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On the tail asymptotics of the area swept under the Brownian storage graph

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In this paper, the area swept under the workload graph is analyzed: with $\{Q(t)\colon\ t \geq 0\}$ denoting the stationary workload process, the asymptotic behavior of

$$\pi_{T(u)}(u) := \mathbb{P}\left(\int_0^{T(u)} Q(r)\,\mathrm{d}r > u\right)$$

is analyzed. Focusing on regulated Brownian motion, first the exact asymptotics of $\pi_{T(u)}(u)$ are given for the case that $T(u)$ grows slower than $\sqrt{u}$, and then logarithmic asymptotics for (i) $T(u) = T\sqrt{u}$ (relying on sample-path large deviations), and (ii) $\sqrt{u} = \mathrm{o}(T(u))$ but $T(u) = \mathrm{o}(u)$. Finally, the Laplace transform of the residual busy period are given in terms of the Airy function.

Keywords: area; Laplace transform; large deviations; queues; workload process

1. Introduction

Queueing models form an important branch within applied probability, having applications in production, storage, and inventory systems, as well as in communication networks. At the same time, there is a strong link with various models that play a crucial role in finance and risk theory, see, for instance, [11].

In more formal terms, the workload process of a queue is commonly defined as follows. Let $(X(t))_{t\in\mathbb{R}}$ be a stochastic process, that is often assumed to have stationary increments; without loss of generality we assume it has zero mean. Let $c > 0$ be the drain rate of the queue. Then the corresponding workload process $(Q(t))_{t\in\mathbb{R}}$ is defined through

$$Q(t) = \sup_{s\leq t} X(t) - X(s) - c(t-s).$$

A sizable body of literature is devoted to the analysis of the probabilistic properties of this workload process, both in terms of its stationary behavior and its transient characteristics.

One of the key metrics of the queueing system under consideration is the mean stationary workload. In many situations, this cannot be computed explicitly, and one then often resorts to simulation. A commonly used estimator is

$$\bar{Q}_T := \frac{1}{T}\sum_{i=1}^{T} Q(i);$$

one could set up the situation such that at time 0 the queue has already run for a substantial amount of time, such that one can safely assume the workload is in stationarity. In the simulation literature, this type of estimators (and related ones) have been analyzed in detail; see, for example, [3]. Results are in terms of laws of large numbers and central limit theorems.

Recently, attention shifted to the large deviation properties of the above type of estimators. It is observed that the subsequent observations are in general dependent, which considerably complicates the analysis. More specifically, standard large-deviations techniques do not apply here; the Gärtner–Ellis theorem [8], that allows only a mild dependence between the increments, is therefore not of any use. Even in cases in which the correlation of the stationary workload exhibits roughly exponential decay (being a manifestation of the queue's input process having short-range dependent properties), it turns out that the probability of the sample mean $\bar{Q}_T$ deviating from the mean stationary workload, say q, under quite general circumstances, does *not* decay exponentially.

Let us consider a few more detailed results. In a random walk setting (i.e., in which $Q(0) = 0$ and $Q(t+1) = \max\{Q(t) + Y(t), 0\}$ for an i.i.d. sequence $Y(t)$), Meyn [13,14] proved an intriguing (asymmetric) result. 'Below the mean' there is, under mild regularity assumptions, exponential decay, in that

$$\limsup_{T\to\infty} \frac{1}{T} \log \mathbb{P}(\bar{Q}_T \le a) < 0$$

for each $a < q$, whereas 'above the mean' there is 'subexponential decay', that is,

$$\lim_{T\to\infty} \frac{1}{T} \log \mathbb{P}(\bar{Q}_T \ge a) = 0$$

for each $a > q$. Subsequently, Duffy and Meyn [10], proved that the right scaling was *quadratic*, in the sense that in their setting $T^{-2}\sum_{i=1}^{T} Q(i)$ satisfies a large deviations principle with a nontrivial rate function. The square can intuitively be understood from the fact that one essentially considers the right scaling for the *area* under the graph of the workload.

The above motivates the interest in tail probabilities of the type

$$\pi_{T(u)}(u) := \mathbb{P}\left(\int_0^{T(u)} Q(t)\,\mathrm{d}t > u\right)$$

for various types of interval lengths $T(u)$, and $u \to \infty$; here the workload is assumed to be in stationarity at time 0. As indicated above, for $T(u)$ be in the order of $\sqrt{u}$ and the queue's input process having i.i.d. increments, the tail probability $\pi_{T(u)}(u)$ decaying roughly like $\exp(-\alpha\sqrt{u})$ for some $\alpha \in (0, \infty)$. On the other hand, for the case $u = \mathrm{o}(T(u))$ it is seen that $\pi(u)$ tends to 1 for u large.

The queueing system we consider in this paper is *reflected* (or: regulated) *Brownian motion*, also referred to as *Brownian storage*; this means that the driving process $(X(t))_{t\in\mathbb{R}}$ is a (standard) Brownian motion. In more detail, our contributions are the following.

- We first, in Section 3, consider the *short timescale* regime, that is, we assume $T(u) = \mathrm{o}(\sqrt{u})$. The main intuition here is that, in this regime, with overwhelming probability the queue does not idle in $[0, T(u)]$, and as a consequence, $Q(s)$ behaves as $Q(0) + X(s) - cs$ for $s \in [0, T(u)]$. This essentially enables us to compute the so-called *exact asymptotics* of $\pi_{T(u)}(u)$, that is, we find an explicit function $\varphi(u)$ such that $\pi_{T(u)}(u)/\varphi(u) \to 1$ as $u \to \infty$.
- The second contribution concerns the *intermediate timescale* regime, in which $T(u)$ is proportional to $\sqrt{u}$. As a function of this proportionality constant, we determine in Section 4 the decay rate
$$-\alpha = \lim_{u\to\infty} \frac{1}{\sqrt{u}} \log \pi_{T(u)}(u),$$
such that $\pi_{T(u)}(u)$ roughly looks like $\exp(-\alpha\sqrt{u})$ for u large. A crucial observation is that the probability under study can be translated into a related probability in the so-called many-sources regime. This means that sample-path large deviations for Brownian motion can be applied here, for example, Schilder's theorem. Apart from determining the decay rate, also the associated most likely path is identified, complementing results in [10].
- Section 5 considers the *long timescale*, that is $\sqrt{u} = \mathrm{o}(T(u))$ but $T(u) = \mathrm{o}(u)$. Relying on the intuition that essentially one 'big' busy period causes the rare event under consideration, we prove that (like in the intermediate timescale regime) $\pi_{T(u)}(u)$ roughly decays like $\exp(-\alpha\sqrt{u})$ for some constant $\alpha > 0$. The proof techniques are reminiscent of those used to establish an analogous property in the M/M/1 queue [5].
- We then consider in Section 6 the integral over the *remaining busy period* (rather than a given horizon $T(u)$), again with Brownian motion input (cf. the results for 'traditional' single-server queues in [7]). It turns out to be possible to explicitly compute its Laplace transform, in terms of the so-called Airy function, which also enables closed-form expressions for the corresponding mean value.

