On cuspidal unipotent representations

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Chapter 3

Lusztig’s classification of unipotent representations

3.1 Deligne-Lusztig characters

We start this chapter by reviewing a fundamental theory on the representation of finite groups of Lie type. We consider a connected reductive group $G$ over $\mathbb{F}_q$ with connected centre. The geometric Frobenius morphism Frob induces a Frobenius action on $G$ which will be denoted by $F$. The $F$-invariants $G^F = G(F_q)$ will be denoted by $G$. It is a finite group of Lie type.

Deligne and Lusztig constructed certain virtual characters of $G$, which we call Deligne-Lusztig characters. Such a character can be constructed from every $F$-stable maximal torus $T$ and an irreducible character of the $F$-invariants $T^F$ of this torus. Based on these characters Lusztig achieved a classification of irreducible representations of $G^F$. In particular, if $T$ is anisotropic, the Deligne-Lusztig virtual characters are cuspidal. We now review the Deligne-Lusztig characters and discuss Lusztig’s classification. Our main reference is Carter’s classic book [15].

Let $G$ be a connected reductive group over $\mathbb{F}_q$ with connected centre, and let $T \subset G$ be an $F$-stable maximal torus. Then $(G, T)$ determines a root datum $(X, R_0, Y, R_0^\vee)$. Fix a Borel subgroup $B = TU$. Let $\tilde{X} = \text{Lang}^{-1}(U)$, where $\text{Lang} : G \to G, g \mapsto g^{-1}F(g)$ is Lang’s mapping. We denote by $H^i_c(X, \mathbb{Q}_l)$ be the $i$-th $l$-adic cohomology group with compact support of $\tilde{X}$.

Observe that $\tilde{X}$ simultaneously has a left $G^F$-action and a right $T^F$-action because
for $x \in \tilde{X}$, we have
\[
\text{Lang}(gx) = x^{-1}g^{-1}F(g)F(x) = x^{-1}F(x) \in U, \text{ if } g \in G^F \\
\text{Lang}(xt) = t^{-1}x^{-1}F(x)F(t) \in t^{-1}Ut = U, \text{ if } t \in T^F.
\]

As a consequence both $G^F$ and $T^F$ act on $H^i_c(\tilde{X}, \overline{Q_l})$. These two actions commute. For every $g \in G^F$, and every character $\theta : T^F \to \overline{Q_l}^\times$ of $T^F$, where $l \neq q$ is prime, define a virtual character
\[
R_{T, \theta}(g) = \sum_{i \geq 0} (-1)^i \text{Tr}(g, H^i_c(\tilde{X}, \overline{Q_l})\theta).
\]

(3.1)

Here the subscript $\theta$ indicates the $\theta$-isotypic component. It is an important result that $R_{T, \theta}$ is independent of the choice of the Borel subgroup $B$ (see [15]). The virtual characters $R_{T, \theta}$ constructed above are the Deligne-Lusztig virtual characters.

### 3.1.1 Lusztig’s series

Lusztig did a great amount of work on the representation theory of finite groups of Lie type [29, 56, 58]. Even restricting to his classification of irreducible characters is worth writing a monograph (e.g. [58]). Here, I am just able to very briefly introduce his ideas on “Jordan decomposition of characters” with reference [15]. This is the tool that he used to compute the characters of all irreducible representations of $G^F$ and to obtain the classification results.

Denote $\text{Irr}(G)$ the collection of isomorphism classes of irreducible representations of $G := G(F_q)$. The classification of $\text{Irr}(G)$ starts with the parametrisation of the Deligne-Lusztig characters of $G$. We recall that in the ordinary representation theory of a finite group\(^1\), an irreducible representation is completely determined by its character. The character is a function defined on the conjugacy classes of the finite group, and the number of irreducible characters is equal to the number of conjugacy classes of the group itself. As a contrast, Deligne-Lusztig characters of $G$ are parametrised by the conjugacy classes of the “dual group” $G^\vee = (G^\vee)^F$. Here the group $G^\vee$ is determined by the root datum $(\Sigma, R_0^\vee, X, R_0)$ dual to the root datum of $G$. By slightly abuse of notation we still denote the Frobenius action on $G^\vee$ by $F$.

We consider two sets. The first one is $\nabla(G, F)$, consisting of all pairs $(T, \theta)$, where $T$ is an $F$-stable maximal torus of $G$ and $\theta \in \text{Irr}(T^F)$. Define an equivalence relation, called geometrically conjugate on $\nabla(G, F)$ by the rule that $(T, \theta) \sim (T', \theta')$ if and only if there exists an element $x \in G$ such that $T' = xT$, where $xT = xT x^{-1}$, and

\(^1\)Here “ordinary” means the characteristic of the ground field of the representation vector space is not a divisor of the order of the finite group.
\( \theta' = x\theta, \) where \( x\theta(t) = \theta(xtx^{-1}), \) for any \( t \in T = T^F. \) The set of equivalence classes of geometric conjugacy is denoted by \( \nabla(G, F)/G. \)

Secondly, we consider the set \( \nabla(T^\vee, s^\vee) \) where \( T^\vee \) be an F-stable maximal torus of \( G^\vee \) and \( s^\vee \in T^\vee. \) The equivalence relation of \textit{geometrical conjugacy} on \( \nabla(T^\vee, s^\vee) \) is defined as follows: Two pairs \( (T^\vee, s^\vee), (T'^\vee, s'^\vee) \) are said to be geometrically conjugate, if and only if there exists an element \( y \in G^\vee \) such that \( T'^\vee = yT^\vee y^{-1} \) and \( s'^\vee = ys^\vee y^{-1}. \) The totality of equivalence classes is denoted by \( \nabla(G^\vee, F)/G^\vee. \)

**Proposition 3.1** (Parametrisation of the Deligne-Lusztig characters). (1) If the F-stable tori \( T, T' \) of \( F \) are not \( G \)-conjugate, then \( R_{T,\theta} \) and \( R_{T',\theta'} \) are orthogonal (with respect to the inner product as defined in Theorem 7.3.4 loc. cit.).

(2) If \( (T, \theta) \) and \( (T', \theta') \) are not geometrically conjugate, then \( R_{T,\theta} \) and \( R_{T',\theta'} \) have no irreducible constituents in common.

These two propositions are Corollary 7.3.7 and Theorem 7.3.8 in [15] respectively.

We now state the classification of \( \text{Irr}(G). \) For a semisimple element \( s^\vee \in G^\vee \) we write \( [s^\vee] \) to the \( G^\vee \)-conjugacy class containing \( s^\vee. \) The \textit{Lusztig series of} \( G \) \( (\text{not of } G^\vee) \) associated with \( s^\vee \) is defined to be the set

\[
\text{Lu}(G, [s^\vee]) = \text{Lu}(G, s^\vee) = \{\chi \in \text{Irr}(G) \mid \langle \chi, R_{T^\vee, s^\vee} \rangle \neq 0 \text{ for some } (T^\vee, s^\vee) \in \nabla(G^\vee, F)\}.
\]

**Theorem 3.1** (Classification of \( \text{Irr}(G) \)). The collection of irreducible characters of \( G \) is partitioned as

\[
\text{Irr}(G) = \sqcup_{[s^\vee]} \text{Lu}(G, [s^\vee]),
\]

where \( [s^\vee] \) runs through all \( G^\vee \)-conjugacy classes of semisimple elements.

The constituent of series \( \text{Lu}(G, 1) \) are called \textit{unipotent representations}.

### 3.1.2 Cuspidal unipotent representations

In the general theory of representations of reductive groups, an important role is played by the so-called \textit{cuspidal} representations. The same phenomenon happens for the class of unipotent representations. This idea of cuspidality origins in Harish-Chandra’s work on Lie groups. We now recall the definition of cuspidal representations.

Let \( G \) be given as above. Assume moreover that \( G \) has a split \( BN \)-pair structure which satisfies the commutator relations (see [15]). Let \( P = MU_P \) be a standard parabolic subgroup of \( G \) where \( U_P \) is the unipotent radical of \( P. \) An irreducible
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character \( \rho \) of \( G \) is called cuspidal if \( \sum_{n \in U_P} \rho(ng) = 0 \) for all proper standard parabolic subgroup \( P = MU_P \) and all \( g \in G \).\(^2\) An irreducible representation is said to be cuspidal if its character is cuspidal. The importance of cuspidal representations can be seen at least from the following

**Proposition 3.2.** For every irreducible representation \( \pi \) of \( G \), there exists a standard parabolic \( P = MU_P \) and a cuspidal irreducible character \( \delta \) of \( M \), such that \( \pi \) occurs as a subrepresentation of the parabolic induction \( \text{Ind}^G_P \delta \). The pair \( (P, \delta) \) is uniquely determined by \( \pi \) up to conjugacy. Here, we extend \( \delta \) trivially over \( U_P \).

