On cuspidal unipotent representations

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Chapter 5

A uniqueness property

5.1 Cuspidal formal degrees

Let $G$ be a connected, absolutely simple algebraic group, which is defined and quasi-split over $k$, and split over the maximal unramified extension $k_{nr}$ of $k$. Denote $G = G(k_{nr})$. We recall that the pure inner forms of $G$ is parametrised by $H^1(F, G) \simeq \mathcal{O} / (1 - F) \mathcal{O}$. Our main interest is to study the category $\mathcal{U}^c(G_{\text{Fu}})$ of cuspidal unipotent representations of $G_{\text{Fu}}$ with $G$ a classical group. In notations of Carter [15], the root systems of absolutely simple classical groups are of the types

$$A_n, 2A_n, B_n(n \geq 2), C_n(n \geq 3), D_n(n \geq 4), D_n(n \geq 4).$$

(The triality $3D_4$, is regarded as a quasi-split exceptional group.) The connected reductive group of adjoint type corresponding to these root systems are (in the same order) the projective general linear group $\text{PGL}_{n+1}$, projective unitary group $\text{PU}_{n+1}$, special orthogonal group $\text{SO}_{2n+1}$, projective conformal symplectic group $\text{PCSp}_{2n}$, projective conformal transform group $\text{P}((\text{CO}_{2n})^0)$ and $\text{P}((\text{CO}_{2n})^0)$.

To obtain cuspidal unipotent representations of classical groups, the conditions on the ranks as stated in Section 3.2.3 should be satisfied. For every classical group not of type A, we can find a parahoric subgroup $\mathcal{P}$ which admits a cuspidal unipotent representation. Such a parahoric subgroup is determined by the rule that the reductive quotient $\mathcal{P} / \mathcal{P}_+$ satisfies the rank conditions stated in Section 3.2.3. Hence the subdiagram corresponding to $\mathcal{P} / \mathcal{P}_+$ of the extended affine Dynkin diagram of $G(u)$ (the inner form of $G$ determined by the cocycle $u$), is determined by two non-negative integers $a, b$.

We have defined the arithmetic diagram of an affine extended Hecke algebra $\mathcal{H}(\mathcal{R}, m)$ associated with the root datum $\mathcal{R}$ and the parameter function $m$. On the other
hand, our $\mu$-function $\mu_{m-,m+}(\mathcal{R})$ reflects the spectral properties of $\mathcal{H}(\mathcal{R}, m)$ via the “spectral parameters” $(m-, m+)$. These parameters lie in a space $\mathcal{V}$, which is defined as

$$\mathcal{V} = \{(m-, m+) : 4m_\pm \in \mathbb{Z}, \text{ and } 2(m_- - m_+) \in \mathbb{Z}\}.$$  

We shall decompose $\mathcal{V}$ into six disjoint subsets $\mathcal{V}^X$ with $X \in \{I, II, III, IV, V, VI\}$, in response to the six classical groups $PU_{2n}, PU_{2n+1}, SO_{2n+1}, PCSp_{2n}, P(CO_{2n})^0$, and $P((CO_{2n})^0)$. Firstly we introduce some notations: for $0 < m_\pm \in \mathbb{Z} \pm \frac{1}{4}$, we write

$$m_\pm = \kappa_\pm + \frac{2\epsilon_\pm - 1}{4} \quad (5.1)$$

with $\epsilon_\pm \in \{0, 1\}$ and $\kappa_\pm \in \mathbb{Z}_{\geq 0}$. Define $\delta_\pm \in \{0, 1\}$ by $\kappa_\pm \in \delta_\pm + 2\mathbb{Z}$ (so $\delta_\pm$ indicates the parity of $\kappa_\pm$). Now we can state the decomposition as follows:

$$\begin{align*}
(m_-, m_+) &\in \mathcal{V}^I \quad \text{iff } m_\pm \in \mathbb{Z}/2 \text{ and } m_- - m_+ \notin \mathbb{Z}, \\
(m_-, m_+) &\in \mathcal{V}^II \quad \text{iff } m_\pm \in \mathbb{Z}/2 \text{ and } m_- - m_+ \in \mathbb{Z}, \\
(m_-, m_+) &\in \mathcal{V}^III \quad \text{iff } m_\pm \in \mathbb{Z} \text{ and } m_- - m_+ \notin 2\mathbb{Z}, \\
(m_-, m_+) &\in \mathcal{V}^IV \quad \text{iff } m_\pm \in \mathbb{Z} \text{ and } m_- - m_+ \in 2\mathbb{Z}, \\
(m_-, m_+) &\in \mathcal{V}^V \quad \text{iff } m_\pm \in \mathbb{Z} \pm \frac{1}{4} \text{ and } \delta_- \neq \delta_+, \\
(m_-, m_+) &\in \mathcal{V}^VI \quad \text{iff } m_\pm \in \mathbb{Z} \pm \frac{1}{4} \text{ and } \delta_- = \delta_+.
\end{align*}$$

In contrast to the arithmetic diagram, we introduce the “spectral diagram” of $\mathcal{H}(\mathcal{R}, m)$, where the root datum $\mathcal{R} = (X, R_0, Y, R_0^\vee)$. Let us first describe the root system of the spectral diagram. We introduce an indicating function $n_m : R_0 \to \{1, 2\}$ which is $W_0$-invariant, defined by the rule that $n_m(\alpha) = 2$ if and only if $m(1 - \alpha) \neq m(\alpha)$, else $n_m(\alpha) = 1$. We observe that $n_m(\alpha) = 2$ implies that $\alpha^\vee \in 2Y$.

Let $R_m$ be the subset of $X$ defined as $R_m = \{n_m(\alpha)\alpha : \alpha \in R_0\}$. One can check that $R_m$ is a reduced root system whose Weyl group is again $W_0(R_0)$, and also

$$ZR_m \subseteq ZR_0 \subseteq X \subseteq wt(R_m) \subseteq wt(R_0).$$

Let $F_m$ be a base of $R_m$, and let $\Omega_Y$ be the finite abelian group $Y/Z(R^\vee_m)$. Define an extended affine Weyl group $W^\vee$ as the semi-direct product of the (unextended) affine Weyl group $W(R_m)$ and $\Omega_Y$.

**Definition 5.1.** We define the spectral diagram $\Gamma_s$ of $\mathcal{H}(\mathcal{R}, m)$ to be the extended affine Dynkin diagram of $(R^{(1)}_m, F^{(1)}_m)$, with the group $\Omega_Y$ acting as diagram automorphisms. The vertices are labelled by a function $m^\vee_R : R^{(1)}_m/W^\vee \to \mathbb{Z}$ defined as follows: for an element $a^\vee = n_m(\alpha)\alpha + b$ of $R^{(1)}_m$, define a $W^\vee$-invariant signature function $\epsilon : R^{(1)}_m/W^\vee \to \{\pm 1\}$ by $\epsilon(a^\vee) = (-1)^{b(n_m(\alpha) - 1)}$. Finally we put

$$m^\vee_R(\alpha^\vee) := n_m(\alpha) m_{\epsilon(\alpha^\vee)}(\alpha).$$
Recall that \(\mu_{m_-, m_+}(\mathcal{R})\) is normalised by a normalisation factor \(d_{m_-, m_+} := \tau_{m_-, m_+}(1)\), where \(\tau_{m_-, m_+}\) is the trace of \(\mathcal{H}(\mathcal{R}, m)\). For the Iwahori-Hecke algebra \(\mathcal{H}^{IM}\) with parameters \(m_{\pm}\), the normalisation factor \(d_{m_-, m_+}^{IM}\) is defined to be the formal degree of the discrete series representation \(\mathcal{H}^{IM}_{m_-, m_+}\).

