The quench action approach to out-of-equilibrium quantum integrable models
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In this chapter we describe the Bragg pulse for the Tonks-Girardeau gas by means of the QA approach and the Fermi-Bose mapping. Our aim is to model the dynamics of the famous quantum Newton’s cradle and to increase our understanding of the underlying physics. The striking outcome of the experiment in Ref. [24] was the absence of thermalization after a Bragg-pulse protocol was applied to a 1D Bose gas in an integrability-breaking harmonic trap. It is still an issue of debate whether and how this system relaxes, and in particular whether integrability or simply energy and momentum conservation plays the crucial role in this process. Other notable experimental papers on Bragg pulses are [207, 208, 79].

After defining an experimentally relevant pulse protocol, we consider the Bragg pulse on a ring as a quench. We derive the exact overlaps with Tonks-Girardeau eigenstates in the limit of short pulses. Using the QA method, we derive the postpulse time evolution of the density, the density-density correlator and the momentum distribution. The GGE is shown to give the correct late-time prediction. Then we extend our analysis to asymmetric pulses (as in Ref. [79]), to pulses at finite temperature and to short pulses in a harmonic trap. For the latter the Fermi-Bose mapping, which is shown to be a good approximation of the thermodynamic limit, is applied. We will find a rapid relaxation in the harmonic trap that is governed by the GGE and unrelated to the physics of the trap.

To the best of our knowledge, this is the first time that a microscopic description of Bragg pulses and the postpulse dynamics in an interacting quantum integrable model at large system size is studied. In Ref. [209] a pulse protocol of two Bragg pulses on a two-state system is investigated and in Ref. [79] observations of the momentum distribution after an asymmetric Bragg pulse are related to the dynamical structure factor at finite temperature via linear-response theory. This chapter is based on results published in [6].
6. Modelling the Bragg pulse

6.1 Pulse protocol

We model the Bragg pulse as a one-body cosine potential $V(x) = V_0 \cos(qx)$ that couples to the bosonic density. The momentum of the pulse $q \geq 0$ is quantized, $q = \frac{2\pi}{L} n_q$, $n_q \in \mathbb{N}$. At time $t = 0$ the pulse is turned on and at time $t = T_0$ it is turned off again. The time-evolution operator for the Bragg pulse is given by

$$\hat{U}_B(V_0, T_0) = \exp \left( -i \left[ \hat{H} + V_0 \int_0^L dx \cos(qx)\hat{\rho}(x) \right] T_0 \right), \quad (6.1)$$

where $\hat{H}$ is the Lieb-Liniger Hamiltonian (2.1) on a ring of size $L$. We will work primarily in the Tonks-Girardeau limit $c \to \infty$. Extending our results to finite $c$ is possible in principle, but hard in practice. We will be using exact overlaps that are not known for finite $c$, and also the Bose-Fermi mapping that we will use is not applicable at finite $c$.

Starting from a generic pre-pulse state $|\psi\rangle$, the state directly after the Bragg pulse is then $|\psi(V_0, T_0)\rangle = \hat{U}_B(V_0, T_0) |\psi\rangle$. The post-pulse time evolution can be viewed as a quench with initial state $|\psi(V_0, T_0)\rangle$, whose overlaps with Tonks-Girardeau eigenstates are not known in general. However, for pulses with very short duration and very high amplitude we have found analytical expressions of the exact overlaps. The evolution operator of such pulses is defined as

$$\hat{U}_B(A) = \lim_{T_0 \to 0} \hat{U}_B(A/T_0, T_0) = \exp \left( -i A \int_0^L dx \cos(qx)\hat{\rho}(x) \right) \quad (6.2a)$$

$$= \exp \left( -i AL \frac{1}{2} (\hat{\rho}_q + \hat{\rho}_{-q}) \right), \quad (6.2b)$$

where $\hat{\rho}_\pm q$ are Fourier modes of the density operator (see Appendix A). It is an infinitesimally short pulse with infinite amplitude such that the ‘area’ is at a finite value $A$. We will call it the instantaneous Bragg pulse.

6.1.1 The Bragg pulse in experiments

In this Chapter we focus on instantaneous Bragg pulses, in the first place because we can treat them analytically and in the thermodynamic limit. One could wonder how physical and how interesting such a pulse is. Let us therefore consider the experiments in Refs [24, 79] to see what is the relevant range of parameters. Due to the Stark effect the atoms couple to the electric field squared of the laser beam. The light has a wavelength of $q/2$ that is of the order of $\lambda_F$, the Fermi momentum of the TG gas. In Ref. [24] $^{87}\text{Rb}$ atoms with an average 1D particle density of $n_{1D} \approx 1.2 \times 10^7 \text{ m}^{-1}$ are pulsed twice for $23 \mu s$, with a $33 \mu s$ gap between the pulses. In our model (with $\hbar = 2m = n = 1$) this corresponds to a pulse duration $T_0 \approx 1.3$. Note that this is for $\gamma \approx 1$, whereas we work in the TG limit. From the intensity of the laser pulse (11 W cm$^{-2}$) and the polarizability caused by the 3.2 THz detuned pulse [210], one finds that in our model $V_0$ is of
the order of 0.8. The parameters in Ref. [79], for \(\gamma \approx 45\), give values of \(T_0\) and \(V_0\) that are similar. We conclude that for the instantaneous Bragg pulse to be a realistic model of experiments, we should use \(q \approx 2\lambda_F\) and \(A = V_0 T_0 \sim 1\).

## 6.2 Matrix elements and overlaps

The matrix elements of the time-evolution operator of the instantaneous pulse can be computed exactly. Using Eqs (2.4) and (2.23) for the Tonks-Girardeau Bethe state and the generic commutation between a bosonic fields and the density operator,

\[
\hat{f}[\rho(x)]\Psi(x_1) = \Psi(x_1)\hat{f}[\rho(x) + \delta(x - x_1)],
\]

one easily arrives at

\[
\langle \mu | \hat{U}_B(A) | \lambda \rangle = \int_0^L d^N x \sum_{P \in S_N} (-1)^{|P|} \prod_{j=1}^N e^{ix_j(\lambda_j - \mu_{Pj}) - iA \cos(qx_j)}
\]

\[
= L^N \det_N \left( I_{\lambda_j - \mu_k} (-iA) \delta_{\lambda_j, \mu_k \mod q} \right)_{j,k},
\]

where \(I_n(z)\) is the modified Bessel function of the first kind (see Appendix A.3) and in the second equality we used the identity

\[
I_n(z) = \frac{1}{\pi} \int_0^\infty dy \cos(ny)e^{z \cos(y)},
\]

which holds for integer \(n\). The Kronecker delta is nonzero if \(\lambda_j\) and \(\mu_k\) differ by a multiple of the Bragg momentum \(q\). This implies that the full matrix element is only nonzero if for all \(\mu_k\) there is at least one \(\lambda_j\) such that \(\lambda_j = \mu_k \mod q\). This is intuitively clear considering the density operator in Fourier space, \(\hat{\rho}_q = \sum_k \hat{\Psi}^\dagger_{k+q} \hat{\Psi}_k\), and its action on a Bethe state:

\[
\hat{\rho}_q | \lambda \rangle = \frac{1}{L} \sum_{j=1}^N \{|\lambda_1, \ldots, \lambda_{j-1}, \lambda_j + q, \lambda_{j+1}, \ldots, \lambda_N\}\rangle,
\]

where states with two equal rapidities are zero due to Pauli exclusion. The instantaneous Bragg pulse repeatedly moves rapidities a distance \(\pm q\).

The \((N \times N)\)-matrix in Eqs (6.4) can be organized in (at most) \(n_q\) blocks, each block formed by the rapidities whose differences are a multiple of \(q\). Define a set of ‘reference rapidities’ that label the different blocks, \(k_\alpha = \frac{2\pi}{L} (\xi_N + \alpha)\) for \(\alpha = 1, \ldots, n_q\) and where \(\xi_N = -1/2\) if \(N\) is even and zero for odd \(N\). Let each block consist of \(\ell_\alpha \geq 0\) rapidities, and \(\sum_{\alpha=1}^{n_q} \ell_\alpha = N\). A state can then be represented by ‘alternative quantum numbers’ \(\beta_{\alpha,j} \in \mathbb{Z}\), with \(j = 1, \ldots, \ell_\alpha\), that
are related to the rapidities via \( \lambda_{\alpha,j} = k_{\alpha} + q \beta_{\alpha,j} \). In terms of these alternative quantum numbers the matrix elements are

\[
\langle \{\beta_{\alpha,j}\}|\hat{U}_B(A)|\{\beta'_{\alpha,j}\} \rangle = L^N \prod_{\alpha=1}^{n_q} \det_{\ell_\alpha} \left[ (I_{\beta_{\alpha,j}}-\beta'_{\alpha,k}+j-k(-iA))_{k,j} \right].
\] (6.7)

As an example, the (partial) matrix element for a cluster of three coupled rapidities is given by

\[
\langle \{\beta_1, \beta_2, \beta_3\}|\hat{U}_B(A)|\{\beta'_1, \beta'_2, \beta'_3\} \rangle = L^3 \det \begin{pmatrix}
I_{\beta_1-\beta'_1} & I_{\beta_2-\beta'_1+1} & I_{\beta_3-\beta'_1+2} \\
I_{\beta_1-\beta'_2-1} & I_{\beta_2-\beta'_2} & I_{\beta_3-\beta'_2+1} \\
I_{\beta_1-\beta'_3-2} & I_{\beta_2-\beta'_3-1} & I_{\beta_3-\beta'_3}
\end{pmatrix},
\] (6.8)

where we suppressed the cluster index \( \alpha \) and the argument of the modified Bessel functions. When we send \( \beta_3, \beta'_3 \to \infty \) while keeping \( \beta_3 - \beta'_3 \) finite, the other rapidities effectively do not ‘feel’ the third rapidity anymore. In the matrix for the cluster, the off-diagonal elements of the third row and column indeed vanish, while the diagonal element remains finite. As expected, in the matrix element the third rapidity decouples from the other two. Also note that the matrix elements are parity symmetric, \( \langle \{\beta_{\alpha,j}\}|\hat{U}_B(A)|\{\beta'_{\alpha,j}\} \rangle = \langle \{-\beta_{\alpha,j}\}|\hat{U}_B(A)|\{-\beta'_{\alpha,j}\} \rangle \). This is a consequence of the parity invariance of the Bragg pulse.