Let us now concentrate on the Lusztig family of unipotent characters. Recall the Deligne-Lusztig virtual characters \( R_{T, \theta} \). We write \( R_{T, 1} \) if \( \theta \) is the trivial character.

**Definition 3.1.** An irreducible representation \( \delta \) of \( G \) is called unipotent if \( \delta \) occurs as a component of \( R_{T, 1} \). We write \( \text{Irr}_{\text{unip}}(G) \) for the set of unipotent representations of a group \( G \).

Recall that \( G \) has connected centre. An irreducible character of \( G \) is called semi-simple if its average value on regular unipotent elements is nonzero (in which case the average value is 1 or \(-1\)).\(^3\) Every Lusztig series contains a unique semisimple character. In his 1984 monograph \([58]\) Lusztig defined the Jordan decomposition of characters and gave the classification of irreducible characters of \( G \).

**Proposition 3.3.** There exists a bijection \( \iota : \text{Lu}(G^F \hookrightarrow s_\cdot) \rightarrow \text{Lu}(C_{G^F}(s_\cdot)^F, 1) \), such that: If \( \chi_{ss} \in \text{Lu}(G^F, s_\cdot) \) is the unique semisimple character, then for any irreducible character \( \chi \in \text{Lu}(G^F, s_\cdot) \), the degree of \( \chi \) is given by \( \chi(1) = \chi_{ss}(1) \cdot \iota(\chi)(1) \).

The bijection \( \iota \) is called the Jordan decomposition of characters by Lusztig, analogous to the Jordan decomposition of elements of the group.

Using the Jordan decomposition of characters, Lusztig reduced the problem of determining the degree and multiplicity of an irreducible representation of any finite reductive \( G \) to the computation of \( \chi_u(1) \) \([58]\). One can further reduce this problem to the case where \( G \) is an adjoint simple group (see \([31]\)). The degrees and multiplicities of unipotent characters in simple groups are listed in \([15, \S \S 13.5-13.9]\).

We can reduce even further the study of unipotent representations to those cuspidal ones. For cuspidal unipotent representations, Howlett-Lehrer theory gives information on the decomposition of representations of the form \( \text{Ind}^G_P \delta \) where \( \delta \) is a cuspidal representation of the Levi factor \( M \) of the parabolic subgroup \( P = MU_P \).

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\(^2\)Functions on \( G \) satisfying this equation is referred to as a cusp form.

\(^3\)If \( p \) is a good prime for \( G \) (meaning that it does not divide any of the coefficients of roots expressed as linear combinations of simple roots), then, an irreducible character is semisimple if and only if its order is not divisible by \( p \).
of \( G \). The main objective in [58] is to deal with the case that the series \( Lu(G, s^\vee) \) for which \( C_{G^\vee}(s^\vee) \) is not contained in a Levi subgroup, because, if the centraliser is contained in a Levi subgroup of \( G^\vee \), then the bijection \( \iota \) in the above theorem can be realised using a Deligne-Lusztig induction map \( R^G_L \) from some Levi subgroup \( L \subset G \) to \( G \) (see for instance [31]).

### 3.2 Unipotent representations of \( p \)-adic groups

Let \( k \) be a non-archimedean local field with finite residue field \( \mathbb{F}_q \), and let \( k_{nr} \) be a maximal unramified extension of \( k \). Denote the ring of integers of \( k_{nr} \) by \( \mathcal{O}_{nr} \). This ring has a unique maximal ideal \( p_{nr} \). The residue field of \( k_{nr} \) is an algebraic closure of \( \mathbb{F}_q \) which we denote by \( \overline{\mathbb{F}}_q \). Recall that the geometric Frobenius Frob is the \( k \)-automorphism of \( k_{nr} \) whose inverse induces the automorphism \( x \mapsto x^q \) (mod \( p_{nr} \)), \( \forall x \in \overline{\mathbb{F}}_q \).

Suppose \( G \) is a connected, absolutely simple \( k \)-group, and \( G \) is split over \( k_{nr} \). We write \( G = G(k_{nr}) \). In this section, such a \( G \) is the main group we are interested in. The geometric Frobenius Frob gives rise to an automorphism \( F \) of \( G \). We have \( G(k) = G^F \). This is because \( G(k) \) is the fixed points of the action of the Galois group \( \text{Gal}(k_{nr}/k) \) on \( G \), and this Galois group is topologically generated by Frob.

From [11, 12] we know that there is a maximal \( k_{nr} \)-split torus \( S \subset G \) defined over \( k \) and maximally \( k \)-split. We fix such a maximal torus \( S = S(k_{nr}) \) of \( G \) and let \( S_0 = S(\mathcal{O}_{nr}) \) be the maximal compact subgroup of \( S \). Let \( G' \) be the derived group of \( G \). Denote \( G_1 \) the subgroup of \( G \) generated by \( G' \) and \( S_0 \). We note that \( G_1 \) is the simultaneous kernel of the group of unramified complex characters of \( G \).

After changing the base field of \( G \) to \( k_{nr} \), we have a model over \( \mathcal{O}_{nr} \). By the “modulo \( p_{nr} \) reduction” \( \pi: \mathcal{O}_{nr} \to \overline{\mathbb{F}}_q \) we obtain a reductive group \( G \) defined over the residue field \( \overline{\mathbb{F}}_q \) of \( k \). Let \( B \) be a Borel subgroup of \( G \). The group \( B := \pi^{-1}(B) \cap G_1 \) is a standard Iwahori subgroup. A subgroup of \( G \) is called a Iwahori subgroup if it is conjugate to a standard Iwahori subgroup.

A parahoric subgroup of \( G \) is a finite union of double cosets of an Iwahori group. Every parahoric subgroup of \( G \) is contained in the group \( G_1 \). One can also define the parahoric subgroup using the Bruhat-Tits building of \( G \) [40]. Let \( X_*(S) \) be the cocharacter lattice of \( S \). The vector space \( V := \mathbb{R} \otimes X_*(S) \) is an \( F \)-stable apartment of the Bruhat-Tits building of \( G \). Let \( C \subset V \) be an \( F \)-stable alcove. A parahoric subgroup of \( G \) is a group of the form \( \text{Stab}_G(F_P) \cap G_1 \), where \( F_P \subset C \) is a facet. Now we define a parahoric subgroup of \( G \) to be a subgroup conjugate to a standard parahoric subgroup.
Recall that a pro-$p$-group is the projective limit of a $p$-group. More precisely, a profinite group $H$ is a pro-$p$-group if for every open normal subgroup $H_1$ of $H$, the quotient group $H/H_1$ is a finite $p$-group. Let $\mathcal{P}$ be a parahoric subgroup of $G$, the pro-unipotent radical (or pro-$p$-radical) $\mathcal{P}_+$ is the maximal normal pro-$p$-subgroup of $\mathcal{P}$. The quotient $\overline{\mathcal{P}}$ for $\mathcal{P}/\mathcal{P}_+$ is the group of $\mathbb{F}_q$-points of a connected reductive group over $\mathbb{F}_q$.

An example of a parahoric subgroup is $G(\mathfrak{o}_{nr})$, the group of $\mathfrak{o}_{nr}$-points of $G$. It is a maximal compact subgroup which can be decomposed as $G(\mathfrak{o}_{nr}) = \sqcup_{w \in W_0} \mathcal{B} w \mathcal{B}$.

We remark that if $G$ is simply connected, the totality of maximal parahoric subgroups of $G(k)$ form $l+1$ conjugacy classes ($l$ is the rank of $G$), corresponding to the vertices of a chosen alcove $C$.

Now we fix an $F$-stable alcove $C$ in the apartment $V$. The group $W := N_G(S)/S_0$ is called the extended affine Weyl group of $G$. There is a normal subgroup $W_a$ of $W$ generated by reflections in the walls of $C$. Let $S_a$ be the collection of these reflections, then $(W_a, S_a)$ is a Coxeter system. One can verify that the quadruple $(G, \mathcal{B}, N_G(S), S_a)$ satisfies the axioms of $BN$-pairs, with $B$ the Iwahori subgroup $\mathcal{B}$ and $N = N_G(S)$. We have, as a consequence of the $BN$-pair structure of $G$, the affine Bruhat decomposition $G = B W \mathcal{B} = \sqcup_{w \in W} \mathcal{B} w \mathcal{B}$.

