On cuspidal unipotent representations

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Chapter 2

Preliminaries

2.1 Local $\gamma$ factors

To tell the story, I will start with a fundamental example: the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{\text{prime } p} \frac{1}{1 - p^{-s}}$$

with $s \in \mathbb{C}$ and $\text{Re}(s) > 1$. It is a classical result that $\zeta(s)$ can be extended to be a meromorphic function on the whole complex plane, with a simple pole at $s = 1$, with residue 1.

We define $\xi(s) = \Gamma_{\mathbb{R}}(s)\zeta(s)$ with $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$, where the $\Gamma$-function is given by

$$\Gamma(s) = \int_{0}^{\infty} e^{-t}t^{s-1}dt.$$

The $\Gamma$ function is defined for $\text{Re}(s) > 1$ and extended to a meromorphic function on the whole $\mathbb{C}$ by analytic continuation. It is also a classical result that $\zeta(s)$ obeys the following remarkable functional equation:

$$\xi(s) = \xi(1 - s).$$

Now comes the question: What is the meaning of the factor $\Gamma_{\mathbb{R}}(s)$?

One can find a beautiful answer to this question in Tate’s famous thesis. Recall the Fourier transform on $\mathbb{R}$ is defined as $\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i x \xi} dx$ (where $dx$ is the usual Lebesgue measure on $\mathbb{R}$). Then the Gaussian function $e^{-\pi x^2}$, $x \in \mathbb{R}$ is equal
to its own Fourier transform. Tate pointed out that the factor $\Gamma_{\mathbb{R}}(s)$ comes from the Mellin transform of the Gaussian function:

$$\int_{\mathbb{R}^\times} e^{-\pi x^2} |x|^s \frac{dx}{|x|} = \pi^{-s/2} \Gamma(s/2),$$

(2.2)

where $|x|^{-1} \, dx$ is the Haar measure on $\mathbb{R}^\times$.

Tate provided us a more satisfactory picture of the theory by treating the $p$-adic analogue. Write $\mathbb{Q}_v$ for the $v$-adic completion of the field of rational numbers $\mathbb{Q}$ at a place $v \in \{\text{Prime numbers}\} \cup \{\infty\}$. Here $\mathbb{Q}_\infty = \mathbb{R}$. Tate considered the Mellin transform (also called zeta integral) in every local field $\mathbb{Q}_v$:

$$\zeta(f, s) = \int_{\mathbb{Q}_v^\times} f(x)|x|^s \, d^\times x, \text{ Re}(s) > 0,$$

(2.3)

for those functions $f$ having good analytic properties (e.g. absolutely integrable). Here, $d^\times x$ is the Haar measure on the multiplicative group $\mathbb{Q}_v^\times$, and $| \cdot |_v$ is the absolute value, normalised by $d(b x) = |b|_v \, dx$. (The multiplicative Haar measure on $\mathbb{R}^\times$ and on $\mathbb{Q}_p$ are $|x|^{-1} \, dx$ and $(1 - p^{-1})^{-1}|x|_p^{-1} \, dx$, respectively, where $dx$ is the translation-invariant Haar measure on the additive group $(\mathbb{Q}_v, +)$. The Haar measure on $\mathbb{R}$ is normalised such that $\hat{f}(x) = f(-x)$, and the Haar measure on $\mathbb{Q}_p$ is normalised to give $\mathbb{Z}_p$ measure 1.)

Tate defined the Fourier transform in each local field $\mathbb{Q}_v$. For this purpose we need to fix an additive unitary character

$$\chi : \mathbb{Q}_v \to S^1 = \{ z \in \mathbb{C} : |z| = 1 \}.$$

On $\mathbb{R}$, we take $\chi_{\mathbb{R}}(x) = e^{2\pi i x}$ and on $\mathbb{Q}_p$, we take $\chi_{\mathbb{Q}_p}(x) = e^{2\pi i b}$ for every $x \in b + \mathbb{Z}_p$. (Recall that $\mathbb{Q}_p = \{ \sum_{n=-k}^\infty a_n p^n, 0 \leq a_n < p \}$ for some $k < \infty$, so every $x \in \mathbb{Q}_p$ can be written as $x = b + \mathbb{Z}_p$ for some $b \in \mathbb{Q}$. Thus $\chi_{\mathbb{Q}_p}$ is well-defined.) The Fourier transform on $\mathbb{Q}_v$ of a function $f$ (enjoying good analytic properties, e.g. absolutely integrable) is defined by

$$\hat{f}(y) = \int_{\mathbb{Q}_v} f(x) \bar{\chi}(xy) \, dx,$$

(2.4)

where $\bar{\chi}$ is the complex conjugate of $\chi$.

Observe that the Gaussian function $e^{-\pi x^2}$ is equal to its own Fourier transform. Its $p$-adic counterpart is the indicator function $1_{\mathbb{Z}_p}$ of the ring of $p$-adic integers.

We now compute

$$\zeta_p(1_{\mathbb{Z}_p}, s) = \int_{\mathbb{Q}_p^\times} 1_{\mathbb{Z}_p}(x_p)|x_p|^s \, d^\times x_p = \sum_{n=0}^\infty \int_{p^n \mathbb{Z}_p^\times} |x_p|^s \, d^\times x_p$$

$$= \sum_{n=0}^\infty p^{-ns} = \frac{1}{1 - p^{-s}}.$$
Here, in the second equation, we use that \( \mathbb{Q}_p^\times = \bigcup_{n \in \mathbb{Z}} p^n \mathbb{Z}_p^\times \) and \( x \in \mathbb{Z}_p \) if and only if \( \text{ord}_p(x) \geq 0 \). By virtue of the Euler product of Riemann’s \( \zeta \), we see that if we put \( \Gamma_p(s) = (1 - p^{-s})^{-1} \), then \( \xi(s) = \prod_v \Gamma_v(s) \) (\( v \) runs through \{Prime numbers\} \( \cup \{\infty\} \)). From this Tate [94] derived that \( \xi \) satisfies the functional equation \( \xi(s) = \xi(1 - s) \).

In this proof it is used in a fundamental way that \( \Gamma(s)|x|^s \) and \( \Gamma(s)|x|^{1-s} \) are Fourier transforms of each other.

**Remark 2.1.** (1) The additive unitary characters \( \chi \) defined above are trivial on the corresponding ring of integers. Every continuous additive unitary character on \( \mathbb{Q}_p \) is given by \( \psi_a(x) = \chi(ax) \) for some \( a \in \mathbb{Q}_p \). The kernel of \( \psi_a \) is \( |a|_p \mathbb{Z}_p \). The conductor of \( \psi_a \) is defined to be its kernel, or simply the value \( |a|_p \).

(2) Observe that \( |x|^s : \mathbb{Q}_v^\times \to \mathbb{C}^\times \) is a multiplicative character of \( \mathbb{Q}_v^\times \). In view of this, the Mellin transform is also called the multiplicative Fourier transform, and the Fourier transform defined above is referred to as the additive Fourier transform.

Let \( k \) be a local field and let \( d^\times x \) be the Haar measure on \( k^\times \). We normalise the Haar measure by \( \hat{f}(x) = f(-x) \) where \( \hat{f} \) is the Fourier transform. For \( \text{Re}(s) > 0 \), define

\[
\zeta(f, s) = \int_{k^\times} f(x)|x|^s d^\times x, \tag{2.5}
\]

where the norm \( |\cdot| \) is defined by the rule that \( d(bx) = |b|dx \) for the additive Haar measure \( dx \). The local gamma factor \( \gamma_k(s) \) of \( k \) is defined by the functional equation

\[
\zeta(f, s) = \gamma_k(s)\zeta(\hat{f}, 1 - s). \tag{2.6}
\]

The factor \( \gamma_k(s) \) is independent of \( f \) (cf. [94]). One can compute that

\[
\gamma_k(s) = \frac{\pi^{-s/2}\Gamma(s/2)}{\pi(s-1/2)\Gamma((1 - s)/2)},
\]

by taking \( f(x) = e^{-\pi x^2} \), and for a \( p \)-adic field \( k \) with residue cardinality \( q \), by taking \( f \) to be \( 1_{\mathfrak{m}_k} \), we can obtain

\[
\gamma_k(s) = d^{s-\frac{1}{2}} \frac{1 - q^{-1-s}}{1 - q^{-s}},
\]

where \( d \) is the discriminant of \( k \).

