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Criterion for condensation in kinetically constrained one-dimensional transport models

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We study condensation in one-dimensional transport models with a kinetic constraint. The kinetic constraint results in clustering of immobile vehicles; these clusters can grow to macroscopic condensates, indicating the onset of dynamic phase separation between free-flowing and arrested traffic. We investigate analytically the conditions under which this occurs and derive a necessary and sufficient criterion for phase separation. This criterion is applied to the well-known Nagel-Schreckenberg model of traffic flow to analytically investigate the existence of dynamic condensates. We find that true condensates occur only when acceleration out of jammed traffic happens in a single time step, in the limit of strong overbraking. Our predictions are further verified with simulation results on the growth of arrested clusters. These results provide analytic understanding of dynamic arrest and dynamic phase separation in one-dimensional traffic and transport models.

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I. INTRODUCTION

A wide range of driven many-particle systems including traffic flow [1], active colloids [2] and shaken granular gases [3] exhibit interesting collective large-scale phenomena. The interactions between the driven constituent particles cause collective behavior such as collective slowing down that can eventually emerge into macroscopic phenomena affecting major parts of the system. In traffic, the interactions of vehicles that avoid collisions lead to nontrivial, strongly nonlinear flow behavior: increasing the number of vehicles does not necessarily result in an increase of the throughput. To understand and predict this behavior, it is important to know how the dynamics organize in space. Do all particles slow down gradually or do only certain particles slow down while others still move? An intriguing question in this context is whether the interactions between cars lead to macroscopic separation into arrested and moving traffic, a transition analogous to an equilibrium phase transition. The accumulation of a macroscopic fraction of all particles into a cluster is called condensation in real space. Recently, such condensation phenomena have been studied in one-dimensional transport models [4–6]. Whether and how dynamic condensation occurs remains a largely open question and can be addressed analytically only in certain exactly solvable models. Most work has focused on the exactly solvable zero-range process (ZRP) [7,8] and related models [9,10]. In these models the dynamics of particles is typically specified per lattice site: particles can accumulate on a given site while hopping from one site to the next. It has been shown analytically that these systems exhibit condensation and symmetry breaking, even in one dimension. The situation is, however, different in traffic models, where vehicles must follow each other and cannot accumulate on any site. In these models, the vehicle dynamics are set by a kinetic constraint between neighboring particles that guarantees vehicles do not collide. The question is whether in these models, condensate transitions can still occur, and how they can be analytically predicted based on the microscopic interactions of the vehicles or particles. Although condensation and phase separation phenomena have been investigated numerically in commonly used traffic models [1,11,12] and some attempts have been made to connect traffic models with the ZRP [13] and related models [14], a general analytic treatment is lacking so far.

Here we present just such an analytic criterion of condensate formation in traffic models, applicable both to traffic and to more general driven systems. Our analysis is based on rate equations for the generation, growth, shrinkage, and split up of traffic jams. We establish a criterion for the occurrence of extensively large jams as a function of the microscopic interactions of the traffic model. Because of the generality of the equations used, the criterion is generally applicable to traffic and transport models. We gain insight into the dynamics of jam creation, lifetime, and size. We apply this criterion to the well-known Nagel-Schreckenberg (NS) [15] model and related models of traffic flow to investigate analytically if and how condensation occurs, hence answering the longstanding question of dynamic condensates in this model [16]. We find that a true condensate forms only in the special case where the acceleration out of jammed traffic occurs in a single acceleration step, in the limit of high braking probability. In all other cases, arrested clusters either dissolve or split up. We illustrate the growth dynamics of these arrested clusters using long simulation runs and demonstrate the similarity to the dynamics of a diffusion process. These simulation results validate our analytic criterion and elucidate the formation kinetics of condensates.

II. CRITERION

A. Class of models

We consider kinetically constrained one-dimensional transport models that are defined by mass-conserving local dynamical rules. The vehicles move unidirectionally over a discrete lattice in discrete time. A vehicle can move freely when it is out of the interaction range of other vehicles, but becomes kinetically constrained when it closely approaches another

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vehicle: it must slow down to avoid a collision. This kind of constraint is also present in more general models of transport of particles. The kinetic constraint can, for example, be a hard-core repulsion between neighboring particles. "Softer" constraints with longer range are also possible as long as the order of the particles is conserved. Once the kinetic constraint is released, particles accelerate with a certain probability back to free flow. We call the free-flowing particles "active," and the kinetically constrained particles "inactive." To derive the criterion for condensation, we consider the bulk of an infinitely extended lattice.

B. Derivation of criterion

Inactive particles form clusters due to their dynamic interaction. Here we define a cluster as a sequence of particles in the same state (active or inactive). A typical particle configuration consists of several coexisting active and inactive clusters as shown in Fig. 1. These clusters can grow and eventually reach macroscopic size. We call a cluster a condensate, if in the limit of infinite system size, the cluster contains an infinite number of particles. Here we allow short-lived interruptions in the sequence of inactive particles that exist on a time scale much shorter than the typical time scale of growth or shrinkage of the cluster. We call such small and short-lived interruptions bubbles, in analogy to fluctuations in an equilibrium liquid phase.

