The quench action approach to out-of-equilibrium quantum integrable models

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Conventions, identities and known results

A.1 Fourier transforms

Fourier transform for the Lieb-Liniger model

The Fourier conventions for the Lieb-Liniger model are

$$f_k = \mathrm{FT}\big[f\big](k) = \frac{1}{L}\int_0^L dx\, e^{ikx} f(x)\,, \tag{A.1a}$$

$$f(x) = \mathrm{FT}^{-1}\big[f_k\big](x) = \sum_k f_k\, e^{-ikx}\,, \tag{A.1b}$$

where k is quantized such that $k = \frac{2\pi}{L}n$, $n \in \mathbb{Z}$. Other useful relations are

$$\int_0^L \mathrm{d}x\, e^{\frac{2\pi i x}{L}(n'-n)} = \begin{cases} 0 & \text{if } n' \neq n \\ L & \text{if } n' = n \end{cases}\,, \text{ with } n, n' \in \mathbb{Z}\,, \tag{A.2a}$$

$$\sum_{n=-\infty}^{\infty} e^{2\pi i n\left(\frac{x-y}{L}\right)} = \sum_{n\in\mathbb{Z}} \delta\left(\frac{x-y}{L} - n\right) = L\,\delta(x-y)\,. \tag{A.2b}$$

Fourier transform for the spin-1/2 XXZ model

The Fourier conventions for the spin-1/2 XXZ model are

$$\hat{f}(k) = \mathrm{FT}\big[f\big](k) = \int_{-\pi/2}^{\pi/2} \mathrm{d}\lambda e^{2ik\lambda} f(\lambda)\,, \quad k \in \mathbb{Z}\,, \tag{A.3a}$$

$$f(\lambda) = \mathrm{FT}^{-1}\big[\hat{f}\,\big](\lambda) = \frac{1}{\pi}\sum_{k\in\mathbb{Z}} e^{-2ik\lambda}\hat{f}(k)\,, \quad \lambda \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2})\,. \tag{A.3b}$$

Fourier transform for the spin-1/2 XXX model

The Fourier conventions for the spin-1/2 XXX model are

$$\hat{f}(k) = \mathrm{FT}\big[f\big](k) = \int_{-\infty}^{\infty} \mathrm{d}\lambda\, e^{ik\lambda} f(\lambda)\,, \quad k \in \mathbb{R}\,, \tag{A.4a}$$

$$f(\lambda) = \mathrm{FT}^{-1}\big[\hat{f}\,\big](\lambda) = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} e^{-ik\lambda} \hat{f}(k)\,, \quad \lambda \in \mathbb{R}\,. \tag{A.4b}$$

A.2 Matrix elements of the momentum distribution for the TG gas

It is well known [12, 220] that the finite-size matrix elements of the one-body density matrix for the Tonks-Girardeau gas are given by

$$\frac{1}{L^N}\langle \boldsymbol{\mu}|\hat{\Psi}^\dagger(x)\hat{\Psi}(0)|\boldsymbol{\lambda}\rangle = \frac{1}{L}\mathrm{det}_N(V_1+V_2) - \frac{1}{L}\mathrm{det}_N(V_1)\ , \tag{A.5a}$$

where the matrices are defined by

$$(V_1)_{j,k} = \frac{1}{iL(\lambda_j-\mu_k)}\left(e^{iL(\lambda_j-\mu_k)} + 1 - 2e^{ix(\lambda_j-\mu_k)}\right)\ , \tag{A.5b}$$

$$(V_2)_{j,k} = \ e^{-ix\mu_j}\ , \tag{A.5c}$$

the latter being a rank-one matrix.

In relation to the content of Chapter 6, the momentum distribution of a state that is created with a Bragg pulse should be independent from the spatial modulation of the Bragg pulse. One can check that this is indeed the case by invoking Graf's addition formula (A.12) for infinite sums of a product of two modified Bessel functions.

