The quench action approach to out-of-equilibrium quantum integrable models

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The overlaps of the Néel-to-XXZ quench

Strictly speaking, a derivation of the overlaps between an initial state and the Bethe eigenstates of the postquench Hamiltonian are not part of the quench action approach. But overlap formulas from which smooth overlap coefficients can be derived in the thermodynamic limit, do form an essential condition for applicability of the quench action method. Here, we briefly summarize how the basic building blocks of the Néel-to-XXZ quench were derived in Refs [2, 3] and how from this the overlap formula for the BEC-to-LL quench was proven in Ref. [158].

B.1 Derivation of the overlaps

Of course, the overlaps between a spin-chain state built from well-defined spin projections at each site and Bethe states of the XXZ Hamiltonian are known via the form of the wave function, Eq. (2.54b). For example (take N even),

$$\begin{aligned}\langle\psi_0|\boldsymbol{\lambda}\rangle &= \frac{1}{\sqrt{2}}\sum_{Q\in S_M}(-1)^{[Q]}\exp\left\{-i\sum_{j=1}^{M}2j\,p(\lambda_{Q_j}) - \frac{i}{2}\sum_{\substack{j,k=1\\k>j}}^{M}\theta_2(\lambda_{Q_k}-\lambda_{Q_j})\right\}\\ &\quad+\frac{1}{\sqrt{2}}\sum_{Q\in S_M}(-1)^{[Q]}\exp\left\{-i\sum_{j=1}^{M}(2j-1)\,p(\lambda_{Q_j}) - \frac{i}{2}\sum_{\substack{j,k=1\\k>j}}^{M}\theta_2(\lambda_{Q_k}-\lambda_{Q_j})\right\}\end{aligned} \tag{B.1}$$

is the unnormalized overlap between the Bethe state wave function and the zero-momentum Néel state. It proves convenient to first consider "off-shell overlaps", for which the $\lambda_j \in \boldsymbol{\lambda} = \{\lambda_j\}_{j=1}^{M}$ in the wave function do not obey Bethe equations. They are arbitrary complex numbers. The number of terms in this expression grows exponentially in system size, which is a real bottleneck for a treatment in the thermodynamic limit.

In Refs [175, 176] a determinant expression for these off-shell overlaps was found, given by

$$\langle\psi_0|\boldsymbol{\lambda}\rangle = \sqrt{2}\left[\prod_{j=1}^{M}\frac{s_{\lambda_j,+\eta/2}}{s_{2\lambda_j,0}}\frac{s^M_{\lambda_j,-\eta/2}}{s^M_{\lambda_j,+\eta/2}}\right]\left[\prod_{j>k=1}^{M}\frac{s_{\lambda_j+\lambda_k,\eta}}{s_{\lambda_j+\lambda_k,0}}\right]\det{}_M(\delta_{jk}+U_{jk})\,, \tag{B.2a}$$

$$U_{jk} = \frac{s_{2\lambda_k,\eta}s_{2\lambda_k,0}}{s_{\lambda_j+\lambda_k,0}s_{\lambda_j-\lambda_k,\eta}}\left[\prod_{\substack{l=1\\l\neq k}}^{M}\frac{s_{\lambda_k+\lambda_l,0}}{s_{\lambda_k-\lambda_l,0}}\right]\left[\prod_{l=1}^{M}\frac{s_{\lambda_k-\lambda_l,-\eta}}{s_{\lambda_k+\lambda_l,+\eta}}\right]\left(\frac{s_{\lambda_k,+\eta/2}}{s_{\lambda_k,-\eta/2}}\right)^{2M}\,, \tag{B.2b}$$

where we used the short-hand notation $s_{\alpha,\beta} = \sin(\alpha + i\beta)$ and U is an $(M \times M)$-matrix. This formula is valid for any M complex numbers $\boldsymbol{\lambda}$, as long as $\lambda_j \neq \lambda_k$ for any pair j, k (due to singularities in the prefactor and zeroes in the determinant). The latter is unfortunate, as we would like to focus on parity-invariant Bethe states. The reason is that odd charges have vanishing expectation value on the zero-momentum Néel state [126], while their eigenvalues on parity-invariant Bethe states are also trivially zero [see Eq. (2.72)].