If $\Omega_C$ is the stabiliser of $C$ in $W$, then $W$ is the semi-direct product of $W_a \rtimes \Omega_C$. The group $\Omega_C$ is isomorphic to the quotient $X_+(S)/Z \Sigma_0'$, where $\Sigma_0 = \Sigma_0(G, S)$ is the root system attached to $(G, S)$, and $Z \Sigma_0'$ is the co-root lattice. Hence $\Omega_C$ is a finite abelian group. It is known that $\Omega_C$ is also isomorphic to $N_G(\mathcal{B})/\mathcal{B}$ [28, 75]. In what follows, we shall simply write $\Omega$ for $\Omega_C$.

The group $W_0 := N_G(S)/S$ is called the finite Weyl group of $G$. It is a subgroup of $W$, and $W$ is equal to the semi-direct product $W_0 \rtimes X_+(S)$.

At present we recall the notion of depth-zero representation. Let $\pi$ be a smooth irreducible representation of $G$ (By results of J. Bernstein [3] such a representation $\pi$ is admissible). We say that $\pi$ is a depth-zero representation if the restriction of $\pi$ on some $F$-stable parahoric subgroup $\mathcal{P}$ of $G$ contains nonzero $\mathcal{P}_+$-fixed vectors.

Let $\mathcal{P}$ be a parahoric subgroup. From the definition we see that both $\mathcal{P}, \mathcal{P}_+$ are $F$-stable. The Frobenius $F : \mathcal{P} \to \mathcal{P}$ induces an action of $F$ on $\overline{\mathcal{P}}$ which agrees with the $\mathbb{F}_q$-rational structure on $\overline{\mathcal{P}}$. Since $\mathcal{P}_+$ is a connected normal subgroup of $\mathcal{P}$, by Lang’s theorem for connected pro-algebraic groups, we have

$$\mathcal{P}^F / \mathcal{P}_+^F = (\mathcal{P} / \mathcal{P}_+)^F = \overline{\mathcal{P}}^F.$$

Let $\pi_{\mathcal{P}} : \mathcal{P}^F \to \overline{\mathcal{P}}^F$ be the natural projection. Given any representation $\delta$ of $\overline{\mathcal{P}}^F$, we lift it to a representation $\hat{\delta}$ of $\mathcal{P}^F$ via the projection $\pi_{\mathcal{P}}$. We say that a smooth representation $\rho$ of $G^F$ contains $\delta$ if the restriction $\rho|_{\mathcal{P}^F}$ contains an isotypic part corresponding to $\hat{\delta}$ (regarded as a representation of $\mathcal{P}^F$).
We now state the definition of \textit{unipotent representations} of $G^F$. We keep our notations of groups.

\textbf{Definition 3.2.} (1) A \textit{unipotent representation} $\pi$ of $G^F$ is a depth-zero representation containing a \textit{cuspidal unipotent} representation $\delta$ of $\mathbb{P}^F$ for some $F$-stable parahoric $\mathbb{P}$ of $G$. We denote by $\text{Irr}(G^F; \mathbb{P}, \delta)$ the set of isomorphism classes unipotent representations of $G^F$.

(2) If $\mathbb{P}$ is an $F$-stable parahoric subgroup of $G$, and $\delta$ is a cuspidal unipotent representation of $\mathbb{P}$, the pair $(\mathbb{P}, \delta)$ is said to be a \textit{unipotent type} of $G$. If in addition $\mathbb{P}$ is a maximal parahoric subgroup, such a pair is called a \textit{maximal unipotent type} of $G$.

Given a maximal unipotent type $(\mathbb{P}, \delta)$ of $G$, we notice that the compact induced representation $c\text{-}\text{Ind}_{\mathbb{P}}^{G_F} \sigma$ is a direct sum of finitely many irreducible supercuspidal representations. Also, if there is another $F$-stable parahoric subgroup $\mathbb{P}'$ of $G$ and a unipotent cuspidal irreducible representation $\delta'$ over $\mathbb{C}$ of the finite group $\mathbb{P}'^F$, then, the sets $\text{Irr}(G^F; \mathbb{P}, \delta)$ and $\text{Irr}(G^F; \mathbb{P}', \delta')$ are either disjoint or coincides ([61]). They are equal if and only if there exists $g \in G^F$ which conjugates $\mathbb{P}$ to $\mathbb{P}'$ and $\delta$ to a representation isomorphic to $\delta'$.

\textbf{Remark 3.1.} The case where $\mathbb{P}$ is an Iwahori subgroup $\mathbb{B}$ and $\delta = 1$ is the trivial representation was considered long time ago by Borel [7] and Matsumoto [66]. Casselman [18] proved the result that the irreducible representations in $\text{Irr}(G^F; \mathbb{B}, 1)$ are precisely the irreducible subquotients of the minimal unramified principal series. Lusztig has classified the unipotent representations of connected adjoint simple groups over a $p$-adic field, as described in the Introduction.

\subsection*{3.2.1 Hecke algebras as endomorphism algebras}

Let $\mathbb{B}$ be an Iwahori subgroup of $G = G(k)$ and let $\mathcal{H}(G, \mathbb{B})$ be the algebra of compactly supported functions $f : G \to \mathbb{C}$ which are bi-$\mathbb{B}$-invariant. For a general reductive group $G$, Iwahori and Matsumoto determined the structure of $\mathcal{H}(G, \mathbb{B})$ by giving a presentation of it using an explicit set of generators and relations. We call $\mathcal{H}(G, \mathbb{B})$ the \textit{Iwahori-Hecke algebra}. If we normalise the Haar measure such that $\text{meas}(\mathbb{B}) = 1$, then the indicator functions of the double cosets $\{T_x = 1_{\mathbb{B}, x, \mathbb{B}} : x \in W\}$ is a $\mathbb{C}$-basis for $\mathcal{H}(G, \mathbb{B})$ where $W$ is the extended affine Weyl group of $G$. ([7])

In case that $G$ is \textit{split} over $k$ with a maximal split torus $T$. Write $Y$ for the cocharacter lattice of $T$. It can be proved that the centre $Z_{\mathbb{B}}$ of the Iwahori-Hecke algebra $\mathcal{H}(G, \mathbb{B})$ is isomorphic to $\mathcal{H}(G, K_1)$ for some hyperspecial compact subgroup $K_1$ of $G$. Indeed, they are both isomorphic to $\mathbb{C}[Y]^{W_0}$, the Weyl group invariants of the group algebra $\mathbb{C}[Y]$. This is the Hecke algebra version of Satake’s
isomorphism.

Let $K$ be a compact open subgroup of $G$. Let $(\rho, V)$ be a smooth representation of $K$. Recall that we write $\text{c-Ind}_K^G \rho$ for the compactly induced representation of $G$. Here $\text{c-Ind}_K^G \rho = \text{Ind}_K^G \rho$ since $K$ is compact. We shall explain in this section that $\text{End}_G(\text{c-Ind}_K^G \rho)$, the endomorphism algebra of $\text{c-Ind}_K^G \rho$, can be interpreted as a convolution algebra of $\rho$-spherical functions ([82]).

The Hecke algebra of $\rho$-spherical functions $H_\rho(G, K)$ is the vector space of locally constant functions $\phi : G \to \text{End}_\mathbb{C}(V)$ satisfying $\phi(x_1 gx_2) = \rho(x_1)\phi(g)\rho(x_2)$ for all $x_1, x_2 \in K$ and all $g \in G$, and the support of $\phi$ is a finite union of $K$-double cosets. $H_\rho(G, K)$ is an associative $\mathbb{C}$-algebra under convolution:

$$1 \star f = \sum_{x \in G/K} \phi_1(x)\phi_2(x^{-1}g).$$

Now we can give an explicit isomorphism between $H_\rho(G, K)$ and $\text{End}_G(\text{c-Ind}_K^G \rho)$: Define $f_v \in \text{c-Ind}_K^G \rho$ with $\text{supp} f_v = K$ and $f_v(k) = \rho(k)v$. Consider the maps

$$\text{End}_G(\text{c-Ind}_K^G \rho) \to H_\rho(G, K), \phi \mapsto t_\phi$$

$$H_\rho(G, K) \to \text{End}_G(\text{c-Ind}_K^G \rho), t \mapsto t_\phi$$

defined as

$$t_\phi(g)(v) = t(f_v)(g), \quad g \in G, v \in V,$$

$$t_\phi(f)(g) = \sum_{x \in G/K} \phi(x)f(x^{-1}g), \quad f \in \text{c-Ind}_K^G \rho, \quad g \in G.$$

One can check that these maps give inverse vector space isomorphisms. Then, one checks that these maps are algebra homomorphisms, and hence $\text{End}_G(\text{c-Ind}_K^G \rho)$ is isomorphic to $H_\rho(G, K)$ as $\mathbb{C}$-algebras.

