The quench action approach to out-of-equilibrium quantum integrable models
Wouters, B.M.

Citation for published version (APA):
Wouters, B. M. (2015). The quench action approach to out-of-equilibrium quantum integrable models

General rights
It is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), other than for strictly personal, individual use, unless the work is under an open content license (like Creative Commons).

Disclaimer/Complaints regulations
If you believe that digital publication of certain material infringes any of your rights or (privacy) interests, please let the Library know, stating your reasons. In case of a legitimate complaint, the Library will make the material inaccessible and/or remove it from the website. Please Ask the Library: http://uba.uva.nl/en/contact, or a letter to: Library of the University of Amsterdam, Secretariat, Singel 425, 1012 WP Amsterdam, The Netherlands. You will be contacted as soon as possible.
The Lieb-Liniger quench

In this chapter the first implementation of the quench action method for a quench to a truly interacting, Bethe Ansatz solvable quantum system is described. We study an interaction quench in the Lieb-Liniger model, in particular a quench from the ground state of free bosons \( (c = 0) \) to the regime of repulsive interactions, \( c > 0 \). As the initial state is a canonical Bose-Einstein condensate (BEC), we call this protocol the BEC-to-LL quench. This type of quench is of experimental interest. As explained in Section 1.2, the Lieb-Liniger model with tunable parameters can approximately be realized in optical lattices and magnetic microtraps. By for example modifying the transverse confinement or the scattering length, an interaction quench can be simulated. We believe that the results obtained here are applicable in both ring-like geometries [74] and box-like potentials [75].

Interaction quenches have been studied before in Refs [145, 146, 147, 148, 34, 149, 150, 151, 152]. In particular, in Ref. [34] the BEC quench to the fermionized Tonks-Girardeau gas \( (c = \infty) \) was studied numerically by means of the form-factor approach. The overlaps with the initial state were found, \( \langle \lambda | \psi_0 \rangle \sim \prod_{\lambda_j > 0} \lambda_j^{-1} \), and the time evolution of the density-density correlator was computed numerically. While finalizing our quench action implementation, the time evolution in the TG regime was computed analytically in Ref. [152] using brute-force methods.

The outline of this chapter is as follows. After introducing the quench protocol, we briefly review the inaptness of the standard GGE for this quench and a way to partially circumvent this problem by means of q-bosons. We implement the quench action approach: we find the exact overlaps, derive a GTBA equation and obtain an analytical expression of the saddle-point state. Then we analyze the physical properties of the saddle point and reproduce the time evolution of the density-density correlator of Ref. [152]. To the best of our knowledge this constitutes the first example of exact results in the thermodynamic limit for a quench to a truly interacting system. This chapter is based on results published in Ref. [1].
4. The Lieb-Liniger quench

4.1 The quench protocol

We investigate relaxation behavior in the Lieb-Liniger model by studying the quench from the free boson gas ($c = 0$) to the regime of finite repulsive interaction $c > 0$. The system will be prepared in the ground state of the free boson gas. As we have seen, for this $N$-particle initial state all rapidities are the same: $\lambda_j = 0, \forall j = 1, \ldots, N$. From Eq. (2.13) we saw that this noninteracting Bose-Einstein condensate $|\psi_0\rangle$ has a constant wave function in position space:

$$\langle x|\psi_0\rangle = \psi_0(x) = \frac{1}{L^{N/2}},$$

(4.1)

where the state is normalized to one. At time $t = 0$, we suddenly turn on interparticle interactions; the time evolution is thus from that moment onwards driven by the Lieb-Liniger Hamiltonian (1.2) in the repulsive regime: $c > 0$.

The problematic nature of the double sum (3.2) over the $N$-particle sector of the Hilbert space was illustrated in Ref. [34] by estimating the number of terms in the sum. Since we are ultimately interested in the behavior of a gas of a large number of particles, suppose we would like to calculate the correlation function for $N = 100$. The standard brute-force approach is to insert a resolution of the identity in $\hat{O}$ and calculate the form-factors using a Slavnov determinant [153, 154, 113]. A cutoff in quantum numbers of the order $10N$ would require us to calculate $(10N)^3 \sim 10^{420}$ terms.

4.1.1 The GGE and the approach with q-bosons

Instead of the usual difficulties with handling the conservation laws or calculating the generalized chemical potentials, the standard application of the GGE for this quench problem is impossible due to infinities of the even local charges on the initial state. This was first observed in Refs [155, 151]. Since the first-quantized expressions of the charges contain products of delta functions that only vanish when evaluated on a wave function that satisfies the cusp condition in Eq. (2.15), and since this condition depends on the value of $c$, the charges are infinite on the initial state. For two particles, explicit computation shows that the overlaps decrease with $\lambda^{-2}$, making all even charges $\hat{Q}_{2m}$ with $m > 1$ strictly infinite. The only finite charge is the energy, $\langle \psi_0|\hat{Q}_2|\psi_0\rangle = c^{N(N-1)}/L^N$, which follows immediately from the first-quantized form of the Hamiltonian, Eq. (1.2).

In Ref. [151] these infinities are regularized using $q$-bosons, which is a model of deformed bosons hopping on a lattice [156, 157]. The lattice model has a continuum limit to the Lieb-Liniger model and the charges are finite on the state that corresponds to the BEC state in the continuum limit. In other words, the $q$-boson model serves as a regulator of the infinities of the charges. An expansion for the density of quasimomenta in the $q$-boson model is derived on the basis of the values of the first few charges. In the continuum limit this density scales to the usual rapidity density and one obtains a prediction for the tails of the
4.2 Exact overlaps and the thermodynamic limit

saddle-point distribution of the BEC-to-LL quench,

\[ 2\pi \rho^\text{sp}(\lambda) = \frac{n^4 \gamma^2}{\lambda^4} + \frac{n^6 \gamma^3(24 - \gamma)}{4\lambda^6} + \ldots \]  \hspace{1cm} (4.2)

Using the known overlaps for the quench to the Tonks-Girardeau gas [34], a prediction for the saddle-point of this specific quench by means of the quench action method was derived independently in Ref. [151].