Now assume M even. To perform the limit to parity-invariant off-shell states, set $\lambda_j = \tilde{\lambda}_j + \epsilon_j$ for $j = 1,\ldots,M/2$ and $\lambda_j = -\tilde{\lambda}_{j-M/2} + \epsilon_{j-M/2}$ for $j = M/2+1,\ldots,M$. Here, the parameters $\tilde{\lambda}_j$, $j = 1,\ldots,M/2$, are arbitrary complex numbers. Then the limit $\{\epsilon_j\}_{j=1}^{M/2}$ is taken in a controlled way. This means that the columns and rows of the matrix U are reordered such that (2×2)-blocks of pseudo-parity-invariant couples $(\lambda_j,\, \lambda_{j+M/2})$ are formed. The components of the matrix are expanded in terms of parameters $\delta_k = s_{2\epsilon_k,0}/s_{0,\eta}$ for $k = 1,\ldots,M/2$ that scale to zero in the parity-invariant limit. Only the terms that cancel in singularities in the prefactor are kept and the result is an overlap formula for off-shell parity-invariant Bethe states:

$$\begin{aligned}\langle\psi_0|\tilde{\boldsymbol{\lambda}}\rangle =& \langle\psi_0|\{\pm\tilde{\lambda}_j\}_{j=1}^{M/2}\rangle \\ &= \langle\psi_0|\{\tilde{\lambda}_j+\epsilon_j\}_{j=1}^{M/2}\cup\{-\tilde{\lambda}_j+\epsilon_j\}_{j=1}^{M/2}\rangle\Big|_{\{\epsilon_j\to 0\}_{j=1}^{M/2}} \\ &= \left[\gamma\det{}_{M/2}(G^+_{jk}) + \mathcal{O}\left(\{\epsilon_j\}_{j=1}^{M/2}\right)\right]_{\{\epsilon_j\to 0\}_{j=1}^{M/2}} = \gamma\det{}_{M/2}(G^+_{jk})\,,\end{aligned} \tag{B.3a}$$

where the prefactor is given by

$$\gamma = \sqrt{2}\left[\prod_{j=1}^{M/2}\frac{s_{\tilde{\lambda}_j,+\eta/2}s_{\tilde{\lambda}_j,-\eta/2}}{s^2_{2\tilde{\lambda}_j,0}}\right]\left[\prod_{\substack{j>k=1\\ \sigma=\pm}}^{M/2}\frac{s_{\tilde{\lambda}_j+\sigma\tilde{\lambda}_k,+\eta}s_{\tilde{\lambda}_j+\sigma\tilde{\lambda}_k,-\eta}}{s^2_{\tilde{\lambda}_j+\sigma\tilde{\lambda}_k,0}}\right]\,, \tag{B.3b}$$

and the matrix G^+_{jk} reads

$$G^+_{jk} = \delta_{jk}\left(N s_{0,\eta} K_{\eta/2}(\tilde{\lambda}_j) - \sum_{l=1}^{M/2} s_{0,\eta} K^+_\eta(\tilde{\lambda}_j, \tilde{\lambda}_l)\right) + s_{0,\eta} K^+_\eta(\tilde{\lambda}_j, \tilde{\lambda}_k) \tag{B.3c}$$

$$+ \delta_{jk} \frac{s_{2\tilde{\lambda}_j,+\eta}\mathfrak{A}_j + s_{2\tilde{\lambda}_j,-\eta}\bar{\mathfrak{A}}_j}{s_{2\tilde{\lambda}_j,0}} + (1-\delta_{jk}) f_{jk}\,, \qquad j,k = 1,\ldots,M/2,$$