2. Notation and model description

Let the stochastic process $\{B(t)\colon t \in \mathbb{R}\}$ be a standard Brownian motion (i.e., $\mathbb{E}B(t) = 0$ and $\mathbb{V}\mathrm{ar}B(t) = t$); $\mathscr{N}$ denotes a standard Normal random variable.

In this paper, we consider a fluid queue fed by $B(\cdot)$ and drained with a constant rate $c > 0$. Let $\{Q(t)\colon t \in \mathbb{R}\}$ denote the stationary buffer content process, that is, the unique stationary solution of the following *Skorokhod problem*:

S1 $Q(t) = Q(0) + B(t) - ct + L(t)$, for $t \geq 0$;
S2 $Q(t) \geq 0$, for $t \geq 0$;
S3 $L(0) = 0$ and L is nondecreasing;
S4 $\int_0^\infty Q(s)\,\mathrm{d}L(s) = 0$.

We recall that the solution to the above Skorokhod problem is

$$Q(t) = \sup_{s \le t} \bigl(B(t) - B(s) - c(t-s)\bigr).$$

The primary focus of this paper concerns the tail asymptotics

$$\pi_{T(u)}(u) := \mathbb{P}\biggl(\int_0^{T(u)} Q(r)\,\mathrm{d}r > u\biggr)$$

for functions $T(\cdot): \mathbb{R} \to \mathbb{R}_+$.

3. Short timescale

In this section, we focus on the analysis of $\pi_{T(u)}(u)$ as $u \to \infty$ and $T(u) = \mathrm{o}(\sqrt{u})$. The main intuition in this timescale is that with overwhelming probability the queue does not idle in $[0, T(u)]$. Therefore, $Q(r)$ essentially behaves as $Q(0) + B(r) - cr$ for $r \in [0, T(u)]$, so that $\pi_{T(u)}(u)$ looks like (u large)

$$\mathbb{P}\biggl(\int_0^{T(u)} \bigl[Q(0) + B(r) - cr\bigr]\,\mathrm{d}r > u\biggr).$$

This idea is formalized in the following theorem.

Theorem 1. *Let $T(u) = \mathrm{o}(\sqrt{u})$. Then, as $u \to \infty$,*

$$\pi_{T(u)}(u) = \exp\biggl(-\frac{2cu}{T(u)} - \frac{1}{3}c^2 T(u)\biggr)\bigl(1 + \mathrm{o}(1)\bigr).$$

The following lemma plays an important role in the proof of Theorem 1.

Lemma 1. *For any $T(\cdot): \mathbb{R} \to \mathbb{R}_+$, as $u \to \infty$,*

$$\mathbb{P}\biggl(\int_0^{T(u)} \bigl[Q(0) + B(r) - cr\bigr]\,\mathrm{d}r > u\biggr) = \exp\biggl(-\frac{2cu}{T(u)} - \frac{1}{3}c^2 T(u)\biggr)\bigl(1 + \mathrm{o}(1)\bigr).$$

Proof. Recalling that we assumed that the workload process is in steady-state at time 0, it is well-known that

$$\mathbb{P}\bigl(Q(0) > u\bigr) = \exp(-2cu), \tag{1}$$

see, for example, Section 5.3 in [12]. The distributional equality, for $T(u) > 0$,

$$\int_0^{T(u)} B(t)\,\mathrm{d}t \stackrel{\mathrm{d}}{=} \biggl(\frac{T(u)}{3}\biggr)^{1/2} \mathscr{N} \tag{2}$$

implies

$$\mathbb{P}\Big(\int_0^{T(u)}\big[Q(0)+B(r)-cr\big]\,\mathrm{d}r>u\Big)=\mathbb{P}\Big(T(u)Q(0)+\frac{T(u)^{3/2}}{\sqrt{3}}\mathscr{N}>u+\frac{1}{2}cT(u)^2\Big).$$

Denote

$$A_1(u):=\frac{\sqrt{3}(u+(1/2)c(T(u))^2)}{(T(u))^{3/2}}.$$

Integrating with respect to the distribution of $\mathscr{N}$, and using (1), we obtain that

$$\begin{aligned}&\mathbb{P}\Big(T(u)Q(0)+\frac{(T(u))^{3/2}}{\sqrt{3}}\mathscr{N}>u+\frac{1}{2}c\big(T(u)\big)^2\Big)\\&\quad=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathbb{P}\Big(Q(0)>\frac{u}{T(u)}+\frac{1}{2}cT(u)-\Big(\frac{T(u)}{3}\Big)^{1/2}x\Big)\mathrm{e}^{-x^2/2}\,\mathrm{d}x=I_1+I_2,\end{aligned}$$

with

$$\begin{aligned}I_1&:=\frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{2cu}{T(u)}-c^2T(u)\Big)\int_{-\infty}^{A_1(u)}\exp\Big(-\frac{x^2}{2}-2c\Big(\frac{T(u)}{3}\Big)^{1/2}x\Big)\,\mathrm{d}x;\\I_2&:=\frac{1}{\sqrt{2\pi}}\int_{A_1(u)}^{\infty}\exp\Big(-\frac{x^2}{2}\Big)\,\mathrm{d}x.\end{aligned}$$

Integral I_1: First, rewrite

$$I_1=\frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{2cu}{T(u)}-\frac{1}{3}c^2T(u)\Big)\int_{-\infty}^{A_1(u)}\exp\Big(-\Big(\frac{x}{\sqrt{2}}+A_2(u)\Big)^2\Big)\,\mathrm{d}x,$$

where $A_2(u)=c\sqrt{2T(u)/3}$. Using the substitution $y:=x+A_2(u)$, we obtain

$$\begin{aligned}I_1&=\frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{2cu}{T(u)}-\frac{1}{3}c^2T(u)\Big)\int_{-\infty}^{A_1(u)+A_2(u)}\exp\Big(-\frac{y^2}{2}\Big)\,\mathrm{d}y\\&=\exp\Big(-\frac{2cu}{T(u)}-\frac{1}{3}c^2T(u)\Big)\big(1+\mathrm{o}(1)\big)\end{aligned}$$

as $u\to\infty$.

Integral I_2:

$$\begin{aligned}I_2&=\frac{1}{\sqrt{2\pi}A_1(u)}\exp\Big(-\frac{(A_1(u))^2}{2}\Big)\big(1+\mathrm{o}(1)\big)\\&=\frac{(T(u))^{3/2}}{\sqrt{6\pi}(u+(1/2)c(T(u))^2)}\exp\Big(-\frac{3\sqrt{6\pi}(u+(1/2)c(T(u))^2)}{2(T(u))^3}\Big)^2\big(1+\mathrm{o}(1)\big)\\&=\mathrm{o}\Big(\exp\Big(-\frac{2cu}{T(u)}-\frac{1}{3}c^2T(u)\Big)\Big)\end{aligned}$$

as $u \to \infty$, where we used that $\mathbb{P}(\mathcal{N} > x) \sim \frac{1}{\sqrt{2\pi x}} \exp(-x^2/2)$ as $x \to \infty$.

This completes the proof. □

Proof of Theorem 1. We establish upper and lower bound separately.