4.2 Exact overlaps and the thermodynamic limit

In order to implement the quench action approach for the BEC-to-LL quench, the first challenge is the computation of the overlaps between the initial BEC state and Bethe eigenstates. The only Bethe states with nonzero overlap are the parity-invariant states such that for each positive rapidity its negative counterpart is also present. Taking \( N \) even, we denote such Bethe states by \(|\tilde{\lambda}\rangle = |\{\lambda_j\}_{j=1}^{N/2} \cup \{-\lambda_j\}_{j=1}^{N/2}\rangle\) where all \( \lambda_j \) are taken to be positive. Parity invariance is imposed by conservation of momentum and all other odd charges during the quench. This can be observed by computing matrix elements of the conserved charges \( \hat{Q}_m \) between the BEC state and the Bethe state, \( 0 = \langle \psi_0 | \hat{Q}_{2m+1} | \lambda \rangle = \langle \psi_0 | \lambda \rangle \sum_{j=1}^{N/2} \lambda_j^{2m+1} \) for any \( m \geq 0 \).

In Ref. [1] the following expression for the exact normalized overlap of the initial BEC state with a parity invariant Bethe state was conjectured:

\[ \langle \tilde{\lambda} | \psi_0 \rangle = \frac{1}{\sqrt{N!}} \frac{\sqrt{(cL)^{-N/2} \prod_{j=1}^{N/2} \lambda_j \sqrt{\frac{\lambda_j^2}{c^2} + \frac{1}{4}}}}{\det_{j,k=1}^{N/2} \tilde{G}^+_{jk} \det_{j,k=1}^{N/2} \tilde{G}^-_{jk}}. \]  \hspace{1cm} (4.3a)

The matrices \( \tilde{G}^\pm_{jk} \) are similar to the Gaudin matrix \( G_{jk} \) of the Lieb-Liniger model [12, 115):

\[ \tilde{G}^\pm_{jk} = \delta_{jk} \left( L + \sum_{l=1}^{N/2} \tilde{K}^+(\lambda_j, \lambda_l) \right) - \tilde{K}^\pm(\lambda_j, \lambda_k), \]  \hspace{1cm} (4.3b)

where \( \tilde{K}^\pm(\lambda, \mu) = \tilde{K}(\lambda - \mu) \pm \tilde{K}(\lambda + \mu) \) and \( \tilde{K}(\lambda) = 2c/(\lambda^2 + c^2) \).

In Ref. [1] the conjectured overlap formula was checked analytically up to \( N = 6 \) and numerically up to \( N = 10 \). It was also asserted that it reproduces the known \( c \to \infty \) limit of Ref. [34] and in the zero-density limit \( L \to \infty \) with \( N \) fixed, Eq. (4.3a) can be proven for any \( N \). It is remarkable that the zero-density part already contains the extensive part of the overlap, which completely fixes the saddle-point distribution. Subsequently, using known overlaps for the spin-1/2 XXZ chain (see Section 5.3) and the scaling limit to the Lieb-Liniger model, in Ref. [158] the overlap formula (4.3a) was proven for any \( N, L \) and \( c > 0 \). In Section B.2 we briefly review this proof.
4. The Lieb-Liniger quench

4.2.1 The thermodynamic limit

In order to derive a GTBA equation in the quench action approach the overlaps must be in a form that can be treated in the thermodynamic limit. The goal of this section is to transform the overlap formula (4.3a) into an expression for large system size, up to corrections $O(N^{-1})$.

Let us first focus on the explicit $\lambda$-dependent part of the overlaps in Eq. (4.3a). The thermodynamic limit for the Lieb-Liniger model was extensively described in Section 2.1.6. By means of the Euler-Maclaurin summation formula a sum over rapidities of a Bethe state can be recast as an integral over the density function. For any smooth function $f$ of the rapidities, we have

$$\lim_{N \to \infty} \sum_{j=1}^{N} f(\lambda_j) = L \int_{-\infty}^{\infty} d\lambda \rho(\lambda)f(\lambda) + O(N^0) \ .$$

(4.4)

where the smooth density function $\rho$ represents the macroscopic description of an ensemble of Bethe states (see Section 2.1.6). Subleading corrections depend on the specific choice of Bethe state in the ensemble, in particular the microscopic arrangement of rapidities.

For large system size, let us choose a reference Bethe state $\lambda$ that scales to $\rho$ and for which the finite-size term in Eq. (4.4) is zero,

$$\lim_{N \to \infty} \sum_{j=1}^{N} f(\lambda_j) = L \int_{-\infty}^{\infty} d\lambda \rho(\lambda)f(\lambda) + O(1/N) \ .$$

(4.5)

In practice, this corresponds to absorbing the finite-size term of the overlap coefficients, $s_{(n,c)}$ in Eq. (3.15), into the measure of the functional integral. In principle any Bethe state that scales to the density function $\rho$ can be selected as reference state, for example the state with maximally flat in-box distributions.

