The quench action approach to out-of-equilibrium quantum integrable models

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Quantum quenches and the quench action approach

We start this chapter with a brief, technical introduction to global quantum quenches and the generalized Gibbs ensemble. In particular, we review the derivation of the generalized thermodynamic Bethe equation of the GGE after a quench to the Lieb-Liniger model. Then, in Section 3.3, we give a detailed derivation of the quench action approach and a careful definition of the conditions on observables and overlaps in order for the method to work. Using knowledge of the thermodynamic Bethe Ansatz, we obtain a saddle-point equation for the representative state and the postquench time evolution as a sum over excitations. The explicit analysis is for the Lieb-Liniger model, but it is easily generalizable to any Bethe Ansatz solvable quantum system. In Section 3.3.2 certain claims of the quench action approach are verified in an explicit first application. A field quench in the transverse-field Ising chain, which is a noninteracting theory, is studied. By comparison to known results it is shown that there exists a Hamiltonian eigenstate that correctly reproduces the time evolution and equilibrium value of physical observables (typically multipoint correlators).

The literature for Sections 3.1 and 3.2 can be found in the text. The results presented in Section 3.3 are based on Refs [53, 124], with minor extensions by the author of this thesis.

3.1 Quantum quenches

As said in Chapter 1, for a global quantum quench [25, 26] a system is prepared in an initial state $|\psi_0\rangle$, which is usually the ground state of some Hamiltonian $\hat{H}'$. At time $t = 0$ a parameter of the theory is instanteously changed, such that the unitary time evolution of the state after the quench is $|\psi(t)\rangle = \exp(-i\hat{H}t)|\psi_0\rangle$. The system is no longer in an eigenstate and observables will evolve in time. In general, the initial state is a superposition of many eigenstates of $\hat{H}$ and the number of states in this superposition grows exponentially with system size.

Now suppose the postquench Hamiltonian is quantum integrable and the spectrum is given in terms of Bethe states $|\boldsymbol{\lambda}\rangle$. At time t after the quench, the state of the system can then be expanded in the basis of Bethe states,

$$|\psi(t)\rangle = \sum_{\boldsymbol{\lambda}} e^{-i\omega_{\boldsymbol{\lambda}} t} \langle \boldsymbol{\lambda}|\psi_0\rangle |\boldsymbol{\lambda}\rangle \ , \tag{3.1}$$

where the sum runs over all Bethe states in the Hilbert space. For Lieb-Liniger the Hilbert space is infinite and one needs a truncation to make sense of this sum. For the spin chain, the size of the Hilbert space is 2^N. The postquench time-dependent expectation value of a generic operator $\hat{\mathcal{O}}$ is exactly given by the double sum

$$\langle\psi(t)|\,\hat{\mathcal{O}}\,|\psi(t)\rangle = \sum_{\boldsymbol{\lambda},\boldsymbol{\lambda}'} e^{-S^*_{\boldsymbol{\lambda}}-S_{\boldsymbol{\lambda}'}} e^{i(\omega_{\boldsymbol{\lambda}}-\omega_{\boldsymbol{\lambda}'})t} \langle\boldsymbol{\lambda}|\hat{\mathcal{O}}|\boldsymbol{\lambda}'\rangle \ , \tag{3.2}$$

where the quantities $S_{\boldsymbol{\lambda}} = -\log\langle\boldsymbol{\lambda}|\psi_0\rangle$ are called overlap coefficients. For both models, when system size increases, one needs to take into account an exponentially growing number of states to obtain an accurate estimation of the expectation value. This makes the double sum over the full Hilbert space highly problematic, as the number of its terms grows exponentially with system size.

Note that the initial Hamiltonian $\hat{H}'$ is not necessarily integrable for the above analysis to be applicable. One could imagine turning off an integrability-breaking term. This is for example done in Refs [46, 35]. Also, the initial state does not have to be an eigenstate of some Hamiltonian. As long as the initial state is a 'thermodynamically large' sum of eigenstates of the postquench Hamiltonian, one can consider it as a quantum quench. This is the case for the Bragg pulse, see Chapter 6.

3.1.1 Relaxation

In the thermodynamic limit a generic initial state is an infinite superposition of energy eigenstates. Due to dephasing in Eq. (3.2), observables of such a closed, out-of-equilibrium, many-body quantum system are expected to relax to an equilibrium value. Strictly speaking this only happens for systems of infinite size. The quantum version of the Poincaré recurrence theorem [125] states that for a quantum mechanical system with discrete energy levels one can always find a time at which a time-evolved state vector becomes arbitrarily close to the initial state vector (in terms of Hilbert space norm of the difference between two vectors). For the integrable models we will study the spectrum is discrete at finite size, meaning that at finite size there is no infinite-time equilibrium value for observables in the model. Instead, expectation values will approximately keep oscillating. In practice, this is often overcome by considering the infinite-time time average of observables, defined as

$$\overline{\hat{\mathcal{O}}} = \lim_{T\to\infty} \frac{1}{T} \int_0^T dt\, \hat{\mathcal{O}}(t) \ . \tag{3.3}$$

As will be shown in the next chapter, the quench action method is applicable only in the strict thermodynamic limit. For the quenches we study and for infinite system size, the spectrum of the models is continuous and expectation values of local observables evolve towards a constant equilibrium value (the time average in Eq. (3.3) converges to this equilibrium value in the thermodynamic limit). Henceforth, we assume this is the case and forget about time averaging.

It is well known that this postquench equilibrium is, in principle, given by the diagonal ensemble. For large time all nondiagonal modes in Eq. (3.2) have dephased and one is left with

$$\lim_{t\to\infty} \lim_{\text{th}} \langle\psi(t)|\, \hat{\mathcal{O}}\, |\psi(t)\rangle = \lim_{\text{th}} \sum_{\boldsymbol{\lambda}} e^{-S^*_{\boldsymbol{\lambda}} - S_{\boldsymbol{\lambda}}} \langle \boldsymbol{\lambda}|\hat{\mathcal{O}}|\boldsymbol{\lambda}\rangle \,. \tag{3.4}$$

As said, the number of terms in this sum grows exponentially with system size. Using the diagonal ensemble is therefore not very practical as a way to obtain equilibrium predictions for general quench problems.