We are interested in the category of nondegenerate $H_\rho(G, K)$-modules. If $(\pi, E)$ is a smooth representation of $G$, then for $a \in E, v \in V$ and $f \in C_c^\infty(G, \text{End}_\mathbb{C}(V))$, we define the operator $\pi'(f)$ on $E \otimes V$ as

$$\pi'(f)(a \otimes v) = \int_G (\pi(x)a \otimes f(x)v) \, d\mu.$$

Define

$$e_\rho(x) = \frac{\rho(x)1_K(x)}{\text{meas}(K)},$$

$$\pi'(e_\rho)(a \otimes v) = \text{meas}(K)^{-1} \int_K (\pi \otimes \rho)(k)(a \otimes v) \, dk.$$

then, $e_\rho$ is the identity of

$$H_\rho(G, K) = e_\rho \ast C_c^\infty(G, \text{End}_\mathbb{C}(V)) \ast e_\rho.$$
Therefore, \( \pi'(e_\rho) \) projects \( E \otimes V \) onto \( (E \otimes V)^K \). Since for every \( f \in H_\rho(G, K) \) we have \( f = e_\rho \ast f \ast e_\rho \), then, \( (E \otimes V)^K \) is \( \pi'(f) \)-invariant. We denote by \( (\pi', (E \otimes V)^K) \) the representation of \( H_\rho(G, K) \) defined by

\[
 f \mapsto \pi'(f)|(E \otimes V)^K.
\]

This representation is finite dimensional if \( \pi \) is admissible.

**Proposition 3.4.** [67] Assume \( K, \rho, V \) as above. There is a bijection between isomorphism classes of irreducible smooth representations \((\pi, E)\) of \( G \) such that \((\pi', (E \otimes V)^K)\) is nonzero, and isomorphism classes of irreducible \( H_\rho(G, K) \)-modules.

From now on we shall identify the \( \rho \)-spherical Hecke algebra \( H_\rho = H_\rho(G, K) \) with the endomorphism algebra \( \text{End}(c\text{-Ind}_G^K \rho) \). One important example is when \( G = G(k) = G^F \) as above, with a parahoric subgroup \( K = \mathfrak{P}^F \), and the representation is a cuspidal unipotent representation \( \delta \). In this case, Morris has found a presentation of \( H_\delta \). Lusztig [61] also gave the same result, which we will review in next section. Moreover we have

**Proposition 3.5** (Morris [68], Moy-Prasad [70], Lusztig [61]). There is a natural bijection between the set of isomorphism classes of irreducible admissible unipotent representations of \( G \), and the set of isomorphism classes of finite dimensional simple \( H_\delta \)-modules.

### 3.2.2 The Kazhdan-Lusztig parameters

Here we review the Kazhdan-Lusztig parameters, with emphasis on Lusztig’s classification on unipotent representations of \( p \)-adic groups.

Let \( G \) be a connected, absolutely simple algebraic group of adjoint type, which is defined and quasi-split over \( k \) and split over the maximal unramified extension \( k_{nr} \) of \( k \). We write \( G = G(k_{nr}) \) for the \( k_{nr} \)-points of \( G \). Denote \( \mathfrak{g} \) the complex dual group of \( G \). In the present case \( \mathfrak{g} \) is simply connected.

If in addition \( G \) is split over \( k \), let \( \mathcal{B} \) be an Iwahori subgroup of \( G(k) \). It is known for a long time that \( \text{End}(c\text{-Ind}_{\mathcal{B}}^{G(k)} \text{triv}) \) is isomorphic to an affine Hecke algebra \( \mathcal{H}^{\text{IM}} \), whose Iwahori-Matsumoto presentation has equal parameter \( q \) (see [50]). Kazhdan and Lusztig [52] gave a complete classification of simple \( \mathcal{H}^{\text{IM}} \)-modules using equivariant \( K \)-theory. We first mention their result.

Let \( \mathcal{F} \mathcal{B} \) be the flag variety of all the Borel subgroups of \( G \). Given a unipotent \( u \in \mathfrak{g} \), define

\[
 \mathcal{F} \mathcal{B}_u = \{ B \in \mathcal{B} : u \in B \}
\]
to be the subvariety of Borel subgroups containing $u$. If a semisimple element $s \in G$ satisfying
\[ sus^{-1} = u^q, \]  
(3.2)
where $q$ is the cardinality of the residue field of $k$, we shall define $\mathcal{FB}^s$ analogously to $\mathcal{FB}_u$. Put $\mathcal{FB}^s = \mathcal{FB}_u \cap \mathcal{FB}^s$. According to [57] $\mathcal{FB}^s_u$ is not empty.

Let $C_G(s, u)$ be the simultaneous centraliser of $(s, u)$ in $G$. Put
\[ A(s, u) = C_G(s, u)/ZG(C_G(s, u)^\circ), \]
where $Z(G)$ is the centre of $G$ and $C_G(s, u)^\circ$ is the identity component of $C_G(s, u)$. If $s, u$ satisfy, there is a natural action of $A(s, u)$ on the variety $\mathcal{FB}^s_u$ (3.2).

A Kazhdan-Lusztig parameter is an ordered triple $(s, u, \rho)$ where $s \in G$ is semisimple, $u \in G$ is unipotent, satisfying (3.2), and $\rho \in \text{Irr}(A(s, u))$ which appears in the natural action of $A(s, u)$ on the $K_0$-group of $\mathcal{FB}^s_u$. The main result of [52] says that

**Proposition 3.6.** Suppose the Hecke algebra parameter $q$ is not a root of unity. There is a natural bijection between the set of equivalence classes of simple $\mathcal{H}^{\text{IM}}$-modules and the set of $G$ orbits of Kazhdan-Lusztig parameters.

A highly remarkable feature of the Kazhdan-Lusztig classification is the fact that the subsets of tempered and discrete series representations can be characterised in terms of geometric properties of the corresponding Kazhdan-Lusztig parameters.

Lusztig managed to generalise these results to obtain a complete classification of the irreducible unipotent representations of the inner forms of an unramified $k$-group $G$ ([61, 62]). This work is based on a series of his prior results.

To obtain this classification, Lusztig described first in detail a canonical bijection between the set of all equivalence classes of irreducible unipotent representations of the inner forms of $G$, and the set of irreducible modules over a certain direct sum of (extended) affine Hecke algebras obtained from algebras $H_\delta$ as considered in the previous section. On the geometric side, Lusztig envisaged that parameters which correspond to this set of irreducible unipotent characters is simply given by the same set of pairs $(s, u)$ as arise in the set of Kazhdan-Lusztig parameters for $G$, but this time we need to consider triples $(s, u, \rho)$ where $\rho$ runs over the complete set of irreducible representations of the finite group $C_G(s, u)/C_G(s, u)^0$ (as opposed to the original Kazhdan-Lusztig result, where one only considers the set of irreducibles of this group which descend to $A(s, u)$ and which appear in $K_0(\mathcal{FB}^s_u)$). We will see a bit later that the set of conjugacy classes of Kazhdan-Lusztig pairs $(s, u)$ for $G$ can be identified with the set of $G$-orbits of unramified local Langlands parameters for $G$, and the set of equivalence classes of “Lusztig triples” $(s, u, \rho)$ as just described is
then in natural bijection with the set of local systems on the $G$-orbits of unramified Langlands parameters.

Meanwhile, the description on the representation theoretic side is much more involved, and the matching of irreducible unipotent representations with the local systems is miraculous.

As an intermediate step in the classification procedure, Lusztig also used the graded version of affine Hecke algebras whose representations are relatively easier to deal with. Consequently, instead of the unipotent element $u$ occurring in the Kazhdan-Lusztig parameter, a nilpotent element $N$ in the Lie algebra $\text{Lie}(\mathfrak{g})$ of the dual group $\mathfrak{g}$ satisfying

$$\text{Ad}(s)N = qN, \quad s \in \mathfrak{g} \text{ is semisimple},$$

will appear in this classification parameter. Below we will turn to introduce Lusztig’s classification in the case where the quasi-split inner form $G_{qs}$ of $G$ is split over $k$. The main references I use are Lusztig’s original article [61] with a feature review by Morris.