Adding a set of $m$ particle-hole excitations $\{\lambda^h_k \rightarrow \lambda^p_k\}_{k=1}^{m}$ to the reference state such that $\lim_{N \to \infty}(m/N) = 0$ splits the set of rapidities in “nonexcited rapidities” $\{\lambda'_j\}_{j=1}^{N-m}$ and in a set of particle excitations $\{\lambda'^p_j\}_{j=1}^{m}$. The set of holes $\{\lambda^h_j\}_{j=1}^{m}$ contains fictitious rapidities which represent the empty slots left by the particle excitations. Note that we use a prime for the rapidities of the excited state, as the tilde is reserved for parity-invariant Bethe states. As explained in Section 2.1.7, the rapidities of the reference state and the excited state are related through the back-flow function $F'$,

$$\lambda'_j = \lambda_j - \sum_{k=1}^{m} \frac{F(\lambda_j|\lambda'^p_k, \lambda'^h_k)}{L} + O(N^{-2}) \ ,$$

(4.6)

where $F(\lambda|\lambda'^p_k, \lambda'^h_k)$ obeys Eq. (2.49) for a single excitation $\{\lambda^h_k \rightarrow \lambda^p_k\}$. We can thus write for parity-invariant Bethe states $\lambda$ and $\lambda'$ differing by $m$ particle-hole
4.2. Exact overlaps and the thermodynamic limit

Excitations

\[
\log \left( \prod_{j=1}^{\frac{N}{2}} \frac{\lambda_j'}{c} \sqrt{\left( \frac{\lambda_j'}{c^2} + \frac{1}{4} \right)} \right) = L \int_0^\infty d\lambda \rho(\lambda) \log \left( \frac{\lambda}{c} \sqrt{\frac{\lambda^2}{c^2} + \frac{1}{4}} \right) + \frac{m}{2} \int_0^\infty d\lambda \rho(\lambda) \frac{1 + 8 \frac{\lambda^2}{c^2}}{\lambda(1 + 4 \frac{\lambda^2}{c^2})} \left( \lambda |\lambda_k'\lambda_k' - l_l\right)
\]

\[
+ \log \left( \frac{\lambda_k'}{c} \sqrt{\left( \frac{\lambda_k'}{c^2} + \frac{1}{4} \right)} \right) - \log \left( \frac{\lambda_k''}{c} \sqrt{\left( \frac{\lambda_k''}{c^2} + \frac{1}{4} \right)} \right) \] + \mathcal{O}(N^{-1}),
\]

Let us now consider the two determinants in Eq. (4.3a). They can be rewritten as

\[
\det_{j,k=1}^{\frac{N}{2}} G_{jk}^\pm = L^N \prod_{j=1}^{\frac{N}{2}} \left( 1 + \frac{1}{L} \sum_{l=1}^{N} K^+(\lambda_j - \lambda_l) \right)
\]

\[
\times \det_{j,k=1}^{\frac{N}{2}} \left( \delta_{jk} - \frac{\tilde{K}^\pm(\lambda_j - \lambda_k)}{L + \sum_{l=1}^{\frac{N}{2}} \tilde{K}^+(\lambda_k - \lambda_l)} \right) \] \quad (4.8)

From the Bethe equations in the thermodynamic limit, Eq. (2.40), we have

\[
1 + \frac{1}{L} \sum_{l=1}^{\frac{N}{2}} \tilde{K}^+(\lambda_j - \lambda_l) = 2\pi \rho_t(\lambda_j) + \mathcal{O}(N^{-1}), \] \quad (4.9)

where \( \rho_t = \rho + \rho_h \). Furthermore, the matrix on the right-hand side of Eq. (4.8) becomes an integral operator on the real line:

\[
\delta_{jk} - \frac{\tilde{K}^\pm(\lambda_j - \lambda_k)}{L + \sum_{l=1}^{\frac{N}{2}} \tilde{K}^+(\lambda_k - \lambda_l)} \rightarrow 1 - \frac{\tilde{K}^\pm}{2\pi},
\]

where \( \left( \tilde{K}^\pm \rho \right) (\lambda) = \int_{-\infty}^{\infty} d\mu \tilde{K}^\pm(\lambda - \mu) \rho(\mu) \) \( g(\mu) \). \quad (4.10)

The determinants of integral operators can be evaluated as Fredholm determinants \[159\]. To conclude, in the thermodynamic limit the ratio of determinants in Eq. (4.3a) transforms into a ratio of two Fredholm determinants (denoted by \( \text{Det} \)),

\[
\lim_{\text{th}} \left( \frac{\det_{j,k=1}^{\frac{N}{2}} G_{jk}^+}{\det_{j,k=1}^{\frac{N}{2}} G_{jk}^-} \right) = \frac{\text{Det} \left( 1 - \frac{\tilde{K}^+}{2\pi} \right)}{\text{Det} \left( 1 - \frac{\tilde{K}^-}{2\pi} \right)} + \mathcal{O}(N^{-1}). \quad (4.11)
\]
4. The Lieb-Liniger quench