We investigate the conditions under which condensation occurs in the stationary state in an infinitely extended system, by analyzing the growth dynamics of inactive clusters. The control parameters are the global particle density ρ and the fluctuation parameter(s). We deduce the dynamics of clusters of particles from the microscopic dynamics of the particles.

There are several competing processes that lead to growth or shrinkage of clusters. Clusters can grow one by one by vehicles leaving or entering at the boundaries (see cluster 3 in Fig. 1). Clusters can also split up into two by vehicles changing their state inside a cluster. Finally, two clusters can merge when the cluster that separates them shrinks to zero.

Below we will analyze these cluster processes in detail to find the condition for condensation. The idea is as follows: 1) Inactive clusters must be unable to split up to become infinitely large. 2) Inactive clusters must grow, i.e., their growth rate must be at least as large as their shrink rate. The growth of existing clusters is, however, reduced by any new inactive cluster that forms upstream; such a new inactive cluster takes up particles and reduces the inflow to existing downstream inactive clusters.

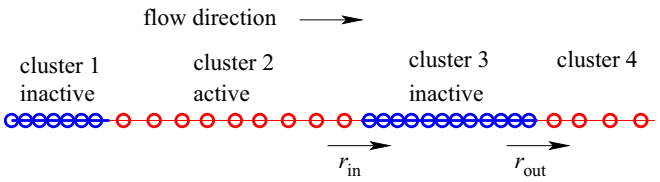

FIG. 1. (Color online) Schematic of vehicle transport in mass transport models. The dynamic interactions lead to clustering of immobile (inactive) particles. These inactive clusters coexist with clusters of mobile (active) particles. A few coexisting clusters are shown, as well as the inflow and outflow rate r_{in} and r_{out} of particles at the upstream and downstream boundary of an inactive cluster. The average distance between particles in active clusters must be larger than the interaction distance, while the distance between inactive particles is typically smaller than the interaction distance.

These two conditions have to be met independently: because the split-up rate of clusters scales with the cluster size, while the growth rate of clusters does not (it is always limited to maximum 1 particle per time step), the two processes cannot balance, and both conditions must be fulfilled simultaneously.

1. Splitting up of inactive clusters

We first investigate the split up of inactive clusters. Split up occurs when the distance between inactive vehicles increases spontaneously releasing the kinetic constraint. Such split up is detrimental for condensation. Below we identify two alternative conditions that prevent split up. One of these must be satisfied to guarantee split up does not occur.

First, if the density inside active clusters is maximum, $\rho_{ina} = \rho_{max}$, so that density fluctuations inside the cluster do not occur, then split up cannot occur. For hard-core repulsion, we have $\rho_{max} = 1$.[1]

Second, if ρ_{ina} is lower than ρ_{max}, density fluctuations do exist, but condensation will still occur if these density fluctuations are short-lived, i.e., no stable active cluster can form within an inactive cluster. This is the case when the density of inactive clusters is much larger than that of active ones, i.e., when $\rho_{act}/\rho_{ina} \to 0$. In this case any active "bubble" requires an infinite amount of space; that much space is not available inside inactive clusters, and as a result inactive clusters do not split up.

We thus obtain the following condition for condensation:

$$\rho_{ina} = \rho_{max} \ \text{OR} \ \frac{\rho_{act}}{\rho_{ina}} \to 0, \tag{1}$$

where the limit here and in all equations below is taken with respect to the fluctuation parameter for a given particle density ρ. We note that this condition also implies that inactive clusters cannot merge.

2. Growth versus creation of inactive clusters

We now consider the processes that grow and shrink the inactive cluster due to in- and outflow of single vehicles. An inactive cluster grows due to vehicles entering at the upstream boundary at rate r_{in}, while it shrinks due to vehicles leaving the cluster at the downstream boundary with rate r_{out}; see cluster 3 in Fig. 1. These two processes grow and shrink the inactive cluster, respectively, with rates r_+ and r_-. Because by definition $\Delta t = 1$, rates equal probabilities, and we can write

$$r_+ = r_{in}(1 - r_{out}), \tag{2}$$

$$r_- = (1 - r_{in})r_{out}. \tag{3}$$

[1]It is possible to construct systems that have a range of densities that make fluctuations impossible. In this case, by ρ_{max} we mean any density in this range.