In Ref. [54] the thermodynamic limit of matrix elements with a countable number of excitations was computed by decomposing the elements in a Fredholm determinant and a finite-size determinant which takes care of the excitations. The result is

$$\langle\rho|\hat{\Psi}^\dagger(x)\hat{\Psi}(0)|\rho,\{\lambda_j^h\to\lambda_j^p\}_{j=1}^m\rangle = \frac{1}{N^m} e^{i\frac{x}{2}\sum_{j=1}^m(\lambda_j^p-\lambda_j^h)} \times \dots \tag{A.6a}$$
$$\dots \times \left\{\mathrm{Det}(1+K'\rho)\det_m\left[W'\big(\lambda_i^h,\lambda_j^p\big)\right] - \mathrm{Det}(1+K\rho)\det_m\left[W\big(\lambda_i^h,\lambda_j^p\big)\right]\right\}.$$

The Fredholm determinants are denoted by Det, where $(K\rho)(\lambda,\mu) = K(\lambda,\mu)\rho(\mu)$ and the kernels are given by ($n = N/L$)

$$K'(\lambda,\mu) = K(\lambda,\mu) + ne^{-i\frac{x}{2}(\lambda+\mu)}\,, \qquad K(\lambda,\mu) = -4n\frac{\sin\left(\frac{x}{2}(\lambda-\mu)\right)}{\lambda-\mu}\,. \tag{A.6b}$$

The function W is determined via the integral equations

$$W(\lambda,\mu) + \int_{-\infty}^{\infty} d\nu\, K(\lambda,\nu)\rho(\nu)W(\nu,\mu) = K(\lambda,\mu)\,. \tag{A.6c}$$

The other kernel W' is defined similarly, with K replaced by K'. As expected the matrix elements for m particle-hole excitations scale like N^{-m}. The dimension of the operator is 1/length. In the kernel K' the second term has this dimension and if one takes the difference between the primed and unprimed determinants most of the terms cancel and only the terms with dimension 1/length survive. In the finite-size situation this is obvious, see Eqs (A.5).

A.3 Identities for Bessel functions

The Bessel functions of the first kind $J_\nu(z)$ of order ν and argument z are solutions to Bessel's differential equation and given by the convergent series

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{1}{2}z)^{\nu+2k}}{k!\,\Gamma(\nu+k+1)}\,, \tag{A.7}$$

where Γ is Euler's gamma function. The modified Bessel function of the first kind is defined by $I_\nu(z) = i^{-\nu} J_\nu(iz)$. In this work the order will always be an integer. In that case we have the useful identies $I_n(-z) = (-1)^n I_n(z)$, $\overline{I_n(z)} = I_n(\overline{z})$ and $I_{-n}(z) = I_n(z)$. Another useful identity is

$$\sum_{n\in\mathbb{Z}} I_n(z) e^{i\theta n} = e^{z\cos(\theta)}\,, \tag{A.8}$$

for any $z \in \mathbb{C}$ and $\theta \in \mathbb{R}$. In Ref. [211] the following identity is proven:

$$B_{k,s}(z) = i^{-s} \sum_{n\in\mathbb{Z}} n^k I_n(iz) I_{n-s}(-iz)\,, \tag{A.9}$$

where the coefficients are given by the recursive relation

$$B_{k+1,s}(z) = s\,B_{k,s}(z) + \frac{z}{2}\left(B_{k,s+1}(z) + B_{k,s-1}(z)\right)\,, \qquad B_{0,s}(z) = \delta_{s,0}\,, \tag{A.10}$$

and which holds for any integers $k \geq 0$, $s \in \mathbb{Z}$ and complex number $z \in \mathbb{C}$. For us the following coefficients are relevant:

$$B_{k,0}(z) = 0\,, \qquad \text{if } k \text{ odd}\,, \tag{A.11a}$$

$$B_{0,0}(z) = 1\,, \tag{A.11b}$$

$$B_{2,0}(z) = \frac{1}{2}z^2\,, \tag{A.11c}$$

$$B_{4,0}(z) = \frac{3}{8}z^4 + \frac{1}{2}z^2\,, \tag{A.11d}$$

$$B_{6,0}(z) = \frac{5}{16}z^6 + \frac{15}{8}z^4 + \frac{1}{2}z^2\,. \tag{A.11e}$$

Graf's addition formula

Graf's addition formula for Bessel functions of the first kind with integer order is given by (see Section 11.3 of Ref. [212])

$$J_n\left(\sqrt{z_1^2 + z_2^2 - 2z_1 z_2 \cos\theta}\right)\left(\frac{z_1 - z_2 e^{-i\theta}}{z_1 - z_2 e^{i\theta}}\right)^{n/2} = \sum_{m\in\mathbb{Z}} J_{n+m}(z_1) J_m(z_2) e^{im\theta}\,. \tag{A.12}$$