4.2.2 The overlap coefficients

Finally, the thermodynamic limit of the logarithm of the overlap of the BEC state \(|\psi_0\rangle\) with a generic parity-invariant Bethe state \(|\tilde{\lambda}'\rangle\), with \(m\) particle-hole excitations with respect to a reference state, is given by

\[
\lim_{\text{th}} \log \langle \tilde{\lambda}' | \psi_0 \rangle = -\frac{L}{2} \int_0^\infty d\lambda \rho(\lambda) \log \left[ \frac{\lambda^2}{c^2} \left( \frac{\lambda^2}{c^2} + \frac{1}{4} \right) \right]
\]

\[
-\frac{Ln}{2} \left[ \log \left( \frac{c}{n} \right) + 1 \right] + \frac{1}{2} \log \left[ \frac{\text{Det} \left( 1 - \frac{\hat{K}_+}{2\pi} \right)}{\text{Det} \left( 1 - \frac{\hat{K}_-}{2\pi} \right)} \right]
\]

\[
- \sum_{k=1}^m \left[ \int_0^\infty d\lambda \rho(\lambda) \frac{1 + 8 \frac{\lambda^2}{c^2}}{\lambda \left( 1 + 4 \frac{\lambda^2}{c^2} \right)} F(\lambda | \lambda_{p_k}^p, \lambda_{h_k}^h)
\]

\[
+ \log \left( \frac{\lambda_{p_k}^p \sqrt{(\lambda_{p_k}^p/c)^2 + 1/4}}{\lambda_{h_k}^h \sqrt{(\lambda_{h_k}^h/c)^2 + 1/4}} \right) \right]. \quad (4.12)
\]

The integrals are over the half-infinite real line due to parity invariance. We can split this expression for the overlap coefficients in an extensive part which depends only on the smooth root density \(\rho\),

\[
S[\rho] = \frac{L}{2} \left( \int_0^\infty d\lambda \rho(\lambda) \log \left[ \frac{\lambda^2}{c^2} \left( \frac{\lambda^2}{c^2} + \frac{1}{4} \right) \right] + n \left[ \log \left( \frac{c}{n} \right) + 1 \right] \right), \quad (4.13)
\]

and in a nonextensive differential overlap \(\delta s_e[\rho]\), as defined in Eq. (3.18b). The latter does depend on details of the excitations \(e = \{\lambda_{p_k}^h \rightarrow \lambda_{n_k}^p\}^m_{k=1}\) and contains information about the time evolution after the quench via Eq. (3.20). It is given by

\[
\delta s_e[\rho] = \sum_{k=1}^m \left[ \int_0^\infty d\lambda \rho(\lambda) \frac{1 + 8 \frac{\lambda^2}{c^2}}{\lambda \left( 1 + 4 \frac{\lambda^2}{c^2} \right)} F(\lambda | \lambda_{p_k}^p, \lambda_{h_k}^h)
\]

\[
+ \log \left( \frac{\lambda_{p_k}^p \sqrt{(\lambda_{p_k}^p/c)^2 + 1/4}}{\lambda_{h_k}^h \sqrt{(\lambda_{h_k}^h/c)^2 + 1/4}} \right) \right], \quad (4.14)
\]

where the Fredholm determinants do not depend on the particle-hole excitations and are absorbed into the measure of the functional integral in Eq. (3.18a).

4.3 GTBA equation and its analytic solution

All ingredients of the saddle-point equation (3.19) are available now. The quench action is given by \(S_{QA}[\rho] = 2S[\rho] - \frac{1}{2} S_{YY}[\rho]\), where the thermodynamic overlap coefficients are given in Eq. (4.13) and the Yang-Yang entropy in Eq. (2.43).
4.3. GTBA equation and its analytic solution

Note the factor $\frac{1}{2}$ in front of the entropy due to the fact that in the Hilbert space sum only parity-invariant Bethe states are included, which halves the number of degrees of freedom.

To impose the normalization condition on $\rho$ a Lagrange multiplier $h$ is added to the quench action. As in Ref. [12] this can be viewed, in the spirit of the free energy, as a generalized chemical potential. The functional integral now reads

$$
\int \mathcal{D}\rho \, e^{-S_{\text{QA}}[\rho]} \to \int_{-i\infty}^{i\infty} dh \int \mathcal{D}\rho \, e^{-S_{\text{QA}}[\rho]} e^{-\frac{Lh}{2} \left( n - \int_{-\infty}^{\infty} d\lambda \, \rho(\lambda) \right)}.
$$

(4.15)

Taking the functional derivative with respect to the smooth density function $\rho$ leads to the generalized TBA equation for the BEC-to-LL quench, which is

$$
\log[a(\lambda)] = g(\lambda) - h - \int_{-\infty}^{\infty} \frac{d\mu}{2\pi} K(\lambda - \mu) \log \left[ 1 + a^{-1}(\mu) \right],
$$

(4.16a)

with $\eta = \rho_h/\rho$, kernel $K$ defined in Eq. (2.22) and ‘driving term’ $g$ given by

$$
g(\lambda) = \log \left[ \frac{\lambda^2}{c^2} \left( \frac{\lambda^2}{c^2} + \frac{1}{4} \right) \right].
$$

(4.16b)

The convolution in the saddle-point equation comes from the variation of the Yang-Yang entropy with the thermodynamic Bethe equation as constraint. The integral equation (4.16a) forms together with the thermodynamic Bethe equation (2.40) and the normalization constraint $\int_{-\infty}^{\infty} d\lambda \, \rho(\lambda) = n$ a set of coupled integral equations. In the next section we will derive an analytic solution for it.

4.3.1 The analytic solution

For convenience everything is written in terms of the dimensionless variable $x = \lambda/c$. The dimensionless kernel is $K(x) = \frac{2}{x^2 + 1}$ and the GTBA equation translates into

$$
\log[a(x)] = \log(\tau^2) - \log \left[ x^2 \left( x^2 + 1/4 \right) \right]
+ \int_{-\infty}^{\infty} K(x - y) \log \left[ 1 + a(y) \right] \frac{dy}{2\pi},
$$

(4.17)

where $a(x) = \rho/\rho_h$ and $\tau$ is related to the Lagrange multiplier $h$ via $\tau = e^{h/2}$. The functions $a$ and $\rho$ are directly connected. Due to the thermodynamic form of the Bethe equations (2.40), taking the derivative $\frac{\tau}{2} \partial_{\tau}$ of Eq. (4.17) leads to the identity $2\pi \rho(x) = \frac{\tau}{2} \partial_{\tau} a(x)/(1 + a(x))$.