To address this problem differently, an important question then is whether and how an infinite-size Bethe Ansatz-solvable quantum system relaxes to a steady state. I.e., whether and how equilibrium expectation values of typical physical operators can effectively be computed on a specific thermodynamic Bethe state, called the steady state and denoted by $|\boldsymbol{\rho}^{\psi_0}\rangle$:

$$\lim_{t\to\infty} \lim_{\text{th}} \langle\psi(t)|\, \hat{\mathcal{O}}\, |\psi(t)\rangle = \left\langle \boldsymbol{\rho}^{\psi_0} \right|\, \hat{\mathcal{O}}\, \left|\boldsymbol{\rho}^{\psi_0}\right\rangle \,. \tag{3.5}$$

Both the generalized Gibbs ensemble and the quench action approach answer this question by the affirmative, at least for a large class of quenches and typical physical observables. We will now discuss both methods in detail, and in particular what their prediction for the postquench steady state is.

3.2 The generalized Gibbs ensemble

The generalized Gibbs ensemble [39, 27, 28, 37] maximizes the entropy under the constraint that all relevant conserved charges are fixed to their initial values. A brief overview of important literature on this subject was given in Chapter 1. Here we will present the technical form of the GGE conjecture and its consequences for a Bethe Ansatz solvable model.

3.2.1 Conjecture of the generalized Gibbs ensemble

An obvious question is what are the 'relevant' conserved charges. Every quantum model has a Hilbert-space size number of conserved charges, but apart from particle number, momentum and energy they are usually very nonlocal. An essential property of quantum integrable models is the existence of conserved charges that are sparse on the basis of the basic degrees of freedom [18]. The number of these charges is unbounded when system size goes to infinity. The local charges that were defined in Section 2.3 for Lieb-Liniger and the XXZ chain are part of this set, but it is not clear whether or not they are exhaustive. Surely, the

presence of local conserved charges heavily constrains the time evolution of local observables after a quench. A conventional formulation of the GGE conjecture is therefore as follows: the postquench equilibrium expectation values of local observables is determined by a GGE based on all local conserved charges. We will use this formulation of the GGE conjecture throughout this thesis.

Let us make the conjecture more concrete. For the Lieb-Liniger model and the spin-1/2 XXZ model, infinite sets of local charges $\{\hat{Q}_m\}_{m=1}^{\infty}$ were defined in Section 2.3. Given a local observable $\hat{\mathcal{O}}$, the GGE conjecture predicts

$$\lim_{t\to\infty} \lim_{\text{th}} \langle\psi(t)|\,\hat{\mathcal{O}}\,|\psi(t)\rangle = \lim_{a\to\infty} \lim_{\text{th}} \frac{\operatorname{Tr}\left(\hat{\mathcal{O}} e^{-\sum_{m=1}^{a}\beta_m \hat{Q}_m}\right)}{\operatorname{Tr}\left(e^{-\sum_{m=1}^{a}\beta_m \hat{Q}_m}\right)}\,, \tag{3.6a}$$

where the trace is over the full Hilbert space and the generalized chemical potentials $\{\beta_m\}_{m=1}^{\infty}$ associated with the charges are determined by the expectation values of the conserved charges on the initial state,

$$\lim_{\text{th}} \frac{1}{N}\langle\psi_0|\hat{Q}_n|\psi_0\rangle = \lim_{a\to\infty} \lim_{\text{th}} \frac{1}{N} \frac{\operatorname{Tr}\left(\hat{Q}_n e^{-\sum_{m=1}^{a}\beta_m \hat{Q}_m}\right)}{\operatorname{Tr}\left(e^{-\sum_{m=1}^{a}\beta_m \hat{Q}_m}\right)} \tag{3.6b}$$

for $n \geq 1$. The limit $a \to \infty$ after taking the thermodynamic limit indicates that we take infinitely many local conservation laws into account.

3.2.2 The GGE for the Lieb-Liniger model

So far, the traces in Eqs (3.6) have the same computational complexity as the sum over the Hilbert space in the diagonal ensemble. However, in the context of the thermodynamic Bethe Ansatz the GGE simplifies enormously in the limit of infinite system size. At the level of the root density ρ in the Lieb-Liniger model, the GGE translates into a saddle-point equation whose solution maximizes the Yang-Yang entropy in Eq. (2.43), as was shown in Refs [46, 47]. This follows from the GGE conjecture (3.6a) in the thermodynamic limit,

$$\langle\rho^{\psi_0}|\hat{\mathcal{O}}|\rho^{\psi_0}\rangle = \frac{1}{\mathcal{Z}_{\text{GGE}}}\int \mathcal{D}\rho\, \mathcal{O}[\rho]\, e^{-G[\rho]}\,, \tag{3.7}$$

where the trace over the Hilbert space was replaced by a functional integral over the root density, $\mathcal{Z}_{\text{GGE}} = \int \mathcal{D}\rho\, e^{-G[\rho]}$ and G is the generalized Gibbs free energy defined as $G[\rho] = \sum_{m=1}^{\infty}\beta_m Q_m[\rho] - S_{\text{YY}}[\rho]$. The quantity $\mathcal{O}[\rho]$ is a (complex) number and can be defined as $\mathcal{O}[\rho] = \lim_{\text{th}}\langle\boldsymbol{\lambda}|\hat{\mathcal{O}}|\boldsymbol{\lambda}\rangle$, where $|\boldsymbol{\lambda}\rangle$ is a representative state that scales to ρ in the thermodynamic limit. In Section 3.3 we will discuss the transition to a functional integral in technical detail in the context of the quench action approach. As was proven in Ref. [47], the functional integral has a unique saddle point specified by a generalized thermodynamic Bethe Ansatz

(GTBA) equation,

$$\log[\eta(\lambda)] = \sum_{m=1}^{\infty} \beta_m \lambda^m - \int_{-\infty}^{\infty} \frac{d\mu}{2\pi} K(\lambda-\mu) \log\left[1+\eta^{-1}(\mu)\right], \tag{3.8}$$

where $K(\lambda) = 2c/(\lambda^2+c^2)$ and $\eta(\lambda) = \rho_h(\lambda)/\rho(\lambda)$.