In steady state, $r_+ > r_-$ is not possible due to particle conservation. A steady state with cluster rates $r_+ < r_-$ is possible but implies that all inactive clusters have a finite size and lifetime. This leaves us with $r_+ = r_-$ as the only possible condition with a condensate in the steady state. This means, for condensation to occur, the difference $\Delta r = r_- - r_+$ must vanish relative to the absolute value of r_+ or r_- that sets the typical time scale of the system. Hence

$$\frac{r_- - r_+}{r_-} = \frac{\Delta r}{r_-} \to 0. \tag{4}$$

The task is now to find an expression for Δr in terms of basic dynamical quantities. We rewrite Δr using Eqs. (2) and (3) to relate it to the in- and outflow rate of particles,

$$\Delta r = (1 - r_{\text{in}})r_{\text{out}} - r_{\text{in}}(1 - r_{\text{out}}) = r_{\text{out}} - r_{\text{in}}. \tag{5}$$

Here the inflow rate r_{in} of the inactive cluster (cluster 3 in Fig. 1) is given by the average flow rate through the upstream active cluster (cluster 2 in Fig. 1), i.e., the average velocity $\tilde{v}$ of particles in the active cluster times their average density $\tilde{\rho}$. Because the boundary between both clusters moves itself with (negative) velocity v_c, this increases the relative velocity of inflowing particles to $(\tilde{v} - v_c)$, and the inflow rate becomes

$$r_{\text{in}} = \tilde{\rho}(\tilde{v} - v_c). \tag{6}$$

The average density $\tilde{\rho}$ itself depends on the outflow rate of the next upstream inactive cluster (cluster 1 in Fig. 1): the outflow rate, r_{out}, of cluster 1 equals the velocity, v_{act}, of active particles, times the density, $\tilde{\rho}$.[2] This allows us to find a corresponding relation for the boundary between cluster 1 and cluster 2, which we rewrite to obtain for the density $\tilde{\rho}$ in cluster 2:

$$\tilde{\rho} = \frac{r_{\text{out}}}{v_{\text{act}} - v_c}. \tag{7}$$

Due to fluctuations, new inactive clusters may form inside the active cluster (technically splitting up the active cluster). This reduces the velocity of cars in this region. To obtain an expression for the resulting average velocity $\tilde{v}$ in this region, which now consists of active and inactive clusters, we introduce the fraction $\tilde{f}$ of inactive particles. We can then write

$$\tilde{v} = \tilde{f} v_{\text{ina}} + (1 - \tilde{f}) v_{\text{act}}. \tag{8}$$

By inserting Eqs. (6)–(8) in Eq. (5), we find that

$$\Delta r = r_{\text{out}} - \frac{r_{\text{out}}}{v_{\text{act}} - v_c}[\tilde{f}(v_{\text{ina}} - v_c) + (1 - \tilde{f})(v_{\text{act}} - v_c)] \tag{9}$$

$$= r_{\text{out}} \tilde{f} \frac{v_{\text{act}} - v_{\text{ina}}}{v_{\text{act}} - v_c}, \tag{10}$$

which relates Δr to the car velocities and outflow rates. Finally, we express the fraction $\tilde{f}$ of inactive particles in terms of the creation rate u per particle of inactive clusters, their average lifetime, T, and their average length, n. In steady state this fraction is

$$\tilde{f} = uTn. \tag{11}$$

Using Eqs. (10) and (11), our criterion for the growth rate of clusters [Eq. (4)] then becomes

$$\frac{\Delta r}{r_-} = \frac{r_{\text{out}} uTn(v_{\text{act}} - v_{\text{ina}})}{r_-(v_{\text{act}} - v_c)} \to 0, \tag{12}$$

which simplifies to

$$\frac{r_{\text{out}} uTn}{r_-} \to 0, \tag{13}$$

because $v_{\text{act}} > v_{\text{ina}}$ and $v_c < 0$, and all velocities are finite. Equation (13) provides the second criterion for condensation. It ensures that the growth rate of inactive clusters is at least as large as their shrink rate, so that inactive clusters can be stable.[3]

We thus arrive at a twofold criterion for condensate formation, consisting of Eqs. (1) and (13). The first equation guarantees that inactive clusters do not split up; the second equation assures that the growth of inactive clusters is not hindered by the formation of new inactive clusters. Together, these two equations provide a necessary and sufficient condition for condensation.

III. APPLICATION TO TRAFFIC MODEL

We now apply the criterion, Eqs. (1) and (13), to specific traffic models to demonstrate the occurrence or absence of dynamic condensates. In particular, we focus on the Nagel-Schreckenberg model of traffic flow [15], a well-studied simple model that captures much of the behavior of real traffic.

A. Nagel-Schreckenberg model

The NS model is a one-dimensional cellular automaton model with discrete time and space. The road consists of a regular lattice of L sites, occupied by N cars with average density $\rho = N/L$. Cars move with integer velocity over the lattice and are updated synchronously. The velocity v_i of car i can be at most the maximum velocity $v_{\max}$ and becomes constrained when the distance to the next car $d_i < v_i$. The following dynamical update rules for the NS-model are applied in parallel to all N cars:

1 *Acceleration*: $v_i \to \min(v_i + 1, v_{\max})$.
2 *Avoiding collisions*: If $d_i < v_i$ then $v_i = d_i$.
3 *Randomization*: Decrease v_i obtained in the previous steps by 1, to a minimum of 0, with probability p.
4 *Position update*: $x_i \to x_i + v_i$, $d_i \to d_i - v_i + v_{i+1}$.