### 2.2 Local class field theory

From now on, except when otherwise stated, \( k \) will be a non-archimedean local field with a discrete valuation such that the residue field is finite. We fix a separable
algebraic closure $\overline{k}$ of $k$ and write $k_{\text{nr}}$ to the maximal unramified extension inside $\overline{k}$.

The ring of integers $\mathfrak{o}_k$ of $k$ is a local ring with a unique maximal ideal $\mathfrak{p}_k$. Choose a generator $\varpi_k$ of $\mathfrak{p}_k$. This generator is called the uniformiser of $k$. The residue field of $k$ is $\mathfrak{o}_k/\mathfrak{p}_k \simeq \mathbb{F}_q$ where $q$ is some power of the residue characteristic $p = \text{char}(\mathbb{F}_q)$.

The residue field of $k_{\text{nr}}$ is an algebraic closure of $\mathbb{F}_q$, which we denote by $\overline{\mathbb{F}}_q$. The Galois group $\text{Gal}(k_{\text{nr}}/k)$ is endowed with the Krull topology: a basis of open sets around an element $\tau \in \text{Gal}(k_{\text{nr}}/k)$ consists of the subsets

$$U_F(\tau) = \{g \in \text{Gal}(k_{\text{nr}}/k) : \tau |_F = g|_F\}$$

for subextensions $F$ of finite degree over $k$. Under this topology, $\text{Gal}(k_{\text{nr}}/k)$ is profinite. The inertia group $I_k$ of $k$ is defined to be the kernel of the homomorphism $\text{Gal}(\overline{k}/k) \to \text{Gal}(\overline{k}/k_{\text{nr}})$. In other words $I_k$ is the subgroup of $\text{Gal}(\overline{k}/k)$ corresponding to $k_{\text{nr}}$ under (infinite) Galois correspondence.

There is a canonical isomorphism defined by restriction to the ring of integers $\mathfrak{o}_{\text{nr}}$ of $k_{\text{nr}}$ between the Galois groups $\text{Gal}(k_{\text{nr}}/k) \simeq \text{Gal}(\mathbb{F}_q/\mathbb{F}_q)$. The latter group is also profinite since it is the inverse limit of the family of finite Galois groups:

$$\text{Gal}(\mathbb{F}_q/\mathbb{F}_q) = \lim_{\rightarrow n} \text{Gal}(\mathbb{F}_q^n/\mathbb{F}_q) = \hat{\mathbb{Z}}.$$

where $\hat{\mathbb{Z}}$ is the profinite completion of $\mathbb{Z}$. This is because every finite Galois group $\text{Gal}(\mathbb{F}_q^n/\mathbb{F}_q)$ is a cyclic group generated by the corresponding Frobenius automorphism

$$\text{Fr} : \mathbb{F}_q^n \to \mathbb{F}_q^n; \quad x \mapsto x^q.$$

Correspondingly, we have a topological generator of $\text{Gal}(k_{\text{nr}}/k)$, called the arithmetic Frobenius. On the other hand, the inverse of Fr is denoted by Frob and called the geometric Frobenius. It is also a topological generator of $\text{Gal}(k_{\text{nr}}/k)$.

Following Tate, we choose the geometric Frobenius as the generator [4, 95].

### 2.2.1 The Weil group

The Weil group $W_k$ of $k$ is defined to be the pull back of $\mathbb{Z}$ in $\text{Gal}(k_{\text{nr}}/k)$ under the epimorphism $\text{Gal}(k_{\text{nr}}/k) \to \hat{\mathbb{Z}}$. Thus $W_k$ is a dense subgroup of $\text{Gal}(k_{\text{nr}}/k)$. The Weil group fits into the following exact sequences

$$1 \to I_k \to \text{Gal}(k_{\text{nr}}/k) \to \hat{\mathbb{Z}} \to 1$$

$$1 \to I_k \to W_k \to \mathbb{Z} \to 1$$
We see that $W_k/I_k \simeq \mathbb{Z} = \langle \text{Frob} \rangle$. The topology in $W_k$ is not the subspace topology from $\text{Gal}(k_{nr}/k)$, but a finer topology: the inertia group is endowed with the profinite topology induced from $\text{Gal}(k_{nr}/k)$ and is open in the Weil group.

For a topological group $G$ we denote by $G^c$ the closure of its commutator subgroup and $G^{ab} = G/G^c$ the maximal abelian quotient of $G$. The main theorem of local class field theory says:

**Theorem 2.1** (Reciprocity law of local class field theory). Let $W_k$ be the Weil group of a non-archimedean local field $k$. There is an isomorphism of topological groups, called the Artin reciprocity map:

$$\text{Art}_k : \mathfrak{a}_k^\times \simeq W_k^{ab}$$

sending the ring of units $\mathfrak{a}_k^\times$ to $I_k^{ab}$, and a uniformiser $\omega_k$ of the maximal ideal $\mathfrak{p}_k$ to the geometric Frobenius element $\text{Frob}$.

Some remarks are in order.

**Remark 2.2.** (1) The Artin reciprocity law is a powerful tool to describe the abelian extensions of both local and global fields. In the local case, the properties mentioned above, plus with some other properties (See [77]) determine the Artin reciprocity map. If $E/k$ is a finite extension of local fields, the local Artin reciprocity law $\text{Art}_{E/k}$ (which is the composition of $\text{Art}_k$ with $W_k^{ab} \hookrightarrow \text{Gal}(\bar{k}/k)^{ab} \to \text{Gal}(E/k)$) is stated in terms of the local Artin symbol or the norm residual symbol. And $\text{Art}_E$ and $\text{Art}_k$ satisfy some functorial properties.

(2) We observe that a homomorphism $\rho : W_k \to \mathbb{C}^\times$ is continuous if and only if its restriction to the inertia group $I_k$ is continuous. But since $I_k$ is endowed with the usual profinite topology, the image $\rho(I_k)$ will be finite. This implies that a homomorphism $W_k \to \mathbb{C}^\times$ is continuous with respect to the Euclidean topology of $\mathbb{C}^\times$ if and only if it is continuous with respect to the discrete topology of $\mathbb{C}^\times$.

Let us denote by $\mathcal{A}_1(k)$ the set of equivalence classes of irreducible smooth complex representations of $\mathfrak{a}_k^\times = \text{GL}_1(k)$. Notice that each representation in $\mathcal{A}_1(k)$ is 1-dimensional because $\mathfrak{a}_k^\times$ is abelian. On the Galois side, let $\mathcal{G}_1(k)$ be the collection of continuous homomorphism $W_k \to \text{GL}_1(\mathbb{C}) = \mathbb{C}^\times$ where $\mathbb{C}^\times$ is endowed with discrete topology. We can restate the main theorem local class field theory as follows:

**Theorem 2.2** (Local Langlands conjecture for $\text{GL}_1$). For any non-archimedean local field $k$, there is a natural bijection between the sets $\mathcal{A}_1(k)$ and $\mathcal{G}_1(k)$. We can normalise the bijection such that the uniformiser of $k$ corresponds to the geometric Frobenius: $\omega_k \leftrightarrow \text{Frob}$.
Langlands’ philosophy is to generalise the correspondence to reductive groups. The correspondence he conjectured between $\mathcal{A}_n$, the set of isomorphism classes of irreducible smooth complex representations of $\text{GL}_n$, and $\mathcal{S}_n$, the set of (isomorphic) $n$-dimensional continuous automorphic representations of $W_k$, is the source of many exciting results. The conjectured correspondence $\mathcal{A}_n \to \mathcal{S}_n$ is finite-to-one. An important problem is to parametrising the set $\mathcal{A}_n$, as the first step to classify the representations in $\mathcal{A}_n$. For this objective, the pioneers introduced the notion of Langlands parameters and $L$-packets.

