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DOI
10.1103/PhysRevA.89.033605

Publication date
2014

Document Version
Final published version

Published in
Physical Review A

Citation for published version (APA):
Finite-temperature correlations in the Lieb-Liniger one-dimensional Bose gas

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(Received 19 August 2013; revised manuscript received 26 November 2013; published 6 March 2014)

We address the problem of calculating finite-temperature response functions of an experimentally relevant low-dimensional, strongly correlated system: the integrable one-dimensional Bose gas with a repulsive δ-function interaction (the Lieb-Liniger model). Focusing on the dynamical density-density function, we present a Bethe ansatz-based method allowing for its accurate evaluation in finite but large systems, over broad ranges of momenta, frequencies, temperatures, and interaction parameters which are difficult to access using other methods. This allows us to quantify the reshaping of the zero-temperature critical behavior by thermal fluctuations, in experimentally accessible regimes.

DOI: 10.1103/PhysRevA.89.033605 PACS number(s): 67.85.De, 05.30.Jp

I. INTRODUCTION

Important examples of strongly correlated systems occur in reduced dimensionality [1], in which the nonperturbative effects of interactions break any single-particle picture and can lead to quantum critical states. In particular, bosonic quantum gases confined to one-dimensional (1D) channels have recently been subjected to intense theoretical and experimental investigations [2]. On the theoretical side, the physical observables correlations [1]. Alternately, methods based on integrability have allowed for the computation of ground-state (zero-temperature) dynamical correlations at arbitrary energy for continuum gases with contact interactions [4].

In experimental situations [5–10] thermal fluctuations cannot be discounted; since typical measurements (using, e.g., Bragg spectroscopy [8,11]) require response functions away from the low-energy universal limit, the theoretical determination of correlations at finite temperature, energy, and momentum scales is a crucial but difficult problem. In the context of the 1D Bose gas this has up to now only been partially addressed [12–18].

In this paper we focus on the dynamical density-density response of the integrable Lieb-Liniger 1D Bose gas [19] at finite temperatures. We present a Bethe ansatz-based approach valid for interactions and temperatures covering physically interesting regimes. The nontrivial line shapes obtained give quantitative predictions for eventual matching with experimental data.

II. THE MODEL

The Hamiltonian of the 1D Bose gas (Lieb-Liniger model [19]) is (setting \( \hbar^2 = 2m = 1 \))

\[
H = -\sum_{i=1}^{N} \partial_{x_i}^2 + 2c \sum_{i>j}^{N} \delta(x_i - x_j) - \mu N, \tag{1}
\]

where \( x_i \) denotes the position of the \( i \)th atom and \( \mu \) is the chemical potential. The coupling \( c \) is related to the scattering length [20]. At finite temperatures, the gas is characterized by two parameters: the interaction strength \( \gamma = c/n \) and temperature \( T \), where \( n = N/L \) is the 1D density. Hereafter we set \( n = 1 \) and \( k_p = 1 \). Hamiltonian (1) is exactly diagonalizable (in each sector of fixed particle number \( N \)) by the Bethe ansatz [19]. Imposing periodicity, eigenstates (labeled by quantum numbers \( \{I_j\}_{j=1}^{N} \)) are fully characterized by rapidities solving the Bethe equations [19]

\[
\lambda_j + \sum_{k=1}^{N} \phi(\lambda_j - \lambda_k) = \frac{2\pi}{L} I_j, \quad j = 1, \ldots, N. \tag{2}
\]

Here \( \phi(\lambda) = 2 \arctan(\lambda/c) \) is the two-particle phase shift. The momentum and energy are

\[
P_k = \sum_{j=1}^{N} \lambda_j, \quad E_k = \sum_{j=1}^{N} \lambda_j^2. \tag{3}
\]

The ground state is formed by a Fermi-sea-like configuration of quantum numbers [19]. Low-lying excitations can be classified in terms of particles and holes, with these following their respective dispersion relations \( \omega_\pm(k) \) [19]. At finite temperatures, the equilibrium state is (similarly to a free fermionic gas) a “melted” Fermi sea with smoothly varying densities of the particles and holes [21].