The analytic solution is derived as an expansion around the Tonks-Girardeau point $\tau = 0$. In the limit $\tau \to 0$ the driving term is large and negative for all fixed $x > \tau$ and the contribution from the convolution term is subleading. At lowest order in $\tau$, the function $a$ reads

$$
a^{(0)}(x) = \frac{\tau^2}{x^2(x^2 + 1/4)}.
$$

(4.18)
4. The Lieb-Liniger quench

The next order is computed by plugging this result into the convolution integral on the right-hand side of the saddle point equation (4.17). Using the relation

$$
\int_{-\infty}^{\infty} \frac{1/\pi}{(x-y)^2 + 1} \log \left[ y^2 + \alpha^2 \right] dy = \log \left[ x^2 + (\alpha + 1)^2 \right],
$$

(4.19)

which is obtained by taking the derivative with respect to $\alpha$, the function $a$ obeys up to order $\tau^2$

$$
\log[a(x)] = \log(\tau^2) - \log \left[ x^2 (x^2 + 1/4) (x^2 + 1) (x^2 + 9/4) \right]
+ \int_{-\infty}^{\infty} K(x-y) \log \left[ y^2 (y^2 + 1/4) + \tau^2 \right] \frac{dy}{2\pi}.
$$

(4.20)

This is solved by rewriting $y^2(y^2 + 1/4) + \tau^2 = (y^2 + y_-^2)(y^2 + y_+^2)$ where $y_{\pm} = \frac{1}{\sqrt{8}} \sqrt{1 \pm \sqrt{1 - 64\tau^2}}$, again with Eq. (4.19) and by expanding $y_{\pm}$ to lowest order in $\tau$. One obtains

$$
a(x) = \frac{\tau^2 \left( x^2 + (1 + y_-^2) \right) \left( x^2 + (1 + y_+^2) \right)}{x^2(x^2 + 1/4)(x^2 + 1)(x^2 + 9/4)}
$$

(4.21a)

$$
= \frac{\tau^2}{x^2(x^2 + 1/4)} \left[ 1 + \frac{4\tau(1 + \tau)}{x^2 + 1} \right].
$$

(4.21b)

Hence, the function $a(x)/a^{(0)}(x)$ up to first order in $\tau$ reads

$$
\frac{a^{(1)}(x)}{a^{(0)}(x)} = 1 + \frac{4\tau}{x^2 + 1}.
$$

(4.22)

This procedure can be repeated indefinitely and leads to an expression up to generic order $\tau^N$:

$$
a^{(N)}(x) = a^{(0)}(x) \sum_{n=1}^{N+1} \binom{2n}{n-1} \prod_{j=2}^{n} \frac{\tau}{x^2 + (j/2)^2}.
$$

(4.23)

The saddle point equation (4.17) is solved by summing over all orders in $\tau$,

$$
a(x) = \lim_{N \to \infty} a^{(N)}(x) = \sum_{n=1}^{\infty} \binom{2n}{n-1} \prod_{j=0}^{n} \frac{\tau}{x^2 + (j/2)^2}
$$

(4.24a)

$$
= \frac{2\pi \tau}{x \sinh(2\pi x)} I_{1-2ix}(4\sqrt{\tau}) I_{1+2ix}(4\sqrt{\tau}),
$$

(4.24b)

where $I_n(z)$ is the modified Bessel function of the first kind of order $n$ (see Section A.3). The result is an analytic expression for the saddle-point state of the BEC-to-LL quench,

$$
rho^{sp}(\lambda) = -\frac{\gamma}{4\pi} \frac{1}{1 + a_{sp}(\lambda)} \frac{\partial a_{sp}(\lambda)}{\partial \gamma},
$$

(4.25a)

$$
a_{sp}(\lambda) = \frac{2\pi/\gamma}{\frac{\lambda}{\cosh(2\pi \lambda)}} I_{1-2i\lambda}(4\sqrt{\tau}) I_{1+2i\lambda}(4\sqrt{\tau}),
$$

(4.25b)
where the normalization constraint leads to $\tau = \gamma^{-1} = n/c$. A nontrivial check of the result is provided by the fact that $\rho^{sp}$ reproduces the initial energy density

$$\int_{-\infty}^{\infty} d\lambda \rho^{sp}(\lambda) \lambda^2 = \gamma n^3 = \lim_{\text{th}} L^{-1} \langle \psi_0 | H | \psi_0 \rangle .$$

(4.26)

In Fig. 4.1 saddle-point densities for different postquench interaction values $\gamma$ are plotted and compared to the thermal state with corresponding particle and energy density. The thermal state is defined by the TBA equation, which is of the form of Eq. (4.16a), with thermal driving term $g(\lambda) = \lambda^2 / T$ and with rescaled chemical potential $h \rightarrow h/T$ [12]. Note that the saddle point of the BEC-to-LL quench has a polynomial $\lambda^{-4}$ tail, which is the cause of the divergencies of the local charges and which is very different from the exponential tails of the thermal state.