The only source of stochasticity in the model is the fluctuation parameter p that reflects the drivers' individual freedom to decelerate below $v_{\max}$. We define car i as freely flowing (active) if before the randomization step (3), $v_i = v_{\max}$. While at low density, most cars move freely, at high density

[2] Here we have used that in steady state, the outflow rates of inactive clusters 1 and 3 are the same.

[3] We note that in order to derive Eq. (13), we assumed high density. For $\rho < \frac{r_{\text{out}}}{v_{\text{act}} - v_c}$, it follows from Eqs. (7) and (8) that $r_{\text{in}} < r_{\text{out}}$, so no condensation can occur.

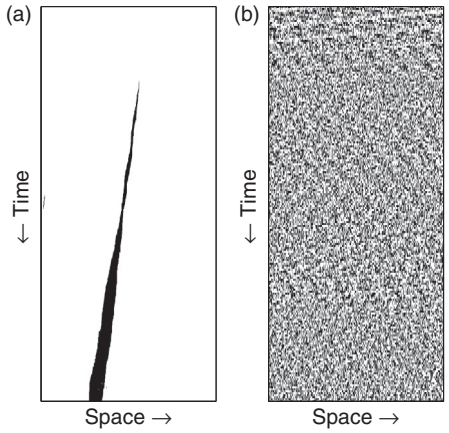

FIG. 2. Space-time diagram of vehicles in the two deterministic limits of the NS model, for $p = 0.9998$ (a) and $p = 0.0002$ (b). Inactive vehicles are indicated in black. The horizontal axis represents the car index. The simulations are performed at densities 20% above the transition density ρ_{tra}.

or large braking probability, p, jams (inactive clusters) form. The density above which stable jams form was estimated by Gerwinski and Krug as $\rho_{tra} = (1 - p)/(v_{max} + 1 - 2p)$ [17].

From simulations the idea has emerged that no sharp transition between free-flow and jammed traffic occurs for finite stochasticity [18–20]. This implies that there is no condensate. Condensates might, however, form in the deterministic limits $p \to 1$ and $p \to 0$ of the model. Numerically, no such condensation has been found in the limit $p \to 0$. However, our own recent numerical results suggests that condensation does occur in the limit $p \to 1$, at least for $v_{max} = 2$ [12]. To appreciate the strikingly different behavior in the two deterministic limits, we show space-time diagrams constructed from simulations for $v_{max} = 2$ in Fig. 2. In the limit $p \to 1$, a condensate forms as illustrated by the thick black line in Fig 2(a). A jam nucleates and grows into a condensate that contains all excess particles above the critical density. In contrast, in the limit $p \to 0$, there are many small jams [Fig. 2(b)] that do not coalesce, and no macroscopic condensate forms. Some jams disappear and new jams are created. Below we investigate analytically the formation of condensates for all different cases of p, starting with $v_{max} = 2$.

B. Nagel-Schreckenberg with $0 < p < 1$ ($v_{max} = 2$)

Simulations suggest that for finite stochasticity, $0 < p < 1$, there is no condensate. Indeed, we will show that in this case, the second condition [Eq. (13)] is not fulfilled. To see this, we first note that for finite p, vehicles slow down randomly, and the average velocity is smaller than v_{max}. Hence, the inflow rate of jams, r_{in}, is smaller than 1. Since we can rewrite $r_{out}/r_- = 1/(1 - r_{in})$ using Eq. (3), we conclude that the first factor in Eq. (13), $r_{out}/r_- > 0$.

Furthermore, also $u > 0$: due to velocity fluctuations at finite p, the distances between cars varies and cars can come within the interaction range with finite probability. Hence, new jams are formed even at arbitrarily low density and $u > 0$.

Because the remaining factors in Eq. (13), T and n, are always larger than zero (a jam always exists for at least one time step and consists of at least one car), we conclude that Eq. (13) is not fulfilled and thus there is no condensate. This is in agreement with the consensus in the literature about the absence of a sharp transition between free-flowing and jammed traffic for $0 < p < 1$ [18–20].

C. Nagel-Schreckenberg in the limit $p \to 1$ ($v_{max} = 2$)

In the limit $p \to 1$, cars almost always overbrake. To determine whether condensation occurs in this limit, we analyze the scaling of all quantities in Eqs. (1) and (13) as a function of the vanishing distance to the deterministic point: $\Delta p = 1 - p \to 0$.