### 6.2.1 Overlaps with Bragg-pulsed ground state

During most of this chapter the pre-pulse system will be in the ground state at zero temperature. In Section 6.5 the extension to a finite-temperature pre-pulse state will be discussed. The Tonks-Girardeau \( N \)-particle ground state \(|\psi_{\text{GS}}\rangle\) has rapidities \( \lambda^\text{GS}_j = \frac{2\pi}{L} \left(-\frac{N+1}{2} + j\right) \), for \( j = 1, \ldots, N \). It is a Fermi sea with Fermi momentum \( \lambda_F = \lambda^\text{GS}_N = \frac{\pi}{L}(N-1) \). The state after an instantaneous Bragg pulse is denoted by \(|\psi_{q,A} \rangle = \hat{U}_B(A)|\psi_{\text{GS}}\rangle\) and can be viewed as the initial state of a quantum quench. We will now discuss the specific overlaps for this quantum quench.

**The case \( q > 2\lambda_F \)**

For large Bragg momentum \( q > 2\lambda_F \) the matrix in Eqs (6.4) is diagonal. The rapidities do not ‘feel’ each other under the action of the Bragg pulse and the overlap decomposes into single-particle overlaps. To specify states, it suffices to work with a set of integers \( \{\beta_j\}_{j=1}^{N} \), where \( \beta_j \) is the number of times particle \( j \) was hit by \( \hat{\rho}_q \), minus the number of times of was hit by \( \hat{\rho}_{-q} \). The rapidities are given by

\[
\lambda_j = \frac{2\pi}{L} \left(-\frac{N+1}{2} + j + n_q \beta_j\right),
\] (6.9)
6.2. Matrix elements and overlaps

where this time the ground-state rapidities were used as reference rapidities. The
overlaps are very simple,
\[ \langle \{ \beta_j \}_j^N | \psi_{q,A} \rangle = L^N \prod_{j=1}^N I_{\beta_j}(-iA). \]  
(6.10)

They have the correct normalization, since \( \sum_{n \in \mathbb{Z}} |I_n(z)|^2 = 1. \)

The case \( q \leq 2\lambda_F \)

When \( q \leq 2\lambda_F \), not all rapidities can move independently under the action of \( \hat{\rho}_\pm q \). The ground state is made up of clusters of rapidities that cannot get passed each other, because this would mean they hit each other and this is precluded because of Pauli exclusion. Given \( N \) and \( q \), there are two types of clusters. The length of both cluster types is given by
\[ \ell_{\text{long}} = \left\lfloor \frac{N-1}{n_q} \right\rfloor + 1, \quad \ell_{\text{short}} = \left\lfloor \frac{N-1}{n_q} \right\rfloor \]  
(6.11)

rapidities. Note that when \( q > 2\lambda_F \), i.e., \( n_q > N-1 \), we have \( \ell_{\text{long}} = 1 \) and \( \ell_{\text{short}} = 0 \), which is expected since in that case all rapidities can move independently. The number of long and short clusters are, respectively,
\[ n_{\text{long}} = N - n_q(\ell_{\text{long}} - 1), \quad n_{\text{short}} = \begin{cases} n_q \ell_{\text{long}} - N & \text{if } n_q \leq N-1, \\ 0 & \text{if } n_q > N-1, \end{cases} \]  
(6.12)

and the total number of clusters is always given by \( n_{\text{long}} + n_{\text{short}} = n_q \). The overlaps are products of \( n_q \) determinants, as in Eqs (6.4).

6.2.2 Combinatorial viewpoint

It is possible to understand the overlaps from a combinatorial viewpoint. Expanding the Bragg operator, the intial state can be written as
\[ |\psi_{q,A} \rangle = \sum_{\beta=-\infty}^\infty \sum_{n=1,|\beta|}^{\infty} \frac{(-iAL/2)^n}{n!} \left( \frac{n}{n+\beta} \right) \left( \frac{n+\beta}{2} \right) (\hat{\rho}_q)^{n+\beta} (\hat{\rho}_{-q})^{n-\beta} |\psi_{\text{GS}} \rangle. \]  
(6.13)

When \( q > 2\lambda_F \), rapidities will never collide onto each other when one repeatedly acts with \( \hat{\rho}_\pm q \) on the ground state. They are independent. In the case of two particles, for example, the density operator is given by \( \hat{\rho}_q = \hat{\rho}_q^{(1)} \otimes I + I \otimes \hat{\rho}_q^{(2)} \), where the first term acts trivially on the second particle and vice versa, and one can show that
\[ e^{-iAL}(\hat{\rho}_q + \hat{\rho}_{-q}) = e^{-iAL}(\hat{\rho}_q^{(1)} \otimes I + \hat{\rho}_{-q}^{(1)} \otimes I) e^{-iAL}(I \otimes \hat{\rho}_q^{(2)} + I \otimes \hat{\rho}_{-q}^{(2)}). \]  
(6.14)

This formula is easily generalized to any number of particles. The total overlap thus becomes a product of one-particle overlaps, and the only thing we need to
compute is the overlap for one particle. Looking at Eq. (6.13), for one particle we immediately find \( L^n (\hat{\rho}_q)^{n+\beta} (\hat{\rho}_{-q})^{n-\beta} |\psi_{GS}\rangle = |\beta\rangle \) and the one-particle overlaps become

\[
\langle \beta | \psi_{q,A} \rangle / L = \sum_{n=|\beta|, +}^{\infty} \frac{(-iA/2)^n}{n!} \left( \frac{n}{n+\beta} \right) = I_{\beta}(-iA),
\]

resulting in the overlap formula (6.10) for large momentum \( q > 2\lambda_F \).

When \( q < 2\lambda_F \) a combinatorial computation of the overlaps is much harder. Rapidities now sit in clusters, meaning that the problem cannot be reduced to single-particle overlaps. Let us look at the partial overlap of a single 2-cluster with alternative quantum numbers \( \beta_1 \) and \( \beta_2 \). If we take the initial rapidity of the first particle to be smaller than the second rapidity, we have to require \( \beta_1 \leq \beta_2 \) because the particles cannot get passed each other. Now, for fixed \( n \) and \( \beta = \beta_1 + \beta_2 \) in Eq. (6.13) there are \( n \) operators acting on the 2-cluster. Each operator either increases or decreases the distance between the two particles by one. The initial distance is zero and it cannot become negative. This combinatoric problem is linked to Catalan’s triangle (which is half Pascal’s triangle) and for a final distance \( \Delta \beta = \beta_2 - \beta_1 \geq 0 \), the combinatorial factor is

\[
L^{n-2} \langle \beta_1, \beta_2 | (\hat{\rho}_q)^{n+\beta} (\hat{\rho}_{-q})^{n-\beta} |\psi_{GS}\rangle = \binom{n}{\frac{n}{2} - \lfloor \frac{\Delta \beta}{2} \rfloor} - \binom{n}{\frac{n}{2} - \lceil \frac{\Delta \beta}{2} \rceil} = \delta_{n,0},
\]

which can be simplified because \( n \) and \( \Delta \beta \) are either both even or both odd, i.e. \( \lfloor \frac{n}{2} \rfloor - \lfloor \frac{\Delta \beta}{2} \rfloor = \frac{n - \Delta \beta}{2} \). Note that this factor is independent of \( \beta \). It is possible to perform the infinite sum over \( n \) in Eq. (6.13) and to obtain a closed expression for this overlap. For a cluster of two rapidities the partial overlap becomes

\[
\langle \beta_1, \beta_2 | \mathcal{O}_{Bragg} |\psi_{GS}\rangle / L^2 = I_{\beta_1}(-iA) I_{\beta_2}(-iA) - I_{\beta_1-1}(-iA) I_{\beta_2+1}(-iA),
\]

for any \( \beta_1, \beta_2 \in \mathbb{Z} \) such that \( \beta_1 \leq \beta_2 \). The result agrees with the matrix elements in Eqs (6.4). Using the identity in Eq. (A.9) for \( k = 0 \), one can check that these partial overlaps are normalized correctly:

\[
\sum_{\beta_1=-\infty}^{\infty} \sum_{\beta_2=\beta_1}^{\infty} |\langle \beta_1, \beta_2 | \mathcal{O}_{Bragg} |\psi_{GS}\rangle|^2 = L^4.
\]

This combinatorial calculation provides more insight in the overlap formulas found in this section. Extending this argument to larger clusters should in principle be possible.