$$f_{jk} = \mathfrak{A}_k\left(\frac{s_{2\tilde{\lambda}_j,+\eta}s_{0,\eta}}{s_{\tilde{\lambda}_j+\tilde{\lambda}_k,0}s_{\tilde{\lambda}_j-\tilde{\lambda}_k,+\eta}} - \frac{s_{2\tilde{\lambda}_j,-\eta}s_{0,\eta}}{s_{\tilde{\lambda}_j-\tilde{\lambda}_k,0}s_{\tilde{\lambda}_j+\tilde{\lambda}_k,-\eta}}\right) \tag{B.3d}$$

$$- \bar{\mathfrak{A}}_j\left(\frac{s_{2\tilde{\lambda}_j,-\eta}s_{0,\eta}}{s_{\tilde{\lambda}_j-\tilde{\lambda}_k,0}s_{\tilde{\lambda}_j+\tilde{\lambda}_k,-\eta}} + \frac{s_{2\tilde{\lambda}_j,-\eta}s_{0,\eta}}{s_{\tilde{\lambda}_j+\tilde{\lambda}_k,0}s_{\tilde{\lambda}_j-\tilde{\lambda}_k,-\eta}}\right)$$

$$+ \mathfrak{A}_k\bar{\mathfrak{A}}_j \frac{s_{2\tilde{\lambda}_j,-\eta}s_{0,\eta}}{s_{\tilde{\lambda}_j-\tilde{\lambda}_k,0}s_{\tilde{\lambda}_j+\tilde{\lambda}_k,-\eta}}\,. \tag{B.3e}$$

Here, $K^+_\eta(\lambda,\mu) = K_\eta(\lambda-\mu) + K_\eta(\lambda+\mu)$, $K_\eta(\lambda) = \frac{s_{0,2\eta}}{s_{\lambda,+\eta}s_{\lambda,-\eta}}$, $\mathfrak{A}_k = 1 + \mathfrak{a}_k$ and $\bar{\mathfrak{A}}_k = 1 + \mathfrak{a}_k^{-1}$, with

$$\mathfrak{a}_k = \left[\prod_{\substack{l=1\\ \sigma=\pm}}^{M/2} \frac{s_{\tilde{\lambda}_k-\sigma\tilde{\lambda}_l,-\eta}}{s_{\tilde{\lambda}_k-\sigma\tilde{\lambda}_l,+\eta}}\right]\left(\frac{s_{\tilde{\lambda}_k,+\eta/2}}{s_{\tilde{\lambda}_k,-\eta/2}}\right)^{2M}\,. \tag{B.4}$$

The reduced overlap formula depends on only $M/2$ complex numbers $\{\tilde{\lambda}_j\}_{j=1}^{M/2}$ and the matrix is half the originial size. Imposing the Bethe equations is equivalent to setting $\mathfrak{A}_k = \bar{\mathfrak{A}}_k = 0$ for $k = 1,\ldots,M/2$. Many terms in the off-shell overlap cancel and using the Gaudin norm formula (2.59) one arrives at the normalized on-shell overlap formula for the Néel-to-XXZ quench, Eqs (5.20) in Section 5. Note that the additional factors $s_{0,\eta}$ in G^+_{jk}, $1 \le j,k \le M/2$, cancel against the prefactor $(s^M_{0,\eta})^{1/2}$ in the norm formula.

Taking the scaling limit to the isotropic point and taking a finite number of rapidities at infinity into account, one also arrives at the overlap formula (5.101) for the Néel-to-XXX quench.

B.2 Scaling limit to Lieb-Liniger model

Working in the spin-1/2 XXZ model, the scaling limit to the Lieb-Liniger model is defined by [11, 221, 222, 165]

$$\eta = i\pi - i\epsilon\,, \qquad N = cL/\epsilon^2\,, \qquad \lambda_j \to \epsilon\lambda_j/c\,, \qquad \epsilon \to 0\,. \tag{B.5}$$