Combined with the thermodynamic Bethe Eq. (2.40), the solution to this GTBA equation is a root density ρ^{GGE}. The claim of the GGE conjecture is that for any local operator $\hat{\mathcal{O}}$ this density of rapidities reproduces the postquench equilibrium expectation value,

$$\langle \rho^{\psi_0} | \hat{\mathcal{O}} | \rho^{\psi_0} \rangle = \langle \rho^{\mathrm{GGE}} | \hat{\mathcal{O}} | \rho^{\mathrm{GGE}} \rangle . \tag{3.9}$$

This formalism was applied to a chain of harmonic oscillators, which is equivalent to free bosons, in Ref. [36]. Using the momentum occupation numbers as conserved charges, it was shown that the GTBA equation produces the same GGE prediction as the ordinary traces in Eqs (3.6). For quenches to truly interacting systems, the practical difficulty with this formalism is obtaining the values of the chemical potentials. Inverting Eqs (3.6b) is a very hard problem, even if one truncates the GGE to a few charges (see e.g. Refs [126, 127]).

A similar computation can be performed for the spin-1/2 XXZ chain. This case will be discussed extensively in Chapter 5.

3.3 The quench action approach

In this section we introduce the quench action approach, whose logic predicts the postquench expectation values of typical physical operators at any time after a quantum quench to a Bethe Ansatz solvable model. It is a variational method that is exact in the thermodynamic limit, as it overcomes the problem of the exponentially large sum in Eq. (3.2) through a saddle-point approximation. The quench action logic is based on first principles and does not rely on the GGE assumption.

The basic input of the method are the overlaps between the initial state and the eigenstates of the postquench Hamiltonian. We will argue that for large system size, due to an interplay between overlaps and entropy, a single Hamiltonian eigenstate dominates the time evolution of typical physical operators while all other states are suppressed exponentially in system size. This representative state is determined via a saddle-point approximation, based on a generalized Thermodynamic Bethe Ansatz [46, 47]. In the thermodynamic limit, postquench equilibrium expectation values are reproduced by expectation values on this saddle-point state, while the full time evolution is recovered via summation only over excitations in the vicinity of the saddle point. In principle, this solves the problem of the exponentially diverging sum.

3.3.1 Derivation of the saddle-point formulas

Consider a generic quantum quench in the Lieb-Liniger model with initial state $|\psi_0\rangle$ and assume the exact finite-size overlaps $\langle \boldsymbol{\lambda}|\,\psi_0\rangle$ are known.

Weak, smooth operators

A first step is to restrict our analysis to typical physical operators, i.e, operators that can be related to (potentially) measurable quantities in actual experiments. Examples are products of spin operators located on a finite interval of a spin chain, or density and one-particle operators in the Lieb-Liniger model. It is known [128, 129, 89, 130] that the norm of matrix elements of a typical physical operator $\langle \boldsymbol{\lambda}|\hat{\mathcal{O}}|\boldsymbol{\lambda}'\rangle$ decreases rapidly when the Bethe states $|\boldsymbol{\lambda}\rangle$ and $|\boldsymbol{\lambda}'\rangle$ become very different, meaning that the sets of corresponding quantum numbers have many elements not in common. It is hard to give a generic quantitative description of this decrease at finite size, as it depends on state and operator. However, in the thermodynamic limit we can make this statement precise by defining a 'weak operator'. We define an operator $\hat{\mathcal{O}}$ to be weak if, for any Bethe state $|\boldsymbol{\lambda}\rangle$,

$$\lim_{\text{th}} \hat{\mathcal{O}}|\boldsymbol{\lambda}\rangle = \lim_{\text{th}} \sum_{\mathbf{e}} c_{\hat{\mathcal{O}}}(\boldsymbol{\lambda},\mathbf{e})|\boldsymbol{\lambda},\mathbf{e}\rangle\,, \tag{3.10}$$

where the coefficients $c_{\hat{\mathcal{O}}}(\boldsymbol{\lambda},\mathbf{e})$ are complex numbers, the Bethe state $|\boldsymbol{\lambda},\mathbf{e}\rangle$ is obtained from $|\boldsymbol{\lambda}\rangle$ by m particle-hole replacements $\mathbf{e} = \{\lambda_j^h \to \lambda_j^p\}_{j=1}^m$ and the number of replacements is subleading in system size, $\lim_{\text{th}} \frac{m}{N} = 0$. The thermodynamic limit in Eq. (3.10) should always be understood as in a sum over the full Hilbert space, as we will shortly see. Keep in mind that particle-hole replacements are to be understood in terms of replacing occupied quantum numbers, as all rapidities will shift. Finite products of weak operators are also weak, so multiple-point or time-split operators constructed from weak operators share the weakness property. An example of this is the density-density correlator in the Lieb-Liniger model, which will be studied in the next chapter.