With $v_{max} = 2$ and $p \to 1$, free-flowing traffic has average velocity $v_{max} - p = 1$, and jammed traffic has velocity 0. Hence, cars accelerate in a single step out of the inactive cluster, and the outflow rate equals the probability of acceleration, $r_{out} = \Delta p$. According to Eq. (7), it then follows that the density in active clusters scales as $\rho_{act} \sim \Delta p$. Meanwhile, the density of a jam, ρ_{ina}, is bounded from below due to the finite interaction range, and must be higher than $1/v_{max}$. Consequently, the second part of Eq. (1) is fulfilled, meaning that inactive clusters do not split up.

To check the second part, Eq. (13), we note that because $r_{in} \to 0$ in the limit $p \to 1$, we can approximate $r_- = (1 - r_{in})r_{out} \approx r_{out}$. We thus find that

$$\frac{r_{out} u T n}{r_-} \to u T n, \tag{14}$$

reducing the criterion to the scaling of u, T, and n.

The scaling of u can be estimated as follows: The distance between cars behaves as a diffusion process. Hence, we can estimate the creation rate u of new jams from the time τ it takes for the root mean square of the change Δd of the distance d between subsequent cars to grow to the average distance itself: $\Delta d \approx \langle d \rangle$. For a random walker, the number of changes necessary to accumulate a change of $\langle d \rangle$ is $\langle d \rangle^2$,

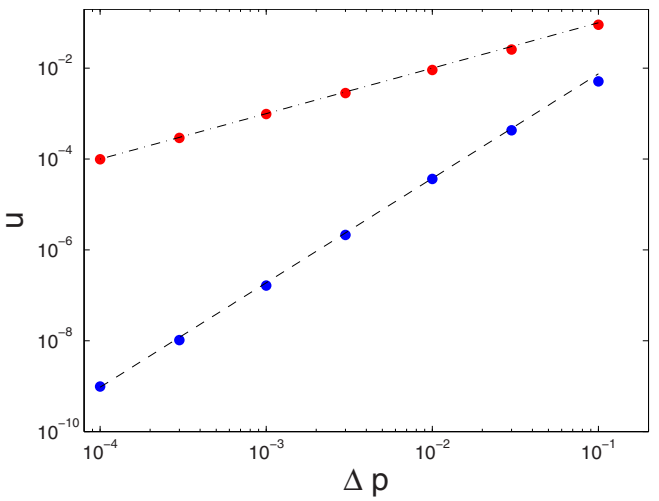

FIG. 3. (Color online) Creation rate per car of new jams as a function of the distance Δp to the deterministic point $p \to 0$ [red (upper) points, $\Delta p = p$] and $p \to 1$ [blue (lower) points, $\Delta p = 1 - p$]. The dashed lines have slope 1 [red (upper) data points] and slope 2.3 [blue (lower) data points].

while for ballistic motion, the number of changes is $\langle d \rangle$. We will allow for a general power $\langle d \rangle^{\beta}$. Because the time to change the distance between two cars by one is of order $(\Delta p)^{-1}$, we obtain

$$\tau \sim \Delta p^{-1-\beta}. \tag{15}$$

Because $u \approx 1/\tau$, we obtain $u \approx \Delta p^{\beta+1}$. With simulations we find $u \sim \Delta p^{2.3\pm0.1}$ (Fig. 3), and hence $\beta = 1.3$, an exponent between random walk and ballistic motion. The quantity u thus vanishes on approach of the deterministic point.

We now consider the scaling of n. A divergence of n by definition means that condensation occurs, since n indicates the number of cars in a jam. Therefore, the maximum scaling of n that does not *a priori* indicate condensation, is that n is constant. We will take this maximum scaling, and will show below that, nevertheless, condensation occurs.

The scaling of T, the average lifetime of jams, can be estimated from the average time it takes for a car to accelerate out of a jam; this time diverges as $1/r_{\text{out}} = 1/\Delta p$. Hence, the average jam lifetime scales as $T \sim \mathcal{O}(\Delta p^{-1})$ if n is constant; any faster decrease would imply that the number of cars in a jam grows and thus again that a condensate forms.

With the scaling obtained for u, n, and T, Eq. (14) becomes

$$uTn \sim \Delta p^{\beta}. \tag{16}$$

This quantity goes to zero in the limit $\Delta p \to 0$, thus meeting the requirement for condensation. We therefore expect condensation to occur in the limit $p \to 1$, in agreement with the simulation results shown in Fig. 2.

It is interesting to investigate the time dependence of the condensation process. In Fig. 4 we plot the number of jams and the growth of the largest jam as a function of time. For p close to 1, the number of cars in the largest jam increases with a power of $1/2$, while the number of jams decreases accordingly. This power-law scaling is reminiscent of the diffusive dynamics of the random-walk process, in which the probability of attachment of a car equals that of detachment. Indeed, we have shown above that a necessary criterion for condensation is $r_+ = r_-$, i.e., inactive clusters increase or decrease with equal probability. This analogy between the size of jams and the position of a random walker was pointed out before by Nagel and Paczuski for the cruise control limit of the NS model [21], and our analytical model predicts it as a necessary condition. We thus find that our criterion concludes correctly on the existence of dynamic condensates and predicts the dynamics of their growth through a random-walk process.