![Figure 4.1: Density function $\rho^{sp}(\lambda)$ in Eqs (4.25) for different $\gamma$'s for the saddle point state (solid lines) and for the thermal state (dashed lines). Inset: Scaled functions $\rho_s(x) = 2\sqrt{\gamma}\rho^{sp}(2n\sqrt{\gamma}x)$ and $\rho_s^{th}(x)$ which approach the semi-circle $\frac{2}{\pi}\sqrt{1-x^2}$ in the limit $\gamma \rightarrow 0$.](image-url)

**4.3.2 Asymptotics of the saddle-point state**

The asymptotics for $\lambda \gg n$ and fixed finite $\gamma > 0$ of the exact solution (4.25) is

$$2\pi\rho^{sp}(\lambda) = \frac{n^4\gamma^2}{\lambda^4} + \frac{n^6\gamma^3(24 - \gamma)}{4\lambda^6} + \frac{n^8\gamma^4(464 - 120\gamma + \gamma^2)}{16\lambda^8} + \ldots$$

(4.27)

whose first two terms agree with the asymptotic expansion found in the q-boson approach [151], Eq. (4.2). The saddle point of the BEC quench into the Tonks-
4. The Lieb-Liniger quench

Girardeau gas, $\gamma \to \infty$, yields

$$\rho^{sp}(\lambda) = \frac{1}{2\pi} \frac{4n^2}{\lambda^2 + 4n^2},$$

(4.28)

while in the limit of an infinitesimal quench, $\gamma \to 0$, the saddle point recovers a semi-circle distribution (although always with a $\lambda^{-4}$ tail for any finite $c > 0$)

$$\rho^{sp}(\lambda) \sim \frac{1}{\pi \sqrt{\gamma}} \sqrt{1 - \frac{\lambda^2}{4\gamma n^2}} \Theta(4\gamma n^2 - \lambda^2),$$

(4.29)

where $\Theta$ is the Heaviside step function. This limit is recovered via the uniform-expansion limit of the modified Bessel function of the first kind [160, 161, 162] and the scaled density $\rho_s(x) = 2\sqrt{\gamma}\rho^{sp}(2n\sqrt{\gamma}x)$, which has a proper limit $\gamma \to 0$. For technical details, see Ref. [1]. As expected, this limit reproduces the leading term of the ground state distribution of the Lieb-Liniger model in the small-$\gamma$ expansion [163].

4.4 Physical properties of the steady state

In principle, the saddle-point distribution allows us to compute the postquench equilibrium expectation value of any operator that obeys the conditions of the quench action approach. However, obtaining the thermodynamic limit of matrix elements can be daunting, if not impossible. In this section we discuss the evaluation of two types of local observables on the thermodynamic state $\rho^{sp}$.

Following the method of Ref. [164], whose results were generalized in Ref. [165], we study the static density moments defined by

$$g_K = \langle \rho^{sp} | : \hat{\rho}(0)/n : \hat{\rho}(\mu) \rangle_K,$$

for $m \in \mathbb{N}^+$. The second moment is then given by

$$g_2 = \frac{2}{n^2 c} \int_{-\infty}^{\infty} \frac{d\mu}{2\pi} \frac{\rho(\mu)}{\rho_t(\mu)} \left[ 2\pi \rho_t(\mu) \mu^2 - h_1(\mu) \mu \right],$$

(4.31)

while expressions for higher moments can be found in Refs [164, 165]. Results for the expectation values of $g_2$ and $g_3$ on the saddle-point state are plotted in Fig. 4.2 and compared to the prediction of the thermal state at corresponding particle and energy density. These results clearly display an absence of thermalization long after the quench, while for $\gamma \to 0$ (small quenches) the results converge as expected.
4.5. Physical properties of the steady state

Figure 4.2: Expectation values $g_2$ and $g_3$ as function of $\gamma$ on the exact saddle point state (solid lines) and on the thermal one (dashed lines). Asymptotic behavior (black dashed lines) $g_2 \sim 8/(3\gamma)$, $g_3 \sim 32/(15\gamma^2)$ for $\gamma \to \infty$ as in [151] and $g_2 \sim 1 - \sqrt{\gamma}/2$, $g_3 \sim 1 - 3\sqrt{\gamma}/2$ for $\gamma \to 0$. Insets: Same plot (in logarithmic scale) for different ranges of $\gamma$.

Of more experimental relevance is the static structure factor, or density-density correlator, $S(x) = \langle \lambda_{sp} | \hat{\rho}(x) \hat{\rho}(0) | \lambda_{sp} \rangle$ and its Fourier transform [see Eqs (A.1)] using the Lehmann representation

$$S(k) = L \sum_{\mu} |\langle \lambda_{sp} | \hat{\rho}(0) | \mu \rangle|^2 \delta_{k,\mu} ,$$

(4.32)

where the state $| \lambda_{sp} \rangle$ is a representative state that scales to the saddle-point distribution in the thermodynamic limit. By means of the algebraic Bethe Ansatz the exact matrix elements are known [110] and summed into correlations by the ABACUS algorithm [166]. To compute averages with a representative state, the method in Ref. [167] was followed. In Fig. 4.3 the static structure factor and its Fourier transform are plotted for different interaction strengths $\gamma$.

One interesting feature of the quench to the Tonks-Girardeau gas is the one-body density matrix of the saddle point, $\langle \rho^{sp} | \hat{\psi}^\dagger(x) \hat{\psi}(0) | \rho^{sp} \rangle = e^{-2n|x|}$. This can be proven analytically by writing the usual difference of Fredholm determinants (see Section A.2) as a minor. The fermionic one-body density matrix, which is the Fourier transform of $\rho^{sp}(x)$, is of exactly the same form. It is not clear whether there is a deeper reason for this similarity.
4. The Lieb-Liniger quench

Figure 4.3: Density-density correlation on the saddle-point state in momentum space \( k \) (in units of \( k_F = \pi n \)) and (inset) in real space \( x \in [0, L] \). At \( x \to 0 \) the numerical results approach the analytical ones for \( g_2 \) (open dots). Curves are obtained by joining data from system sizes \( N = 64 \) (small \( k \)), 32 and 8 (large \( k \)). Error bars and shaded region are respectively estimates of finite-size discretization errors and missing intensity based on sumrule saturation levels (see Ref. [167]). At \( k = 0 \) numerical data suggests \( S(k) \to 1/2 \) for any interaction strength, which agrees with the Bogoliubov prediction for small \( \gamma \) in Ref. [148].