### 6.3 Quench action analysis

In this section we solve the instantaneous Bragg pulse by means of the quench action approach. Initially, we consider pulses with large momenta \( q > 2\lambda_F \). In
Section 6.3.2 we will discuss the complications of small-momentum pulses in the thermodynamic limit.

As a consequence of working in the Tonks-Girardeau regime, there are many microstates with exactly the same overlap. We can rephrase the overlap (for \( q > 2\lambda_F \)) as

\[
\langle \{ \beta_j \}_{j=1}^N | \psi_{q,A} \rangle = L^N \prod_{\alpha=-\infty}^{\infty} [I_\alpha(-iA)]^{n_\alpha},
\]

where \( n_\alpha \) is the number of rapidities \( j \) with \( \beta_j = \alpha \) and \( \alpha \in \mathbb{Z} \). In the thermodynamic limit these numbers are given by

\[
n_\alpha = L \int_{\alpha q - \lambda_F}^{\alpha q + \lambda_F} d\lambda \, \rho(\lambda).
\]

The numbers \( n_\alpha \) are in the same spirit as the out-box distributions of Chapter 2.1.6, although these ‘boxes’ do not shrink to zero in the thermodynamic limit. The normalized overlap coefficients \( S_{\{\beta_j\}} = -\log(\langle \{ \beta_j \}_{j=1}^N | \psi_{q,A} \rangle / L^N) \) therefore have a well-defined thermodynamic limit,

\[
S[\rho] = \lim_{\text{th}} \Re S_{\{\beta_j\}} = -L \sum_{\alpha=-\infty}^{\infty} \int_{\alpha q - \lambda_F}^{\alpha q + \lambda_F} d\lambda \, \rho(\lambda) \log ||I_\alpha(-iA)||
\]

\[
= L \int_{-\infty}^{\infty} d\lambda \, \rho(\lambda) \sum_{\alpha=-\infty}^{\infty} [\theta(\lambda - \alpha q - \lambda_F) - \theta(\lambda - \alpha q + \lambda_F)] \log ||I_\alpha(iA)||,
\]

where \( \theta \) is the Heaviside step function and we used that \( ||I_n(-z)|| = ||I_n(z)|| \). Note that in the thermodynamic limit \( \lambda_F = \pi n \), where \( n \) is the average particle density. The noncontinuous integrand will serve as the driving term of the GTBA equations. Note that in the second line we implicitly assume that \( \rho(\lambda) = 0 \) when \( \lambda \notin [\lambda - \alpha q - \lambda_F, \lambda - \alpha q + \lambda_F] \) for any \( \alpha \in \mathbb{Z} \). The reason is that for Bethe states that do not obey this condition, the overlap is exactly zero (rapidities will never end up in those regions) and therefore \( S[\rho] = \infty \). These states are therefore infinitely suppressed in the quench action saddle-point equations. Another way of seeing this is that originally the functional integral in the quench action approach is a sum over states with nonzero overlaps and these states are not in that sum.

Even when you restrict the support of the density function to these intervals, in this ensemble of states there are still many microstates that have zero overlap with the Bragg-pulsed ground state. The reason is that when a rapidity \( \lambda_{GS}^j \) has moved to an interval \( \alpha = \beta_j \), it is not in the other intervals \( \alpha \neq \beta_j \) and therefore leaves a hole there. This alters the form of the Yang-Yang entropy significantly.
6. Modelling the Bragg pulse

Given the fillings \( \{ n_\alpha \}_{\alpha = -\infty}^{\infty} \), the finite-size entropy is

\[
e^{S_{YY,(n_\alpha)}} = \left( \begin{array}{c} N \\ n_0 \end{array} \right) \left( \begin{array}{c} N - n_0 \\ n_1 \end{array} \right) \cdots \left( \begin{array}{c} N - n_0 - n_1 - n_2 \\ n_1 \end{array} \right) \cdots, \tag{6.22a}
\]

and

\[
= \frac{N!}{\prod_{\alpha = -\infty}^{\infty} (n_\alpha)!}, \tag{6.22b}
\]

which leads to a modified Yang-Yang entropy for the Bragg pulse from the ground state,

\[
S_{YY}[\rho] = -L \int_{-\infty}^{\infty} d\lambda \rho(\lambda) \log[2\pi\rho(\lambda)], \tag{6.23}
\]

where we used the Tonks-Girardeau Bethe equation \( 2\pi[\rho(\lambda) + \rho_h(\lambda)] = 1 \). The variation of the quench action should be restricted to densities for which \( \rho(\lambda) = 0 \) when \( \lambda \notin [\lambda - \alpha q - \lambda_F, \lambda - \alpha q + \lambda_F] \) for any \( \alpha \in \mathbb{Z} \). Also, a Lagrange multiplier \( h \) is added to fix the total particle density to \( n = N/L \). The resulting GTBA equation is not an integral equation because of the Tonks-Girardeau limit,

\[
2 \sum_{\alpha = -\infty}^{\infty} [\theta(\lambda - \alpha q - \lambda_F) - \theta(\lambda - \alpha q + \lambda_F)] \log [I_\alpha(iA)] - \log \left( \frac{1}{2\pi\rho^{sp}(\lambda)} \right) + 1 - h = 0. \tag{6.24}
\]

The saddle-point density for the instantaneous pulse and large Bragg momentum \( q > 2\lambda_F \) is (for the normalization one finds \( h = 1 \))

\[
\rho^{sp}(\lambda) = \begin{cases} 
\frac{1}{2\pi} |I_\alpha(iA)|^2 & \text{if } \alpha q - \lambda_F \leq \lambda \leq \alpha q + \lambda_F, \ \alpha \in \mathbb{Z}, \\
0 & \text{otherwise}.
\end{cases} \tag{6.25}
\]

In Fig. 6.1 the saddle-point densities for different instantaneous Bragg pulses are plotted. Both for increasing \( q \) and increasing \( A \) the densities shift towards larger rapidities.

6.3.1 Consistency check with local conserved charges

As a consistency check for the saddle-point state we obtained, the expectation values of the local conserved charges on the initial state are computed at finite system size. We show that in the thermodynamic limit the results of this partial GGE approach are in agreement with the saddle-point density from the QA approach. We call it partial because no maximization of entropy is required, as the constraints from the local charges determine the saddle point already.

Remember that the rapidities of an \( N \)-particle Bethe state are parametrized by integers \( \{ \beta_j \}_{j=1}^{N} \) via \( \lambda_j(\beta_j) = \lambda_j^{GS} + q\beta_j \), for any \( j = 1, \ldots, N \). The \( \lambda_j^{GS} \) were the ground-state rapidities. The Bragg-pulsed ground state is parity invariant,
6.3. Quench action analysis

Figure 6.1: The saddle-point root densities for an instantaneous Bragg pulse with momentum (a) \( q = \frac{1}{2} \lambda_F \), (b) \( q = 2 \lambda_F \) and (c) \( q = 3 \lambda_F \), and for different amplitudes \( A \).

Putting the expectation values of odd local charges to zero. For the even charges we find at finite system size

\[
\langle \psi_{q,A} | \hat{Q}_{2m} | \psi_{q,A} \rangle = \sum_{j=1}^{N} \sum_{\beta \in \mathbb{Z}} |I_\beta(iA)|^2 (\lambda_j(\beta))^{2m} \tag{6.26a}
\]

\[
= \sum_{j=1}^{N} \sum_{\beta \in \mathbb{Z}} |I_\beta(iA)|^2 \sum_{a=0}^{m} \left( \frac{2m}{2a} \right) (\lambda_j^{GS})^{2(m-a)} (q\beta)^{2a} \tag{6.26b}
\]

\[
= \sum_{j=1}^{N} \sum_{a=0}^{m} \left( \frac{2m}{2a} \right) (\lambda_j^{GS})^{2(m-a)} q^{2a} B_{2a,0}(A), \tag{6.26c}
\]

where the coefficients \( B_{2a,0} \) come from the sum over the order of the Bessel functions and are known recursively \([211]\). For details, see Appendix A.3. The sum over particles \( j \) can be performed, after which the thermodynamic limit can be taken,

\[
\lim_{\text{th}} \langle \psi_{q,A} | \hat{Q}_{2m} | \psi_{q,A} \rangle = N \sum_{a=0}^{m} \left( \frac{2m}{2a} \right) \frac{(n\pi)^{2(m-a)}}{2(m-a)+1} \frac{q^{2a}}{B_{2a,0}(A)}, \tag{6.27}
\]

where \( n = N/L \) is the average particle density. For example, the energy density pumped into the system by an instantaneous Bragg pulse is given by

\[
\lim_{\text{th}} \left( \langle \psi_{q,A} | \hat{Q}/N | \psi_{q,A} \rangle - \langle \psi_{GS} | \hat{Q}/N | \psi_{GS} \rangle \right) = \frac{q^2 A^2}{2}. \tag{6.28}
\]

One easily finds that \( L \int_{-\infty}^{\infty} d\lambda \rho_{sp}(\lambda) \lambda^{2m} = \lim_{\text{th}} \langle \psi_{q,A} | \hat{Q}_{2m} | \psi_{q,A} \rangle \) by performing the integral and recasting the infinite sum into the coefficients \( B_{2a,0} \) again. Since the local conserved charges (if well defined) uniquely determine the saddle point, this is a confirmation of the correctness of the QA saddle point.
6. Modelling the Bragg pulse

6.3.2 The case $q < 2\lambda_F$

The starting point of the quench action approach are the overlaps with some initial state. A necessary condition for the logic to work is that for large system sizes the exponentially dominant part of the overlaps is universal, in the sense that it only depends on the smooth root density $\rho$ and not on the microscopic arrangement of the rapidities of Bethe states. In Chapter 3 we have called such overlaps smooth.