Upper bound: We distinguish between the case that the queue has idled before $T(u)$, and the case the buffer has been nonnegative all the time. We thus obtain $\pi_{T(u)}(u) = P_1(u) + P_2(u)$, where

$$P_1(u) := \mathbb{P}\biggl(\int_0^{T(u)} Q(r)\,\mathrm{d}r > u, L\bigl(T(u)\bigr) = 0\biggr),$$
$$P_2(u) := \mathbb{P}\biggl(\int_0^{T(u)} Q(r)\,\mathrm{d}r > u, L\bigl(T(u)\bigr) > 0\biggr).$$

Due to S1 and Lemma 1, as $u \to \infty$,

$$\begin{aligned} P_1(u) &\le \mathbb{P}\biggl(\int_0^{T(u)} \bigl[Q(0) + B(r) - cr\bigr]\,\mathrm{d}r > u\biggr) \\ &= \exp\biggl(-\frac{2cu}{T(u)} - \frac{1}{3}c^2 T(u)\biggr)\bigl(1 + \mathrm{o}(1)\bigr). \end{aligned} \tag{3}$$

Moreover, for any $T(u) > 0$,

$$P_2(u) \le \mathbb{P}\biggl(\sup_{s,t\in[0,T(u)]} \bigl[B(t) - B(s) - c(t-s)\bigr] > \frac{u}{T(u)}\biggr),$$

realizing that for some epoch in $[0, T(u)]$ the workload has exceed level $u/T(u)$, whereas for another epoch it has been 0. According to the Borell inequality [2], Theorem 2.1, in conjunction with the self-similarity of Brownian motion, $P_2(u)$ is majorized by

$$\begin{aligned} &2\exp\biggl(-\frac{((u/T(u)) - \mathbb{E}[\sup_{s,t\in[0,T(u)]} B(t) - B(s) - c(t-s)])^2}{2T(u)}\biggr) \\ &\quad\le 2\exp\biggl(-\frac{((u/T(u)) - cT(u) - \sqrt{T(u)}\mathbb{E}[\sup_{s,t\in[0,1]} B(t) - B(s)])^2}{2T(u)}\biggr), \end{aligned}$$

which is negligible with respect to (3) as $u \to \infty$. This completes the proof of the upper bound.

Lower bound: In view of

$$\mathbb{P}\biggl(\int_0^{T(u)} Q(t)\,\mathrm{d}t > u\biggr) \ge \mathbb{P}\biggl(\int_0^{T(u)} \bigl[Q(0) + B(r) - cr\bigr]\,\mathrm{d}r > u\biggr),$$

due to Lemma 1 the proof is complete. □

4. Intermediate timescale

In this section, we consider the case of $T(u)$ being proportional to $\sqrt{u}$: we set $T(u) = T\sqrt{u}$ for some $T > 0$. The main result of this section is given in the following theorem, that describes the asymptotics of the probability that the area until time $T\sqrt{u}$ exceeds Mu. It uses the following notation:

$$\varphi(T, M) := \begin{cases} \frac{2}{3}\sqrt{6}c\sqrt{cM}, & \text{if } \sqrt{6M/c} < T; \\ 2cM/T + c^2T/3, & \text{else.} \end{cases}$$

In this regime the intuition is that, in order to build up an area of at least u, for relatively small values of T the queue does not idle with high probability, leading to an expression for the decay rate that involves both M and T. If, on the contrary, T is somewhat larger, then the most likely path is such that the queue starts off essentially empty at time 0, to return to 0 before $T\sqrt{u}$, thus yielding a decay rate that just depends on M.

Theorem 2. *For all $T, M > 0$, it holds that*

$$-\lim_{u\to\infty} \frac{1}{\sqrt{u}} \log \mathbb{P}\left(\int_0^{T\sqrt{u}} Q(r)\,\mathrm{d}r \geq Mu\right) = \varphi(T, M). \tag{4}$$

We first observe that the probability under study can be translated into a related probability in the so-called *many-sources regime*, as will be shown in Lemma 2. Let $B^{(i)}(\cdot)$ be a sequence of independent standard Brownian motions. Define

$$\overline{B^{(n)}}(t) := \frac{1}{n}\sum_{i=1}^{n} B^{(i)}(t), \qquad Q^{(n)}(t) := \sup_{s\leq t}\big(\overline{B^{(n)}}(t) - \overline{B^{(n)}}(s) - c(t-s)\big).$$

Lemma 2. *For each $T, M > 0, n \in \mathbb{N}$*

$$\mathbb{P}\left(\int_0^T Q^{(n)}(r)\,\mathrm{d}r > M\right) = \mathbb{P}\left(\int_0^{Tn} Q(r)\,\mathrm{d}r > Mn^2\right). \tag{5}$$

Proof. Observe that the left-hand side of (5) equals

$$\begin{aligned} &\mathbb{P}\left(\frac{1}{n}\int_0^T \sup_{s\leq r}\left(\sum_{i=1}^{n} B^{(i)}(r) - B^{(i)}(s) - cn(r-s)\right)\mathrm{d}r > M\right) \\ &= \mathbb{P}\left(\frac{1}{n}\int_0^T \sup_{s\leq r}\big(B(rn) - B(sn) - cn(r-s)\big)\,\mathrm{d}r > M\right) \\ &= \mathbb{P}\left(\frac{1}{n}\int_0^T \sup_{s\leq rn}\big(B(rn) - B(s) - crn + cs\big)\,\mathrm{d}r > M\right). \end{aligned}$$

Using the substitution $v := rn$, we obtain that

$$\mathbb{P}\Big(\int_0^T Q^{(n)}(r)\,\mathrm{d}r > M\Big) = \mathbb{P}\Big(\int_0^{Tn} \sup_{s\le v}\big(B(v)-B(s)-cv+cs\big)\,\mathrm{d}v > Mn^2\Big).$$

This completes the proof. □

In our analysis, we use the following notation:

$$\psi(M,a,s) := \frac{(M+(1/2)cs^2-as)^2}{(2/3)s^3} + 2ac.$$

The proof of Theorem 2 is based on the following lemmas.

Lemma 3. *For each $M, T > 0$ it holds that*

$$\inf_{a\ge 0}\inf_{s\in(0,T]} \psi(M,a,s) = \varphi(T,M).$$

The optimizing (a,s) equals

$$(a^\star, s^\star) = \begin{cases} (0, \sqrt{6M/c}), & \textit{if } \sqrt{6M/c} < T; \\ (M/T - cT/6, T), & \textit{else.} \end{cases}$$

Proof. Straightforward computation. □

Define

$$p_n(T,M,a) := \mathbb{P}\Big(\int_0^T Q^{(n)}(r)\,\mathrm{d}r \ge M \Big| Q^{(n)}(0) = a\Big).$$

Lemma 4. *For each $T, M, a > 0$*

$$\limsup_{n\to\infty} \frac{1}{n} \log p_n(T,M,a) \le -\inf_{s\in[0,T]} \frac{(M+(1/2)cs^2-as)^2}{(2/3)s^3}.$$