**Remark 2.3.** So far, there are many great results along this line. For instance, the Langlands correspondence for $\text{GL}_n$ over $p$-adic fields, for $\text{GL}_2$ over both local fields and global function fields have been established. See for example [4, 13, 45, 42, 85].

### 2.2.2 An example of split tori

By a $k$-torus $T$ of rank $n$ we mean a linear algebraic group defined over $k$, and isomorphic to the direct product of $n$ copies of the multiplicative group $G_m$ over $\overline{k}$. If $T$ is isomorphic to $(G_m)^n$ over $k$, we say that $T$ is split over $k$.

Let $T$ be a $k$-torus of rank $n$ which is split over $k$, and let $T = T(k)$ be its $k$-rational points. Then $T$ is $k$-isomorphic to the direct product of $n$ copies of $k^\times$. Put $X = \text{Hom}(T, G_m), Y = \text{Hom}(G_m, T)$. They are both free abelian groups of rank $n$, and are called the character and cocharacter lattices of $T$, respectively.

There is a perfect pairing $\langle \cdot, \cdot \rangle : X \times Y \to \mathbb{Z}$, given by $\chi(\rho(b)) = b^{(X, \rho)}$ for all $b \in k^\times$. In view of this, we obtain a canonical isomorphism

$$Y \otimes k^\times \simeq T, \quad \rho \otimes b \mapsto \rho(b). \quad (2.7)$$

Consequently, by evaluating $\lambda \in Y$ at the uniformiser $\varpi$, we obtain an isomorphism of abelian groups between $Y$ and $T(k)/T(0_k)$.

Since $T$ is abelian, its irreducible complex representations are all one-dimensional. We have

$$\text{Hom}(T, \mathbb{C}^\times) = \text{Hom}(Y \otimes k^\times, \mathbb{C}^\times)$$

$$= \text{Hom}(k^\times, X \otimes \mathbb{C}^\times)$$

$$\simeq \text{Hom}(W_k, X \otimes \mathbb{C}^\times).$$

The second equation uses the duality between $X$ and $Y$, and the last step we use the Artin reciprocity isomorphism.

Inspired by the isomorphism (2.7), we opt to regard $X \otimes \mathbb{C}^\times$ as the $\mathbb{C}$-points of a torus $T^\vee$. This tensor product makes sense because $T$ is abelian and thus its Pontryagin dual is also a group. We call such a torus $T^\vee$ the complex dual group of $T$ and write $T^\vee = T^\vee(\mathbb{C})$. The character and cocharacter lattices of $T^\vee$ are
Y = \text{Hom}(T^\vee, G_m) \text{ and } X = \text{Hom}(G_m, T^\vee) \text{ respectively.}

**Proposition 2.1** (local Langlands correspondence for split tori). *Let T and T^\vee be given as above. There is a bijective correspondence*

\[ \sigma : \text{Hom}(T, \mathbb{C}^\times) \to \text{Hom}(W_k, T^\vee). \]  

**(2.8)**

*The image \( \sigma(\chi) \) of \( \chi \in \text{Hom}(T, \mathbb{C}^\times) \) is called the Langlands parameter of \( \chi \).*

### 2.3 Root data

In this section we review some structure theory of linear algebraic groups. We assume the basic definitions of linear algebraic groups and their important subgroups including tori, parabolic subgroups and the Levi decompositions, for instance, one can refer to [91, 93, 92].

A root datum consists of

(i) two free abelian groups \( X^* \) and \( X_* \) of finite type, in duality by a perfect pairing

\[ \langle , \rangle : X^* \times X_* \to \mathbb{Z}; \]

(ii) a finite subset \( \Sigma_0 \subset X^* \) and a finite subset \( \Sigma_0^\vee \subset X_* \), with a bijection from \( \Sigma_0 \) onto \( \Sigma_0^\vee \), denoted by \( \alpha \mapsto \alpha^\vee \);

(iii) endomorphism \( s_\alpha \in \text{End}(X^*) \) and \( s_\alpha^\vee \in \text{End}(X_*) \) attached to \( \alpha \in \Sigma_0 \) and \( \alpha^\vee \in \Sigma_0^\vee \), respectively, defined by

\[ s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha, \quad s_\alpha^\vee(y) = y - \langle \alpha, y \rangle \alpha^\vee. \]

These endomorphisms satisfy the following two requirements:

(RD1) For all \( \alpha \in \Sigma_0 \) we have \( \langle \alpha, \alpha^\vee \rangle = 2; \)

(RD2) For all \( \alpha \in \Sigma_0 \) we have \( s_\alpha(\Sigma_0) \subset \Sigma_0 \) and \( s_\alpha^\vee(\Sigma_0^\vee) \subset \Sigma_0^\vee \).

We will denote a root datum as a quadruple \( (X^*, \Sigma_0, X_*, \Sigma_0^\vee) \).

Given a root datum \( (X^*, \Sigma_0, X_*, \Sigma_0^\vee) \), the lattice \( \mathbb{Z}\Sigma_0 \) is the subgroup of \( X^* \) generated by \( \Sigma_0 \). The root datum is said to be *semisimple* if \( \mathbb{Z}\Sigma_0 \subset X^* \) has maximal rank, and *toral* if \( \Sigma_0 \) is empty.

We now explain how to obtain a root datum from a connected reductive linear algebraic group \( G \) (defined over \( \bar{k} \)). We shall identify \( G \) with the group \( G(\bar{k}) \) of \( \bar{k} \)-points. Let \( T \) be a maximal torus of \( G \). Let \( X^* = \text{Hom}(T, G_m) \) and \( X_* = \text{Hom}(G_m, T) \) respectively.
Hom\((\mathbb{G}_m, T)\) be the character and cocharacter lattices of \(T\), respectively. The perfect pairing
\[
\langle \ , \ \rangle : X^* \times X_* \rightarrow \mathbb{Z}
\]
is defined by
\[
\chi\langle \rho(b) \rangle = b^{(\chi, \rho)}, \quad \forall b \in k^*.
\]
This pairing is due to the fact that \(\text{End}(\mathbb{G}_m) = \mathbb{Z}\).

Since \(G\) is reductive, the torus \(T\) determines a finite subset \(\Sigma_0 \subset X^*\) satisfying the axioms of a root system. Let \(E = \mathbb{R} \otimes \mathbb{Z} X^*\). Within the usual Euclidean metric of \(\mathbb{R}\), we can make \(E\) into a finite dimensional Euclidean vector space and \(\Sigma_0\) becomes a root system of \(E\). So we can obtain the dual root system \(\Sigma_0^\vee\) in the dual vector space of \(E\). Then \((X^*, \Sigma_0, X_*, \Sigma_0^\vee)\) is a root datum.

**Theorem 2.3.** For every root datum \((X^*, \Sigma_0, X_*, \Sigma_0^\vee)\) with reduced root system \(\Sigma_0\), there exist a pair of linear algebraic groups \((G, T)\) where \(G\) is connected and reductive and \(T\) its maximal torus. The pair \((G, T)\) is unique up to isomorphism.