Nonsmoothness of the overlaps

For the case $q < 2\lambda_F$ with $|\psi_{q,A}\rangle$ as initial state, this smoothness condition is not met. Different microstates with the same macroscopic description $\rho$ lead to a different thermodynamic overlap coefficient $S[\rho]$, as defined below Eq. (3.18a).

The handwaving argument goes as follows. When $q < 2\lambda_F$ the rapidities form clusters and rapidities in the same cluster cannot get passed each other under the action of the Bragg operator. This changes the structure of the overlaps significantly. When describing a Bethe state in terms of root densities $\rho$ one only fixes the number of rapidities in infinitesimal intervals $[\lambda, \lambda + d\lambda]$. Information about which rapidities are grouped in clusters is lost in this description. As a consequence, the leading exponential-in-system-size factor of two states with the same $\rho$ but with a different cluster structure could be different (in particular when there is a thermodynamically large number of rapidities with a disagreeing cluster structure). It is then impossible to write down a well-defined quench action that is a functional of the densities $\rho$ only.

In order to give the argument more substance, let us give an explicit example. Take the Bragg momentum to be $q = \lambda_F$. There are $n_q = N/2$ clusters and all are of length two. A macroscopic description of Bethe states is provided by a set of integers $n_a$, $a \in \mathbb{Z}$, which denote the number of rapidities in an interval $[2aq - \lambda_F, 2aq + \lambda_F]$. We take all $n_a$ to be even and we impose macroscopic symmetry on the state, i.e. $n_a = n_{-a}$. Furthermore, the rapidities are evenly distributed over the intervals. In particular, we assume even distribution over the left and right part of an interval $[2aq - \lambda_F, 2aq + \lambda_F]$, both parts containing exactly $n_a/2$ rapidities. This is a perfectly viable macroscopic description of a Bethe state.

Now define two Bethe states with the above macroscopic density and a different substructure in terms of clusters. All clusters are of length two and specified by integers $\beta_{\alpha,1}$ and $\beta_{\alpha,2}$, for $j = 1, \ldots, N/2$, where the first integer specifies the position of the left rapidity in the chain and the second integer determines the right rapidity. For the first state $|\beta\rangle_1$, take all coupled rapidities to be maximally close to each other, i.e. $\beta_{\alpha,1} = \beta_{\alpha,2} - 1$ for $j = 1, \ldots, N/2$. For the second state $|\beta\rangle_2$ we demand that coupled rapidities are arranged symmetrically around $\lambda = 0$, i.e. $\beta_{\alpha,1} = - (\beta_{\alpha,2} + 1)$ for $j = 1, \ldots, N/2$. The respective overlaps are
then necessarily given by

\[
1\langle \beta | \psi_{q,A} \rangle = \prod_{a=-\infty}^{\infty} \left[ I_{2a}(-iA)I_{2a}(-iA) - I_{2a-1}(-iA)I_{2a+1}(-iA) \right]^{n_a/2},
\]

(6.29a)

\[
2\langle \beta | \psi_{q,A} \rangle = \prod_{a=0}^{\infty} \left[ I_{a}(-iA)I_{-a}(-A) - I_{-a-1}(-iA)I_{a+1}(-iA) \right]^{n_a/2}.
\]

(6.29b)

In Fig. 6.2 the quantities \( \frac{1}{N} \log |1,2\langle \beta | \psi_{q,A} \rangle| \) are plotted as a function of system size \( N \) for a typical exponential distribution of \( n_a \). In the thermodynamic limit, these quantities converge to the leading (in system size) exponential behavior of the overlaps. They clearly converge to a different value depending on the microstate. This shows that the leading exponential part is nonuniversal and does not only depend on the macroscopic description in terms of root densities. The overlaps are not smooth.

In the above argument we chose an exponential distribution and not a power law, because the latter would give infinite expectation values for certain local charges and this is not the case (see next paragraph). In the space of root densities, the power-law decaying densities are very far away from the quench action maximum. Nonetheless, also power-law distributions show the same nonsmoothness of overlaps.

![Figure 6.2: The extensive exponential part of the overlaps \( \frac{1}{N} \log |1,2\langle \beta | \psi_{q,A} \rangle| \) between the Bragg pulsed state \( |\psi_{q,A}\rangle \) and Bethe states \( |\beta\rangle_1 \) and \( |\beta\rangle_2 \) is given as a function of system size \( N \). The distribution \( n_a \) is exponentially decaying, i.e. \( n_a = 2^{|a|} \) for \( |a| < p \) and \( n_a = 0 \) for \( |a| \geq p \), where \( p \) is some positive integer. The total number of particles is given by \( N = 3 \cdot 2^p - 4 \). The Bethe states \( |\beta\rangle_1 \) and \( |\beta\rangle_2 \) differ by how the rapidities are arranged in terms of clusters. In the thermodynamic limit, the extensive exponential parts of the overlaps of the two states are clearly different, signifying that they do not only depend on the macroscopic description in terms of root densities. The data were obtained for \( A = 1 \) and \( p = 32 \).](image)
6. Modelling the Bragg pulse

Using the local charges for the case $q < 2\lambda_F$

Despite the inapplicability of the quench action approach for small Bragg momenta $q < 2\lambda_F$, the expectation values of the local conserved charges can give us the saddle-point state.

Let us consider the most simple example of $N$ even and $n_q = N/2$, where the rapidities of Bethe states with nonzero overlap with $|\psi_{q,A}\rangle$ are structured in $N/2$ clusters of length two. The expectation value of an even conserved charge (the odd charges are still zero) on the initial state is given by

$$
\langle \psi_{q,A}|\hat{Q}_{2m}|\psi_{q,A}\rangle = \sum_{a=1}^{N/2} \sum_{\beta_1 \in \mathbb{Z}} \sum_{\beta_2 \geq \beta_1} |\langle \beta_1, \beta_2|\psi_{q,A}\rangle|^2 \left[ (\lambda_{a,1}(\beta_1))^{2m} + (\lambda_{a,2}(\beta_2))^{2m} \right], \quad (6.30a)
$$

where $|\beta_1, \beta_2|\psi_{q,A}\rangle$ is the partial overlap of a single 2-cluster, i.e. a 2-by-2 determinant, and

$$
\lambda_{a,1}(\beta) = \frac{2\pi}{L} \left( -\frac{N + 1}{2} + a \right) + q\beta, \quad (6.30b)
$$

$$
\lambda_{a,2}(\beta) = \frac{2\pi}{L} \left( -\frac{1}{2} + a \right) + q\beta, \quad (6.30c)
$$

where $a = 1, \ldots, N/2$. These are the rapidities associated with cluster $a$.