Here L is the Lieb-Liniger system size and c the interaction parameter. It is a limit $\Delta \downarrow -1$ and simultaneously towards an infinite spin chain, where the number of down spins becomes the number of bosons in the Lieb-Liniger model, N_{LL}. The Néel state scales to a state with an infinite number of bosons. To obtain a state in the XXZ model that scales to the BEC state, a q-raised Néel state is constructed by applying global $U_q(s\ell_2)$ spin raising operators to the Néel state: $|\psi_N^{(n)}\rangle = (S_q^+ \tilde{S}_q^+)^n |\psi_N\rangle$. Here $|\psi_N\rangle$ is the pure Néel state of system size N and the global $U_q(s\ell_2)$ operators are

$$S_q^+ = \lim_{\lambda\to-\infty}\left(\frac{q^{-N/2}\sinh(\lambda)C(\lambda)}{\sinh(\eta)}\right), \quad \tilde{S}_q^+ = \lim_{\lambda\to-\infty}\left(\frac{q^{+N/2}\sinh(\lambda)C(\lambda)}{\sinh(\eta)}\right), \tag{B.6}$$

where $q = e^\eta$ and C is the usual annihilation operator in the Algebraic Bethe Ansatz [12]. The q-raised Néel state $|\psi_N^{(n)}\rangle$ has magnetization $2n$ and a total number of $2m = N/2 - 2n$ down spins. In the scaling limit the raising operators scale to staggered spin operators, $\sum_{j=1}^N (-1)^j S_j^\pm$. Therefore, these operators act on the pure Néel state as if they are raising/lowering operators, since half of the terms act trivially. So, $|\psi_N^{(n)}\rangle$ scales to a state with $2m$ uniformly-distributed down spins, which corresponds to the state with $N_{LL} = 2m$ bosons and the least spatial structure: the BEC state.

To get the overlaps between the q-raised Néel state and a parity-invariant Bethe state with $2m$ rapidities, one uses the original off-shell overlaps (B.3) and the fact that the operators S_q^+, $\tilde{S}_q^+$ send rapidities to infinity, see Eq. (B.6):

$$\langle\psi_N^{(n)}|\{\pm\lambda_j\}_{j=1}^m\rangle = \lim_{\{\mu_j\to\infty\}_{j=1}^n}(-1)^n\prod_{j=1}^n\frac{\sinh^2(\mu_j)}{\sinh^2(\eta)}\langle\psi_N|\{\pm\lambda_j\}_{j=1}^m\cup\{\pm\mu_j\}_{j=1}^n\rangle. \tag{B.7}$$

Putting the rapidities on shell leads to a normalized overlap:

$$\frac{\langle\psi_N^{(n)}|\{\pm\lambda_j\}_{j=1}^m\rangle}{\|\psi_N^{(n)}\|\|\{\pm\lambda_j\}_{j=1}^m\|} = \frac{[2n]_q!}{\|\psi_N^{(n)}\|}\left[\prod_{j=1}^m\frac{\sqrt{\tanh(\lambda_j+\frac{\eta}{2})\tanh(\lambda_j-\frac{\eta}{2})}}{2\sinh(2\lambda_j)}\right]\sqrt{\frac{\det_m(\hat{G}_{jk}^+)}{\det_m(\hat{G}_{jk}^-)}}, \tag{B.8a}$$

where $\hat{G}_{jk}^\pm$ are defined in Eq. (5.101b), $\|\{\pm\lambda_j\}_{j=1}^m\|$ is the usual Gaudin norm and

$$\|\psi_N^{(n)}\| = (2n)!\sqrt{\binom{N/2}{2n}}, \qquad [2n]_q! = \prod_{j=1}^{2n}\frac{q^j - q^{-j}}{q - q^{-1}}\,. \tag{B.8b}$$

The scaling limit of Eq. (B.8a) to the Lieb-Liniger model finally leads to the overlaps in Eq. (4.3a), where one should take into account that there are $2^{N_{LL}}$ states like the q-raised Néel state that all scale to the BEC state and there is a factor $2^{-N_{LL}/2}$ in norm difference. For details, see [158].

In Ref. [3] similar formulas we derived for M and N_{LL} odd, and it was proven that non-parity-invariant Bethe states have zero overlap with both the zero-momentum Néel state and the BEC state. The second observation is crucial for the quench action analysis and not automatically guaranteed by the vanishing expectation values of the odd local charges.