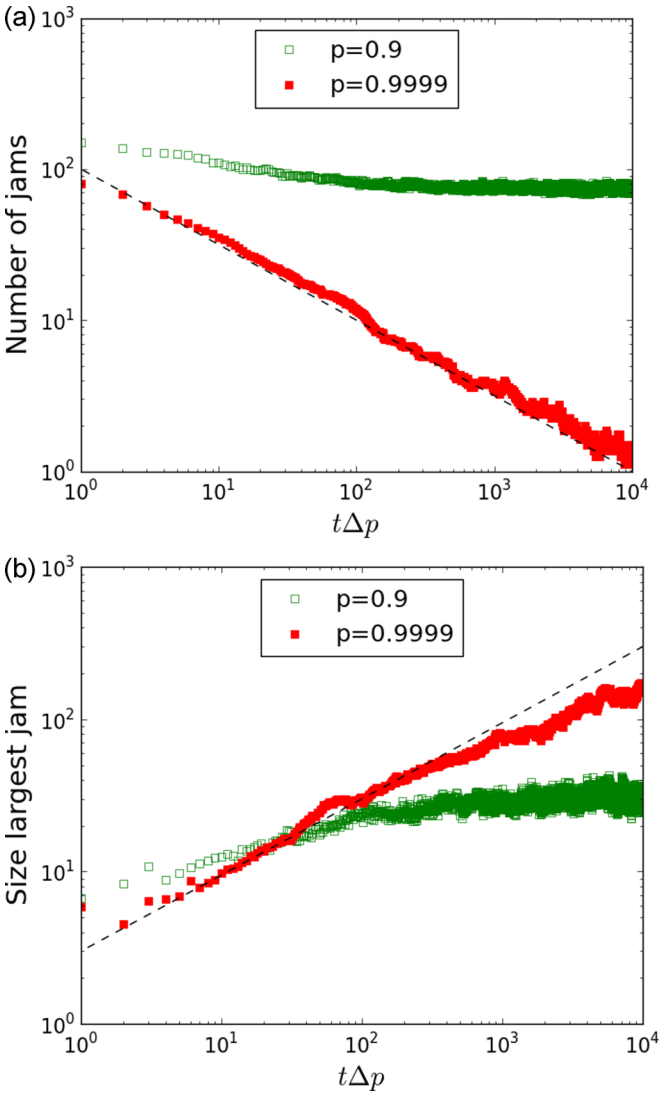

FIG. 4. (Color online) Time evolution of jams for p approaching 1. (a) Number of cars in the largest jam versus time. The dashed line has slope $-1/2$. (b) Number of jams versus time. The dashed line has slope $1/2$. The jam size diverges, and the number of jams decreases over time indicating condensate formation.

D. Nagel-Schreckenberg in the limit $p \to 0$ ($v_{\max} = 2$)

Simulations suggest that in this limit, no condensate forms [22,23]. We will address this issue with the criterion starting with Eq. (13). For $p \to 0$, the braking probability p is vanishingly small. As a result, the outflow rate of jams $r_{\text{out}} = 1 - p$. Using $r_{\text{in}} = r_{\text{out}} - \Delta r$, we can hence approximate $r_- = r_{\text{out}}(1 - r_{\text{in}}) \approx p + \Delta r$. *A priori* we do not know which term dominates the scaling of r_- when p vanishes: p or Δr. If Δr determines the scaling, we immediately see that the left-hand side of Eq. (12): $\Delta r/r_- = \Delta r/\Delta r \neq 0$ and there is no condensation. If p determines the scaling, we can simplify Eq. (13) as follows: Because the outflow rate is close to unity, the density of free flow is high and any random slow down of a car immediately causes the upstream neighbor to become kinetically constrained. Because this happens with probability p, the jam creation rate per car is $u \sim p$, as is also shown by the simulation results in Fig. 3. With $u \sim p$, $r_- \sim p$ and $r_{\text{out}} \approx 1$, Eq. (13) becomes

$$\frac{r_{\text{out}} uTn}{r_-} \sim Tn. \tag{17}$$

Since both $T > 0$ and $n > 0$, we conclude that there is no condensation in the limit $p \to 0$, in agreement with the simulation results [22,23].