The first thing to note is the ordering in the sum over $\beta$’s. For computations it is convenient to get rid of this ordering. This is possible, since $|\beta_1, \beta_2\rangle = -|\beta_2 + 1, \beta_1 - 1\rangle$ for any $\beta_1, \beta_2 \in \mathbb{Z}$. They represent the same Bethe state because both expressions have the same rapidities, e.g. $\lambda_{a,1}(\beta_2 + 1) = \lambda_{a,2}(\beta_2)$, with the only difference that they are interchanged (hence the minus sign). A similar argument holds for the matrix elements of typical physical operators, such as conserved charges, which are always symmetric under the interchange of rapidities. Therefore one can get rid of the ordering in Eq. (6.30a),

$$
\sum_{\beta_1 \in \mathbb{Z}} \sum_{\beta_2 \geq \beta_1} \rightarrow \frac{1}{2} \sum_{\beta_1 \in \mathbb{Z}} \sum_{\beta_2 \in \mathbb{Z}}. \quad \text{This argument can easily be generalized to clusters of any length.}
$$

Given a cluster of length $p$, we would like to know the Hilbert-space sum over a generic power $(\beta_j)^m$ of one of the $\beta$’s in the cluster. This is useful in computing the local charges for the case $q < 2\lambda_F$. One finds,

$$
\sum_{\beta_1 \in \mathbb{Z}} \cdots \sum_{\beta_p \in \mathbb{Z}} |\langle \beta_1, \beta_2, \ldots, \beta_p|\psi_{q,A}\rangle|^2 (\beta_j)^m
$$

$$
= (p - 1)! \sum_{\beta_j \in \mathbb{Z}} \sum_{a=1}^{p} |I_{\beta_j - p - 1 + j + a}(iA)|^2 (\beta_j)^m \quad (6.31a)
$$

$$
= (p - 1)! \sum_{a=1}^{p} m \binom{m}{b} (p + 1 - j - a)^{m-b} B_{b,0}(A), \quad (6.31b)
$$

126
6.3. Quench action analysis

where the coefficients $B_{k,s}$ are defined in Appendix A.3. In the first line we decomposed the $p$-dimensional determinants and we used that $B_{0,s}(z) = \delta_{s,0}$, which kills most of the terms in the decomposition. This formula is valid for any rapidity $j = 1, \ldots, p$ in the cluster.

Using this and taking the thermodynamic limit, one obtains the following surprising result (for the case $n_q = N/2$):

\[
\lim_{\text{th}} \langle \psi_{q,A} | \hat{Q}_{2m} | \psi_{q,A} \rangle = \frac{N}{2} \sum_{c=0}^{2m} \binom{2m}{c} (-1)^{2m-c} \frac{(n\pi)^{2m-c} q^c}{2m-c+1} \left( \sum_{b=0}^{c} \binom{c}{b} B_{b,0}(A) + B_{c,0}(A) \right) = N \sum_{a=0}^{m} \binom{2m}{2a} \frac{(n\pi)^{2(m-a)} q^{2a}}{2(m-a)+1} B_{2a,0}(A). \tag{6.32a}
\]

This is precisely the expectation value of charge $\hat{Q}_{2m}$ for $q > 2\lambda_F$, given in Eq. (6.27). We have obtained this result for the simple case $n_q = N/2$ where only 2-clusters are present, but the derivation is generalizable to any cluster length and also to situations of mixed long and short clusters. Therefore, the expectation value of local conserved charges on the initial state is of the form of Eq. (6.27) for any Bragg momentum $q \geq 0$. Since the saddle-point density is uniquely determined by the conserved charges in this case, we conclude that the density $\rho^{sp}$ for any $q \geq 0$ is given by

\[
\rho^{sp}(\lambda) = \frac{1}{2\pi} \sum_{\alpha \in \mathbb{Z}} \left[ \theta(\lambda - \alpha q + \lambda_F) - \theta(\lambda - \alpha q - \lambda_F) \right] |I_{\alpha}(iA)|^2 , \tag{6.33}
\]

where $\theta$ is the Heaviside step function. This result is a continuous extension of the saddle point for large momenta in Eq. (6.25). In Fig. 6.1 examples of densities were plotted. This is an interesting result. We have shown that the quench action approach did not work when $q < 2\lambda_F$ due to nonsmoothness of the overlaps. However, using the local conserved charges the saddle point for long times after the Bragg pulse is still obtainable and a smooth extension of the densities for large momenta $q > 2\lambda_F$.

One may speculate whether the time evolution for $q < 2\lambda_F$ is still given by a summation over excitations around the saddle point, as in Eq. (3.20). One question is how to model the excitations, and in particular what to do with the partial overlap coefficients $\delta s_{e}[\rho^{sp}]$. In the above example of $n_q = N/2$ each particle-hole pair is entangled with another rapidity and $\delta s_{e}[\rho^{sp}]$ are determinants of $(2 \times 2)$-matrices. It is not clear however with which rapidity this excitation is entangled, or how to handle this entanglement in a sum over rapidities.

In Section 6.6 we present an efficient finite-size computation of the time evolution of the one-body density matrix. There appears to be no qualitative distinction between momenta larger or smaller $2\lambda_F$. This suggests that some averaging takes place such that all particle-hole pairs are effectively not entangled with
other rapidities and one can simply use Eq. (6.34) for the partial overlap coefficients. In particular in the thermodynamic limit, where there is always space in the sea of holes for unentangled excitations, this seems to be a valid assumption. Henceforth we assume that the time evolution for any \( q \geq 0 \) can be computed with the QA approach and partial overlaps coefficients as in Eq. (6.34). A non-trivial consistency check would be to compare the result at \( t = 0 \) with a direct computation, as will be done in what follows.

### 6.3.3 Time evolution

In this section we present the quench action results for the time evolution of the density, density-density and momentum distribution after an instantaneous Bragg pulse in the thermodynamic limit. Our starting point is Eq. (3.20). In the Tonks-Girardeau regime the rapidities are trivial functions of the quantum numbers, so there is no backflow resulting from particle-hole excitations. This means that \( \delta S_{\text{YY},e}[\rho] = 0 \) and \( \delta \omega_e[\rho] = \sum_{j=1}^{m} [(\mu_j^+)^2 - (\mu_j^-)^2] \). Furthermore,

\[
\delta s_e[\rho] = -\sum_{j=1}^{m} \log \left( \frac{I_{\alpha(\mu_j^+)}(-iA)}{I_{\alpha(\mu_j^-)}(-iA)} \right),
\]

where \( \alpha(\mu) \in \mathbb{Z} \) such that \( \alpha(\mu)q - \lambda_F \leq \mu \leq \alpha(\mu)q + \lambda_F \). For \( q < 2\lambda_F \) this mapping is not unique and one must take all possible integers into account. Putting everything together, we find for a generic operator \( \hat{O} \),

\[
\lim_{\text{th}} \langle \psi_{q,A}(t)|\hat{O}|\psi_{q,A}(t) \rangle = \text{Re} \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{\mu_j^-} \prod_{j=1}^{m} \frac{I_{\alpha(\mu_j^+)}(-iA)}{I_{\alpha(\mu_j^-)}(-iA)} e^{-i[(\mu_j^+)^2 - (\mu_j^-)^2]t} \langle \rho^{\text{sp}}|\hat{O}|\rho^{\text{sp}},\{\mu_j^- \rightarrow \mu_j^+\}_{j=1}^{m} \rangle,
\]

where the argument of the Bessel functions was suppressed. Normally there is a factor \( 1/(m!)^2 \) in the above formula, as was explained in Ref. [54], accounting for the indistinguishability of both particles and holes. In the above sum over states of the Hilbert states however, the particles are one-to-one associated with the respective holes, hence the factor \( 1/m! \).

**Time evolution of the density operator**

The finite-size matrix elements of the density operator are

\[
\langle \lambda | \hat{\rho}(x) | \lambda \rangle = \frac{N}{L}, \quad \langle \bar{\lambda} | \hat{\rho}(x) | \lambda \rangle = \frac{e^{ix(\lambda_1 - \bar{\lambda}_1)}}{L}, \quad (\lambda_1 \neq \bar{\lambda}_1),
\]

where in the nondiagonal case only one rapidity is different \( \lambda_1 \neq \bar{\lambda}_1 \). This means that only the terms \( m = 0,1 \) are nonzero in Eq. (6.35). Also, for the overlaps to be nonzero the difference between the particle and hole rapidity has
6.3. Quench action analysis

to be a multiple of \( q \). So \( \mu^+ = \mu^- + q \beta \), where \( \beta \in \mathbb{Z} \setminus 0 \). The resulting time evolution of the density operator is

\[
\lim_{\text{th}} \langle \psi_{q,A}(t) | \hat{\rho}(x) | \psi_{q,A}(t) \rangle \\
= n + \Re \frac{1}{2\pi} \sum_{\alpha \in \mathbb{Z}} \sum_{\beta \in \mathbb{Z}, \beta \neq 0} I_\alpha(iA)I_{\alpha+\beta}(-iA) \int_{\alpha q - \lambda_F}^{\alpha q + \lambda_F} d\mu^- e^{-i(2\beta q \mu^- + q^2 \beta^2) t} e^{ixq\beta}
\]

\[
= n + \Re \frac{1}{\pi} \sum_{\beta \in \mathbb{Z}, \beta \neq 0} i\beta I_\beta(i2A \sin(q^2 \beta t)) e^{ixq\beta} \frac{\sin(2q \beta t \lambda_F)}{2q \beta t},
\]

(6.37a)

(6.37b)

where in the last line we used an identity

\[
\sum_{\alpha \in \mathbb{Z}} I_\alpha(iA)I_{\alpha+\beta}(-iA) e^{ic\alpha} = i\beta I_\beta(-i2A \sin(c/2)) e^{-ic\beta/2},
\]

(6.38)

which is valid for any \( A \in \mathbb{R}, c \in \mathbb{R} \) and \( \beta \in \mathbb{Z} \), and is derived from Graf’s addition formula [212] for the modified Bessel functions of the first kind (see Section A.3).