There is a correspondence from the arithmetic side to the spectral side of \(\mathcal{H}\). On the arithmetic side, we have the formal degree \(f_{\text{deg}}(\sigma)\) coming from a maximal unipotent type \((P, \sigma)\). We observe that \(f_{\text{deg}}(\sigma)\) can be factorised as the product of a rational constant independent of \(q\), and a \(q\)-rational number \(f_{\text{deg}, q}(\sigma) \in \mathbb{M}\). We have the following result:

**Proposition 5.1.** [74, Theorem 2.28] Let \((\mathcal{H}, \tau^d)\) be a normalised affine Hecke algebra, and let \(\delta\) be a generic discrete series character of \(\mathcal{H}\). Suppose the generic central character of \(\delta\) is \(W_0^\tau \in W_0 \setminus T\) of \(\delta_\sigma\) with \(\tau^*\) a generic residual point. There exists a nonzero rational constant \(c_\delta\) (independent of \(q\)) such that

\[
f_{\text{deg}}(\delta)(v) = c_\delta \mu(\{\tau\})(v, \tau^*(v)).
\]

**Remark 5.1.**

1. The expression \(\mu(\{\tau\})(v, \tau^*(v))\) can be understood as the residue of the rational function \(\mu(\tau)(v)\) at the residual point \(\tau^*(v)\).

2. We shall call a residual point \(\tau^*\) of \(\mathcal{H}^{IM}\) cuspidal if there is an inner form \(G(u)\) of \(G\), and a cuspidal unipotent representation \(\sigma_u\) of \(G(u)\), such that the \(q\)-rational part of \(f_{\text{deg}}(\sigma_u)\) is equal to the \(q\)-rational part of the residue \(\mu^{IM, (\{\tau\})}(v, \tau^*(v))\).

On the other side, the \(\mu\)-function and the normalised trace \(\tau\) summarise the spectral properties of \(\mathcal{H}\). In view of Section 4.6.2, we observe that if we can find a rank 0 STM

\[
\phi : (\Xi, \tau^d_0) \sim (\mathcal{H}, \tau^d),
\]

with \(\text{Im} \phi_T = \tau^*\) a generic residual point for \((\mathcal{H}, \tau^d)\), then there exists a rational constant \(\lambda\) independent of \(q\), such that \(d_0(\mathcal{H}) = \lambda \mu(\{\tau\})(v, \tau^*(v))\). In particular, if \(\mathcal{H} = \mathcal{H}^{IM}_{m_-, m_+}\) (normalised by \(d_{m_-, m_+}^{IM}\)) then we will have

\[
f_{\text{deg}, q}(\delta) = d_{q}^{IM}.
\]

Here, the discrete series representation \(\delta\) of \(\mathcal{H}\) coming from the maximal unipotent type \((P_{a, b}, \sigma)\).

The correspondence of parameters \((a, b) \leftrightarrow (m_-, m_+)\) is given by the following
Some remarks follow.

We define

\[
d^\text{IM}_{m_-,m_+} = \tau^\text{IM}_{m_-,m_+} (1) :=\begin{cases} 
    d^{(2A)}_a(q) d^{(2A)}_b(q) & \text{if } (m_-,m_+) \in \mathcal{V}^1 \\
    d^A_a(q) d^B_b(q) & \text{if } (m_-,m_+) \in \mathcal{V}^\text{II} \\
    d^B_a(q) d^B_b(q) & \text{if } (m_-,m_+) \in \mathcal{V}^\text{III} \\
    d^A_a(q) d^A_b(q) & \text{if } (m_-,m_+) \in \mathcal{V}^\text{IV} \\
    d^{(2A)}_a(q) d^B_b(q^2) & \text{if } (m_-,m_+) \in \mathcal{V}^V \\
    d^{(2A)}_a(q) d^B_b(q^2) & \text{if } (m_-,m_+) \in \mathcal{V}^\text{VI} 
\end{cases}
\tag{5.4}
\]

as in [75, Equation (33)]. The pair \((a,b) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}\) appearing above corresponds to the pair of parameters \((m_-,m_+)\) via the following rules:

\[
\begin{align*}
1/2 + a, 1/2 + b &= \{|m_+ - m_-|, m_+ + m_-|\} \text{ if } (m_-,m_+) \in \mathcal{V}^I \\
2a, 1 + 2b &= \{|m_+ - m_-|, m_+ + m_-|\} \text{ if } (m_-,m_+) \in \mathcal{V}^\text{II} \\
1 + 2a, 1 + 2b &= \{|m_+ - m_-|, m_+ + m_-|\} \text{ if } (m_-,m_+) \in \mathcal{V}^\text{III} \\
2a, 2b &= \{|m_+ - m_-|, m_+ + m_-|\} \text{ if } (m_-,m_+) \in \mathcal{V}^\text{IV} \\
1/2 + a, 1 + 2b &= \{|m_+ - m_-|, m_+ + m_-|\} \text{ if } (m_-,m_+) \in \mathcal{V}^V \\
1/2 + a, 2b &= \{|m_+ - m_-|, m_+ + m_-|\} \text{ if } (m_-,m_+) \in \mathcal{V}^\text{VI} 
\end{align*}
\tag{5.5}
\]

Some remarks follow.