4.5 Time evolution towards the steady state

Within the quench action logic, the time evolution is recoverable by considering excitations around the saddle point state according to Eq. (3.20). For the special case of a quench to the Tonks-Girardeau gas the density operator creates only one particle-hole excitation, and the complete time evolution of the density-density correlation is, due to the parity constraint, found by simply summing over all symmetric two-particle-hole excitations.

According to Eq. (3.20), in the quench action formalism the full postquench time evolution of an observable is encoded in a sum over excitations in the vicinity of the saddle point. In this section we consider the dynamics of the density-density operator in the special case of a quench to the Tonks-Girardeau gas, \( c = \infty \).

Using the first-quantized version of the density operator \( \hat{\rho}(x) = \sum_{j=1}^{N} \delta(x - x_j) \) and the standard expression for the wave function for the Tonks-Girardeau gas, see Eq. (2.23), we compute the finite-size matrix elements of the normal-
ordered density-density operator.

\[
\langle \tilde{\lambda} | \hat{\rho}(x) \hat{\rho}(0) | \lambda \rangle = \frac{1}{N! L^N} \int_0^L d^N x \left( \sum_P (-1)^{|P|} \prod_{j=1}^N e^{ix_j \lambda P_j} \right) (4.33a)
\]

\[
\times \left( \sum_{P'} (-1)^{|P'|} \prod_{k=1}^N e^{-ix_k \lambda P'_k} \right) \sum_{l,m=1}^N \delta(x - x_l) \delta(x_m) = \frac{1}{(N - 2)! L^N} \sum_{P,P'} (-1)^{|P| + |P'|} e^{i x (\lambda P_1 - \bar{\lambda} P'_1)}
\]

\[
\times \left( \prod_{j=3}^N \int_0^L dx_j e^{i x_j (\lambda P_j - \bar{\lambda} P'_j)} \right). (4.33b)
\]

The product of \( N - 3 \) integrations is only nonzero if at most two rapidities differ between the bra and ket state. This is a well-known property of the density operator in the Tonks-Girardeau regime. Due to the parity constraint, only terms with either zero or two (symmetric) particle-hole excitations are present in the Hilbert space sum. So let us focus on states which differ by two rapidities:

\[
\bar{\lambda}_j = \lambda_j \quad \text{for} \quad j = 3, 4, \ldots, N,
\]

and two rapidities \( \lambda_1 \) and \( \lambda_2 \) are different from \( \bar{\lambda}_1 \) and \( \bar{\lambda}_2 \). The integrals are nonzero only if \( P_j = P'_j \) for \( j = 3, 4, \ldots, N \) and if the permutation \( P \in S_N \) is such that

\[
P_j \in \{3, 4, 5, \ldots, N\} \quad \text{for} \quad j = 3, 4, 5, \ldots, N. (4.35)
\]

There are \( 2 \cdot (N - 2)! \) permutations obeying this condition, namely \((N - 2)!\) permutations for which \( P_1 = 1 \) and \( P_2 = 2 \), and another \((N - 2)!\) permutations for which \( P_1 = 2 \) and \( P_2 = 1 \). The permutation \( P' \) is almost completely fixed by \( P \), the only possible choices are

\[
(P'_1 = P_1 \quad \text{and} \quad P'_2 = P_2) \quad \text{or} \quad (P'_1 = P_2 \quad \text{and} \quad P'_2 = P_1). (4.36)
\]

Using the truncated permutations, we obtain

\[
\langle \tilde{\lambda} | \hat{\rho}(x) \hat{\rho}(0) | \lambda \rangle = \frac{1}{L^2} \sum_{P_{\text{truncated}}} \left( e^{i x (\lambda P_1 - \bar{\lambda} P_1)} - e^{i x (\lambda P_2 - \bar{\lambda} P_2)} \right) (4.37a)
\]

\[
= \frac{1}{L^2} \left( e^{i x \lambda_1} - e^{i x \lambda_2} \right) \left( e^{-i x \bar{\lambda}_1} - e^{-i x \bar{\lambda}_2} \right). (4.37b)
\]

Similarly, the diagonal matrix elements in the thermodynamic limit become

\[
\lim_{\text{th}} \langle \lambda | \hat{\rho}(x) \hat{\rho}(0) | \lambda \rangle = n^2 - \int_{-\infty}^{\infty} d\lambda \rho(\lambda) e^{i x \lambda} \bigg| \lambda \bigg|^2. (4.38)
\]
Another way to derive these matrix elements is through the Fourier transform of the density operator, which acts on a Bethe state (only for Tonks-Girardeau)

[\hat{\rho}_p | \lambda \rangle = \frac{1}{L} \sum_{j=1}^{N} | \lambda_1, \ldots, \lambda_{j-1}, \lambda_j + p, \lambda_{j+1}, \ldots, \lambda_N \rangle).