Proof. The proof is based on the Schilder's sample-path large-deviations principle [8,12]. Define the path space

$$\Omega := \Big\{ f:\mathbb{R}\to\mathbb{R}, \text{continuous}, f(0)=0, \lim_{t\to\infty}\frac{f(t)}{1+|t|} = \lim_{t\to-\infty}\frac{f(t)}{1+|t|} = 0 \Big\},$$

equipped with the norm

$$\|f\|_\Omega := \sup_{t\in\mathbb{R}} \frac{f(t)}{1+|t|}.$$

For a given function f, we have that the corresponding workload is given through $q[f](t) := \sup_{s \le t}(f(t) - f(s) - c(t-s))$. In addition,

$$\mathscr{S} := \left\{ f \in \Omega \colon q[f](0) = a, \int_0^T q[f](r)\,\mathrm{d}r \ge M \right\}.$$

The set $\mathscr{S}$ is closed; the proof of this property can be found in the Appendix. Hence, due to Schilder's theorem, we have that

$$\limsup_{n\to\infty} \frac{1}{n} \log p_n(T, M, a) \le -\inf_{f \in \mathscr{S}} \mathbb{I}(f), \tag{6}$$

with

$$\mathbb{I}(f) := \begin{cases} \frac{1}{2}\int_{\mathbb{R}} \left(f'(r)\right)^2 \mathrm{d}r, & f \in \mathscr{A}, \\ \infty, & \text{otherwise}, \end{cases}$$

where $\mathscr{A}$ denotes the space of absolutely continuous functions with a square integrable derivative.

Now we show that

$$-\inf_{f \in \mathscr{S}} \mathbb{I}(f) = -\inf_{f \in \mathscr{T}} \mathbb{I}(f), \tag{7}$$

where

$$\mathscr{T} := \left\{ f \in \Omega \colon \exists s \in [0, T] \colon \int_0^s f(r)\,\mathrm{d}r \ge M + \frac{1}{2}cs^2 - as \right\}.$$

To this end, first observe that $\mathscr{T} \subseteq \mathscr{S}$, so that $-\inf_{f \in \mathscr{T}} \mathbb{I}(f) \le -\inf_{f \in \mathscr{S}} \mathbb{I}(f)$; we are therefore left with proving the opposite inequality. Now fix for the moment a path f. Bearing in mind f is an absolutely continuous function, the following procedure yields a path $\bar{f} \in \mathscr{T}$ with $\mathbb{I}(f) = \mathbb{I}(\bar{f})$. First, we let

$$m[f] := \int_0^T 1_{\{q[f](u) > 0\}}(u)\,\mathrm{d}u,$$

denote the amount of 'nonidle time' corresponding to the path f in $[0, T]$. Then define

$$i[f](r) := \inf\left\{ s \in [0, T] \colon \int_0^s 1_{\{q[f](u) > 0\}}(u)\,\mathrm{d}u > r \right\}$$

for $r \in [0, m[f]]$, and

$$j[f](r) := \inf\left\{ s \in [0, T] \colon \int_0^s 1_{\{q[f](u) = 0\}}(u)\,\mathrm{d}u > r \right\}$$

for $r \in [0, T - m[f]]$.

Now we construct the path $\bar{f}$ by shifting all the idle periods of $q[f]$ to the end of the interval $[0,T]$. That is, for $r \in [0, m[f]]$, let

$$\bar{f}(r) := q[f]\big(i[f](r)\big) + cr - a,$$

and for $r \in [m[f], T]$, let

$$\bar{f}(r) := q[f]\big(i[f]\big(m[f]\big)\big) + cr + f\big(j[f]\big(r - m[f]\big)\big) - cj[f]\big(r - m[f]\big).$$

We also set

$$\bar{f}(r) := 0 \qquad \text{for } r < 0 \quad \text{and} \quad \bar{f}(r) := \bar{f}(T) \qquad \text{for } r > T.$$

It is clear that $\mathbb{I}(f) = \mathbb{I}(\bar{f})$ (because we just permuted subintervals of $[0,T]$, which does not affect the rate function), while the constructed path $\bar{f}$ is now in $\mathscr{T}$. Conclude that $-\inf_{f\in\mathscr{S}} \mathbb{I}(f) \le -\inf_{f\in\mathscr{T}} \mathbb{I}(f)$, as desired.

We are therefore left with computing $-\inf_{f\in\mathscr{T}} \mathbb{I}(f)$. Let $\varepsilon > 0$. Clearly, $\mathscr{T} \subseteq \bigcup_{s\in[0,T]} \mathscr{T}^s$, with

$$\mathscr{T}^s := \left\{ f \in \Omega\colon \int_0^s f(r)\,\mathrm{d}r > M + \frac{1}{2}cs^2 - as - \varepsilon \right\}.$$

This implies that

$$-\inf_{f\in\mathscr{T}} \mathbb{I}(f) \le -\inf_{s\in[0,t]} \inf_{f\in\mathscr{T}^s} \mathbb{I}(f). \tag{8}$$

Observe that set $\mathscr{T}^s$ is open, and combine this with Schilder's theorem and (2):

$$\begin{aligned} -\inf_{f\in\mathscr{T}^s} \mathbb{I}(f) &\le \lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{1}{n} \int_0^s \sum_{i=1}^n B^{(i)}(r)\,\mathrm{d}r > M + \frac{1}{2}cs^2 - as - \varepsilon \right) \\ &= \lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\mathscr{N} > \sqrt{\frac{3n}{s^3}} \left(M + \frac{1}{2}cs^2 - as - \varepsilon \right) \right). \end{aligned}$$

Using that $\mathbb{P}(\mathscr{N} > x) \le (\sqrt{2\pi}x)^{-1} \exp(-x^2/2)$, we obtain

$$-\inf_{f\in\mathscr{T}^s} \mathbb{I}(f) \le -\frac{(M + (1/2)cs^2 - as - \varepsilon)^2}{(2/3)s^3}. \tag{9}$$

Thus the claim follows from combining (6), (7), and (8) with (9). □

Proof of Theorem 2. Due to Lemma 2 it suffices to find the logarithmic asymptotics

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\int_0^T Q^{(n)}(r)\,\mathrm{d}r \ge M \right).$$

We establish the upper and lower bound separately.

Upper bound: Recall $\mathbb{P}(Q^{(n)}(0) \geq a) = \mathrm{e}^{-2nca}$ by virtue of (1). For any $\varepsilon > 0$ and an arbitrary integer N,

$$\begin{aligned}\mathbb{P}\biggl(\int_0^T Q^{(n)}(r)\,\mathrm{d}r \geq M\biggr) &= \int_0^\infty 2nc\mathrm{e}^{-2ncv\varepsilon} p_n(T, M, v\varepsilon)\,\mathrm{d}v \\ &\leq \sum_{k=0}^{\infty} 2nc\mathrm{e}^{-2nck\varepsilon} p_n\bigl(T, M, (k+1)\varepsilon\bigr) \\ &\leq \sum_{k=0}^{N-1} 2nc\mathrm{e}^{-2nck\varepsilon} p_n\bigl(T, M, (k+1)\varepsilon\bigr) + 2nc \cdot \frac{\mathrm{e}^{-2ncN\varepsilon}}{1-\mathrm{e}^{-2nc\varepsilon}}.\end{aligned}$$