E. Nagel-Schreckenberg with $v_{\max} > 2$

It is frequently assumed that the NS model behaves qualitatively similar when changing $v_{\max}$ [24,25]. Here we will

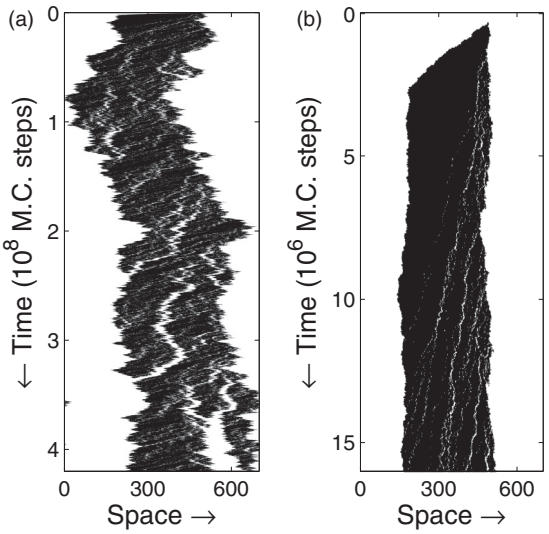

FIG. 5. Space-time diagram of vehicles in the NS-model with $p = 0.9998$ and $v_{\max} = 3$, entire simulation run (a), and enlarged section at early times (b). Inactive cars are shown in black. Horizontal axis represents car index; to follow the evolution of jams, we plot the space-time diagram in a frame comoving with the speed of jams. The emerging white regions inside the jam indicate a split up of the original jam that becomes more pronounced at later times. The enlarged section in (b) shows that this split up emerges at the downstream boundary of the jam.

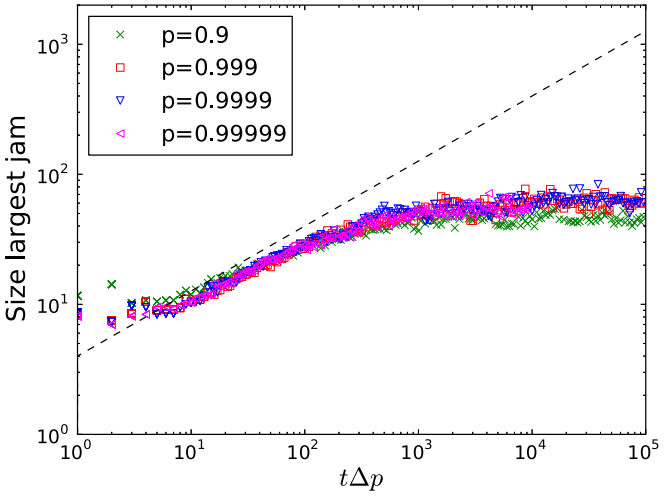

FIG. 6. (Color online) Time evolution of the number of cars in the largest jam for $v_{\max} = 3$, for values of p approaching 1. The dashed line has slope 1/2. The jam size does not diverge; the data for all p approaching 1 overlap.

investigate this analytically and find that in the limit $p \to 1$ there is a qualitative difference. Surprisingly, for $v_{\max} > 2$, there is no condensation in this limit, in contrast to $v_{\max} = 2$.

To see this, we note that for $v_{\max} = 2$ acceleration from jam to free flow occurs in a single step. Therefore, an accelerating car immediately leaves the jam, keeping the density of the jam finite. In contrast, for $v_{\max} > 2$, the acceleration needs multiple steps in the limit $p \to 1$. A car leaving the jam is still part of the jam until it reaches the maximum velocity. This lowers the density of jams, and leaves Eq. (1) unfulfilled. In the spaces created inside the jam, new free flow can emerge that splits up the jam. This mechanism prevents the formation of an infinitely large jam.

We demonstrate the split up of jams in the space-time diagram obtained in simulations; see Fig. 5. The simulation starts from random initial car positions; after a jam has nucleated, it grows, but shortly after that, the first free flow starts to appear inside the jam. This becomes most obvious in Fig. 5(b), where we show a magnified section at early times. All free-flow "bubbles" inside the jam clearly emerge at the downstream boundary of the jam. This free flow is persistent and covers larger regions at later times. These pictures demonstrate that there is no single macroscopic condensate for $v_{\max} > 2$.

We confirm the absence of condensation for $v_{\max} = 3$ numerically by studying the number of cars in the largest jams in simulations; see Fig. 6. In contrast to $v_{\max} = 2$ [Fig. 4(b)], the number of cars no longer diverges as p approaches 1, i.e., the data are not approaching anymore the asymptotic line. The curves for all p overlap, demonstrating the absence of condensation, and the qualitative difference to $v_{\max} = 2$.

To complete the analytical discussion of the NS model we shortly comment on the limits $0 < p < 1$ and $p \to 0$ for $v_{\max} > 2$. In both cases, the argument is similar to that of $v_{\max} = 2$. For $0 < p < 1$, fluctuations in velocity create fluctuations in distances between free-flowing cars. As a result, the creation rate of jams $u > 0$. In the limit $p \to 0$ the creation rate of jams vanishes with p, $u \sim p$, but the growth rate of jams vanishes just as quickly, so there is no condensation.

In summary, the surprising conclusion of our analytical treatment of the NS model is that only in the case $v_{\max} = 2$ (limit $p \to 1$) is there a true condensate transition.