From the definition of root datum we deduce that \((X_*, \Sigma_0^\vee, X^*, \Sigma_0)\) is also a root datum. This root datum is said to be dual to the original one. By Theorem 2.3 we see that there is uniquely a pair of linear algebraic groups \((G^\vee, T^\vee)\) attached to the dual root datum. We say that \(G^\vee\) is dual to \(G\).

### 2.4 Rational forms

The main reference I used for this section is [65]. Throughout this section, \(k\) is a non-archimedean local field.

Let \(G\) be a connected \(\tilde{k}\)-group. Let \(\text{Aut}_{\text{alg}}(G)\) (resp. \(\text{Aut}_{\text{abs}}(G)\)) be the group of algebraic (resp. all) automorphisms of \(G\). Let \(T \subset B\) be a Cartan pair, i.e. \(T\) a maximal torus of \(G\), containing in a Borel subgroup \(B\). Recall that the choice of a Borel subgroup \(B \supset T\) determines a base \(F_0\) of the root system \(\Sigma_0 := \Sigma(G, T)\). We introduce the based root datum
\[
\Sigma = (X^*(T), F_0, X_*(T), F_0^\vee)
\]
determined by \((G, B, T)\), where \(F_0^\vee\) is a base of \(\Sigma_0^\vee\), which is bijectively corresponding to \(F_0\).

If \(T' \subset B'\) is another Cartan pair, then there exists a unique element \(g T \in G / T\) conjugates \((T', B')\) to \((T, B)\). Applying this fact to the case of \((\sigma T, \sigma B)\), where \(\sigma \in \text{Aut}_{\text{alg}}(G)\), then we can find an element \(g \in G(\tilde{k})\) such that
\[
\text{Int}(g)(\sigma B) = B, \quad \text{Int}(g)(\sigma T) = T.
\]
So $\text{Int}(g) \circ \sigma$ gives an automorphism of $T$ which is only depending on $\sigma$ (because the coset $gT$ is uniquely determined). The morphism $\text{Int}(g) \circ \sigma$ determines an automorphism $\beta_G(\sigma)$ of $X^*(T)$, permuting the elements of $F_0$ (since $\text{Int}(g) \circ \sigma$ fixes $G$). Thus, we obtain a homomorphism $\beta_G : \text{Aut}_{alg}(G) \rightarrow \text{Aut}(\Sigma)$. The kernel of $\beta_G$ is $\text{Int}(G)$, which is isomorphic to the adjoint form $G_{ad}$ of $G$.

**Definition 2.1.** Given a connected reductive algebraic group $G$ over $\bar{k}$, by a $k$-rational structure on $G$, we mean a homomorphism

$$c : \text{Gal}(\bar{k}/k) \rightarrow \text{Aut}_{abs}(G), s \mapsto c_s,$$

obeying the requirement that if $f : G \rightarrow \bar{k}$ is a regular function, then the function $x \mapsto sf(c_{s^{-1}}(x))$ is also regular for each $s \in \text{Gal}(k/k)$. Every $k$-rational structure defines a $k$-rational form $G$ of $G$ such that $G$ can be identified with the fixed points of $G$ under the action of $\text{Im}(c)$. Two rational forms are considered to be equivalent if there is an algebraic automorphism of $G$ that is equivariant for the corresponding Galois actions.

From the definition, if $G$ and $G'$ are two rational forms of $G$, then $G$ and $G'$ are isomorphic after taking scalar extension to $\bar{k}$. Note that $G$ acts by conjugation on the set of its rational forms.

**(ii)** Let $G$ be a reductive connected $\bar{k}$-group ($k$ non-archimedean local field and $\bar{k}$ the algebraic closure of $k$). Fix an épínglage [92, 96] for $G$, and let $u_\alpha : \bar{k} \rightarrow G$ be the homomorphism from this épinglage associated with $\alpha \in F_0$. If $\eta \in \text{Hom}(\text{Gal}(\bar{k}/k), \text{Aut}(\Sigma))$, put $\eta_s = \eta(s), \forall s \in \text{Gal}(\bar{k}/k)$. Let $\rho \in X_*(T)$. The following conditions characterise an action of $\text{Gal}(\bar{k}/k)$ on elements of $G$:

$$s(\rho(b)) = \eta_s(\rho)(sb), \quad s(u_\alpha(b)) = u_{\eta_s(\alpha)}(sb), b \in \bar{k} \quad (2.9)$$

for all such $\rho \in X_*(T), \alpha \in F_0$ and $s \in \text{Gal}(\bar{k}/k)$. The rational form of $G$ determined by the homomorphism $\eta : \text{Gal}(\bar{k}/k) \rightarrow \text{Aut}(\Sigma)$ is called the quasi-split form over $k$ of $G$.

Before proceeding, we recall some terminologies. Every $k$-rational form of $G$ gives a linear algebraic group defined over $k$. Such a group will be called a $k$-group. If a connected $k$-group $H$ contains a maximal torus $T$ which is defined and split over $k$, then $H$ is said to be $k$-split, or we say $H$ is split over $k$. On the other end,
a connected reductive $k$-group is called *anisotropic* over $k$ if it has no nontrivial $k$-split $k$-subtorus.

**Example 2.2.**  
(i) The group $\text{GL}_n$ is split over any field, because it contains the group of diagonal matrices, which is isomorphic to the direct product of $n$ copies of $\text{G}_m$.

(ii) If $q$ is a nondegenerate quadratic form on a $K$-vector space (assume that the field $K$ is *not* of characteristic 2). The special orthogonal group $\text{SO}(q)$ of $q$ is anisotropic over $K$ if and only if $q$ does not represent 0 over $K$.

If a connected $k$-group $H$ contains a Borel subgroup defined over $k$, then $H$ is said to be *quasi-split* over $k$. A connected $k$-group $H$ is said to be *unramified* if it is quasi-split over $k$ and split over some unramified field extension $k'$ of $k$.

We return to the discussion on $k$-rational structures on our $\bar{k}$-group $G$. Fix any rational form of $G$. Suppose $\sigma : \text{Gal}(\bar{k}/k) \to \text{Aut}_{\text{abs}}(G)$ is another rational form. One observes that $s^{-1} \circ \sigma(s)$ is an algebraic automorphism of $G$. To see this, recall that if $f : G \to \bar{k}$ is a regular function, then $x \mapsto s^{-1}f(\sigma(s)x)$ is regular. Applying this to the regular function $y \mapsto s(f(s^{-1}y))$, we see that the function $y \mapsto s^{-1}sf(s^{-1}\sigma(s)y)$ is also regular, as desired.

With this observation, we are able to classify, using the machinery of group cohomology, all rational forms of $G$ over $k$ in terms of a *split form* 

$$d : \text{Gal}(\bar{k}/k) \to \text{Aut}_{\text{abs}}(G)$$

which preserves the Cartan pair $T \subset B$ and our fixed épininglage. In fact, if 

$$c : \text{Gal}(\bar{k}/k) \to \text{Aut}_{\text{abs}}(G)$$

is a rational form, then for each $s \in \text{Gal}(\bar{k}/k)$, the element $\alpha_s := c_s d_{s^{-1}}$ is in $\text{Aut}_{\text{alg}}(G)$ by the observation in the above paragraph. Define an action * of $\text{Gal}(\bar{k}/k)$ on $\text{Aut}_{\text{alg}}(G)$ by the rule that $s \ast \phi = d_s \circ \phi \circ d_{s^{-1}}$. One can check that the map $\alpha : \text{Gal}(\bar{k}/k) \to \text{Aut}_{\text{alg}}(G)$ sending $s$ to $\alpha_s = c_s d_{s^{-1}}$ defined above is a 1-cocycle: $\alpha \in Z^1(k, \text{Aut}_{\text{alg}}(G)) := Z^1(\text{Gal}(\bar{k}/k), \text{Aut}_{\text{alg}}(G))$. Two cocycles are cohomologous if and only if the associated rational forms are equivalent. Hence we obtain a bijection between rational forms of $G$ and $Z^1(k, \text{Aut}_{\text{alg}}(G))$, and between their equivalence classes and $H^1(k, \text{Aut}_{\text{alg}}(G))$. We shall regard rational forms as cocycles.