(1) We retain our notations of groups. The Galois group $\text{Gal}(k_{nr}/k)$ acts on $G$ by the Frobenius automorphism, which we denote by $F$. Lusztig’s classification of unipotent representations of $G^F$ establishes a bijection between the collection $\mathcal{U}(G^F)$ of isomorphism classes of irreducible unipotent representations of $G^F$, and the set of triples $(s, N, \rho)$ where $s$ and $N$ satisfy (3.3), and $\rho$ is an irreducible representation of the component group of the simultaneous centraliser $C_G(s, N)$, on which the centre $Z(G)$ acts trivially.

We will call the triple $(s, N, \rho)$ the Lusztig-Langlands parameters. Note that $\mathfrak{g}$ acts naturally on the collection of the Lusztig-Langlands parameters. We denote by $\mathfrak{G}(\sqrt{q})$ the equivalence classes of Lusztig-Langlands parameters under the $G$ action.

The Galois action on $G$ via $F$ gives rise to a permutation $u$ on the vertices of the affine Dynkin diagram of $G$. (The permutation $u$ should not be confused with the unipotent element $u$ as in a Kazhdan-Lusztig pair $(s, u)$.) Using Lusztig’s notations we shall write $\mathbf{I}$ for the set of vertices of the affine Dynkin diagram of $G$.

It is known that we can always choose an $\text{Gal}(\overline{k}/k)$-stable Iwahori subgroup $\mathcal{B}$ in $G$. In the present case, since $G_{qs}$ is $k$-split and of adjoint type, this implies that the action of $\text{Gal}(\overline{k}/k)$ is given by twisting the standard split action by an element $u \in N_G(\mathcal{B})$. The resulting $k$-rational structure, given by the cocycle $z_u : F \mapsto u$ in $Z^1(F, G)$, is known to be split over an unramified extension of $k$ (see [28]). As explained in [28], the group $\Omega := N_G(\mathcal{B})/\mathcal{B}$ is isomorphic to $H^1(F, G)$ via the map $u \mathcal{B} \mapsto [z_u]$. The action of $F_u = \text{Ad}(u) \circ F_{qs}$ (where $F_{qs}$ denotes the standard Frobenius of the $k$-split group $G_{qs}$) induces a special diagram isomorphism of the
affine Dynkin diagram associated with $G_{qs}$. In this situation, Kottwitz’s theorem states that $\omega =: u_\mathcal{B} \in \Omega$ can be identified canonically with a character $\chi$ of the centre $Z$ of $G$.

Denote $\mathcal{U}(G^{F_u})$ for the collection of isomorphism classes of irreducible unipotent representations of $G^{F_u}$. Lusztig considered the various inner forms of $G$ at the same time. His main result asserts there is a bijection between $\sqcup_{u \in \Omega} \mathcal{U}(G^{F_u})$ and $\mathfrak{S}(\sqrt{q})$.

(2) Consider an $F$-stable parahoric subgroup $\mathcal{P}$ of $G$, and let $\mathcal{P}^F$ be the subgroup of $F$-invariants. We write $N \mathcal{P}$ for the normaliser of $\mathcal{P}$ in $G$. Up to $G$-conjugacy, each parahoric subgroup $\mathcal{P}$ of $G$ corresponds to a $u$-stable subset $J$ of $I$. If the parahoric $\mathcal{P}$ is chosen such that its reductive quotient $\overline{\mathcal{P}}^F$ contains a cuspidal unipotent representation $\iota: \mathcal{E} \hookrightarrow \mathcal{E}$, where $\mathcal{E}$ is the vector space of the representation $\delta$, then for every character $\psi$ of $(N \mathcal{P})^F / \mathcal{P}^F$, the data $(J, \mathcal{E}, \psi)$ is called an arithmetic diagram of type $u$ by Lusztig. Let $\mathfrak{A}_u$ be the totality of the arithmetic diagrams of type $u$. We shall point out that the assumption that $\mathcal{P}$ admits cuspidal unipotent representation is a strong restriction on both $\overline{\mathcal{P}}$ and $\mathcal{P}$, in view of the conditions in Section 3.2.3.

The action of the Frobenius element on the (reduced) building of $G$ is still denote by $F$ (we slightly abuse the notation). We remark that there exists an $F$-stable alcove $C$ in this building, which is invariant under the action of $\Omega$. If $\mathcal{P}$ is $F$-stable, corresponding to the facet $C_\mathcal{P}$, then $C_\mathcal{P}$ remains stable under the Frobenius action $F$, and

$$(N \mathcal{P})^F / \mathcal{P}^F = \Omega^F(\mathcal{P}),$$

where $\Omega(\mathcal{P}) = \text{Stab}_\Omega(C_\mathcal{P})$ is the stabiliser of $C_\mathcal{P}$ inside $\Omega$, and $\Omega^F(\mathcal{P})$ is its $F$-fixed point subgroup. In particular, the permutation action of $\Omega^F(\mathcal{P})$ on $I$ induces a homomorphism of $\Omega^F(\mathcal{P})$ to the group of permutations of the set $(I - J)/u$. We denote by $\Omega^F_1(\mathcal{P})$ (resp. $\Omega^F_2(\mathcal{P})$) the kernel (resp. image) of this homomorphism.

(3) Write $\mathcal{H}(u, I, J, E)$ for the endomorphism algebra $\text{End}_{C^F}(c\text{-Ind}_{\mathcal{P}^F}^{G^F} \mathcal{E})$. This algebra can be decomposed into the direct sum of two-sided ideals $\mathcal{H}^\psi$ corresponding to central idempotents, where $\psi$ runs through the Pontryagin dual $(\Omega^F(\mathcal{P}))^*$ of $\Omega^F(\mathcal{P})$. More explicitly, we have

$$\mathcal{H}(u, I, J, E) = \mathcal{H}^\psi \rtimes \Omega^F(\mathcal{P}).$$

This $\mathcal{H}^\psi$ is an unextended affine Hecke algebra. The affine Coxeter system and parameter function of this unextended affine Hecke algebra can be completely defined in terms of the unipotent type $\overline{\mathcal{P}}^F, \delta$. Lusztig showed that

$$\mathcal{H}(u, I, J, E) = \left( \mathcal{H}^\psi \rtimes \Omega^F_2(\mathcal{P}) \right) \otimes \mathbb{C}[\Omega^F_1(\mathcal{P})].$$

(3.4)
Remark 3.2. Since we assume that $G_{qs}$ is $k$-split, the action of $F$ on $\Omega$ (resp. on $\Omega(\mathcal{P}), \Omega_1(\mathcal{P})$ and $\Omega_2(\mathcal{P})$) is trivial. Hence in (3.4) we can write $\Omega(\mathcal{P})$ instead of $\Omega(\mathcal{P})^F$. Nonetheless, we keep the notations with $F$ so that the results stated here will also hold in the general case.

Each of the algebra $\mathcal{H}^{\psi}$ above is non-canonically isomorphic to a fixed affine Hecke algebra. We write $\mathcal{H}'(u, I, J, E)$ for the Iwahori-Matsumoto presentation of this Hecke algebra. Denote $\text{Irr} \mathcal{H}'(u, I, J, E)$ the set of isomorphism classes of irreducible unital modules of $\mathcal{H}'(u, I, J, E)$. Then we have a one-to-one correspondence between

$$\mathcal{U}(G^F) \leftrightarrow \sqcup_{(J, E, \psi) \in \mathfrak{a}_u} \text{Irr} \mathcal{H}'(u, I, J, E).$$

(3.5)

So far, we have discussed (very briefly) the settings on the group side (the arithmetic side) of Lusztig’s method of classification of unipotent representations. Below we turn to investigate the dual group side (the geometric side). In particular, we shall see how an affine Hecke algebra will arise from the information on the dual group.

(4) We start the geometric side by an important remark. Note that $G$ is simply connected (and simple) because $G$ is of adjoint type (and absolutely simple). In this case, to the data $(\mathcal{O}, \mathcal{F})$ where $\mathcal{O}$ is a nilpotent orbit of a Levi subgroup $\mathcal{M}$ of $G$, and $\mathcal{F}$ an irreducible equivariant cuspidal local system of $\mathcal{O}$ ([59]), Lusztig has constructed an affine Hecke algebra $\mathcal{H}(G, \mathcal{M}, \mathcal{O}, \mathcal{F})$, and a graded affine Hecke algebra $\overline{\mathcal{H}}(G, \mathcal{M}, \mathcal{O}, \mathcal{F})$ over $\mathbb{C}[z]$ ($z$ an indeterminate).