Using this weakness property of typical physical observables, we can restrict the second exponentially large sum in Eq. (3.2) to a much smaller sum over particle-hole excitations:

$$\begin{aligned}
&\lim_{\text{th}}\langle\psi_0(t)|\hat{\mathcal{O}}|\psi_0(t)\rangle \\
&= \lim_{\text{th}} \frac{1}{2}\sum_{\boldsymbol{\lambda}}\sum_{\mathbf{e}} \Big(e^{-S^*_{\boldsymbol{\lambda}}-S_{(\boldsymbol{\lambda},\mathbf{e})}+i[\omega_{\boldsymbol{\lambda}}-\omega_{(\boldsymbol{\lambda},\mathbf{e})}]t}\langle\boldsymbol{\lambda}|\hat{\mathcal{O}}|\boldsymbol{\lambda},\mathbf{e}\rangle \\
&\qquad\qquad + e^{-S^*_{(\boldsymbol{\lambda},\mathbf{e})}-S_{\boldsymbol{\lambda}}+i[\omega_{(\boldsymbol{\lambda},\mathbf{e})}-\omega_{\boldsymbol{\lambda}}]t}\langle\boldsymbol{\lambda},\mathbf{e}|\hat{\mathcal{O}}|\boldsymbol{\lambda}\rangle\Big)\,,
\end{aligned} \tag{3.11}$$

where the overlap coefficients were defined as $S_{\boldsymbol{\lambda}} = -\log\langle\boldsymbol{\lambda}|\,\psi_0\rangle$ and for convenience we symmetrized the excitations over bra and ket.

For the next step a representation of Bethe states in terms of out-box densities and in-box distributions, as outlined in Section 2.1.6, proves useful. We split the

sum over the full Hilbert space in a sum over out-box densities and a sum over in-box distributions,

$$\begin{aligned}&\lim_{\rm th}\langle\psi_0(t)|\hat{\mathcal{O}}|\psi_0(t)\rangle\\ &=\lim_{\rm th}\frac{1}{2}\sum_{\mathbf{n}}\sum_{\mathbf{c}}\sum_{\mathbf{e}}\Big(e^{-S^*_{(\mathbf{n},\mathbf{c})}-S_{(\mathbf{n},\mathbf{c},\mathbf{e})}+i[\omega_{(\mathbf{n},\mathbf{c})}-\omega_{(\mathbf{n},\mathbf{c},\mathbf{e})}]t}\langle\mathbf{n},\mathbf{c}|\hat{\mathcal{O}}|\mathbf{n},\mathbf{c},\mathbf{e}\rangle\\ &\qquad+e^{-S^*_{(\mathbf{n},\mathbf{c},\mathbf{e})}-S_{(\mathbf{n},\mathbf{c})}+i[\omega_{(\mathbf{n},\mathbf{c},\mathbf{e})}-\omega_{(\mathbf{n},\mathbf{c})}]t}\langle\mathbf{n},\mathbf{c},\mathbf{e}|\hat{\mathcal{O}}|\mathbf{n},\mathbf{c}\rangle\Big)\,.\end{aligned}\tag{3.12}$$

Remember that the particle-hole excitations $\mathbf{e}$ can be both out-box and in-box replacements. Let us interchange the sum over $\mathbf{c}$ and $\mathbf{e}$ and focus on the sum over in-box distributions in the thermodynamic limit. In Section 2.1.7 it was shown that the nonextensive energy difference is independent of the in-box configuration, $\lim_{\rm th}[\omega_{(\mathbf{n},\mathbf{c})}-\omega_{(\mathbf{n},\mathbf{c},\mathbf{e})}]=-\delta\omega_{\mathbf{e}}[\rho]$, where $\delta\omega_{\mathbf{e}}[\rho]$ is defined in Eq. (2.51).

We call an operator smooth if up to terms that vanish in the thermodynamic limit the matrix elements are independent of in-box configurations, i.e.,

$$\langle\mathbf{n},\mathbf{c}|\hat{\mathcal{O}}|\mathbf{n},\mathbf{c},\mathbf{e}\rangle=\langle\mathbf{n},\mathbf{c}'|\hat{\mathcal{O}}|\mathbf{n},\mathbf{c}',\mathbf{e}\rangle[1+O(N^{-1})]\,,\tag{3.13}$$

for any $\mathbf{n}$ and $\mathbf{e}$. Most physical operators are smooth, but in Appendix E we will encounter a nonsmooth operator. For the moment, let us assume that in addition to being weak the considered physical operator also obeys the smoothness property. The matrix elements can then be taken out of the sum over $\mathbf{c}$.

Smooth overlaps

It is well known that overlaps between the initial state and a Bethe state, which contain all the information about the initial state before the quench, are wildly fluctuating. Replacing a single quantum number can mean an order-one change in the relative value of the overlap. However, for a subclass of initial states the part of the overlaps that is exponential in system size becomes universal in the thermodynamic limit. We call such overlaps smooth, since at leading order in system size the corresponding extensive overlap coefficients $S_{\mathbf{n},\mathbf{c}}=-\log\langle\mathbf{n},\mathbf{c}|\,\psi_0\rangle$ are independent of in-box configurations, i.e.,

$$S_{(\mathbf{n},\mathbf{c})}=S_{(\mathbf{n},\mathbf{c}')}[1+O(N^{-1})]\,,\tag{3.14}$$

for any distribution of out-box densities $\mathbf{n}$. It should be noted that the smoothness property for overlaps is nontrivial. In Section 6 we will see an example of an initial state whose overlaps with Bethe states in the Tonks-Girardeau regime are nonsmooth.