F. Application to velocity-dependent randomization model

An extension of the Nagel-Schreckenberg model is the velocity-dependent randomization (VDR) model, in which there are *two* fluctuation parameters instead of *one*: p_{f} controls the fluctuations in free flow, while p_{j} controls the fluctuations of jammed traffic. This model takes account of the fact that drivers may behave differently depending on the traffic context, free flowing or jammed. We will show that in this model, where we have two control parameters, one for the creation rate and one for the growth rate of clusters, there is a condensate even in the limit $p_f, p_j \to 0$.

To do so, we use simulations to determine the size of the largest jam as a function of p_j and p_f. To incorporate both p_f and p_j, we modify the NS model update scheme by adding an extra step before the randomization step 3. If car i is jammed ($v_i < v_{\max}$ after step 2) then $p = p_j$, and if car i is free flowing ($v_i = v_{\max}$) then $p = p_f$. Further, the update scheme is identical to the NS update scheme. We plot the size of the largest jam in a two-dimensional contour plot in Fig. 7. This plot shows that a condensate forms if

$$\frac{p_f}{p_j} \to 0. \tag{18}$$

This numerical finding is indeed in line with the qualitative argument that the creation rate of new jams, controlled by p_f,

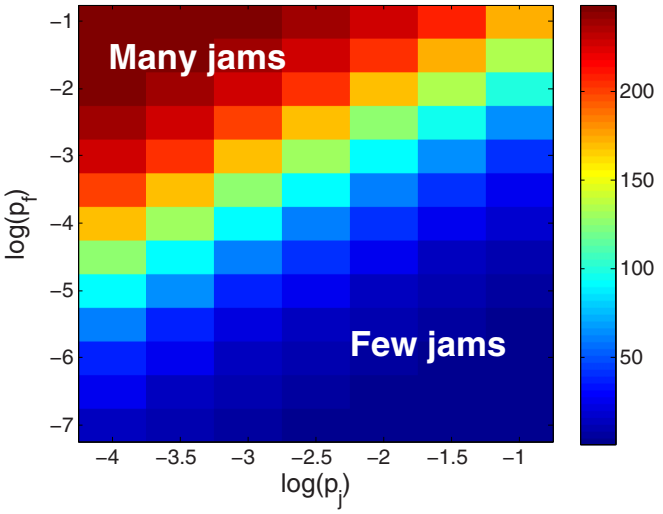

FIG. 7. (Color online) Diagram illustrating the number of cars in the largest jam (see color bar) as a function of p_f and p_j in the velocity-dependent randomization model. Simulations were performed with 1000 cars at density $\rho = 0.4$. The diagram (logarithmic with base 10) is plotted in the region close to $p_f \to 0$ and $p_j \to 0$. Whether a condensate forms depends on how the limit to zero is taken. A few jams signal a condensate forms only in the limit $p_f/p_j \to 0$.

must vanish faster than the growth rate of jams, controlled by p_j. This finding thus again demonstrates the principle of the criterion for condensation.

IV. CONCLUSION

We have derived a criterion for condensation in one-dimensional transport models with a kinetic constraint that causes clustering of immobile particles. Whether this clustering leads to macroscopic phase separation depends on two factors: First, density fluctuations in inactive clusters must be small enough to prevent the split up of inactive clusters. Second, the growth rate of inactive clusters must dominate the formation rate of new inactive clusters since those reduce the inflow of existing clusters downstream.

The latter condition means that condensation is only possible if the growth rate of inactive clusters equals their shrink rate. This establishes a generic analogy of the size of inactive clusters to the position of a random walker, that was previously found by Nagel and Paczuski [21] and Barlovic *et al.* [26] for specific models. With this analogy we can explain the growth dynamics of condensates as well as the distribution of lifetimes and sizes of inactive clusters upon condensation.

We have applied the criterion to the well-known Nagel-Schreckenberg traffic model and find that a condensation transition occurs only in the limit $p \to 1$ with $v_{\max} = 2$. In all other cases traffic jams are finite. We note that nevertheless there can be a discontinuity in the mean velocity of cars [23] or its derivative, reflecting a sudden onset of jams. This situation, where there is a discontinuous transition but no macroscopic condensate, appears different from traditional equilibrium transitions, and is likely related to the nonequilibrium nature of the system.

The applicability to traffic is general. We derive the criterion from basic postulates, therefore it does not depend on the specifics of the model. The challenge in applying the criterion to specific models will be to find the size and density of jams in terms of the overall density and fluctuations [Eqs. (1) and (13)] of the individual model.

A wider applicability of the criterion is demonstrated by analyzing condensation in the velocity-dependent randomization model, in which the creation rate of new traffic jams is controlled by two stochastic parameters, one for the fluctuations of free-flowing traffic (p_f), and one for those of jammed traffic (p_j). Exploring both parameters, we demonstrated a new condensate transition in the limit $p_f, p_j \to 0$ if p_f vanishes faster than p_j, in agreement with the idea behind our criterion for condensation.

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