### 2.4.1 Pure inner forms

The homomorphism $\beta_G : \text{Aut}_{\text{alg}}(G) \to \text{Aut}(\Sigma)$ gives rise to a functorial map $\beta_G^* : Z^1(k, \text{Aut}_{\text{alg}}(G)) \to Z^1(k, \text{Aut}(\Sigma))$. The fibres of $\beta_G^*$ are called *inner classes*.
of the rational forms. Elements in the same inner class are said to be inner forms of each other. Every inner class has a unique equivalence class of quasi-split forms. If \( \eta \in Z^1(k, \text{Aut}(\Sigma)) \), we write \( q_\eta \) for the quasi-split form defined by the rules (2.9). The fibres of \( \beta^*_G \) over \( \eta \) is in bijective correspondence with \( Z^1(k, \mathbf{G}_{ad}) \), where the \( \text{Gal}(\overline{k}/k) \) acts on \( \mathbf{G}_{ad} \) by \( s \ast \phi = q_\eta(s) \circ \phi \circ q_\eta(s^{-1}) \). Consequently, within an inner class, the equivalence classes of inner forms are parametrised by \( H^1(k, \mathbf{G}_{ad}) \). In particular, the privilege element of \( H^1(k, \mathbf{G}_{ad}) \) corresponds to the quasi-split forms in this inner class.

Analogous to the \( \text{Gal}(\overline{k}/k) \)-module structure of \( \mathbf{G}_{ad} \) via the action (2.9) arising from the quasi-split form, we can also give \( \mathbf{G} \) a structure of \( \text{Gal}(k/k) \)-module. Elements of \( Z^1(k, \mathbf{G}) \) are called pure inner forms of the given quasi-split form by Vogan [97]. If the inner class is not specified, we speak of pure rational forms.

**Remark 2.4.** (i) We see that pure rational forms can determine rational forms via the map \( Z^1(k, \mathbf{G}) \to Z^1(k, \mathbf{G}_{ad}) \). However, on the level of cohomology, this map is neither injective nor surjective.

(ii) Kottwitz gives an interpretation of the cohomology set \( H^1(k, \mathbf{G}) \), which we will see in next chapter.

**Example 2.3.** Suppose \( k \) is a non-archimedean local field with finite residue field. Let us describe the \( k \)-forms of \( \text{SL}_n \). We factor \( n \) as \( n = md \). By the theory of central simple algebra, there exists a central division algebra \( \mathbb{D} \) with centre \( k \) and dimension \( m^2 \) over \( k \). Inside \( \mathbb{D} \) sits an unramified extension \( k' \) of \( k \), which is the maximal commutative subalgebra of \( \mathbb{D} \). A basis of \( \mathbb{D} \) over \( k' \) is indexed by the Galois group \( \text{Gal}(k'/k) = \mathbb{Z}/m\mathbb{Z} \). The desired \( k \)-form \( \text{SL}_d(\mathbb{D}) \) of \( \text{SL}_n \) is the set of (reduced) norm one elements in the central simple \( k \)-algebra \( \text{Mat}_d(\mathbb{D}) \).

### 2.4.2 Steinberg’s vanishing theorem

Let \( \mathbf{G} \) be a connected \( k \)-group. Write \( G = \mathbf{G}(k_{nr}) \) for the group of \( k_{nr} \)-points. The Frobenius element \( \text{Frob} \) induces an automorphism \( F \) of \( G \). We have the following result of Steinberg:

**Proposition 2.2** (Steinberg’s vanishing theorem). We have \( H^1(k_{nr}, \mathbf{G}) = 1 \). As a consequence, the natural surjection \( \text{Gal}(\overline{k}/k) \to \text{Gal}(k_{nr}/k) \) induces an isomorphism of Galois cohomology sets: \( H^1(k, \mathbf{G}) = H^1(F, G) := H^1(\text{Gal}(k_{nr}/k), G) \).

From this proposition, we deduce that if a quasi-split form splits over \( k_{nr} \), then every form in its inner class also splits over \( k_{nr} \).
Chapter 2

2.5 Unramified representations

Let $k$ be a non-archimedean local field. Fix a uniformiser $\varpi$ of $k$. Using the Artin reciprocal mapping, we can define a norm $|\cdot|_{W_k}$ on $W_k$ such that any element in the inertia group has norm 1 and $|\text{Frob}|_{W_k} = |\varpi|_k$.

Recall that the quotient of the Weil group $W_k$ by the inertial subgroup $I_k$ is isomorphic to $\text{Frob}^\mathbb{Z}$. Thus, if a continuous homomorphism $\rho$ of $W_k$ is trivial on the inertia group, then $\rho$ is completely determined by $\rho(\text{Frob})$. For example, if $\rho : W_k \to \mathbb{C}^\times$ is trivial on $I_k$, then the image of $\rho$ is bounded if and only if $\rho(\text{Frob})$ has absolute value 1.

Recall that a finite extension $E/k$ is said to be unramified if the relative inertia group $I_{E/k}$ is trivial. We say a continuous homomorphism $\rho$ of $W_k$ unramified if $\rho(I_k) = 1$. And we say a character $\chi$ of a $k$-split torus $T$ unramified if $\chi|_{T(s_k)} = 1$. In next subsection we will recall the notion of unramified representations of reductive groups.

Let $G$ be a connected reductive $k$-group and denote $G = G(k)$ the group of its $k$-points. The topology of $k$ induces a topology of $G$ which makes $G$ a locally compact, totally disconnected group. We follow P. Cartier [16] to call this group of $td$-type.

2.5.1 Spherical functions

We turn to review the basic theory of unramified representation of $G = G(k)$. We assume that $G$ is a connected reductive unramified $k$-group.

Let $K_0$ be a hyperspecial maximal compact subgroup of $G$ [96]. We have the Iwasawa decomposition $G = BK_0 = K_0B$ where $B$ is a Borel subgroup of $G$. Let $T \subset B$ be a maximal $k$-torus, and let $Y$ be its cocharacter lattice. Then $(G, T)$ determines a root system $\Sigma_0 = \Sigma_0(G, T)$. Let $F_0$ be the base of $\Sigma_0$ with respect to $B$. Put

$$\text{wt}_+^Y = \{ \lambda \in Y : \langle \alpha, \lambda \rangle > 0; \forall \alpha \geq 0 \in F_0 \}.$$ 

We then also have a refinement of Cartan decomposition of $G$, namely, $G = \sqcup K_0 \lambda(\varpi) K_0$ where $\lambda$ runs through the cocharacters in $\text{wt}_+^Y$.

Let $K$ be a compact open subgroup of $G$. Denote by $\mathcal{H}(G, K)$ the vector space of functions $f : G \to \mathbb{C}$ of compact support such that $f(k_1 g k_2) = f(g)$ for all $k_1, k_2 \in K, g \in G$. Define a convolution $\ast$ on this vector space by

$$f_1 \ast f_2(x) = \int_G f_1(xg^{-1})f_2(g) \, dg.$$
Then \( \mathcal{H}(G, K) \) is an associative algebra with \( * \) as multiplication. This algebra has 
\( e_K = 1_K / \text{meas}(K) \) as identity, where \( 1_K \) is the characteristic function of \( K \) and 
\( \text{meas}(K) \) is the measure of \( K \). Write \( \mathcal{H}(G) = \bigcup_K \mathcal{H}(G, K) \) where \( K \) runs over 
all compact open subgroups of \( G \). It is easy to see that \( \mathcal{H}(G) = C_c^\infty(G; \mathbb{C}) \), the 
associative algebra of compactly supported, locally constant complex-values functions \( f \) on \( G \), 
with \( * \) as multiplication (because \( G \) has a basis consisting of compact open subgroups in the 
neighbourhood of the identity, so \( \mathcal{H}(G) \subset C_c^\infty(G; \mathbb{C}) \). To see the reverse 
containment relation, observe that every function \( f \in C_c^\infty(G; \mathbb{C}) \) is bi-invariant with respect to 
some compact open subgroup \( K \) of \( G \). Observe that \( \mathcal{H}(G) \) is commutative if and only if 
\( G \) is commutative. However, \( \mathcal{H}(G) \) does not have an identity unless \( G \) is discrete.