The quench action approach crucially depends on the smoothness of the overlaps. Let us assume for the moment this property is obeyed. The overlap coefficients can then be decomposed as

$$S_{\mathbf{n},\mathbf{c}}=S_{\mathbf{n}}+s_{(\mathbf{n},\mathbf{c})}+O(N^{-1})\,,\tag{3.15}$$

where $S_{\mathbf{n}}$ is the extensive part of the overlap coefficients and $s_{(\mathbf{n},\mathbf{c})}$ is finite in the thermodynamic limit. The sum over in-box configurations in Eq. (3.12) can now be performed,

$$\begin{aligned}&\lim_{\mathrm{th}} \sum_{\mathbf{n}} \sum_{\mathbf{c}} \sum_{\mathbf{e}} e^{-S^*_{(\mathbf{n},\mathbf{c})} - S_{(\mathbf{n},\mathbf{c},\mathbf{e})} + i[\omega_{(\mathbf{n},\mathbf{c})} - \omega_{(\mathbf{n},\mathbf{c},\mathbf{e})}]t} \langle \mathbf{n},\mathbf{c}|\hat{\mathcal{O}}|\mathbf{n},\mathbf{c},\mathbf{e}\rangle \qquad (3.16)\\ &= \lim_{\mathrm{th}} \sum_{\mathbf{n}} \sum_{\mathbf{e}} e^{S_{\mathrm{YY},(\mathbf{n},\mathbf{e})} - 2\mathrm{Re}\, S_{\mathbf{n}} - 2\mathrm{Re}\, \bar{s}_{\mathbf{n}} - \delta s_{\mathbf{n},\mathbf{e}} - i\delta\omega_{(\mathbf{n},\mathbf{e})} t} \langle \mathbf{n},\mathbf{0}|\hat{\mathcal{O}}|\mathbf{n},\mathbf{0},\mathbf{e}\rangle\end{aligned}$$

The quantity $\exp\left(S_{\mathrm{YY},(\mathbf{n},\mathbf{e})}\right)$ is the finite-size Yang-Yang entropy defined in Eq. (2.41) and corresponds to the number of terms in the sum over $\mathbf{c}$. Furthermore, $\bar{s}_{\mathbf{n}}$ is the averaged finite part of the overlap coefficients for Bethe states with out-box densities $\mathbf{n}$,

$$\sum_{\mathbf{c}} e^{-s_{\mathbf{n},\mathbf{c}}} = e^{S_{\mathrm{YY},(\mathbf{n},\mathbf{e})} - 2\bar{s}_{\mathbf{n}}}\,, \qquad (3.17)$$

and $\delta s_{\mathbf{n},\mathbf{e}} = -\log\left(\langle \mathbf{n},\mathbf{c},\mathbf{e}|\,\psi_0\rangle \,/\, \langle \mathbf{n},\mathbf{c}|\,\psi_0\rangle\right)$, where terms vanishing in the thermodynamic limit are neglected. Similarly, $\delta\omega_{(\mathbf{n},\mathbf{e})} = \omega_{(\mathbf{n},\mathbf{c},\mathbf{e})} - \omega_{(\mathbf{n},\mathbf{c})}$.

The functional integral

Finally, taking the true thermodynamic limit replaces the sum over out-box densities $\mathbf{n}$ by a functional integral over smooth density functions ρ. One uses the concept of a representative state, as defined at the end of Section 2.1.6. The time-evolved postquench expectation value for infinite system size then becomes

$$\begin{aligned}\lim_{\mathrm{th}} \langle \psi_0(t)|\hat{\mathcal{O}}|\psi_0(t)\rangle = \frac{1}{\mathcal{Z}_{\mathrm{QA}}} \int \mathcal{D}\rho\, e^{-S_{\mathrm{QA}}[\rho]} \qquad (3.18a)\\ \times \frac{1}{2} \sum_{\mathbf{e}} \Big(e^{\delta S_{\mathrm{YY},\mathbf{e}}[\rho] - \delta s_{\mathbf{e}}[\rho] - i\delta\omega_{\mathbf{e}}[\rho]t} \langle \rho|\hat{\mathcal{O}}|\rho,\mathbf{e}\rangle \\ + e^{\delta S_{\mathrm{YY},\mathbf{e}}[\rho] - \delta s^*_{\mathbf{e}}[\rho] + i\delta\omega_{\mathbf{e}}[\rho]t} \langle \rho,\mathbf{e}|\hat{\mathcal{O}}|\rho\rangle \Big)\,.\end{aligned}$$

Defining $S[\rho] = \lim_{\mathrm{th}} \mathrm{Re}\, S_{\boldsymbol{\lambda}}$ as the extensive real part of the overlap coefficients in the thermodynamic limit, the weight of the functional integral is given by the quench action $S_{\mathrm{QA}}[\rho] = 2S[\rho] - S_{\mathrm{YY}}[\rho]$. Furthermore, $\delta S_{\mathrm{YY},\mathbf{e}}[\rho]$ is defined in Eq. (2.52) and $\delta s_{\mathbf{e}}[\rho]$ is the nonextensive part of the overlap coefficients related to the excitations,

$$\delta s_{\mathbf{e}}[\rho] = -\lim_{\mathrm{th}} \log\left[\frac{\langle \mathbf{n},\mathbf{c},\mathbf{e}|\,\psi_0\rangle}{\langle \mathbf{n},\mathbf{c}|\,\psi_0\rangle}\right]\,, \qquad (3.18b)$$

while $\delta\omega_{\mathbf{e}}[\rho]$ is the energy of $|\rho,\mathbf{e}\rangle$ relative to $|\rho\rangle$ defined in Eq. (2.51). The quantity $\delta s_{\mathbf{e}}[\rho]$ is sometimes called the differential overlap. The excitations in $\sum_{\mathbf{e}}$ are obtained by displacing, creating, and annihilating a denumerable number

of strings of the representative state associated with $|\rho\rangle$. It is assumed that the end result is independent from the particular choice made for this representative state. Finally, the averaged nonextensive part of the overlap coefficients $\bar{s}_{\mathbf{n}}$ is absorbed into the functional integral. Since $|\psi_0(t)\rangle$ is normalized, one accounts for this by dividing by the quench action partition function $\mathcal{Z}_{\mathrm{QA}} = \int \mathcal{D}\rho\, e^{-S_{\mathrm{QA}}[\rho]}$.