Figure 6.3: Left panel: Time evolution of the density profile \( \lim_{\text{th}} \langle \psi_{q,A}(t) | \hat{\rho}(x) | \psi_{q,A}(t) \rangle \) after an instantaneous Bragg pulse with \( q = 3 \lambda_F \), \( A = 1 \) and \( n = 1 \), computed by means of the Fermi-Bose mapping with 50 particles. Middle panel: Quench action result of the same quantities. Right panel: Absolute value of the difference between left and middle panel. The small discrepancy can be attributed to finite-size effects (recursion).
6. Modelling the Bragg pulse

It is easy to see that both at zero time and at infinite time the density profile is flat at the value $n$ (the average particle density). Intuitively, these limits are sensible. Remember that we started from the ground state and applied an infinitesimally short Bragg pulse to it. At time $t = 0$ the particles have acquired new momenta due to the Bragg potential that was switched on an infinitesimally short amount of time, but the particles have not yet had the time to move according to their newly acquired momenta. A simple finite-size computation shows indeed that $\langle \psi_{q,A} | \hat{\rho}(x) | \psi_{q,A} \rangle = N/L$. Therefore the initial density should equal the ground-state density profile. And since all the particles have different momenta and scatter with each other, after a long time the density fluctuations dephase and the profile becomes flat again.

In Fig. 6.3 the density profile $\lim_{t \to \infty} \langle \psi_{q,A}(t) | \hat{\rho}(x) | \psi_{q,A}(t) \rangle$ is plotted as a function of time. Shortly after the Bragg pulse strong fluctuations in the density arise. This is also intuitively clear, as the particles get a ‘kick’ from the cosine potential towards the nearest minimum and start accumulating until collisions start to play a role. The early behavior of the density profile is actually well described by Luttinger liquid theory [6]. After an initial strong fluctuation, relaxation sets in at a timescale $\Delta t_{\text{rel}} \sim 0.1$. As is clear from Eqs (6.37), the density approaches its equilibrium values like $t^{-1}$. 

Figure 6.4: The density-density correlator after an instantaneous Bragg pulse at $t = 0$ (red) and $t = \infty$ (blue), for momenta $q = 3\lambda_F$ (a) and $q = 6\lambda_F$ (b). The strong oscillations for $t = \infty$ are related to the nonthermal momentum distribution at late times. We used an average particle density $n = 1$ and $A = 1$. 

\[ \lim_{t \to \infty} \langle \psi_{q,A}(t) | \hat{\rho}(x) | \psi_{q,A}(t) \rangle \]
6.3. Quench action analysis

Time evolution of the density-density operator

Nonzero matrix elements of the density-density operator differ by at most two rapidities. All nonzero matrix elements are given by Eqs (4.37), (4.38) and

$$\langle \Lambda | : \hat{\rho}(x)\hat{\rho}(0) : | \Lambda \rangle = \frac{n^2}{N} \left( 1 + e^{ix(\lambda_1 - \lambda_i)} \right) - \frac{n^2}{N} e^{ix\lambda_1} \int_{-\infty}^{\infty} d\mu \rho(\mu)e^{-ix\mu}$$

$$- \frac{n}{N} e^{-ix\lambda_1} \int_{-\infty}^{\infty} d\mu \rho(\mu)e^{ix\mu} + O \left( \frac{1}{N^2} \right), \quad (6.39)$$

where in the last equation only one rapidity is different: $\lambda_1 \neq \lambda_i$. For simplicity we drop the normal-ordering signs. Using Eq. (6.35), the time evolution of the density-density correlator after an instantaneous Bragg pulse is

$$\lim_{t \to 0} \langle \psi_{q,A}(t) | \hat{\rho}(x)\hat{\rho}(0) | \psi_{q,A}(t) \rangle = \langle \rho^{sp} | \hat{\rho}(x)\hat{\rho}(0) | \rho^{sp} \rangle$$

$$+ \lim_{t \to 0} \left( \langle \psi_{q,A}(t) | \hat{\rho}(x)\hat{\rho}(0) | \psi_{q,A}(t) \rangle_{1ph} + \langle \psi_{q,A}(t) | \hat{\rho}(x)\hat{\rho}(0) | \psi_{q,A}(t) \rangle_{2ph} \right),$$

with

$$\langle \rho^{sp} | \hat{\rho}(x)\hat{\rho}(0) | \rho^{sp} \rangle = \frac{\sin(x\lambda_F)}{\pi x} I_0 \left( i2A \sin \left( \frac{xq}{2} \right) \right), \quad (6.41a)$$

$$\lim_{t \to 0} \langle \psi_{q,A}(t) | \hat{\rho}(x)\hat{\rho}(0) | \psi_{q,A}(t) \rangle_{1ph}$$

$$= \Re \frac{1}{n} \sum_{\beta \in \mathbb{Z}, \beta \neq 0} \sum_{\beta \in \mathbb{Z}} \left\{ \frac{2n \cos(xq\beta/2)}{2} f_\beta(2q\beta t) \right\}$$

$$- \frac{\sin(x\lambda_F)}{\pi x} I_0 \left( i2A \sin \left( \frac{xq}{2} \right) \right) \left[ f_\beta(2q\beta t - x) + f_\beta(2q\beta t + x) \right], \quad (6.41b)$$

$$\lim_{t \to 0} \langle \psi_{q,A}(t) | \hat{\rho}(x)\hat{\rho}(0) | \psi_{q,A}(t) \rangle_{2ph}$$

$$= \Re \frac{1}{\sqrt{\pi}} \sum_{\beta_1 \in \mathbb{Z}, \beta_2 \in \mathbb{Z}} \sum_{\beta_1 \neq 0, \beta_2 \neq 0} e^{ixq\beta_1} \left[ f_{\beta_1}(2q\beta_1 t) f_{\beta_2}(2q\beta_2 t) \right]$$

$$- e^{ix(q\beta_2 - \beta_1)/2} f_{\beta_1}(2q\beta_1 t - x) f_{\beta_2}(2q\beta_2 t + x), \quad (6.41c)$$

where the subscripts 1ph and 2ph stand for the contributions from one and two particle-hole excitations, respectively, and we introduced the auxiliary function $f_\beta(x) = \frac{\sin(x\lambda_F)}{x} I_0[i2A \sin(qx/2)]$. In the limit $t \to 0$ the density-density profile of the ground state is recovered:

$$\lim_{t \to 0} \lim_{t \to 0} \langle \psi_{q,A}(t) | \hat{\rho}(x)\hat{\rho}(0) | \psi_{q,A}(t) \rangle = n^2 - \frac{\sin^2(x\lambda_F)}{\pi^2 x^2}. \quad (6.42)$$

In Fig. 6.4 the density-density correlator at $t = 0$ and $t = \infty$ is plotted for different values of $q$. The late-time oscillations can be understood by thinking in
6. Modelling the Bragg pulse

terms of free particles and realizing that many particles have momenta around $\pm q$ (as we will see shortly). Their wave functions oscillate in space with wave vector $q$, hence the strong oscillations in the density-density profile.

Time evolution of the momentum distribution

The momentum distribution operator was defined in Eq. (2.24) in terms of the one-body reduced density matrix. Since the Bragg pulse breaks translational invariance, it is not possible to dismiss one of the spatial integrals. The matrix elements between Bethe states are

$$\langle \mu | \hat{n}_k | \lambda \rangle = \text{FT} \left[ \langle \mu | \hat{\Psi}^\dagger(x) \hat{\Psi}(0) | \lambda \rangle \right](k) \delta_{P_\mu, P_\lambda},$$

where we used translational invariance of Bethe states and periodicity due to the ring-like geometry. The Fourier transform is defined in Appendix A.

In Ref. [54] the thermodynamic limit of matrix elements of the one-body density matrix $\hat{\Psi}^\dagger(x) \hat{\Psi}(0)$ with a countable number of excitations was computed in terms of Fredholm determinants. The results are reviewed in Appendix A.2. Following steps analogous to Ref. [54] one arrives at a convenient expression for the time evolution of the momentum distribution after an instantaneous Bragg pulse,

$$\lim_{t \to \infty} \langle \psi_{q,A}(t) | \hat{n}_k | \psi_{q,A}(t) \rangle = \text{FT} \left[ \int_{-\pi}^{\pi} \frac{dv}{2\pi} \times \ldots \right] (6.43)$$

$$\ldots \times \text{Det} \left( 1 + \sum_{\beta \in \mathbb{Z}} \frac{I_\alpha(\mu) I_\alpha(\mu+\beta)}{2\pi} e^{-it(q^2\beta^2+2q\beta\mu)+ixq\beta/2+iv\beta} K'(\lambda, \mu + q\beta) \right)$$

$$- \text{Det} \left[ 1 + \sum_{\beta \in \mathbb{Z}} \frac{I_\alpha(\mu) I_\alpha(\mu+\beta)}{2\pi} e^{-it(q^2\beta^2+2q\beta\mu)+ixq\beta/2+iv\beta} K(\lambda, \mu + q\beta) \right](k),$$

where we suppressed the arguments ($\pm iA$) of the Bessel functions, the function $\alpha(\mu)$ is defined below Eq. (6.34) and where Det denotes a Fredholm determinant. It can be computed by discretizing the space of $\lambda$ and $\mu$. The integral over $v$ imposes that only matrix elements with the same momentum for bra and ket states are taken into account.