Now suppose \((\pi, V)\) is a smooth representation of \( G \). For any \( f \in \mathcal{H}(G) \), define 
the element \( \pi(f) \in \text{End}_\mathbb{C}(V) \) by
\[
\pi(f)v = \int_G f(g)\pi(g)v \, dg, \quad v \in V. \tag{2.10}
\]
Then, \( V \) is an \( \mathcal{H}(G) \)-module, and the correspondence \( f \mapsto \pi(f) \) is an algebra 
homomorphism from \( \mathcal{H}(G) \) to \( \text{End}_\mathbb{C}(V) \).

A module \( M \) over a ring \( A \) without unit element is said to be non-degenerate (or unital) if 
\( \mathcal{H}(G).M = M \). (See [3, 16]. We also make the convention that if \( A \) has unit, then any unitary module \( M \) over \( A \) is non-degenerate.) A fundamental fact is that 
the category of nondegenerate \( \mathcal{H}(G) \)-modules is the same as the category of smooth (complex) representations of \( G \) and intertwining operators ([3]).

**Definition 2.2.** Let \( K \) be any compact open subgroup of \( G \). An irreducible smooth 
representation \( V \) of \( G \) is said to be unramified if \( V^K \neq 0 \), i.e. \( V \) has nonzero \( K \)-
fixed vectors.\(^1\)

We will review the basic representation theory of unramified representation in this 
subsection. One can see that unramified representations of \( G \) coincides with irreducible 
\( \mathcal{H}(G, K) \)-modules.

Let us give an example of an unramified representation of \( G \), with an additional 
assumption that \( G \) is split over \( k \). Recall the Iwasawa decomposition \( G = K_0B \) of 
\( G \), where \( K_0 \) is a hyperspecial subgroup. There is a maximal torus \( T \) contained in 
\( B \), and \( B = TN \), where \( N \) is the unipotent radical of \( B \). Let \( \chi \) be an unramified 
character of \( T \). We extend \( \chi \) to \( B \) by \( \chi(tn) = \chi(t), \forall b = tn \in B \) (i.e. extend over 
\( N \) trivially). Define
\[
\text{Ind}_B^G(\delta^{1/2}_B \chi) = \{ f \in C_c^\infty(G; \mathbb{C}): f(bg) = \delta^{1/2}_B(b)\chi(b)f(g), \forall b \in B, \forall g \in G \}. \tag{2.11}
\]
\(^1\)In literature, one might find that the term “\( K \)-spherical” has the same meaning as unramified 
here.
Here $\delta_B : B \to \mathbb{R}_{>0}^\times$ is the modulus function of $B$, defined by $\delta_B(b) = |\det Ad_n(t)|$ for $b = nt \in B$, where $n$ is the Lie algebra of $N$.

The vector space $E = \text{Ind}^G_B(\delta_B^{1/2} \chi)$ possesses a right $G$-action. It is called the *normalised* induced representation of $G$. Since $G = BK_0$, the restriction of this representation to $K_0$ can be identified with the vector space $C^\infty((K_0 \cap B) \setminus K_0, \mathbb{C})$ of smooth and left-$(K_0 \cap B)$-invariant functions on $K_0$. In particular, the subspace of $K_0$-fixed vectors of $E$ has dimension 1. It is easy to verify that the contragredient of $E$ is $\tilde{E} := \text{Ind}^G_B(\delta_B^{1/2} \tilde{\chi}) = \text{Ind}^G_B(\delta_B^{1/2} \chi^{-1})$ (because the contragredient $\tilde{\chi}$ of $\chi$ is its inverse).

If $(\pi, V)$ is any admissible representation of $G$, then for each $f \in \mathcal{H}(G, K_0)$, the operator $\pi(f)$ defined in (2.10) acts on $V^{K_0}$. If we take $V = E$, the normalised induced representation as above, we see that $\pi(f)$ becomes a scalar, because $\dim E^{K_0} = 1$. In view of this, we have a homomorphism of algebras $j : \mathcal{H}(G, K_0) \to \mathbb{C}$, sending $f \in \mathcal{H}(G, K_0)$ to the scalar determined by $\pi(f)$.

### 2.5.2 The Satake transform

Thanks to the Satake isomorphism (or equivalently, Harish-Chandra’s theory of constant terms), we have a nice description of the algebra $\mathcal{H}(G, K_0)$. The *Satake transform* ([64]) of $f \in \mathcal{H}(G, K_0)$ is defined by

$$Sf(a) = \delta_B(a)^{-1/2} \int_N f(an) \, dn,$$

where the Haar measure $dn$ on $N$ is chosen such that $N \cap K_0$ has measure 1. The function $Sf : T \to \mathbb{C}$ is also called the constant term of $f$ along $B$. Note that $Sf \in \mathcal{H}(T, T \cap K_0)$. This latter associative algebra is commutative.

Denote $W_0$ the finite Weyl group of $\Sigma_0$ and $T^\vee = T^\vee(\mathbb{C})$ the dual complex torus. There is an isomorphism of commutative algebras

$$\mathcal{H}(G, K_0) \simeq \mathbb{C}[T^\vee]^W = \mathbb{C}[T^\vee/W_0],$$

which is called the *Satake isomorphism* [16, 36]. Here $\mathbb{C}[T^\vee]$ is the coordinate algebra of the variety $T^\vee$, which is isomorphic to the group algebra of the cocharacter group $Y$ of $T$. In particular, unramified representations of $G$ are in bijection with $T^\vee/W_0$. By [9, Proposition 6.7], we notice that $T^\vee/W_0$ is in bijection with the semisimple conjugacy classes of $G^\vee$. Hence, unramified representations of $G$ are classified by the semisimple conjugacy classes of $G^\vee$. To every isomorphism classes of unramified representation $[\pi]$ of $G$, the corresponding semisimple conjugacy classes $s([\pi])$ in $G^\vee$ is called its Satake parameter.
Chapter 2

Remark 2.5. If $G$ is not split over $k$, then the Galois group will play some role in the Satake parameters (for instance, when $G$ is unramified over $k$, which implies that $G$ is split over the maximal unramified field extension $k_{nr}$ of $k$). Recall that $\text{Gal}(k_{nr}/k) = \langle \text{Frob} \rangle \simeq \mathbb{Z}$. We then need to use an appropriate notion, called Frobenius-conjugacy classes in $G^G = G^\vee \rtimes \text{Gal}(k_{nr}/k)$, to replace the semisimple conjugacy classes in $G^\vee$. This is Langlands’ version of Satake isomorphism, and it is also the origin of his $L$-group (cf. Casselman [22]).

2.5.3 Macdonald’s formula

We retain our notations. Recall that we have the Cartan decomposition

$$G = \bigcup_{\lambda} K_0 \lambda(\varpi) K_0$$

of $G$ where $\lambda$ runs through the dominant cocharacter lattice $\text{wt}^\vee_+$. As a corollary of the Satake isomorphism, we have a $\mathbb{C}$-basis $\{\theta_i\}_{i=1}^r$ of the algebra $\mathcal{H}(G, K_0)$ where $\{\theta_i\}$ is a set of generators of the semi-group of $\text{wt}^\vee_+$.