The saddle-point approximation

The quench action is extensive, real, and in the quenches we will treat also bounded from below due to the concave property of the Yang-Yang entropy [12]. Therefore, convergence of the functional integral is ensured and in the thermodynamic limit a saddle-point approximation of the functional integral becomes exact. The saddle-point distribution ρ^{sp} is determined by the variational equation

$$0 = \left.\frac{\delta S_{\mathrm{QA}}\left[\rho\right]}{\delta\rho(\lambda)}\right|_{\rho=\rho^{\mathrm{sp}}}, \tag{3.19}$$

which can be considered as a generalized Thermodynamic Bethe Ansatz equation [46, 47].

Plugging the saddle point into Eq. (3.18a) leads to the main result of the quench action approach. For a quantum quench whose initial state $|\psi_0\rangle$ produces smooth overlaps with the basis of Bethe states, the expectation value of a weak, smooth operator $\hat{\mathcal{O}}$ in the thermodynamic limit at any time t after the quench is exactly given by

$$\begin{aligned}\lim_{\mathrm{th}}\langle\psi_0(t)|\hat{\mathcal{O}}|\psi_0(t)\rangle = \frac{1}{2}\sum_{\mathbf{e}}\Big(&e^{\delta S_{\mathrm{YY},\mathbf{e}}-\delta s_{\mathbf{e}}-i\delta\omega_{\mathbf{e}}t}\,\langle\rho^{\mathrm{sp}}|\,\hat{\mathcal{O}}\,|\rho^{\mathrm{sp}},\mathbf{e}\rangle\\ &+e^{\delta S_{\mathrm{YY},\mathbf{e}}-\delta s^{*}_{\mathbf{e}}+i\delta\omega_{\mathbf{e}}t}\,\langle\rho^{\mathrm{sp}},\mathbf{e}|\,\hat{\mathcal{O}}\,|\rho^{\mathrm{sp}}\rangle\Big),\end{aligned} \tag{3.20}$$

where the quantities $\delta S_{\mathrm{YY},\mathbf{e}}$, $\delta s_{\mathbf{e}}$ and $\delta\omega_{\mathbf{e}}$ are all evaluated at the saddle point ρ^{sp}. From this formula the power of the quench action approach is obvious. Instead of a double sum over the complete Hilbert space that grows exponentially with system size, the quench action method selects a saddle-point state such that the complete, exact postquench time evolution of an infinitely large system is recovered from a single sum over a denumerable set of excitations in the vicinity of the saddle point.

In particular, due to dephasing it predicts the equilibrium expectation value to which the operator $\hat{\mathcal{O}}$ will relax at long times after the quench,

$$\lim_{t\to\infty}\lim_{\mathrm{th}}\langle\psi_0(t)|\,\hat{\mathcal{O}}\,|\psi_0(t)\rangle = \langle\rho^{\mathrm{sp}}|\,\hat{\mathcal{O}}\,|\rho^{\mathrm{sp}}\rangle\ . \tag{3.21}$$

To summarize, the GTBA Eq. (3.19), whose driving terms are determined by the leading part of the overlap coefficients $S_{\boldsymbol{\lambda}} = -\log\langle\boldsymbol{\lambda}|\psi_0\rangle$ in the thermodynamic limit, gives the exact quench action prediction for the steady state after a quantum quench with initial state $|\psi_0\rangle$.

3.3.2 A first application

Here we review a first application of the quench action approach to field quenches in the transverse-field Ising chain (TFIC), as done in Ref. [53]. The TFIC Hamiltonian has a spectrum of free fermions, so there is no Bethe Ansatz involved. Furthermore, this quench has been studied and largely understood with more conventional methods including the GGE, using the free-fermion momentum occupations as conserved quantities [131, 132, 133, 134, 135, 136, 137, 138, 139, 140]. Here, the time evolution and equilibrium values of physical observables will be predicted via a single Hamiltonian eigenstate. A comparison with the existing results will verify a central claim of the quench action approach, namely the existence of a single Hamiltonian eigenstate that correctly reproduces all dynamics after the quench.

The transverse-field Ising chain

The Hamiltonian of the transverse-field Ising chain is

$$H_h = -J\sum_{j=1}^{L}[\sigma_j^x\sigma_{j+1}^x + h\sigma_j^z]\,, \tag{3.22}$$

which represents a one-dimensional ferromagnetic ($J > 0$) Ising model with transverse magnetic field (parametrized by $h > 0$). The Pauli matrices σ_j^α ($\alpha = x, y, z$) represent the spin-1/2 degrees of freedom at lattice sites $j = 1, 2, \ldots, L$.

It is well known [141, 142] that the TFIC can be diagonalized by a Jordan-Wigner transformation

$$a_{2\ell-1} = \left(\prod_{j=1}^{\ell-1}\sigma_j^z\right)\sigma_\ell^x\,, \qquad a_{2\ell} = \left(\prod_{j=1}^{\ell-1}\sigma_j^z\right)\sigma_\ell^y\,, \tag{3.23}$$

where the a_i are Majorana fermions ($a_i^\dagger = a_i$) and obey the anticommutation relations $\{a_i, a_j\} = 2\delta_{i,j}$, followed by a Bogoliubov transformation

$$a_{2\ell-1} = \frac{1}{\sqrt{L}}\sum_p e^{i\frac{\theta_p}{2}-ip\ell}\left(\alpha_p + \alpha_{-p}^\dagger\right) \tag{3.24a}$$

$$a_{2\ell} = -\frac{i}{\sqrt{L}}\sum_p e^{-i\frac{\theta_p}{2}-ip\ell}\left(\alpha_p - \alpha_{-p}^\dagger\right) \tag{3.24b}$$

with Bogoliubov angle θ_p given by

$$e^{i\theta_p} = \frac{h - e^{ip}}{\sqrt{1 + h^2 - 2h\cos(p)}}\,. \tag{3.24c}$$

The fermionic operators α_p, $\alpha_p^\dagger$ obey the canonical anticommutation relations. The diagonalized Hamiltonian is

$$H_h = \sum_p \varepsilon_h(p)\left[\alpha_p^\dagger\alpha_p - \frac{1}{2}\right]\,, \tag{3.25}$$

with single-particle energy $\varepsilon_h(p) = 2J\sqrt{1+h^2-2h\cos(p)}$. In the sums $\sum_p$ the momenta p are quantized, $p = \frac{2\pi}{L}(n+\frac{1}{2})$, where the integer n lies in the interval $-L/2 \leq n < L/2$.