As a consequence, [8], Lemma 1.2.15, leads to

$$\begin{aligned}&\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}\biggl(\int_0^T Q^{(n)}(r)\,\mathrm{d}r \geq M\biggr) \\ &\quad\leq \max\biggl\{\max_{k=0,\ldots,N-1}\biggl(\lim_{n\to\infty}\frac{1}{n}\log p_n\bigl(T, M, (k+1)\varepsilon\bigr) - 2ck\varepsilon\biggr), -2cN\varepsilon\biggr\}.\end{aligned}$$

Due to Lemma 4, we can further bound this by

$$\begin{aligned}&\max\Bigl\{\max_{k=0,\ldots,N-1}\Bigl(-\inf_{s\in[0,T]}\psi\bigl(M, (k+1)\varepsilon, s\bigr)\Bigr) + 2c\varepsilon, -2cN\varepsilon\Bigr\} \\ &\quad\leq -\min\Bigl\{\inf_{a\geq 0}\inf_{s\in[0,T]}\psi(M, a, s) - 2c\varepsilon, 2cN\varepsilon\Bigr\}.\end{aligned}$$

Now Lemma 3 yields

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}\biggl(\int_0^T Q^{(n)}(r)\,\mathrm{d}r \geq M\biggr) \leq -\min\bigl\{\varphi(T, M) - 2c\varepsilon, 2c\varepsilon N\bigr\}.$$

We establish the upper bound by subsequently letting $N \uparrow \infty$ and $\varepsilon \downarrow 0$.

Lower bound: Let $\varepsilon > 0$. Due to the Skorokhod representation, we have, with $L^{(n)}(\cdot)$ defined in the obvious way,

$$Q^{(n)}(t) = Q^{(n)}(0) + \overline{B^{(n)}}(t) - ct + L^{(n)}(t).$$

Observe that for each $a \geq 0$ and $s \in [0, T]$,

$$\begin{aligned}\mathbb{P}\biggl(\int_0^T Q^{(n)}(r)\,\mathrm{d}r \geq M\biggr) &\geq \mathbb{P}\biggl(\int_0^s Q^{(n)}(r)\,\mathrm{d}r > M\biggr) \\ &\geq \mathbb{P}\biggl(\int_0^s \bigl(Q^{(n)}(0) + \overline{B^{(n)}}(r) - cr\bigr)\,\mathrm{d}r > M\biggr) \\ &\geq \int_a^{a+\varepsilon} 2nc\exp(-2ncv)\mathbb{P}\biggl(\int_0^s \overline{B^{(n)}}(r)\,\mathrm{d}r > \frac{1}{2}cs^2 + M - vs\biggr)\,\mathrm{d}v\end{aligned}$$

$$
\begin{aligned}
&\geq 2nc \exp\bigl(-2nc(a+\varepsilon)\bigr)\mathbb{P}\biggl(\int_0^s \overline{B^{(n)}}(r)\,\mathrm{d}r > \frac{1}{2}cs^2 + M - as\biggr) \\
&\geq 2nc \exp\bigl(-2cn(a+\varepsilon)\bigr)\mathbb{P}\biggl(\mathscr{N} > \sqrt{\frac{3n}{s^3}}\biggl(\frac{1}{2}cs^2 + M - as\biggr)\biggr).
\end{aligned}
$$

Now applying that for $x > 0$,

$$
\mathbb{P}(\mathscr{N} > x) \geq \frac{x^2-1}{\sqrt{2\pi}x^3}\exp\bigl(-x^2/2\bigr),
$$

see, for example, Section 2 in [2], we obtain that for all $a \geq 0$ and $s \in [0, T]$,

$$
\liminf_{n\to\infty} \frac{1}{n}\log\mathbb{P}\biggl(\int_0^T Q^{(n)}(r)\,\mathrm{d}r \geq M\biggr) \geq -\frac{((1/2)cs^2 + M - as)^2}{(2/3)s^3} - 2c(a+\varepsilon).
$$

In order to complete the proof it suffices to let $\varepsilon \downarrow 0$ and to maximize over $a \geq 0$ and $s \in [0, T]$. □

Remark 1. Interestingly, the most likely path $f^\star$ of $\overline{B^{(n)}}(\cdot)$ can be explicitly computed, revealing two separate scenarios.

- Suppose $s^\star = \sqrt{6M/c} < T$. Then the queue (most likely) starts empty at time 0, is positive for a while, drops to 0 at time $s^\star$, and remains empty. The corresponding path $f^\star$ of $\overline{B^{(n)}}(\cdot)$ is, for $r \in [0, s^\star]$,

$$
f^\star(r) = 2cr - \frac{cr^2}{6}\sqrt{\frac{6c}{M}},
$$

  and $f^\star(r) = f^\star(s^\star)$ for $r \in (s^\star, T]$.
- Suppose $s^\star = T < \sqrt{6M/c}$. Then the queue is symmetric in the interval $[0, T]$, and has the value $a^\star$ at times 0 and T. The corresponding path $f^\star$ of $\overline{B^{(n)}}(\cdot)$ is, for $r \in [0, T]$,

$$
f^\star(r) = 2cr - \frac{c}{T}r^2.
$$

It can easily be verified that indeed

$$
\frac{1}{2}\int_0^T \bigl((f^\star)'(r)\bigr)^2\,\mathrm{d}r + 2a^\star c = \varphi(T, M)
$$

as expected.

5. Long timescale

In this section, we consider the case that $T(u)$ is between $\sqrt{u}$ and u. It turns out that we find the same logarithmic asymptotics as in the case that $T(u) = T\sqrt{u}$ for large T (i.e., T larger than

$\sqrt{6M/c}$). In the proof, we first introduce some sort of 'surrogate busy periods' (recall that 'traditional' busy periods do not exist for reflected Brownian motion). Then we show that the event of interest occurs essentially due to a single busy period being 'big' (in terms of the area swept under the workload graph); this is due to the fact that the contribution of a single busy period has a subexponential distribution (viz. roughly a Weibull distribution with shape parameter $\frac{1}{2}$).

Defining

$$\varphi(M) := \frac{2}{3}\sqrt{6}c\sqrt{cM}, \qquad \tilde{\psi}(M,\delta,s) := \frac{(M+(1/2)cs^2-\delta s)^2}{(2/3)s^3},$$

we are in a position to state the main result of the section.

Theorem 3. *Let* $\sqrt{u} = \mathrm{o}(T(u))$ *and* $T(u) = \mathrm{o}(u)$. *Then,*

$$\lim_{u\to\infty} \frac{1}{\sqrt{u}} \log \mathbb{P}\left(\int_0^{T(u)} Q(r)\,\mathrm{d}r > Mu\right) = -\varphi(M).$$

In order to prove Theorem 3, we need to introduce some notation. Let

$$\tau_0 := \inf\{t > 0\colon\ Q(0) + B(t) - ct = 0\}, \qquad \tau(x) := \inf\{t > 0\colon\ x + B(t) - ct = 0\}.$$

Besides, for given $\delta > 0$ and $i = 1, 2, \ldots,$ let

$$\sigma_i := \inf\{t > \tau_{i-1}\colon\ Q(t) \geq 2\delta\}, \qquad \tau_i := \inf\{t > \sigma_i\colon\ Q(t) \leq \delta\}$$

and

$$H_0 := \int_0^{\tau_0} Q(r)\,\mathrm{d}r, \qquad H_i := \int_{\sigma_i}^{\tau_i} Q(r)\,\mathrm{d}r.$$

Observe that $\{H_i\}_{i\in\mathbb{N}}$ constitutes a sequence of i.i.d. random variables, that is in addition independent of H_0; likewise, the $\xi_i := \tau_i - \sigma_i$ are i.i.d. random variables. Moreover, for each $i = 1, 2, \ldots$ we have

$$H_i \stackrel{\mathrm{d}}{=} \int_0^{\tau(\delta)} \big(\delta + B(r) - cr\big)\,\mathrm{d}r.$$

The following lemmas play crucial role in the proof of Theorem 3.