Macdonald [64] gave a nice account of unramified representations of reductive groups by using the structure theory developed by Bruhat and Tits. He studied $p$-adic reductive groups possessing a $BN$-pair structure. And he used the affine root system associated with the $p$-adic group to interpret the Plancherel theorem concerning the Fourier transform of spherical functions.

Recall the induced representation $E = \text{Ind}_{B}^{G}(\delta^{1/2} \chi)$. Let $\phi_\chi \in E^{K_0}$ be the unique element satisfying that $\phi_\chi(1) = 1$. We have a homomorphism: $j_\chi : \mathcal{H}(G, K_0) \to \mathbb{C}$ defined by $\pi(f)\phi_\chi = j_\chi(f)\phi_\chi$. To introduce Macdonald’s formula, we consider the problem of computing $j_\chi(f)$ where $f$ is the indicator function of the double coset $K_0 \lambda(\varpi) K_0$. We identify $\lambda(\varpi) = t \in T$. Since $j_\chi(f)$ is $\pi(f)\phi_\chi$ evaluated at 1 (here $\pi$ is the right regular representation of $G$ on $E$), Macdonald computed that

$$j_\chi(f) = \pi(f)\phi_\chi(1) = \int_{G} f(g)\phi_\chi(g) \, dg = |K_0 tK_0/K_0| \int_{K_0} \phi_\chi(kt) \, dk. \quad (2.12)$$

Let $\Phi_\chi(t) = \int_{K_0} \phi_\chi(kt) \, dt$. Macdonald showed that the function $\Phi_\chi$ is a spherical function associated with $E$, which is equivalent to say that $\Phi_\chi(g) = \int_{K_0} \phi_\chi(kg) \, dk$ is a matrix coefficient $\langle \pi(g)\phi_\chi, \phi_\chi^{-1} \rangle$. Furthermore, he gave an explicit formula for $\Phi_\chi(t)$ for all $t \in T$ which is of the form $\lambda(\varpi)$, where $\lambda \in \text{wt}^\vee_+$.

We will describe Macdonald’s formula for simply connected Chevalley groups. A simply connected Chevalley group $G$ is generated by elements $u_\alpha(b), \alpha \in \Sigma_0 \subset
$Y, b \in k$. (In fact, each $u_\alpha$ is a monomorphism of the additive group $(k, +)$ into $G.$) If $\alpha = \alpha + l$ is an affine root, we then define a subgroup $U_\alpha$ of $G$ by

$$U_\alpha = U_{\alpha + l} = \{u_\alpha(b) : b \in k, \text{ord}_k(b) \geq l\}.$$ 

We have a filtration $U_\alpha \supset U_{\alpha + 1} \supset U_{\alpha + 2} \supset \cdots$. Moreover, one can show that the index $q_\alpha = [U_{\alpha - 1} : U_\alpha]$ is finite for every $\alpha \in R = \Sigma_0 \times \mathbb{Z}$. Note that the composition of reflections $s_{\alpha + 1} \circ s_\alpha$ sends $\alpha + b$ to $\alpha + b + 2$. Thus $q_{\alpha + 2} = q_\alpha$. But it can happen that $q_{\alpha + 1} \neq q_\alpha$. We define $q_{\alpha/2} = q_{\alpha + 1}/q_\alpha$ for every root $\alpha \in \Sigma_0$. Likewise, we can define $q_{\alpha^\vee + b}, q_{\alpha^\vee}/2$.

Denote $T$ a maximal torus of $G$. Let $t = \alpha^\vee(\varpi) \in T$ with $\alpha^\vee \in Y$, and let $s \in X$ be a character of $T$. Macdonald introduced the following $c$ function (analogue of Harish-Chandra’s $c$ function for real semisimple Lie groups):

$$c(\alpha, s) = \begin{cases} 
\frac{1 - q_{\alpha^\vee}^{-1}(s(t))^{-1}}{1 - s(t)^{-1}}, & \text{if } 2\alpha \notin R_{nr}, \\
\frac{(1 + q_{\alpha^\vee/2}^{-1/2} s(t)^{-1})(1 - q_{\alpha^\vee/2}^{-1/2} q_{\alpha^\vee}^{-1/2} s(t)^{-1})}{1 - s(t)^{-2}}, & \text{if } 2\alpha \in R_{nr},
\end{cases}$$

(2.13)

where $R_{nr} = \Sigma_0 \cup \{2\alpha : \alpha^\vee \in \Sigma_0^\vee \cap 2Y\}$, and put $c(s) = \prod_{\alpha \in \Sigma_{0,+}} c(\alpha, s)$, where $R_{0,+}$ is the set of positive roots in $\Sigma_0$ (determined by the Borel subgroup we have used), and $q$ is the cardinality of the residue field of $k$. This $c$ function, regarded as a function in variables $t \in T$, is a rational function.

Let $P_{W_0}(q) = \sum_{w \in W_0} q^{l(w)}$ be the Poincaré polynomial of the finite Weyl group $W_0$. An element $w$ of the Weyl group $W_0$ acts on a character $\chi$ of $T$ by $w_\chi(a) = \chi(w^{-1}aw)$. If $\chi$ is not fixed by any nontrivial element of $W_0$, it is called regular.

We can state Macdonald’s formula now:

**Proposition 2.3.** Assume that $P_{W_0}(q^{-1}) \neq 0$. If $s$ is a regular unramified character of $T$ and $a \in T^{\mathcal{F}_0} := \{a \in T : |\alpha(a)| \leq 1 \forall \alpha \in \mathcal{F}_0\}$, then

$$\Phi_s(a) = \frac{\partial_{R}^{1/2}(a)}{P_{W_0}(q^{-1})} \sum_{w \in W_0} c(ws) ws(a).$$

(2.14)

**Remark 2.6.** We remark that for split groups, in Macdonald’s formula, if $s(t) = q^z$ for some complex number $z$, then we obtain (at least formally) the local $\gamma$ factor $\gamma_k(z)$ of a non-archimedean local field $k$ whose residue cardinality is $q$, up to a rational constant.

We conclude this section with an example.
Example 2.4. Let $G = \text{GL}_2(k)$ and let $K_0 = \text{GL}_2(o_k)$ be a maximal compact subgroup of $G$. The complex dual group of $G$ is $G^\vee = \text{GL}_2(\mathbb{C})$. The trivial representation triv of $G$, which can be obtained by taking $\chi = \delta_B^{-1/2}$ in $E = \text{Ind}_B^G(\delta_B^{1/2} \chi)$, is certainly unramified. The corresponding semisimple conjugacy classes, i.e., the Satake parameter in $G^\vee = \text{GL}_2(\mathbb{C})$ corresponding to the trivial representation, is the class of the matrix

$$s_0 = \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}.$$ 

In fact, the Satake parameter of triv coincides with the Langlands parameter of $\delta_B^{-1/2}$ (regarded as a character of $T$).

If we take $f = 1_{K_0 \lambda \omega \lambda}$, then triv$(f)$ is equal to $\int_G f(g) \, dg = |K_0 g K_0/K_0|$. One can show that this value is equal to the Poincaré polynomial $P_{W_0}(q)$ of the Weyl group $W_0$ ([20]).

A character $\chi : k^\times \to \mathbb{C}^\times$ is said to be smooth if it is trivial on a compact open subgroup of $k^\times$. Since $k^\times = o_k^\times \times \mathbb{Z}^\times$, a smooth $\chi$ consists of a finite order character $\chi_0 : o_k^\times \to \mathbb{C}^\times$ and a complex number $z \in \mathbb{C}/2\pi i \log(q) \mathbb{Z}$ such that $\chi(\omega) = \omega^{-z}$ (where $q$ is the cardinality of the residue field of $k$).