We are interested in dynamical properties in equilibrium at finite temperatures. Although our method in principle applies to any few-point correlator, we focus on the experimentally relevant density-density function

\[
S_T(k,\omega) = \frac{2\pi}{L} \sum_{\lambda'} |\langle \lambda' | \hat{\rho}_k |\lambda_\rho \rangle|^2 \delta(\omega - \omega_{\lambda'} - \omega_{\lambda_\rho}), \tag{4}
\]

where \( \rho_\rho \) is the thermal equilibrium state [21], and the density operator is \( \hat{\rho}(x) = \sum_{j=1}^{N} \delta(x - x_j) \). Its matrix element for any two eigenstates of the system and any value of the interaction parameter is known exactly [22] from the algebraic Bethe ansatz (see Ref. [23] and references therein).

The density-density correlation function at \( T = 0 \) is characterized by a singular behavior along the dispersion lines \( \omega_\pm(k) \) [24,25]. It vanishes below the lower dispersion \( \omega_{\pm}(k) \) and has a power-law singularity around \( \omega_+(k) \). At small momentum...
k smeared but stays approximately within the same region in the temperature. As the temperature increases, the correlation becomes intermediate values of interaction $k$.

Ref. [21]. Choosing a fixed the distribution of rapidities in the thermal state following $1/L$ [27]. Their origin is twofold: from the saddle-point approximation, which was used to derive Eq. (4) [23], and from the evaluation of Eq. (4) in a finite system. We will quantify them later.

Let us now discuss the evaluation of Eq. (4). We first obtain the distribution of rapidities in the thermal state following Ref. [21]. Choosing a fixed $N$, this was then approximated by a closest-matching state $|\lambda_T\rangle^N$. The representation (4) could then be scanned through the Hilbert space of relevant excitations (for this, the ABACUS algorithm [4] was extended to arbitrary excited states). Convergence was measured by the $f$-sum rule [28]

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \omega S_T(k,\omega) = nk^2.$$  

To verify that the computed correlation was indeed thermal, we used the $f$-sum rule combined with detailed balance relation [Eq. (6)] for the intermediate interaction strengths and two values of momentum.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$c = 1$</th>
<th>$T = 1/4 (N = 100)$</th>
<th>0.991</th>
<th>0.975</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$T = 1/2 (N = 64)$</td>
<td>0.991</td>
<td>0.979</td>
<td></td>
</tr>
<tr>
<td>$c = 4$</td>
<td>$T = 1/2 (N = 80)$</td>
<td>0.992</td>
<td>0.982</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$T = 1 (N = 50)$</td>
<td>0.987</td>
<td>0.982</td>
<td></td>
</tr>
<tr>
<td>$c = 16$</td>
<td>$T = 1 (N = 100)$</td>
<td>0.990</td>
<td>0.981</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$T = 2 (N = 64)$</td>
<td>0.997</td>
<td>0.989</td>
<td></td>
</tr>
</tbody>
</table>

Repeating calculations for different system sizes then explicitly showed convergence to the thermodynamic limit.

### III. RESULTS

#### A. Momentum space

The full $k$- and $\omega$-dependent density-density correlation function for various temperatures and interaction strengths is plotted in Fig. 1. Representative $f$-sum rule saturations are presented in Table I. The $\omega$ dependence of the correlation is shown in Fig. 2 where fixed momentum cuts (at $k = k_F$) are plotted. Figure 3 illustrates finite-size effects.

<table>
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</tbody>
</table>

![TABLE I. The levels of saturation of the $f$-sum rule combined with the detailed balance relation [Eq. (6)] for the intermediate interaction strengths and two values of momentum.](image)

![FIG. 2. (Color online) Fixed momentum cuts through the correlation function for $c = 1,4,16,64$ and at increasing values of temperatures from the top to the bottom.](image)
interactions are strong and dominate over the temperature, but difficult to describe, intermediate regime with the discrete nature of the thermal state is noticeable. This when discussing the real-space correlation function. Any thermal fluctuations destroy this feature. We come back to the response in this limit is singular with vanishing width, and small momentum (also around umklapp excitations). The dispersion relation of whose correlations can be obtained from a high-temperature expansion. For example, when the temperature dominates, the system resembles an ideal gas whose correlations can be obtained from a high-T expansion [14]. Here we set our attention on the experimentally relevant, but difficult to describe, intermediate regime with $T/\sqrt{\epsilon} = 1/4, 1/2$ ($T = 0$ curves are shown for reference).