Lemma 5. *For each* $M > 0$ *it holds that*

$$\lim_{\delta\downarrow 0} \inf_{s\geq 0} \tilde{\psi}(M,\delta,s) = \varphi(M).$$

Proof. This proof is a straightforward computation. Note that

$$s^\star(\delta) = \frac{-\delta + \sqrt{\delta^2 + 6Mc}}{c}$$

is the minimizer in $\inf_{s\geq 0}\tilde{\psi}(M,\delta,s)$. Consequently,

$$\lim_{\delta\downarrow 0}\inf_{s\geq 0}\tilde{\psi}(M,\delta,s)=\lim_{\delta\downarrow 0}\frac{(M+(1/2)c(s^\star(\delta))^2-\delta s^\star(\delta))^2}{(2/3)(s^\star(\delta))^3}=\varphi(M).$$

This completes the proof. □

Lemma 6. *For each $M>0$ and $i=0,1,\ldots,$ we have*

$$\limsup_{u\to\infty}\frac{1}{\sqrt{u}}\log\mathbb{P}(H_i>Mu)\leq-\varphi(M).$$

Proof. We start with the analysis of H_i, for $i=1,2,\ldots$. Observe that

$$\mathbb{P}(H_i>Mu)=\mathbb{P}\biggl(\exists s\geq 0\colon\ \frac{1}{u}\int_0^s\bigl(\delta+B(r)-cr\bigr)\,\mathrm{d}r>M,\forall r\in(0,s)\colon\ \delta+B(r)-cr>0\biggr),$$

which is majorized by

$$\mathbb{P}\biggl(\exists s\geq 0\colon\ \frac{1}{u}\int_0^s\bigl(\delta+B(r)-cr\bigr)\,\mathrm{d}r>M\biggr).\tag{10}$$

Substituting $r=\sqrt{u}v$ we obtain that, for u sufficiently large, (10) equals

$$\begin{aligned}&\mathbb{P}\biggl(\exists s\geq 0\colon\ \int_0^{s/\sqrt{u}}\biggl(\frac{\delta}{\sqrt{u}}+\frac{1}{\sqrt{u}}B(\sqrt{u}v)-cv\biggr)\,\mathrm{d}v>M\biggr)\\&\quad=\mathbb{P}\biggl(\exists s\geq 0\colon\ \int_0^{s}\biggl(\frac{\delta}{\sqrt{u}}+u^{-1/4}B(v)-cv\biggr)\,\mathrm{d}v>M\biggr)\\&\quad=\mathbb{P}\biggl(\sup_{s\geq 0}\frac{\int_0^s B(v)\,\mathrm{d}v}{M+(1/2)cs^2-\delta s/\sqrt{u}}>u^{1/4}\biggr).\end{aligned}\tag{11}$$

Now, observe that $Y(s):=\int_0^s B(v)\,\mathrm{d}v/(M+\frac{1}{2}cs^2-\delta s/\sqrt{u})$ has bounded trajectories a.s. Hence, the Borell inequality (see, e.g., [2], Theorem 2.1) leads to the following upper bound of (11):

$$2\exp\biggl(-\inf_{s\geq 0}\frac{(M+(1/2)cs^2-(\delta/\sqrt{u})s)^2}{(2/3)s^3}\Bigl(u^{1/4}-\mathbb{E}\sup_{s\geq 0}Y(s)\Bigr)^2\biggr),$$

where $\mathbb{E}\sup_{s\geq 0}Y(s)$ is bounded (by 'Borell'). Combining the above with Lemma 5, we obtain that

$$\limsup_{u\to\infty}\frac{1}{\sqrt{u}}\log\mathbb{P}(H_i>Mu)\leq-\varphi(M).\tag{12}$$

In order to prove the claim for H_0 observe that

$$\mathbb{P}(H_0>Mu)=\int_0^\infty 2ca\exp(-2ca)\mathbb{P}\biggl(\int_0^{\tau(a)}\bigl(a+B(r)-cr\bigr)\,\mathrm{d}r>Mu\biggr)\,\mathrm{d}a.$$

Thus, by (12), it suffices to proceed along the lines of the proof of the upper bound of Theorem 2. □

Proof of Theorem 3. We establish upper and lower bound separately.

Lower bound: The lower bound follows straightforwardly from Theorem 2 combined with the fact that for sufficiently large u we have (recalling that $\sqrt{u} = \mathrm{o}(T(u))$)

$$\mathbb{P}\biggl(\int_0^{T(u)} Q(r)\,\mathrm{d}r > Mu\biggr) \geq \mathbb{P}\biggl(\int_0^{\sqrt{(6M/c)u}} Q(r)\,\mathrm{d}r > Mu\biggr).$$

Upper bound: Let $\delta > 0$ and denote $N(u) := \inf\{i\colon \tau_i \geq T(u)\}$, $K := 2/\mathbb{E}\xi_i$. Observe that

$$\begin{aligned}\mathbb{P}\biggl(\int_0^{T(u)} Q(r)\,\mathrm{d}r > u\biggr) &\leq \mathbb{P}\Biggl(2\delta T(u) + \sum_{i=0}^{N(u)} H_i > u\Biggr)\\ &\leq \mathbb{P}\Biggl(2\delta T(u) + \sum_{i=0}^{N(u)} H_i > u, N(u) \leq KT(u)\Biggr) + \mathbb{P}\bigl(N(u) > KT(u)\bigr)\\ &\leq \bar{P}_1(u) + \bar{P}_2(u),\end{aligned}$$

with

$$\bar{P}_1(u) := \mathbb{P}\Biggl(\sum_{i=0}^{\lceil KT(u)\rceil} H_i > u - 2\delta T(u)\Biggr), \qquad \bar{P}_2(u) := \mathbb{P}\bigl(N(u) > \lfloor KT(u)\rfloor\bigr).$$

We first analyze $\bar{P}_1(u)$. The idea is to reduce the problem of finding the upper bound of $\bar{P}_1(u)$ to the setting of [9], Theorem 8.3. To this end, pick $\varepsilon > 0$. Due to Lemma 6 there exists a sequence $\{\tilde{H}_i\}_{i=0,1,\ldots}$ of i.i.d. random variables such that for each $x > 0$ and δ sufficiently small,

$$\mathbb{P}(H_i > x) \leq \mathbb{P}(\tilde{H}_i > x) \tag{13}$$

and

$$\mathbb{P}(\tilde{H}_i > x) = p(x)\exp\bigl(-\bigl(\varphi(M) - \varepsilon\bigr)\sqrt{x}\bigr), \tag{14}$$

where $p(\cdot)$ is some O-regularly varying function, that is, $p(x)$ is a measurable function, such that, for each $\lambda \geq 1$

$$0 < \liminf_{x\to\infty} \frac{p(\lambda x)}{p(x)} \leq \limsup_{x\to\infty} \frac{p(\lambda x)}{p(x)} < \infty$$

(see, e.g., [4], Chapter 2, or the Appendix of [9]).