The torus $T = k^\times \times k^\times$ is maximal split in $G = \text{GL}_2(k)$. Let $\chi = (\chi_1, \chi_2)$ be a character of $T$, where $\chi_1, \chi_2$ are both smooth unramified characters of $k^\times$. The induced representation $\text{Ind}_B^G(\delta_B^{1/2} \otimes \chi)$ is called the principal series representation. This representation is well-known.

Proposition 2.4. The representation $\text{Ind}_B^G(\delta_B^{1/2} \otimes \chi)$ of $G$ admits a one dimensional subrepresentation if and only if $\chi_1 \chi_2^{-1}(y) = |y|^{-1}$ for all $y \in k^\times$. It admits a one dimensional quotient representation if and only if $\chi_1 \chi_2^{-1}(y) = |y|$ for all $y \in k^\times$. Apart from the two cases $\chi_1 \chi_2^{-1}(y) = |y|^{\pm 1}$, the representation $\text{Ind}_B^G(\delta_B^{1/2} \otimes \chi)$ is irreducible.

Another application of the Satake transform is to compute the character of $\pi(f)$ ([64, 20]). Recall that for an admissible representation $(\pi, V)$ of $G$, we can associate an operator $\pi(f)$ on $V$ to any $f \in \mathcal{H}(G)$, and the character $\text{Tr}(\pi(f))$ of $\pi(f)$ is a linear functional on $\mathcal{H}(G)$. Let $(\pi, E)$ be the normalised induced representation of $G$ as above. Macdonald computed the character of the operator $\pi(f)$ acting on the 1-dimensional vector space $E^{K_0}$ for any $f \in \mathcal{H}(G)$. Here we focus on the case that $f \in \mathcal{H}(G, K_0)$. Recall that in this case the trace of $\pi(f)$ is just the scalar we
denote by \( j_\chi(f) \) as above. One can show that for \( f \in \mathcal{H}(G, K_0) \),
\[
\text{Tr}(\pi(f)) = j_\chi(f) = \int_G f(g)\phi_\chi(g) \, dg
\]
\[
= \int_T \chi(t)\left( \delta_B^{-1/2}(t) \int_N f(an) \, dn \right) \, dt = \sum_T \chi(t)Sf(t)
\]
where \( Sf \) is the Satake transform of \( f \). The last equation holds, because \( Sf \) has finite support in \( T \), and hence the integral is simply a sum.

### 2.6 Matrix coefficients, formal degrees

Let \( G \) be a group of \( td \)-type. Let \((\pi, V)\) be a smooth representation of \( G \). The dual \((\bar{\pi}', V')\) is the representation defined by \( \bar{\pi}'(g) = \pi(g) \) on the algebraic dual space \( V' \) of \( V \). The contragredient representation \((\bar{\pi}, \tilde{V})\) of \( G \) is the restriction of \( \pi' \) to the subspace of smooth vectors of \( V' \) (i.e. vectors fixed by some open subgroup of \( G \)). There exists a natural pairing

\[
\langle \ , \ \rangle : V \times \tilde{V} \to \mathbb{C}^\times
\]

defined by \( \langle v, \tilde{v} \rangle = \tilde{v}(v) \). The matrix coefficient of \( \pi \) associated with \( v \) and \( \tilde{v} \) is the function \( c_{v, \tilde{v}}(g) = \langle \pi(g)v, \tilde{v} \rangle \) defined on \( G \).

Matrix coefficients are useful in the classification of irreducible smooth representations of groups of \( td \)-type. The idea is to list irreducible representations according to the growth rate of their matrix coefficients ([25]). In particular, in this scheme of classification, the so-called tempered representations are believed to be of nicest growth (see, e.g. the Langlands classification [55] of irreducible admissible \((g, K)\)-modules in terms of tempered representations of the parabolic subgroups).

A smooth representation \((\pi, V)\) of \( G \) is called tempered if all matrix coefficients \( c_{v, \tilde{v}} \) lie in the vector space \( L^{2+\varepsilon}(G) \), for all \( \varepsilon > 0 \).

Suppose \((\pi, V)\) is a smooth irreducible representation of \( G \). We say \( \pi \) is square integrable modulo the centre \( Z \) of \( G \) (we often just say square integrable) if \( \pi|_Z \) is a unitary character, and if for every \( v \in V \) and \( \tilde{v} \in \tilde{V} \), the matrix coefficient \( c_{v, \tilde{v}}(g) \) is square integrable on \( G/Z \) for some (equivalently, any) Haar measure. In order for an irreducible smooth \((\pi, V)\) to be square integrable, it is necessary and sufficient that there is some matrix coefficient \( c_{v_0, \tilde{v}_0}(g) \) which is square integrable over \( G/Z \).

It can be shown that an irreducible representation \((\pi, V)\) of \( G \) is square integrable if and only if \((\pi, V)\) can be embedded into the right regular representation of \( G \) on \( L^2(G/Z) \), the space of square integrable functions on \( G/Z \). The collection of all
irreducible smooth representations which are square integrable modulo the centre of \( G \), is called the \textit{discrete series} of \( G \). Thus, a square integrable representation is also called a discrete series representation. Casselman ([19]) gave criteria of square integrable representations and of tempered representations. We will, instead of group representations, state Casselman’s criteria for representations of Hecke algebras in Section 4.2.

Comparing the given norm on \( V \) with the \( L^2 \)-norm of the image of \( V \) in \( L^2(G/Z) \) leads to the following important notion.

**Proposition 2.5 (Formal degree).** Let \( (\pi, V) \) be a discrete series representation of \( G \). Denote the Haar measure on \( G \) by \( d\mu \). Then \( (\pi, V) \) is unitarisable and there exists a positive real constant \( d\mu(\pi) \) such that

\[
d\mu(\pi) \int_{G/Z} \langle \pi(g)u, \tilde{u} \rangle \langle \pi(g^{-1})v, \tilde{v} \rangle \, d\mu = \langle u, \tilde{v} \rangle \langle v, \tilde{u} \rangle, \quad u, v \in V, \tilde{u}, \tilde{v} \in \tilde{V}.
\]

The constant \( d\mu(\pi) \) is called the \textit{formal degree} of \( \pi \). Notice that it depends on the choice of Haar measure on \( G/Z \), namely they are inverse proportional. In other words, the formal degree is the normalisation factor of the Plancherel measure so that the Parseval identity holds.

An important example of discrete series is the \textit{supercuspidal representation}. A representation \( \pi \) of \( G \) is supercuspidal if every of its matrix coefficients has compact support modulo the centre \( Z \). Recall that for every unipotent radical \( U \) of a proper parabolic subgroup of \( G \), the matrix coefficient \( c_{v, \tilde{v}} \) is called a \textit{cusp form} if \( \int_U c_{v, \tilde{v}}(gn) \, dn = 0 \) for every \( g \in G \), every \( v \in V \) and \( \tilde{v} \in \tilde{V} \), where \( dn \) denotes the Haar measure on the unimodular group \( U \). If \( \pi \) is a smooth supercuspidal representation of \( G \), then every matrix coefficient of \( \pi \) is a cusp form.

To summarise, every supercuspidal representation is square integrable (modulo the centre of the group). In general, irreducible tempered representations occur as direct summands of unitarily parabolically induced square integrable representations.

**Remark 2.7.** To conclude this chapter, we point out that a conjecture of Hiraga, Ichino and Ikeda (abbreviated to HII conjecture) relates the formal degree of discrete series representations of reductive groups, to the adjoint \( \gamma \) factors coming from the functional equation of the corresponding \( L \)-functions. To state their conjecture one needs the notions of Langlands dual group and Langlands parameter, and we shall introduce this HII conjecture in next chapter, after reviewing the necessary notions.