Let $S = S_u S_+$ denote the polar decomposition of the maximal torus $S$ of $G$, with $S_u$ its unitary part and $S_+$ the real split part. Recall that this means $S_u := \text{Hom}(X, \mathbb{R}/\mathbb{Z})$ and $S_+ := \text{Hom}(X, \mathbb{R}_{>0}^\times)$, where $X$ denotes the character lattice of $S$. Thus $S := \text{Hom}(X, \mathbb{C}^\times)$.

Assume that $t' \in S_u$, one then has a connected reductive group $C_G(t')$. Lusztig applied the above constructions to the case that $\mathcal{M} = C_G(t')$ is a Levi subgroup, as we will see.

(5) We denote by $I$ an index set in one-to-one correspondence with a set of simple affine roots determined by $(G, S)$. Then for every $t \in S$, the centraliser $C_G(t)$ is a Levi subgroup of $G$, generated by $S$ and by the root subgroups of $G$ corresponding to those roots $\alpha$ satisfying $\alpha(t) = 1$. We shall choose $t$ such that $\{\alpha : \alpha(t) = 1\}$ is a standard parabolic root sub-system. Let $J \subset I$ be the corresponding subset, so that $R_J = \{\alpha : \alpha(t) = 1\}$. We shall denote $C_G(t)$ by $G_J$. Notice that this implies that $C_G(t) = G_J$ is only depended on $J$.\footnote{One may find that $T_u$ is called the compact part, $T_+$ the hyperbolic part, and different notations may be used in other literature.}
Now assume that the Lie algebra $\text{Lie}(\mathcal{G})$ has a nilpotent orbit which carries a cuspidal local system $\mathcal{F}$ in the sense of Lusztig [61]. Let us temporarily assume that the complement $1 - J$ has at least two elements. Lusztig then constructed a root datum $(\mathcal{R}, \mathcal{L}, \mathcal{R}', \mathcal{L}')$ from the data $(\mathcal{G}_J, \emptyset, \mathcal{F})$. To this root datum he associated an affine Hecke algebra $H_J := H(G_j, \mathcal{O}, \mathcal{F})$. The centre $Z_J$ of $H_J$ can be identified with the algebra of regular functions of the quotient $(\mathcal{W}_0 \setminus \mathcal{S} \times \mathbb{C}^\times)$, i.e. $\text{Spm}(Z_J) = (\mathcal{W}_0 \setminus \mathcal{S}) \times \mathbb{C}^\times$, where the cocharacter lattice of $\mathcal{S}$ is $\mathcal{L}'$. Let $(\mathcal{W}_0 t, v_0), t \in \mathcal{S}$ be a central character of $Z_J$, and denote $\text{Irr}_{\mathcal{W}_0 t, v_0} H_J$ the collection of isomorphism classes of irreducible $H_J$-modules where $Z_J$ acts by the character $(\mathcal{W}_0 t, v_0)$.

(7) For any central character $(\mathcal{W}_0 t, v_0)$ as above, decompose $t = t_c t_+$. To $t_c$, Lusztig associated an element $t_c^\psi \in \mathcal{S}_u$, such that $\mathcal{G} J$ is a Levi subgroup of $C_G(t_c^\psi)$. We then have a graded affine Hecke algebra $\overline{H}(C_G(t_c^\psi), \mathcal{G}_J, \emptyset, \mathcal{F})$ as in (4). The element $\log t_+ \in \text{Lie}(\mathcal{S})$, so we can identify $\text{Lie}(\mathcal{S})$ with $\text{Lie}(Z_J)$. Just as in (6), we can form the set $\text{Irr}_{\mathcal{W}_{t_c,0}, \log t_+, \log v_0} H(C_G(t_c^\psi), \mathcal{G}_J, \emptyset, \mathcal{F})$, where the Weyl group $\mathcal{W}_{t_c,0}$ can be identified with $N_{C_G(t_c^\psi)}(G_j)/\mathcal{G}_J$. Using the results established in previous sections of his paper [61], Lusztig was able to set up a bijection

$$\text{Irr}_{\mathcal{W}_0 t, v_0} H(G_j, \mathcal{G}_J, \emptyset, \mathcal{F}) \leftrightarrow \text{Irr}_{\mathcal{W}_{t_c,0}, \log t_+, \log v_0} \overline{H}(C_G(t_c^\psi), \mathcal{G}_J, \emptyset, \mathcal{F}). \quad (3.6)$$

**Remark 3.3.** We have assumed that $1 - J$ has at least two elements. We can drop this assumption (see [61, 5.18]). In fact, if $1 - J$ has only one element, the algebra $H_J$ is defined to be $\mathbb{C}[v, v^{-1}]$, and the algebra $\overline{H}(C_G(t_c^\psi), \mathcal{G}_J, \emptyset, \mathcal{F})$ is $\mathbb{C}[z]$, and we have $\mathcal{S} = \{1\}$. The bijection above is reduced to a bijection between two sets with one element.

(8) Fix a number $v_0 \in \mathbb{R}_{>0}$. Let $\mathfrak{G}$ denote the collection of triples $(J, \emptyset, \mathcal{F})$, and consider the disjoint union $\bigsqcup_{(J, \emptyset, \mathcal{F}) \in \mathfrak{G}} \text{Irr} H_{J,v_0}$ (the subscript $v_0$ means the Hecke algebra specialised at $v_0$). By changing the orbits of $\mathcal{S}$ under the action of $\mathcal{W}_0$, and applying the results in (4) – (7), Lusztig showed there is a bijection

$$\bigsqcup_{(J, \emptyset, \mathcal{F}) \in \mathfrak{G}} \text{Irr} H(G_j, \mathcal{G}_J, \emptyset, \mathcal{F})_{v_0} = \bigsqcup_{(J, \emptyset, \mathcal{F}) \in \mathfrak{G}} \text{Irr} H_{J,v_0} \leftrightarrow \mathfrak{G}(v_0), \quad (3.7)$$

where the set $\mathfrak{G}(v_0)$ consists of Lusztig-Langlands parameters $(s, N, \rho)$, except that $\mathfrak{q}$ is replaced by $v_0^2$.

One can refine this parametrisation as follows. Recall that the group of characters of $\mathbb{Z}$ (the centre of $\mathcal{G}$) can be identified with $\Omega$. For any $\chi \in \Omega$ we define two sets

- $\bar{\mathfrak{G}}_\chi :=$ the collection of triples $(J, \emptyset, \mathcal{F})$ where $\mathbb{Z}$ acts on each stalk of $\mathcal{F}$ by $\chi$,  
- $\mathfrak{G}(v_0)_\chi :=$ the collection of triples $(s, N, \rho)$ where the restriction of $\rho$ on the image of $\mathcal{G}$ acts by $\chi$. 

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We have natural partitions $\mathcal{G}(v_0) = \sqcup_{\chi} \mathcal{G}(v_0)_{\chi}$ and $\bar{\mathcal{G}} = \sqcup_{\chi} \bar{\mathcal{G}}_{\chi}$, where $\chi$ runs through $\Omega$ at both decomposition.

The triples $(J, O, \mathcal{F})$ in $\bar{\mathcal{G}}_{\chi}$ are called *geometric diagrams of type* $\chi$ by Lusztig. We have the following refinement of the bijection (3.7)

$$\bigsqcup_{(J, O, \mathcal{F}) \in \bar{\mathcal{G}}_{\chi}} \text{Irr} \mathcal{H}_{J, v_0} \leftrightarrow \mathcal{G}(v_0)_{\chi}, \quad (3.8)$$

(9) Finally, we arrive at Lusztig’s arithmetic/geometric correspondence [61, §6, §7].

We shall consider the various inner forms of the group $G$. Since $G$ is of adjoint type, the inner forms of $G$ are parametrised by $H^1(F, G)$ (with $G = G(k_{nr})$). In our present situation, Kottwitz’s theorem says this Galois cohomology set is isomorphic to $\Omega$, which turns out to be the Pontryagin dual of $\mathbb{Z}(G)$.

Lusztig’s arithmetic/geometric correspondence is a combination of the various bijections (3.5), (3.6), (3.7), (3.8) established above. It says if $u \in \Omega$ and $\chi \in (\mathbb{Z})^*$ correspond to each other, then there exists a bijective correspondence

$$\mathfrak{A}_u \leftrightarrow \bar{\mathcal{G}}_{\chi}, \quad (J, E, \psi) \mapsto (J, O, \mathcal{F}) \quad (3.9)$$

in the sense that the affine Hecke algebras $\mathcal{H}(u, I, J, E)$ and $H(G, G_J, O, \mathcal{F})_{\sqrt{q}}$ have the same Iwahori-Matsumoto presentation, and hence are isomorphic.