It is standard that, due to (13), for each $x > 0$,

$$\mathbb{P}\Biggl(\sum_{i=0}^{\lceil KT(u)\rceil} H_i > x\Biggr) \leq \mathbb{P}\Biggl(\sum_{i=0}^{\lceil KT(u)\rceil} \tilde{H}_i > x\Biggr). \tag{15}$$

Now, applying [9], Theorem 8.3 and recalling that $KT(u) = o(u)$, we have, as $u \to \infty$,

$$\mathbb{P}\left(\sum_{i=0}^{\lceil KT(u)\rceil} \tilde{H}_i > u - 2\delta T(u)\right) = \lceil KT(u)\rceil \cdot \mathbb{P}\big(\tilde{H}_0 > u - 2\delta T(u)\big)\big(1 + o(1)\big). \tag{16}$$

Combining (15) and (16) with (14), we obtain that, for each $\varepsilon > 0$,

$$\limsup_{u\to\infty} \frac{1}{\sqrt{u}} \log \bar{P}_1(u) \le -\varphi(M) + \varepsilon;$$

letting $\varepsilon \downarrow 0$, we conclude that we can replace the right-hand side in the previous display by $-\varphi(M)$.

We now focus on $\bar{P}_2(u)$. Observe that

$$\bar{P}_2(u) \le \mathbb{P}\big(S_{\lfloor KT(u)\rfloor} < T(u)\big) \qquad \text{where } S_{\lfloor KT(u)\rfloor} := \tau_0 + \sum_{i=1}^{\lfloor KT(u)\rfloor} \xi_i.$$

Moreover, note that $\xi_i, i = 1, 2, \ldots$ are i.i.d. with

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{P}(\xi_1 \le t) = \frac{\delta}{\sqrt{2\pi t^3}} \exp\big(-(\delta - ct)^2/2t\big)$$

for $t > 0$; see, for example, [15], Section 2.9. Hence, a Chernoff bound argument yields, recalling that $K > 1/\mathbb{E}\xi_i$,

$$\limsup_{u\to\infty} \frac{1}{T(u)} \log \mathbb{P}\big(S_{\lfloor KT(u)\rfloor} < T(u)\big) \le -K \cdot \sup_{\theta<0}\left[\theta\frac{1}{K} - \log \mathbb{E}\exp(\theta\xi_1)\right] < 0.$$

We have found that $\bar{P}_1(u)$ is smaller than a function of the order $\exp(-\beta_1\sqrt{u})$, while $\bar{P}_2(u)$ is smaller than a function of the order $\exp(-\beta_2 T(u))$, for some $\beta_1, \beta_2 > 0$. Now recalling that $\sqrt{u} = o(T(u))$, it follows that the upper bound on $\bar{P}_1(u)$ is smaller than the upper bound on $\bar{P}_2(u)$. As a result,

$$\limsup_{u\to\infty} \frac{1}{T(u)} \log \mathbb{P}\left(\int_0^{T(u)} Q(r)\,\mathrm{d}r > Mu\right) \le \limsup_{u\to\infty} \frac{1}{\sqrt{u}} \log\big(\bar{P}_1(u) + \bar{P}_2(u)\big) = -\varphi(M).$$

This completes the proof. □

6. Residual busy period

In this section, we analyze the integral of the stationary workload for regulated Brownian motion over the *residual busy period*. It turns out to be possible to explicitly compute its Laplace transform, in terms of the so-called Airy function. As a by-product, the corresponding mean value is calculated.

Recall that

$$\tau_0 := \inf\{t \geq 0\colon\ Q(t) = 0\}, \qquad \tau(x) := \inf\{t \geq 0\colon\ x + B(t) - ct = 0\};$$

we also define the integral of the workload until the end of the busy period, conditional on the workload being x at time 0:

$$J(x) := \int_0^{\tau(x)} \big(x + B(t) - ct\big)\,\mathrm{d}t.$$

By

$$\mathrm{Ai}(x) := \frac{1}{\pi}\int_0^\infty \cos\left(\frac{1}{3}t^3 + xt\right)\mathrm{d}t$$

we denote the *Airy function* (see, e.g., [1], Chapter 10.4).

Theorem 4. *For each* $\gamma \geq 0$,

$$\mathbb{E}\exp\left[-\gamma\int_0^{\tau_0} Q(t)\,\mathrm{d}t\right] = \frac{2c}{\mathrm{Ai}((2\gamma)^{-2/3}c^2)}\int_0^\infty \mathrm{e}^{-cx}\mathrm{Ai}\big((2\gamma)^{-2/3}c^2 + (2\gamma)^{1/3}x\big)\,\mathrm{d}x.$$

Proof. Observe that up to time τ_0 we have that $Q(t) = Q(0) + B(t) - ct$. Hence,

$$\begin{aligned}
&\mathbb{E}\exp\left[-\gamma\int_0^{\tau_0} Q(t)\,\mathrm{d}t\right] \\
&= \int_0^\infty \mathbb{P}\left(\exp\left[-\gamma\int_0^{\tau_0} Q(t)\,\mathrm{d}t\right] > u\right)\mathrm{d}u \\
&= \int_0^\infty \mathbb{P}\left(\exp\left[-\gamma\int_0^{\tau_0} \big(Q(0) + B(t) - ct\big)\,\mathrm{d}t\right] > u\right)\mathrm{d}u \\
&= \int_0^\infty\int_0^\infty 2c\exp(-2cx)\mathbb{P}\big(\exp[-\gamma J(x)] > u\big)\,\mathrm{d}x\,\mathrm{d}u \\
&= \int_0^\infty 2c\exp(-2cx)\int_0^\infty \mathbb{P}\big(\exp[-\gamma J(x)] > u\big)\,\mathrm{d}u\,\mathrm{d}x \\
&= \int_0^\infty 2c\exp(-2cx)\mathbb{E}\big[\exp[-\gamma J(x)]\big]\,\mathrm{d}x.
\end{aligned} \tag{17}$$

Following Borodin and Salminen [6], Chapter 2, equation (2.8.1), we have that

$$\mathbb{E}\big[\exp[-\gamma J(x)]\big] = \exp(cx)\frac{\mathrm{Ai}(2^{1/3}\gamma^{-2/3}((1/2)c^2 + \gamma x))}{\mathrm{Ai}((2\gamma)^{-2/3}c^2)}, \tag{18}$$

which combined with (17) completes the proof. □

In the following proposition, we compute the mean value of the integral over the residual busy period, given the workload at time 0 equals x.