The only nondiagonal contribution to the sum in Eq. (3.20) comes from particle-hole excitations consisting of only one parity-invariant pair: \{ \bar{\lambda}_1 = \lambda_p, \bar{\lambda}_2 = -\lambda_p | \lambda_1 = \lambda_h, \lambda_2 = -\lambda_h \}. Their spectrum is the one of free particles \( \delta \omega_\epsilon = 2\lambda_p^2 - 2\lambda_h^2 \) and the back-flow function is trivial for the Tonks-Girardeau gas. The differential overlap coefficients in Eq. (4.14) are therefore independent of the saddle-point density and given by

\[ \delta s_\epsilon = 2 \log (\lambda_p / \lambda_h). \]

Putting everything together, we obtain the full time evolution of the density-density operator,

\[ \lim_{t \to h} \langle \psi_0(t) | \hat{\rho}(x) \hat{\rho}(0) : | \psi_0(t) \rangle \]

\[ = \langle \rho^{sp} | \hat{\rho}(x) \hat{\rho}(0) : | \rho^{sp} \rangle + \lim_{\substack{\lambda_p > 0, \\ \lambda_h > 0}} \sum_{\substack{\lambda_p > 0, \\ \lambda_h > 0}} e^{-\delta s_\epsilon - i \delta \omega_\epsilon t} \frac{4 \sin(\lambda_h x) \sin(\lambda_p x)}{L^2} \] (4.39a)

\[ = n^2 (1 - e^{-4n|x|}) + \left| \int_{-\infty}^{\infty} \frac{dk}{\pi} \frac{nk}{4n^2 + k^2} e^{-2itk^2 + ikx} \right|^2. \] (4.39b)

The sum over holes was replaced by an integral using the Tonks-Girardeau saddle-point density \( \rho^{sp}(\lambda) = \frac{1}{2\pi} \frac{1}{(\lambda/2n)^2 + 1} \), while for the sum over particles the hole density \( \rho_h(\lambda) = \frac{1}{2\pi} - \rho^{sp}(\lambda) \) was used. Note that in the limit \( t \to 0 \) the density-density is a constant. This is natural, since the initial state is a gas of free bosons. Furthermore, \( \lim_{t \to h} \langle \psi_0(t) | \hat{\rho}(0) \hat{\rho}(0) : | \psi_0(t) \rangle = 0 \) for all \( t > 0 \), as expected for hard-core bosons. In Fig. 4.4 the relaxation of the density-density correlator is plotted.

The result in Eqs (4.39) was previously obtained in Eq. [152] via a brute-force calculation of the fermionic four-point correlator on the initial state. This serves as a validation of the quench action method regarding the time evolution of local observables. As said, this time evolution was also studied numerically in Ref. [34]. The analytic result corroborates the numerical findings.

Finally, using a stationary-phase approximation for large times, \( n t \gg x \), one can show that the equilibrium of the density-density correlator is approached with a power law \( t^{-3} \):

\[ \lim_{t \to h} \langle \psi_0(t) | \hat{\rho}(x) \hat{\rho}(0) : | \psi_0(t) \rangle - \langle \rho^{sp} | \hat{\rho}(x) \hat{\rho}(0) : | \rho^{sp} \rangle \sim \frac{x^2}{512\pi n^2 t^3}. \] (4.40)

4.6 Concluding remarks

Our findings are an example of a truly interacting quantum quench where non-standard ergodicity leads to a nonthermal equilibrium. To test the GGE as an alternative description of this equilibrium, one needs (local) conserved charges that are finite on the BEC initial state. Up to this moment it is unclear whether
4.6. Concluding remarks

Figure 4.4: The density-density correlator as a function of $n x \in [0, 2]$, for different times $t$ after the quench. Note that all quantities are expressed in terms of dimensionless variables ($\hbar = 1, 2m = 1$).

the infinities of the canonical local charges point to some general feature of interaction quenches, or whether this is a specific feature of the BEC-to-LL quench. An argument for the first option is that the infinities arise from a change of the cusp condition in Eq. (2.15) and this condition always changes when the interaction parameter is quenched. More generally one could argue that quenching the unbounded interaction strength in a theory without UV cutoff will lead to infinities in the conserved charges.

An implementation of a meaningful GGE without $q$-boson regularization requires either new (local) charges, or linear combinations of the standard ones such that their combination leads to finite expectation values. In Ref. [168] a first step in this direction is made. Using the exact overlaps one could try to determine the chemical potentials $\{\beta_m\}_{m=1}^\infty$ that determine the GGE for the BEC-to-LL quench. The chemical potentials should then obey an equation of the form

$$
\sum_{m=1}^\infty \beta_m \int_{-\infty}^{\infty} d\lambda \rho(\lambda) \lambda^m = \int_{-\infty}^{\infty} d\lambda \rho(\lambda) \log \left[ \frac{\lambda^2}{c^2} \left( \frac{\lambda^2}{c^2} + \frac{1}{4} \right) \right]
$$

for any smooth, normalized density $\rho(\lambda)$. However, $\log(x)$ does not have an expansion around $x = 0$ and $\log(x^2 + 1)$ has a finite radius of convergence. Their are no chemical potentials that solve the above equation, even if one neglects the term $\log(\lambda^2/c^2)$.

An interesting extension of the quenches studied here would be a quench to the attractive Lieb-Liniger model, $c < 0$. Although the ground state in this
4. The Lieb-Liniger quench

regime is ill defined in the thermodynamic limit, quenching into this regime still makes sense. In Ref. [169] the overlap formula (4.3a) was extended to the attractive regime, including string solutions to the Bethe equations. Ultimately, one would like to study quenches from the ground state of a generic $c$ to another interaction parameter $c \neq c'$. The realization of this is still far away however.