The proof of the correspondence is a case by case verification. In our account of Lusztig’s work we have made the assumption of $G$ being a split $k$-group, but we point out that the bijection (3.8) also holds for general connected reductive unramified $k$-groups.

### 3.2.3 Formal degrees of cuspidal unipotent representations

Let $G$ be a connected reductive absolutely simple algebraic group which is defined over $k$ and unramified. Denote $G = G(k)$, and let $(\mathcal{P}, \sigma)$ be a maximal $F_u$-stable unipotent type.

We normalise the Haar measure on $G$ in the same way as [28], such that $\text{vol}(\mathcal{P}^F) = v^a|\mathcal{P}^F_a|$, where $v^2 = q$, and $a \in \mathbb{Z}$ is a suitable integer to keep $v^a|\mathcal{P}^F_a|$ invariant under the substitution $v \mapsto v^{-1}$. Then the formal degree $\text{fdeg}(\sigma)$ of $\sigma$ is given by $\text{fdeg}(\sigma) = |\mathcal{P}^F|^{-1} \deg(\sigma)$. By [75, Definition 2.6], all these formal degrees can be expressed as the product of a non-zero rational constant times the $q$-rational part $\text{fdeg}_q$, where $\text{fdeg}_q$ is a fraction of products of $q$-integers $[n]_q := \frac{v^n - v^{-n}}{v - v^{-1}}$ where $n \in \mathbb{Z}_{>0}$ and $v^2 = q$. 

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Suppose $G$ is an absolutely simple classical group (or isogenous to a classical group). Then $G$ has at most one cuspidal unipotent representation. These cuspidal unipotent representations only occur if the ranks of $G$ meet certain requirements. These conditions, as well as the degrees, can be found in Carter’s book [15, §13.7]. We need the $q$-rational part of these formal degrees later. Therefore, we include them here. In the list below $l$ is the rank of $G$ and $s \in \mathbb{Z}_{\geq 0}$:

1. **Type $A_l$**  
   No cuspidal unipotent characters for classical groups, but it has one cuspidal unipotent character coming from the anisotropic inner form. See the discussion below.

2. **Type $^2A_l$**  
   This group has no cuspidal unipotent character unless $l = (s/2)(s+1) - 1$ for some $s$. In this case, $G$ has one cuspidal unipotent character. Its formal degree is given by
   
   $$d_s^{(2A)}(q) = \prod_{j=1}^{s} (v^{2j-1} + v^{1-2j})^{j-s-1}$$

3. **Type $B_l$**  
   This group has no cuspidal unipotent character except if $l = s(s+1)$ for some $s$. In this case, $G$ has one cuspidal unipotent character, whose formal degree is given by
   
   $$d_s^B(q) = \frac{1}{(v + v^{-1})^{2s}(v^2 + v^{-2})^{2s-1} \ldots (v^{2s} + v^{-2s})}.$$ 

4. **Type $C_l$**  
   The condition on the rank and the formal degree groups of this type are exactly the same as type $B_l$. We shall write $d_s^B(q)$ for this formal degree too.

5. **Type $D_l$**  
   This group has no cuspidal unipotent character except if $l = s^2$ for some even $s$. In this case, $G$ has one cuspidal unipotent character, whose formal degree is given by
   
   $$d_s^D(q) = \frac{1}{(v + v^{-1})^{2s-1}(v^2 + v^{-2})^{2s-2} \ldots (v^{2s-1} + v^{1-2s})}.$$ 

6. **Type $^2D_l$**  
   This group has no cuspidal unipotent character unless $l = s^2$ for some odd $s$. In this case, $G$ has one cuspidal unipotent character, whose formal degree is equal to $d_s^D(q)$.

Besides these classical groups, for the group $G = \text{PGL}_l$, its anisotropic inner form $G(u)$ provides the only cuspidal unipotent representation for type $A_{l-1}$. In fact, any anisotropic absolutely simple unramified group is isomorphic to $\text{PGL}_1(\mathbb{D}) = \mathbb{D}/k^\times$. Here, $\mathbb{D}$ is a division algebra over $k$ with centre equals to $k$, and of degree...
l i.e. \( \dim_k(\mathbb{D}) = l^2 \). There is only one parahoric subgroup \( \mathcal{P} \) of \( G(u) \), namely \( \mathcal{P} = G(u)_1 \) (analogue to the group \( G_1 \) of \( G \)). To obtain the formal degree within our normalisation above, we have to know \( |\mathcal{P}^\mathbb{F}| \). It can be verified that the \( q \)-rational part of this formal degree is \( [l]_q^{-1} \).

### 3.3 Local Langlands correspondence for unipotent representations

#### 3.3.1 Local Langlands parameters

We keep our the notations. The group \( G \) we are interested in is a connected reductive \( k \)-group of adjoint type. The group \( G^\vee = G^\vee(C) \) is called the (complex) dual group of \( G \). Denote \( Z^\vee \) the centre of \( G^\vee \). The absolute Galois group \( \text{Gal}(\bar{k}/k) \) acts on \( G \) via its action on the root datum \( \Sigma = (X^*, \Sigma_0, X_*, \Sigma^\vee_0) \) associated with \( G \) and its maximal torus \( S \). The \( L \)-group of \( G := G(k_{nr}) \) (denoted by \( L^G \)) is defined to be the semi-direct product group \( L^G := G^\vee \rtimes \text{Gal}(\bar{k}/k) \).

A local Langlands parameter is a homomorphism \( \varphi : W_k \times \text{SL}_2(C) \to L^G \) such that:

1. \( \varphi \) is continuous and the restriction of \( \varphi|_{\text{SL}_2(C)} : \text{SL}_2(C) \to G^\vee \) is an algebraic group homomorphism;
2. \( s = \varphi(\text{Frob}) \) is semisimple;
3. \( \text{pr} \circ \varphi|_{W_k} \to \text{Gal}(\bar{k}/k) \) is the canonical projection, where \( \text{pr} : L^G \to \text{Gal}(\bar{k}/k) \).

The group \( G^\vee \) acts on the set of local Langlands parameters by conjugation. Two parameters are considered to be equivalent if they are in the same \( G^\vee \)-orbit.

If \( \varphi \) is trivial on \( I_k \) we say \( \varphi \) is unramified. In this case we can regard \( \varphi \) as a homomorphism \( \varphi : (\text{Frob}) \times \text{SL}_2(C) \to L^G \), thus \( \varphi \) is determined, up to conjugacy, by the elements \( s = \varphi(\text{Frob}), c = \varphi\left(\begin{smallmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{smallmatrix}\right) \) and \( u = \varphi(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}) \). Notice that such an equivalence class of unramified parameters is simply a conjugacy class in \( G^\vee \).

Let \( A_\varphi \) be the centraliser in \( G^\vee \) of the image of the local Langlands parameter \( \varphi \). We say that \( \varphi \) is discrete if \( A_\varphi \) is finite.

**Remark 3.4.** (1) The name of Langlands parameter comes from the local Langlands conjecture which claims that the collection of equivalence classes of tempered irreducible representations of a reductive connected group \( G \) over a local field, is the union of some finite subsets defined by stability properties (called \( L \)-packets), and these subsets are parametrised by the Langlands parameters. The tempered \( L \)-packets are supposed to be the minimal subsets.
with the property that there exists a positive linear combination of the corresponding characters which is a stable distribution of $G$. Since all members of such an $L$-packet have the same Langlands parameter, they give the same Artin $L$-functions for a given finite dimensional representation of $^LG$. In this sense they are $L$-indistinguishable.

(2) Sometimes, if $G$ has special properties, then considering a “finite form” of its $L$-group is more convenient. For example if $G$ is unramified, we denote by $G_{qs}$ its quasi-split inner form. Let $k_0$ be the finite extension of $k$ such that $G_{qs}$ is split on $k_0$. The Galois group action factors through the subgroup $\text{Gal}(\bar{k}/k_0)$. Therefore, the finite Galois group $\text{Gal}(k_0/k)$ acts on $G^\vee$. The finite form of the $L$-group $^LG$ of $G$ is defined to be the semi-direct product $G^\vee \rtimes \text{Gal}(k_0/k)$.