In our choice of the quantization of momenta and the specific form of the Hamiltonian (3.25), we are restricting our analysis to the sector of the Hilbert space with an even number of fermionic excitations. This is the sector of anti-periodic boundary conditions and it is commonly known as the Neveu-Schwarz (NS) sector, as opposed to the Ramon (R) sector with periodic boundary conditions.

At finite size the ground state of the system is the NS sector Fock state vacuum, $|0,h\rangle_{\rm NS}$, which is defined such that $\alpha_p|0,h\rangle_{\rm NS} = 0$ for all p. The TFIC exhibits a quantum phase transition. In the paramagnetic phase $h > 1$ the ground state at infinite system size is still $|0,h\rangle_{\rm NS}$, whereas in the ferromagnetic phase $0 < h < 1$ the ground state is degenerate, $(|0,h\rangle_{\rm NS} \pm |0,h\rangle_{\rm R})/\sqrt{2}$, with $|0,h\rangle_{\rm R}$ the vacuum of the Ramond sector. The paramagnetic ground state is invariant under π rotations around the z axis. This $\mathbb{Z}_2$ symmetry is spontaneously broken in the ferromagnetic phase. The order parameter of this phase transition is $\sigma^x_{\rm tot}$.

Tranverse-field quenches and the saddle point

Let us now study the following quench protocol. The system is prepared in the paramagnetic ground state $|\psi_0\rangle = |0,h_0\rangle_{\rm NS}$ associated with an initial transverse field $h_0 > 0$. At time $t = 0$ the transverse field is suddenly altered to a new value h and the system is unitarily time evolved by the Hamiltonian H_h. This quench was extensively studied in Refs [133, 134, 135, 143]. The free-fermion modes that were defined in Eqs (3.24) are not the same before and after the quench, since the Bogoliubov angle θ_p depends on the value of the transverse field. Using the Bogoliubov transformations the initial state can be expressed in terms of postquench free-fermion modes leading to the following time-evolved state [134],

$$|\psi_0(t)\rangle = \frac{1}{\mathcal{N}} \exp\left(i \sum_p K(p)\, e^{-i2\varepsilon_h(p)} \alpha^\dagger_{-p}\alpha^\dagger_p \right) |0,h\rangle_{\rm NS}\,, \tag{3.26a}$$

where $\mathcal{N}$ is some irrelevant normalization constant and

$$K(p) = \tan\left(\frac{\Delta_p}{2}\right), \qquad \cos(\Delta_p) = \frac{4J^2\left[1+hh_0-(h+h_0)\cos(p)\right]}{\varepsilon_h(p)\varepsilon_{h_0}(p)}\,. \tag{3.26b}$$

The quantity $\Delta_p = \theta_p - \theta_p^{(0)}$ is the difference between Bogoliubov angles after and before the quench.

The reduced density matrix of a subsystem A is defined as $\rho_A(t) = {\rm Tr}_{\bar{A}}\rho(t)$, with density matrix $\rho(t) = |\psi_0(t)\rangle\langle\psi_0(t)|$, and is used to compute the dynamics of local correlators. Let subsystem A consist of the first ℓ lattice sites. Its reduced

density matrix is then expressible in terms of the Majorana fermions [143],

$$\rho_\ell(t) = \frac{1}{2^\ell} \sum_{\{\mu\}} \mathrm{Tr}\left[\rho(t) a_1^{\mu_1} \dots a_{2\ell}^{\mu_{2\ell}}\right] a_{2\ell}^{\mu_{2\ell}} \dots a_1^{\mu_1} , \tag{3.27}$$

with $\mu_j = 0, 1$. Since the initial state is invariant under the aforementioned $\mathbb{Z}_2$ symmetry $a_j \to -a_j$, we have $\mathrm{Tr}[\rho(t) a_j] = 0$ and with Wick's theorem for Jordan-Wigner fermions the reduced density matrix can be expressed in Gaussian form [144],

$$\rho_\ell(t) = \frac{1}{\mathcal{Z}} \exp\left(\frac{1}{4} \sum_{i,j=1}^{2\ell} a_i W_{ij} a_j\right) , \tag{3.28}$$

where W is an antisymmetric matrix obeying $\tanh(W/2) = \Gamma$. The correlation matrix Γ is defined as

$$\Gamma_{ij} = \mathrm{Tr}[\rho(t) a_j a_i] - \delta_{i,j} . \tag{3.29}$$

Note that the time dependence of the matrices W and Γ is left implicit. Combining Eqs (3.24) and (3.26a), it was calculated for the quench from the paramagnetic phase in Ref. [134]. The correlation matrix is of block-Toeplitz form as its 2×2-blocks depend only on the difference between row and column indices, $\Gamma_{2n-1,2j-1} = \Gamma_{2j,2n} = f_{j-n}$, $\Gamma_{2n-1,2j} = g_{n-j}$, where in the thermodynamic limit

$$f_j = \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{-ikj} \sin(\Delta_k) \sin(2\varepsilon_h(k)t) , \tag{3.30a}$$

$$g_j = -i \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{-ikj} \frac{(h - e^{ik})[\cos(\Delta_k) - i \sin(\Delta_k) \cos(2\varepsilon_h(k)t)]}{\sqrt{1 + h^2 - 2h\cos(k)}} . \tag{3.30b}$$

Wick's theorem ensures that any higher order correlation function is computable from the Gaussian reduced density matrix in Eq. (3.28). In particular, taking the limit $t \to \infty$ any equilibrium expectation value is determined.

