becomes linear (see Refs. [26,28], Fig. 1) with the sound velocity given by the isothermal compressibility

\[ u_s = \sqrt{2n(\frac{\mu}{\omega})} \], which can be calculated from the thermodynamic Bethe ansatz [21]. The f-sum rule combined with detailed balance then captures the correlation function, which becomes in this limit [28]

\[ S(0) = \begin{cases} \frac{\mu}{v_i} + O(k^2), & T = 0, \\ \frac{\mu}{c_T} + O(k^2), & T > 0. \end{cases} \] (8)

The static correlator plotted in Fig. 4 agrees with this low momentum prediction and, moreover, for weak interactions, confirms the validity of the Bogolyubov theory.
with our predictions. The value of the correlator at $x = 0$ (squares) is calculated via the Hellmann-Feynmann theorem [16,29] and agrees the largest for $c$ in real space [see Eq. (9)]. The points are the Luttinger liquid correlation (Fig. 5) computed as in Ref. [26]. For the 1D Bose gas at parameter $A$ is a nonuniversal number. The Luttinger parameter $K$ depends on the interaction strength $c$ and can be computed as in Ref. [26]. For the 1D Bose gas at $T = 0$, $A$ can be explicitly computed from the scaling limit of a single, specific matrix element of the density operator [32].

At finite temperatures the relationship is more intricate; at low temperatures the prefactor is, however, expected to be temperature independent [17]. For the temperatures considered here (which go beyond the low-temperature limit) we find that the $T = 0$ prefactor indeed gives predictions consistent with our results (see Fig. 5). The correlation weight in the vicinity of the umklapp excitation is still the same (thus the same prefactor $A$) but is smeared over a finite region in energy and momentum. At $T = 0$ this region shrinks to zero, yielding a power-law decay instead of an exponential.

Throughout this paper we considered a homogeneous gas with a constant density of particles. In an experimental situation, where the presence of an external trapping potential leads to a spatially varying distribution of particles, the correlation function can be well approximated by fixing the density to an average density of particles in the trap.

B. Real space

The Fourier transform of the static correlator yields the pair correlation (Fig. 5)

$$S(x) = \frac{1}{L} \sum_k e^{-ikx} S(k).$$  \hspace{1cm} (9)

Luttinger liquid theory [1,3,26] predicts exponential decay of this function at finite temperatures ($x \gg n^{-1}$),

$$S_{LL}(x) = 1 - \frac{K}{2\pi^2} \left( \frac{\pi T/v_s}{\sinh(\pi T x/v_s)} \right)^2 + A \cos(2\pi x) \left( \frac{\pi T/v_s}{\sinh(\pi T x/v_s)} \right)^{2K} + \ldots,$$  \hspace{1cm} (10)

where $\ldots$ represent terms that decay faster with distance, and where the prefactor $A$ is a nonuniversal number. The Luttinger parameter $K$ depends on the interaction strength $c$ and can be computed as in Ref. [26]. For the 1D Bose gas at $T = 0$, $A$ can be explicitly computed from the scaling limit of a single, specific matrix element of the density operator [32].

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IV. CONCLUSIONS

In this paper we presented results for the finite-temperature correlation function of the 1D Bose gas obtained through a combination of the Bethe ansatz and numerical evaluations of states and matrix elements. The results cover the experimentally relevant regime of intermediate physical parameters (temperature, interaction, energy, and momentum) which is difficult to access through other, analytical or numerical, methods. We showed that for intermediate temperatures the correlation function carries remnants of $T = 0$ characteristics, such as signs of threshold singularities and exponential decay closely resembling the power-law decay. The exact line shape of the correlation is, however, significantly and observably modified. The exact quantitative nature of our results should facilitate fitting with experimental predictions, perhaps paralleling what can be done, for example, in the context of ground-state correlations in spin chains [33]. In fact, besides extensions to other correlators, the method presented here is generalizable to other models solved by the Bethe ansatz, e.g., the XXZ spin chain. We will address this problem in future work.

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[27] Note that, experimentally, the situation is slightly altered by the presence of a trapping potential. This change can be understood within a local density approximation. The response of the 1D Bose gas was studied in Ref. [15]; the trapping potential mostly affects low momenta, and remains small enough to justify studying the dynamics of homogeneous systems as a starting point.