\textbf{Remark 5.2.}  
(i) The list in §13.7 of Carter [15] tells us why the parameters appearing on the left hand sides of the equalities above are expressed as \(1 + 2a, 2b\) etc. These parameters were obtained by Lusztig [56].

(ii) Observe that these equalities determine \(a\) and \(b\) in case II, V and VI, and determines \(a\) and \(b\) up to order in the other cases. Thus the normalisation (5.4) is always well defined.

(iii) The normalisation factors of the last two cases V and VI can be uniformly written as (cf. [75, Equation (36)]):

\[
d^\text{IM}_{m_-,m_+} = \prod_{i=1}^{\lceil |m_--m_+| \rceil} \left( \frac{v^2|m_--m_+|-2i}{1 + q^2|m_--m_+|-2i} \right)^i \prod_{j=1}^{\lceil |m_--m_+| \rceil} \left( \frac{v^2|m_--m_+|-2j}{1 + q^2|m_--m_+|-2j} \right)^j.
\tag{5.6}
\]

These two cases are said to be extra special, since they relate to a kind of STMs based on the extra special correspondence which we will discuss in next chapter.
5.2 The spectral transfer category $\mathcal{C}_{\text{class}}$

We introduce the following notations for unipotent normalised affine Hecke algebras. Put $X_m = \text{wt}(R_m)$, the weight lattice of $R_m$. Together with the dual root system $Y_m$ and the dual lattice $Y_m = \mathbb{Z}R_m$, we have a based root datum $\mathcal{R}_m = (X_m, R_m, Y_m, R_m, F_m)$.

**Definition 5.2.** Consider a normalised Hecke algebra $\mathcal{H}(\mathcal{R}_m, m)$.

1. If $R_m$ is simply-laced, and the parameters $m_+ (\alpha_i)$ are all equal to $b$, we denote this algebra by $R_m[q^b]$.

2. If $R_m = C_r$, we shall write $C_r (m_-, m_+) [q^b]$ for the Hecke algebra associated with the root system of type $C_r$, and if the parameters satisfying (i) $m_+ (\alpha) = b$ for $\alpha$ a type $D$ root of $R_m$, and (ii) $m_- (\beta) = bm_-$ and $m_+ (\beta) = bm_+$ for a type $B$ short root $\beta$ of $R_m$.

3. Finally if $R_m$ is not simply-laced and not of type $C_r$, then we shall denote this algebra by $R_m (m_+ (\alpha_1), m_+ (\alpha_2)) [q^b]$, where $\alpha_1 \in F_m$ is long and $\alpha_2$ is short, and $q^b$ is the base for the Hecke parameters.

**Example 5.1.** Let us consider the projective unitary group, which is of type $\Sigma_0 = 2\Lambda_l$ of rank $l - 1$. Here we separate the odd rank and even rank cases.

(i) For $G = \text{PU}_{2n}(k_{nr})$ (odd rank), we have $\Omega_G = C_{2n}$ since its of adjoint type. This group is $k$-quasi-split. The affine Dynkin diagram of $\Sigma_0 = 1$ is a cycle of $2n$ nodes. Let us name the nodes as $0, 1, \ldots, 2n$, where $0$ denotes the affine extended root. The Frobenius element $F$ action on this affine Dynkin diagram is as described in [35, 79]. Under this action, the nodes $0$ and $n$ are invariant, and the nodes $j$ and $2n - j$ ($j = 1, \ldots, n$) lie in the same orbit. Thus the resulting root system $R_0' = C_n$ (hence $R_0 = B_n$, and on the arithmetic diagram of $(R_0)_{(1)}$, the two extremal nodes are both with parameter $q$, while the other nodes are with $q^2$ (see [15]). Note that here $\Omega^F = C_2$, which swaps the arithmetic diagram.