The fixed momentum curves show the importance of thermal fluctuations in shaping the correlations. We begin by ascertaining the effect of temperature on the mean $\bar{\omega} = \int_{-\infty}^{\infty} d\omega \omega S_T(k, \omega)$ and variance $\sigma^2 = \int_{-\infty}^{\infty} d\omega (\omega - \bar{\omega})^2 S_T(k, \omega)$. As seen from Eq. (6), the mean should increase with temperature; similarly, the correlation should broaden. However, as can be seen from Table II, both effects are small. Even at finite temperatures, in the range we studied, the spread of the fixed momentum cuts is mainly due to interactions.

However, thermal fluctuations have an important and much more subtle effect in smoothening the singularities of the $T = 0$ correlator. Two effects occur: first, a rounding off of the $T = 0$ threshold singularities along the particle and hole modes $\omega_{\pm}(k)$, and second, a broadening of the correlation at very small momentum (also around umklapp excitations). The $T = 0$ response in this limit is singular with vanishing width, and any thermal fluctuations destroy this feature. We come back to this when discussing the real-space correlation function.

Integration over $\omega$ yields the static correlator (Fig. 4)

$$S(k) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} S(k, \omega). \tag{7}$$

In the small momentum limit the dispersion relation of excitations becomes linear (see Refs. [26,28], Fig. 1) with the sound velocity given by the isothermal compressibility $v_s = \sqrt{2n(\frac{N}{2\pi})_T}$, which can be calculated from the thermodynamic Bethe ansatz [21]. The $f$-sum rule combined with detailed balance then captures the correlation function, which becomes in this limit [28]

$$S(0) = \begin{cases} \frac{9}{4} + O(k^2), & T = 0, \\ \frac{2\pi}{3} + O(k^2), & T > 0. \end{cases} \tag{8}$$

The static correlator plotted in Fig. 4 agrees with this low momentum prediction and, moreover, for weak interactions, confirms the validity of the Bogolyubov theory.
is calculated via the Hellmann-Feynmann theorem \[16,29\] and agrees the largest for hydrodynamic predictions varies with the interaction strength and is in real space [see Eq. (9)]. The points are the Luttinger liquid correlation (Fig. 5) is a nonuniversal number. The Luttinger liquid density to an average density of particles in the trap.

## IV. CONCLUSIONS

In this paper we presented results for the finite-temperature correlation function of the 1D Bose gas obtained through a combination of the Bethe ansatz and numerical evaluations of states and matrix elements. The results cover the experimentally relevant regime of intermediate physical parameters (temperature, interaction, energy, and momentum) which is difficult to access through other, analytical or numerical, methods. We showed that for intermediate temperatures the correlation function carries remnants of \( T = 0 \) characteristics, such as signs of threshold singularities and exponential decay closely resembling the power-law decay. The exact line shape of the correlation is, however, significantly and observably modified. The exact quantitative nature of our results should facilitate fitting with experimental predictions, perhaps paralleling what can be done, for example, in the context of ground-state correlations in spin chains \[33\]. In fact, besides extensions to other correlators, the method presented here is generalizable to other models solved by the Bethe ansatz, e.g., the XXZ spin chain. We will address this problem in future work.

## ACKNOWLEDGMENTS

We thank C. Fort, N. Fabbri, L. Fallani, D. Clément, F. H. L. Essler, and R. Konik for stimulating and fruitful discussions. We gratefully acknowledge support from the Foundation for Fundamental Research on Matter (FOM) and from the Netherlands Organisation for Scientific Research (NWO).

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[27] Note that, experimentally, the situation is slightly altered by the presence of a trapping potential. This change can be understood within a local density approximation. The response of the 1D Bose gas was studied in Ref. [15]; the trapping potential mostly affects low momenta, and remains small enough to justify studying the dynamics of homogeneous systems as a starting point.