Proposition 1. *The mean area until the end of the transient busy period, is*

$$\mathbb{E}J(x) = \mathbb{E}\left[\int_0^{\tau(x)} \big(x + B(t) - ct\big)\,\mathrm{d}t\right] = \frac{x^2}{2c} + \frac{x}{2c^2}.$$

Proof. Due to the fact that

$$\mathrm{Ai}(u) = \frac{1}{2\sqrt{\pi}u^{1/4}} \exp\left(-\frac{2}{3}u^{3/2}\right)\left(1 - \frac{5}{48}u^{-3/2} + \mathrm{o}\big(u^{-3/2}\big)\right) \tag{19}$$

as $u \to \infty$, combined with (18), we have that

$$\begin{aligned}\mathbb{E}\big[-\gamma J(x)\big] &= \exp(cx)\left(\frac{1}{1 + (2x/c^2)\gamma}\right)^{1/4} \exp\left[\frac{c^3}{3\gamma}\left(1 - \left(1 + \frac{2\gamma x}{c^2}\right)^{3/2}\right)\right]\big(1 + \mathrm{o}(\gamma)\big) \\ &= 1 - \left(\frac{x^2}{2c} + \frac{x}{2c^2}\right)\gamma + \mathrm{o}(\gamma)\end{aligned}$$

as $\gamma \to 0$. This completes the proof. □

Combining Proposition 1 with (17) (and using the dominated convergence theorem) immediately leads to the following corollary.

Corollary 1.

$$\mathbb{E}\left[\int_0^{\tau_0} Q(t)\,\mathrm{d}t\right] = \frac{1}{2c^3}.$$

We note that, applying more precise expansions in (19), one can get the analogue of Proposition 1 for higher moments of $J(x)$, and (by applying (17)) also formulas for corresponding moments of $\int_0^{\tau_0} Q(t)\,\mathrm{d}t$. These computations are tedious (although standard), and are therefore left out.

7. Discussion and outlook

In this paper, we analyzed the probability that the area swept under the Brownian storage graph between 0 and $T(u)$ exceeds u. We did so for various types of interval lengths $T(u)$, leading to asymptotic results for three timescales ($u \to \infty$). A topic for future research could be to consider a wider class of inputs $\{X(t)\colon t \in \mathbb{R}\}$, for instance, Gaussian processes or Lévy processes. In the former case, there is the major complication that $Q(0)$ is *not* independent of $\{X(t)\colon t > 0\}$, which is a property that we repeatedly used in this paper. In the latter case, we have to make sure that all steps in which we use specific properties of Brownian motion, carry over to the more general

Lévy case. We do anticipate, though, that in case the Lévy-input is light-tailed the asymptotics are in the qualitative sense very similar to those related to the Brownian case (i.e., the same three regimes apply). Another related problem concerns the derivation of a central limit theorem for

$$\frac{1}{\sqrt{T}}\left(\int_0^T Q(t)\,\mathrm{d}t - qT\right),$$

with q the mean stationary workload.

Appendix

In this appendix, we prove that

$$\mathscr{S} = \left\{ f \in \Omega\colon\ q[f](0) = a,\ \int_0^T q[f](s)\,\mathrm{d}s \geq M \right\}$$

is a closed set in the space Ω. To this end, let $f_n \in \mathscr{S}$ be a sequence of functions such that $\|f_n - f\|_\Omega \to 0$, as $n \to \infty$ for some function $f \in \Omega$. We prove our claim by showing that $f \in \mathscr{S}$.

First, we show that for the limiting path f it holds that

$$q[f](0) = -\inf_{s\leq 0}\bigl(f(s) - cs\bigr) = a. \tag{20}$$

First, observe that $g(s) - cs \to \infty$ as $s \to -\infty$, as an immediate consequence of the fact that $|g(s) - cs|/(1 + |s|) \to c$ for all $g \in \Omega$. Consequently, for any such g there is a point s in which g takes its minimum in $[-\infty, 0]$.

Let s_0 be such that $\inf_{s\leq 0}(f(s) - cs) = f(s_0) - cs_0$. Then

$$-a \leq \lim_{n\to\infty} f_n(s_0) - cs_0 = f(s_0) - cs_0.$$

On the other hand, let $\{s_n\}$ be the sequence of points such that $\inf_{s\leq 0}(f_n(s) - cs) = f(s_n) - cs_n$. Observe that $\{s_n\}$ is bounded. If not, then, for each k and $\varepsilon > 0$, we would have

$$\bigl\| f_k(s) - f(s) \bigr\|_\Omega \geq \sup_{s\in\{s_n\}} \frac{|f_k(s) - f(s)|}{1 + |s|} = \sup_{s\in\{s_n\}} \frac{|-a - f(s) - cs|}{1 + |s|} \geq c - \varepsilon.$$

Conclude that there exists an $M > 0$ such that $|s_n| < M$. For n large enough

$$\bigl| f(s_n) - cs_n - \bigl(f_n(s_n) - cs_n\bigr)\bigr| = \bigl| f_n(s_n) - f(s_n)\bigr| \leq \bigl(1 + |s_n|\bigr)\varepsilon \leq (1 + M)\varepsilon,$$

which implies

$$f(s_0) - cs_0 \leq f(s_n) - cs_n \leq f_n(s_n) - cs_n + (1 + M)\varepsilon = -a + (1 + M)\varepsilon.$$

To complete the proof of (20), it is enough to let $\varepsilon \downarrow 0$.

Now we prove that

$$\int_0^T q[f](s)\,\mathrm{d}s \geq M.$$

Observe that

$$\int_0^T \big|q[f_n](s) - q[f](s)\big|\,\mathrm{d}s \leq I_1 + I_2,$$

where

$$I_1 := \int_0^T \big|f_n(s) - f(s)\big|\,\mathrm{d}s, \qquad I_2 := \int_0^T \Big|\inf_{r\leq s}\big(f(r) - cr\big) - \inf_{v\leq s}\big(f_n(v) - cv\big)\Big|\,\mathrm{d}s.$$

Let us examine I_1 first. Due to the fact that $\lim_{n\to\infty}\|f_n - f\|_\Omega = 0$, we have for n large enough that

$$\varepsilon \geq \sup_{s\leq T}\frac{|f_n(s) - f(s)|}{1+|s|} \geq \sup_{s\in[0,T]}\frac{|f_n(s) - f(s)|}{1+s} \geq \sup_{s\in[0,T]}\frac{|f_n(s) - f(s)|}{1+T}. \tag{21}$$

This implies

$$\int_0^T \big|f_n(s) - f(s)\big|\,\mathrm{d}s < T(1+T)\varepsilon.$$

Now consider I_2. Let s_0 be the minimizer in $\inf_{r\in[0,s]}(f(r) - cr)$ and s_n the minimizer in $\inf_{r\in[0,s]}(f_n(r_n) - cr_n)$. Then (21) implies that for n large enough

$$f_n(s_n) - cs_n - \big(f(s_0) - cs_0\big) \leq f_n(s_0) - cs_0 - \big(f(s_0) - cs_0\big) \leq (1+T)\varepsilon.$$

On the other hand

$$f(s_0) - cs_0 - \big(f_n(s_n) - cs_n\big) \leq f(s_n) - cs_n - \big(f_n(s_n) - cs_n\big) \leq (1+T)\varepsilon.$$

It follows that $I_2 \leq T(1+T)\varepsilon$. Now it is enough to let $\varepsilon \downarrow 0$; realizing that for each n we have $\int_0^T q[f_n](s)\,\mathrm{d}s \geq M$, the proof is completed.

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