In Ref. [53] a saddle-point state is derived which exactly reproduces the infinite-time limit of the correlation matrix in Eqs (3.29) and (3.30). The occupation numbers of free-fermion modes are conserved quantities and with Eq. (3.26a) the expectation values on the initial state are easily computed,

$$\tilde{\rho}(k) = \frac{\langle \psi_0 | \alpha_k^\dagger \alpha_k | \psi_0 \rangle}{2\pi} = \frac{1 - \cos(\Delta_k)}{4\pi} , \tag{3.31}$$

where the function $\tilde{\rho}$ is a mode density. For large system size, consider the Hamiltonian eigenstate

$$|\Phi_s\rangle = \prod_{j=1}^{N} \alpha_{\kappa_j}^\dagger \alpha_{-\kappa_j}^\dagger |0, h\rangle_{\mathrm{NS}} , \tag{3.32}$$

where the chosen momenta obey $\kappa_{j+1} - \kappa_j = [L\tilde{\rho}(\kappa_j)]^{-1} + \hat{\mathcal{O}}(L^{-2})$ and $2N/L$ is fixed. At leading order in system size this eigenstate reproduces the mode occupation numbers and differences disappear in the thermodynamic limit $N, L \to \infty$. The associated density matrix $\rho_s = |\Phi_s\rangle\langle\Phi_s|$ is Gaussian due to its product form and fully specified by its correlation matrix, which is defined as in Eq. (3.29). A simple computation with mode expansions teaches that the only nonzero components are

$$(\Gamma_s)_{2n-1,2n-2j} = -\frac{i}{L}\sum_p \frac{e^{-ipj}(h - e^{ip})}{\sqrt{1 + h^2 - 2h\cos(p)}}\left[1 - 2\sum_{i=1}^{N}\delta_{p,\kappa_i}\right] . \tag{3.33}$$

Remember that the information about the initial state, the value of h_0, is contained in the distribution of momenta $\{\kappa_j\}_{j=1}^N$. Via the Euler-Maclaurin sum formula, the sum turns into an integral in the thermodynamic limit. One immediately sees that Γ_s is equal to the late-time correlation matrix of the quantum quench, specified by Eqs (3.30). Accordingly, $\rho_{s,\ell} = \lim_{t\to\infty}\rho_\ell(t)$, where $\rho_{s,\ell}$ is the reduced density matrix of the saddle-point state. This construction proves the existence of a single representative state, the saddle-point state, that reproduces all local correlations in the postquench equilibrium.

It should be observed that the saddle-point state is completely fixed by the values of the conserved mode occupation numbers. This is generally not the case. Usually, the interplay between overlaps and entropy will lead to a true saddle-point equation, as described in the previous section. The importance of the above derivation lies in the existence of a single Hamiltonian eigenstate that reproduces all local correlations long after the quench. A first example of a true saddle-point equation coming from the overlaps will be the topic of the next chapter.

Time evolution

In the formula for the time evolution in the quench action approach, Eq. (3.20), as long as the considered observable is weak the sum over excitations can be replaced by the initial state. Defining $\rho_s(t) = |\Phi_s(t)\rangle\langle\Phi_s(t)|$, we find

$$\lim_{\text{th}}\langle\psi_0(t)|\hat{\mathcal{O}}|\psi_0(t)\rangle = \frac{1}{2}\lim_{\text{th}}\left[\frac{\langle\psi_0|\hat{\mathcal{O}}(t)|\Phi_s\rangle}{\langle\psi_0|\,\Phi_s\rangle} + \frac{\langle\Phi_s|\hat{\mathcal{O}}(t)|\psi_0\rangle}{\langle\Phi_s|\,\psi_0\rangle}\right] \tag{3.34a}$$

$$= \frac{1}{2}\lim_{\text{th}}\left[\frac{\mathrm{Tr}[\rho_s(t)\rho(t)\hat{\mathcal{O}}]}{\mathrm{Tr}[\rho_s(t)\rho(t)]} + \frac{\mathrm{Tr}[\rho(t)\rho_s(t)\hat{\mathcal{O}}]}{\mathrm{Tr}[\rho(t)\rho_s(t)]}\right] \tag{3.34b}$$

$$= \lim_{\text{th}}\mathrm{Tr}[\rho'(t)\hat{\mathcal{O}}] , \tag{3.34c}$$

where the last line serves as a definition of the density matrix ρ'. Since both ρ_s and ρ are Gaussian matrices, so is ρ'. Is Ref. [53] the correlation matrix of ρ' is computed and it is shown that $\lim_{\text{th}}\Gamma'(t) = \Gamma(t)$ for any time t after the quench. This verifies the central claim of the quench action approach: the full postquench

dynamics of local operators can be recovered from a single Hamiltonian eigenstate $|\Phi_s\rangle$ and a denumerable set of excitations around it.

For quenches from the ferromagnetic phase, $h_0 < 0$, the $\mathbb{Z}_2$ symmetry is spontaneously broken and the initial state is given by $(|0,h\rangle_{\mathrm{NS}} + |0,h\rangle_{\mathrm{R}})/\sqrt{2}$, as argued in Refs [134, 135]. As a consequence, the density matrix is not Gaussian. For late times however the reduced density matrix is still given by $\lim_{t\to\infty}\Gamma(t)$, with the correlation matrix as in Eqs (3.30). So also for quenches from the ferromagnetic regime the saddle-point state predicts to correct equilibrium expectation values. The validity of the formalism for the time evolution of local observables is much harder to verify. In Ref. [53] this was done for one specific observable, the order parameter σ^x_{tot}, and only for late times $Jt \gg 1$.