In Fig. 6.5 the time evolution of the momentum distribution is plotted for a pulse with $q = 3\lambda_F$ and $A = 1.5$. Immediately after the Bragg pulse, at $t = 0$, the distribution has infinite peaks at $\pm 3m\lambda_F$, for $m \in \mathbb{Z}$. These are Bragg-pulse copies of the peak $\sim k^{-1/2}$ of the momentum distribution of the ground state. The peaks quickly decay and a ghost-like equilibrium is attained, again with a relaxation time of the order $\Delta t_{\text{rel}} \sim 0.1$. The ghost-like imprint of the Bragg pulse is a clear sign of nonthermalization. We would like to stress the resemblance with the momentum distribution observed in the quantum Newton’s cradle [24] and the fact that the GGE predicts this distribution.
6.4 The asymmetric Bragg pulse

The instantaneous Bragg pulse as defined in Eq. (6.2) is for a pulse symmetric in momentum space. Inspired by the experiments of Ref. [79] we define an asymmetric instantaneous Bragg pulse via the operator

$$\hat{U}^{(\text{asym.})}_B(A) = \exp \left( -\frac{iAL}{2}\hat{\beta}_q \right).$$  \hfill (6.44)

Using the combinatorial viewpoint of Section 6.2.2 it is rather straightforward to compute the overlaps for this Bragg pulse (with $q > 2\lambda_F$) acting on the Tonks-Girardeau ground state with an arbitrary Bethe state:

$$\langle \{\beta_j\}_{j=1}^N|\phi_{q,A}^{(\text{asym.})}\rangle = \prod_{j=1}^N \frac{1}{\beta_j!} \left( -\frac{iA}{2} \right)^{\beta_j}, \quad 0 \leq \beta_j \in \mathbb{Z}. \hfill (6.45)$$

For $q < 2\lambda_F$ it is possible to write down partial overlaps for clusters of two rapidities in terms of numbers of Catalan’s triangle. But they are not easily generalized to larger clusters, nor easily treated in the thermodynamic limit.

The saddle-point state for large momenta is given by

$$\rho^{sp}(\lambda) = \frac{1}{2\pi I_0(A)} \sum_{\alpha=0}^{\infty} \left[ \theta(\lambda - \alpha q + \lambda_F) - \theta(\lambda - \alpha q - \lambda_F) \right] \frac{1}{(\alpha!)^2} \left( \frac{A}{2} \right)^{2\alpha}. \hfill (6.46)$$

Using the local conserved charges on the initial state, it is possible to verify numerically the correctness of this saddle point. One can also show it is extensible to smaller momenta. Note that the odd charges are nonzero this time. For example, the momentum density on the saddle-point state is given by $\langle \rho^{sp}|\hat{P}/N|\rho^{sp}\rangle = \frac{qA}{2} \frac{I_1(A)}{I_0(A)}$. The late-time momentum distribution is like a ghost with only one arm. A precise comparison with the experimental results in Ref. [79] and thereby going beyond linear-response theory would be interesting for future research.

6.5 Bragg pulse at finite temperature

The thermal state at temperature $T$ is given by the root distribution [94]

$$\rho_T(\lambda) = \frac{1}{2\pi} \frac{1}{1 + \exp \left( \frac{\lambda^2 - h_T}{T} \right)}, \hfill (6.47)$$

which is the Fermi-Dirac distribution with chemical potential $h_T$. The chemical potential is determined by the usual normalization condition and at zero temperature $h_0 = (n\pi)^2 = \lambda_F^2$.  

133
Figure 6.5: (a) Finite-size computation using the Fermi-Bose mapping with 50 particles of the time evolution of the momentum distribution after an instantaneous Bragg pulse with $q = 3\lambda_F$, $A = 1.5$ and $n = 1$. (b) Quench action results of the same quantities. The results of the two methods show excellent agreement. The finite-size computation has a minimum grid spacing in $k$ space, causing small differences in the sharpest peaks.

Now think of applying the (symmetric) instantaneous Bragg pulse to the thermal state. Since local charges determine the saddle point for long times after the Bragg pulse and since for local charges the entanglement in clusters is not important, as was shown in Section 6.3.2, we can neglect the clustering here as well when we are interested in the saddle point. The thermodynamic overlap coefficient is then given by

$$S[\rho] = -L \int_{-\infty}^{\infty} d\lambda \rho(\lambda) \sum_{\alpha \in \mathbb{Z}} \log \left[ P(\rho_T|\lambda - \alpha q, \lambda) \right],$$

(6.48)
where \( P(\rho_T|\lambda - \alpha q, \lambda) \) is the probability that when you apply the Bragg pulse to a thermal state with temperature \( T \), a rapidity will be ‘excited’ from position \( \lambda - \alpha q \) to position \( \lambda \). The probability is the usual modified Bessel function of the first kind, multiplied by the probability of finding a rapidity at position \( \lambda - \alpha q \),

\[
P(\rho_T|\lambda - \alpha q, \lambda) = I_\alpha(-iA) \frac{\rho_T(\lambda - \alpha q)}{\rho_{T,tot}(\lambda - \alpha q)} = 2\pi I_\alpha(-iA) \rho_T(\lambda - \alpha q). \tag{6.49}
\]

This completely fixes the thermodynamic overlap coefficient. The normalized saddle point for long times after the Bragg pulse is easily obtained using quench action methods,

\[
\rho_{sp}^T(\lambda) = \frac{1}{2\pi} \sum_{\alpha \in \mathbb{Z}} \frac{\theta(\lambda - \alpha q + \lambda_F) - \theta(\lambda - \alpha q - \lambda_F)}{1 + \exp\left(\frac{\lambda(\alpha q)^2 - h_T}{T}\right)} |I_\alpha(iA)|^2. \tag{6.50}
\]

In the limit \( T \to 0 \) the saddle point in Eq. (6.33) is recovered.

### 6.6 Finite-size analysis with Fermi-Bose mapping

The well-known Fermi-Bose mapping \cite{67} says that the time-dependent wave function of the Tonks-Girardeau gas can be written as the wave function of free spinless fermions, i.e. a Slater determinant, multiplied by factor accounting for the different statistics:

\[
\chi_N(x, t) = \chi_N^{(F)}(x, t) \prod_{1 \leq l < j \leq N} \text{sgn}(x_j - x_l), \tag{6.51}
\]

with \( \chi_N^{(F)}(x, t) = \frac{1}{\sqrt{N!}} \text{det}_{i,j=1}^N [\tilde{\chi}_j(x, t)] \), where the \( \{\tilde{\chi}_j(x, t)\}_{j=1}^N \) are orthonormal single-particle wave functions. Because the fermions are free, each \( \tilde{\chi}_j(x, t) \) obeys the time-dependent Schrödinger equation for a single particle (remember we set \( \hbar = 2m = 1 \)),

\[
i \frac{\partial \tilde{\chi}_j(x, t)}{\partial t} = \left[ -\frac{\partial^2}{\partial x^2} + V(x, t) \right] \tilde{\chi}_j(x, t), \tag{6.52}
\]

valid for any external potential \( V(x, t) \). In Ref. \cite{213} a computationally efficient expression for the bosonic one-body reduced density matrix, \( \hat{\Psi}^\dagger(x) \hat{\Psi}(y) \), in terms of the fermionic single-particle wave functions was derived. It is given by

\[
\langle \chi_N(x, t)|\hat{\Psi}^\dagger(x)\hat{\Psi}(y)|\chi_N(x, t) \rangle = \sum_{l,j=1}^N \tilde{\chi}_l^*(x, t) A_{lj}(x, y, t) \tilde{\chi}_j(x, t), \tag{6.53a}
\]

with \( A \) an \((N \times N)\)-matrix given by \( A(x, y, t) = (P^{-1})^T \text{det}[P] \), and \( P \) also an \((N \times N)\)-matrix,

\[
P_{lj}(x, y, t) = \delta_{lj} - 2 \int_x^y dx' \tilde{\chi}_l^*(x', t) \tilde{\chi}_j(x', t). \tag{6.53b}
\]
6. Modelling the Bragg pulse

From the one-body reduced density matrix the bosonic density (equal to the fermionic density) and the bosonic momentum distribution can be computed easily [see Eq. (2.24)]. The inverse and determinant of only \((N \times N)\)-matrices must be computed, and also the construction of \(P\) can be made efficient. In Refs 35, 214 the Fermi-Bose mapping was applied to the release of the TG gas from a harmonic trap.