### 3.3.2 The Kottwitz isomorphism

Assume from now on that $G := G(k_{nr})$ is split over $k_{nr}$. The Frobenius element induces an automorphism of $Z^\vee$. We denote this action still by $F$ by abuse of notation. The centre $^LZ$ of $^LG$ is equal to $(Z^\vee)_F$. Thus, $^LZ$ is finite if and only if the maximal torus in the centre $Z$ of $G$ is anisotropic over $k$. Assuming this, then a local Langlands parameter $\varphi$ is discrete if and only if its image does not lie in any proper Levi subgroup of $G^\vee$.

Let $\pi_0((Z^\vee)_F)$ be the component group of $(Z^\vee)_F$. The following theorem of Kottwitz allows us to parametrise $H^1(F, G) := H^1(\text{Gal}(k_{nr}/k), G)$ using irreducible representations of $\pi_0((Z^\vee)_F)$.

**Proposition 3.7** (Kottwitz’s isomorphism [28]). There is a canonical bijection between $H^1(F, G)$ and the set $\text{Irr}[\pi_0((Z^\vee)_F)]$ of irreducible complex representations of $\pi_0((Z^\vee)_F)$. In particular, if $(Z^\vee)_F$ is finite, then $H^1(F, G) = \text{Hom}((Z^\vee)_F, \mathbb{C}^\times)$.

The Kottwitz isomorphism can be simplified if $G$ is semisimple. Let $S \subset G$ be a maximal $k_{nr}$-split torus, and let $X_*(S)$ be its cocharacter lattice. Let $X_{sc,*} = X_*(S_{sc})$ be the cocharacter lattice of the inverse image of $T$ in the simply connected cover $G_{sc}$ of $G$ (hence $X_{sc,*}$ is the co-root lattice of $(G, T)$). Denote the finite abelian group $X_*/X_{sc,*}$ by $\Omega$. A remarkable fact is that we can identify $\text{Hom}(Z^\vee, \mathbb{C}^\times)$ with $\Omega$ canonically (see the previous discussion in Section 3.2.2). In this case the Kottwitz isomorphism simplifies as

$$H^1(F, G) \xrightarrow{\sim} \Omega/(1 - F)\Omega. \quad (3.10)$$
3.3.3 Enhanced discrete Langlands parameters

We have introduced Lusztig’s parametrisation of irreducible unipotent characters and witnessed that this parametrisation fits Langlands’ philosophy. In this subsection, we state an enhanced version of the Lusztig-Langlands parameters.

Let \( G \) be a connected unramified semisimple group over \( k \), and \( G := G(k_{nr}) \). Let \( G_{qs} \) be the quasi-split inner form of \( G \), which splits over a finite extension \( k_0 \) of \( k \), and let \( G_{ad} = G/Z(G) \) be the adjoint form of \( G \). Denote by \( (G^\vee)_{sc} \) the simply-connected cover of the derived group of the complex dual group \( G^\vee \) of \( G \). The group \( (G^\vee)_{sc} \) is the complex dual group of \( G_{ad} \). Put \( L(G_{ad}) = (G^\vee)_{sc} \times \text{Gal}(k_0/k) \) and denote \( L(Z_{ad}) \) the centre of \( L(G_{ad}) \). Explicitly, \( L(Z_{ad}) \) is the \( \text{Gal}(k_0/k) \)-invariants in \( (Z^\vee)_{sc} \), the centre of \( (G^\vee)_{sc} \).

Applying the Kottwitz isomorphism to \( G_{ad} \) we obtain an isomorphism

\[
H^1(k; G_{ad}) \simeq \text{Hom}(L(Z_{ad}), \mathbb{C}^\times).
\]

We choose an element \( \zeta_G \in \text{Hom}((Z^\vee)_{sc}, \mathbb{C}^\times) \), whose restriction to \( LZ_{ad} \) corresponds to \( G \) via Kottwitz’s isomorphism. If \( G = G_{qs} \) then we take \( \zeta_{G_{qs}} \) to be the trivial character.

Let \( \varphi : W_k \times \text{SL}_2(\mathbb{C}) \to LG \) be a discrete Langlands parameter of \( G \). Recall that \( A_\varphi \) is the centraliser of the image of \( \varphi \) in \( G^\vee \). The group \( A_\varphi/LZ \) is a finite subgroup of the adjoint group \( (G^\vee)_{ad} = G^\vee/Z(G^\vee) \). Let \( A_\varphi \subset (G^\vee)_{sc} \) be the full pre-image of \( A_\varphi/LZ \). Therefore, \( (Z^\vee)_{sc} \subset A_\varphi \) and \( A_\varphi/(Z^\vee)_{sc} = A_\varphi/LZ \). We define \( \text{Irr}(A_\varphi, \zeta_G) \) to be the set of complex irreducible representations of \( A_\varphi \) whose restriction on \( (Z^\vee)_{sc} \) is \( \zeta_G \cdot \text{id} \).

The pair \((\varphi, \rho)\) with \( \varphi \) a discrete Langlands parameter and \( \rho \in \text{Irr}(A_\varphi, \zeta_G) \) is called an enhanced discrete Langlands parameter (EDLP) for \( G = G(k) \). Note that \( G^\vee \) acts on the collection of EDLP. We write \( L(G/k) \) for the collection of equivalence classes of enhanced discrete Langlands parameters of \( G \).

3.3.4 The HII conjecture

We now turn to the local Langlands correspondence of unipotent representations.

There is a conjecture by three Japanese mathematicians Hiraga, Ichino and Ikeda (HII) on the relation of adjoint \( \gamma \)-factors and formal degrees. We are particularly interested in investigating formal degrees of unipotent representations and try to verify the HII conjecture for unipotent representations for connected adjoint group \( G \) over a non-archimedean local field \( k \).
To state the HII conjecture we need the adjoint $\gamma$ factor of the pair $(\varphi, V)$ where $\varphi$ is a Langlands parameter of $G$ and $V$ is a representation of $L^G$. Following [46], we define the adjoint $\gamma$ factor $\gamma(\varphi, V; s)$ by the functional equation of the local $L$-function $L(\varphi, V; s)$ as follows:

$$
\gamma(\varphi, V; s) = \frac{L(\varphi, \tilde{V}; 1 - s)\epsilon(\varphi, V; s)}{L(\varphi, V; s)}
$$

(3.11)

where $\epsilon(\varphi, V; s)$ is the $\epsilon$-factor ([94]), and $\tilde{V}$ is the contragredient of $V$.

Then we state the following conjecture, combining various results and conjectures due to Lusztig [61, 62], Reeder [80, 81, 78] and HII [46].

**Conjecture 1.** For a connected reductive group $G$ over $k$, we assume that the maximal torus in the centre $Z$ of $G$ is anisotropic. Let $\mathcal{L}(G/k)$ be as above, and let $\text{Irr}^2(G/k)$ be the totality of equivalence classes of irreducible discrete series representations of $G$.

There is a bijection

$$
\text{Irr}^2(G/k) \leftrightarrow \mathcal{L}(G/k), \quad \pi_{(\phi, \rho)} \mapsto (\varphi_\pi, \rho_\pi),
$$

(3.12)

satisfying the following properties:

1. Lusztig’s parametrisation described above for irreducible unipotent representations is the local Langlands parametrisation.

2. If $G$ is split over $k_{nr}$, let $\mathcal{P}$ be a parahoric subgroup of $G$ and let $\mathcal{P}_+$ be its pro-$p$-radical. Put $\overline{\mathcal{P}} = \mathcal{P}/\mathcal{P}_+$. We normalise the Haar measure of $G$ such that $\text{vol}(\mathcal{P}) = q^{-\dim(\overline{\mathcal{P}})/2}|\overline{\mathcal{P}}|$. Then, for any $\pi \in \text{Irr}^2(G/k)$, the formal degree of $\pi$ with respect to the normalised Haar measure as above, is given by

$$
f\text{deg}(\pi) = \frac{\dim(\rho_\pi)}{|A_{\varphi_\pi/LZ}|}q^{-\dim(G)/2}|\gamma(\varphi, \text{Ad}; 0)|
$$

(3.13)

Some remarks follow.

**Remark 3.5.** (i) Our choice of normalisation of Haar measure follows [28]. This choice differs from the one in [46] by a power in $q^{1/2}$.

(ii) Note that we do not claim the conjectural bijection (3.12) should be unique. However, the conditions above are so selective that sometimes $\pi_{(\phi, \rho)}$ is forced to be a particular representation.

(iii) A conjecture on formal degrees in this fashion (though not precisely as the one stated here) is probably firstly appeared (in literature) in Shahidi [88].