Applying this method to the instantaneous Bragg pulse is rather easy. Starting with a long pulse, defined in Eq. (6.1), one has a potential \(V(x, t) = V_0 \cos(qx)\) for \(0 \leq t \leq T_0\) and zero otherwise. In the limit \(T_0 \to 0\) with \(A = V_0 T_0\) fixed, the potential term will dominate over the kinetic term and the Schrödinger equation is approximately solved by \(\tilde{\chi}_j(x, t) \approx e^{-iV_0 T_0 \cos(qx)} \tilde{\chi}_j(x, 0)\) during the pulse, with \(\tilde{\chi}_j(x, 0)\) the wave function of the \(j\)-th particle before the pulse. In the strict limit \(T_0 \to 0\) this becomes exact and the wave function instantaneously jumps at \(t = 0\). For this Bragg pulse acting on the ground state on a ring one immediately finds

\[
\tilde{\chi}_j(x, t) = \frac{1}{\sqrt{L}} \sum_{n=-\infty}^{\infty} (-1)^n I_n(iA) e^{-i(k_j + nq)x - i(k_j + nq)^2t},
\]

valid for all \(t \geq 0\), where we used Eq. (A.8) and the \(\{k_j\}_{j=1}^{N}\) are the momenta of the \(N\)-particle Fermi sea. In Fig. 6.3 the postquench density profile is computed from these single-particle wave functions and Eq. (6.53) for 50 particles. It is also compared with the exact result from the quench action analysis and found to agree extremely well for all times. The tiny discrepancy is a finite-size effect. Similarly, the momentum distribution function in Fig. 6.5 agrees excellently with the quench action result. We conclude that the large-system-size dynamics of the instantaneous Bragg pulse is well captured by a Fermi-Bose mapping for 50 particles.

We can perform the same analysis in a harmonic trap, which is an approximation of the longitudinal potential in the quantum Newton’s cradle [24]. This adds a term \(V_{\text{trap}} = \frac{1}{2} m \omega^2 x^2\) to the Hamiltonian, where for clarity we keep \(m\) as a parameter. The ground state of \(N\) free fermions in a trap is described by the normalized single-particle wave functions

\[
\phi_j(x) = \frac{1}{\sqrt{2^j j!}} \left( \frac{m \omega}{\pi} \right)^{1/4} e^{-m \omega x^2} H_j(\sqrt{m \omega} x),
\]

for \(j = 1, \ldots, N\) and where \(H_j\) are Hermite polynomials. A lengthy calculation leads to the following single-particle wave functions after an instantaneous Bragg pulse [6],

\[
\tilde{\chi}_j(x, t) = \sum_{n=-\infty}^{\infty} (-1)^n I_n(iA) e^{-i n q \cos(\omega t)\left[ x + \frac{n q}{2 m \omega} \sin(\omega t) \right]} \times \ldots
\]

\[
\ldots \times \phi_j\left( x + \frac{n q}{m \omega} \sin(\omega t) \right) e^{-i \omega (j + \frac{1}{2}) t},
\]
6.7. Analysis of results

for \( j = 1, \ldots, N \). In Fig. 6.6 the time evolution of the momentum distribution and density after such a pulse are plotted for 50 particles. As in the quantum Newton’s cradle experiment, the oscillation of the momentum distribution due to the trap is clearly visible. Notice that in the density plot not only the clouds with momenta \( \pm q \) are visible, but also those with \( \pm 2q \). From Eq. (6.56) it is obvious that this oscillation is eternal and after each cycle the initial state is completely recovered. Because of anharmonicity in experimental traps (of the order of 8% in Ref. [24]) the oscillation modes quickly dephase and no such recovery is visible. It is an interesting question for future research whether this exact recovery is also present in a truly interacting model, i.e. for finite \( c \).

Figure 6.6: Left panel: Time evolution of the density profile in a harmonic trap (with \( \omega = 0.2 \)) after a Bragg pulse with \( q = 2\pi \) and \( A = 1.5 \). Note that the cloud of atoms around momenta \( \pm 2q \) is also visible. Right panel: Momentum distribution of the same pulse. Note the very small slice of rapid relaxation at the beginning of each period, where the momentum distribution is sharply peaked (see Fig. 6.8 for details). All results were obtained via the Fermi-Bose mapping with 50 particles.

6.7 Analysis of results

Taking a closer look at the results one sees that at very short times (of the order \( \Delta t \sim 0.1 \)) a rapid relaxation of both density and momentum distribution takes place, after which the much slower oscillation in the trap becomes visible. The rapid relaxation is clearly visible in Fig. 6.6 at the beginning of each oscillation period, where one can discern very sharp peaks in the momentum distribution. In Fig. 6.7 we zoom in on this relaxation for the density and compare it with the QA result on a ring. For the latter we used a local-density approximation to mimick the initial density profile given by \( n_0(x) = \frac{1}{\pi} \sqrt{2N\omega - \omega^2 x^2} \), see Ref. [35]. It
turns out that the densities agree well during the rapid relaxation. Discrepancies originate at the edges of the atom cloud and propagate inwards. This makes sense, as the local-density approximation is worse at the rather steep profile at the edges.

A similar conclusion can be drawn from the evolution of the momentum distribution for early times, as is done in Fig. 6.8. There the early-time momentum distributions in a trap are compared with the same quantities on a ring, both computed by means of the Fermi-Bose mapping for 50 particles. The discrepancies for small momenta \( k \) are a sign of the trap geometry, which becomes visible at larger length scales. Otherwise, for larger momenta the two relaxations are in excellent agreement.

![Figure 6.7](image)

Figure 6.7: **Left panel:** Time evolution of the density profile \( \lim_{t \to \infty} \langle \rho_{q,A}(t) | \rho(x) | \psi_{q,A}(t) \rangle \) after an instantaneous Bragg pulse in a harmonic trap with \( q = \lambda_F, A = 1.5 \) and \( \omega = 0.2 \), computed by means of the Fermi-Bose mapping with \( N = 50 \) particles. **Middle panel:** Quench action result of the same quantities using a local-density approximation with initial density \( n_0(x) = \frac{1}{\pi} \sqrt{2N} \omega \omega^2 x^2 \). **Right panel:** Absolute value of the difference between left and middle panel. Discrepancies originate at the edges of the atom cloud and propagate inwards. This makes sense, as the local-density approximation is worse at the rather steep profile at the edges.

We conclude that the rapid relaxation in the trap is well described by the physics on a ring. In particular, much before the trap-related oscillations commence a temporary equilibrium is attained. This equilibrium is well described by the QA saddle-point state for a pulse on a ring and also in agreement with the GGE prediction on a ring, as we have seen in Section 6.3.1. We observe a separation of relaxation timescales: a rapid relaxation that is unrelated to the trap and governed by the GGE for the underlying integrable model, and a slow oscillation in the trap without thermalization. A rough estimation gives a rapid relaxation within \( 1.6 \mu s \) in the quantum Newton’s cradle experiment. This is much slower than the oscillations in the trap (\( \sim 10 \, \text{ms} \)) and likely very hard to observe.

Since the trap breaks integrability, this temporary equilibrium could be inter-
6.7. Analysis of results

Interpreted as a pre-thermalization plateau [76]. However, an important difference is that the system does not end up in a thermal equilibrium. Note that, in order to keep the average density in the trap constant, we scaled $\omega$ with $N^{-1}$. More particles would thus imply a larger separation of timescales, as the rapid relaxation has the same timescale of order $\Delta t_{rel} \sim 0.1$ in the thermodynamic limit.

For the instantaneous Bragg pulse on a ring we managed to compute all important observables by means of the QA approach. We showed that a finite-size calculation using the Fermi-Bose mapping accurately describes the dynamics after such a pulse and extended this analysis to the harmonic trap. Our main observation is a separation of timescales: a rapid relaxation and a slow oscillation.

To make contact with experiment two obvious extensions should be made. First, the Fermi-Bose mapping can be applied to pulses of a finite duration. The single-particle eigenstates of a cosine trap on a ring are given by Mathieu functions (see e.g. Refs [215, 216]) and further implementation is straightforward [6]. An interesting research question is how the postquench momentum distribution is related to the details of the quench protocol (one or two pulses; duration; intensity of laser; asymmetric Bragg pulses; temperature). An analysis of the quench protocol for a two-state system was given in Ref. [209]. With our methods this can be studied more thoroughly and with a direct link to experiments.

Note that the pulse duration in the quantum Newton’s cradle experiment is much longer than the rapid relaxation we observe. This could imply that the instantaneous Bragg pulse is not a good approximation of the pulses used in experiments, as the dynamics is much faster than the experimental pulse duration. Further investigation is required on this point. A combination of a long pulse in a harmonic trap is also feasible, although the single-particle eigenstates must be approximated numerically.

Second, including interactions by working at finite $c$ is another interesting extension. One question is whether the exact recovery after an oscillation period in a trap is destroyed when interactions are included. More generally, the reason for the absence of thermalization in an integrability-breaking trap remains the fundamental question to be answered. The techniques developed in this chapter, possibly in combination with the findings of Ref. [215], could lead to an opportunity to address this question. For more details about these issues, we refer to Ref. [6].
Figure 6.8: Time evolution of the momentum distribution after a Bragg pulse with $q = 2\pi$ and $A = 1.5$, both in a harmonic trap (with $\omega = 0.2$) and on a ring (with $n = 1$). All results were obtained with the Fermi-Bose mapping for 50 particles. For large momenta $k$ the rapid-relaxation dynamics in a trap and on a ring agree very well. Discrepancies for small $k$ can be attributed to the trap geometry.