Moreover, we can see that $Y = \mathbb{Z}C_n$, and hence $X = \text{wt}(B_n)$. We witness that $\Omega_X := X/\mathbb{Z}R_0' = C_2 = \Omega^F$. So $\mathcal{R}_{qs} = (\text{wt}(B_n), B_n, \mathbb{Z}C_n, C_n)$. Since the parameters of the two extremal nodes of the arithmetic diagram $\Gamma_a (C_n)$ are euqual, we conclude that the root system $R_m = R_0 = B_n$. Thus, the Iwahori-Hecke algebra $\mathcal{H}_{\text{IM}} = B_n (2, 1)[q] = B_n (1, 1/2)[q^2]$.

Consider a more concrete example: take $n = 9$. Then in the affine Dynkin diagram of $\Sigma_0 = (2\Lambda_{17})_{(1)}$, take two subdiagrams of type $2\Lambda_5$, which correspond to a parahoric subgroup $\mathcal{P}_{a,b} = \mathcal{P}_{3,3}$ of $G$, which admits cuspidal unipotent representations. Then the resulting diagram is of type $(C_3)_{(1)}$,
with two extremal nodes with parameter $q^7$, and two other nodes with parameter $q^2$ ([15]). Using the corresponding rule (5.5) for $\mathcal{V}^I$, we deduce that $m_+ = 7/2, m_- = 0$. The resulting affine Hecke algebra is $\mathcal{H}(G/P_{a,b}) =: C_3(0, 7/2)[q^2]$.

(ii) For $G = \text{PU}_{2n+1}(k_{nr})$ (even rank) of adjoint type and $k$-quasi-split, we have $\Sigma_0 = \tilde{2}A_{2n+1}$. One can again find the description of the Frobenius action on its affine Dynkin diagram in [79]. This time we have $R_0 = C_n$, so $R_0 = B_n$, $\Omega_G = C_{2n+1}$, but $\Omega^F = 1$. Let us name the nodes similarly as the odd rank case. Only the node $\bar{0}$ is invariant under the Frobenius action. The resulting arithmetic diagram of is an affine Dynkin diagram of $C_n = R_0$, so $R_0 = B_n$.

Consider a concrete example with $n = 9$. For the unipotent type, we can take a subdiagram of type $2A_5$, which corresponds to a parahoric subgroup $P_{a,b} = P_{3,0}$ of $G$. The resulting diagram is of type $(C_6)^{(1)}$, where the extremal node corresponding to the node $\bar{0}$ is with parameter $q^7$, while the other extremal node is with parameter $q$, and the nodes between them are with parameters all equal to $q^2$. Again using the rule (5.5) for $\mathcal{V}^I$, we see that $m_+ = 2, m_- = 3/2$. Hence the resulting affine Hecke algebra is $\mathcal{H}(G/P_{a,b}) =: C_6(3/2, 2)[q^2]$.

Similar consideration to this example suggests that the root system of type $C$ will occur much more often than others. Hence we would single it out, by defining a category $\mathcal{C}_{\text{class}}$ which has normalised affine Hecke algebras of type $C_l(m_-, m_+) [q^b]$ with normalisation factor $\tau_{m_-, m_+}$ as objects. The rank $l$ is a non-negative integer, and the parameters $m_\pm \in \mathbb{Z}/4$ satisfying $2(m_- - m_+) \in \mathbb{Z}$. The number $b$ is determined by the rule that $b = 1$ if both $m_- \pm m_+ \in \mathbb{Z}$; else $b = 2$.

We turn to consider the morphisms of $\mathcal{C}_{\text{class}}$. Define

$$\eta_+ : m_+ \mapsto -m_+, \quad m_- \mapsto m_-$$

$$\eta : m_+ \mapsto m_- , \quad m_- \mapsto m_+$$

Then the group $\text{Iso} = \langle \eta_+, \eta \rangle$ is isomorphic to the dihedral group of order 8. This group acts on $\mathcal{C}_{\text{class}}$ by spectral isomorphisms (cf. [74, 3.3.3]). We define $\eta_- = \eta \circ \eta_+ \circ \eta$.

The spectral isomorphism group $\text{Iso}$ certainly acts on $\mathcal{C}_{\text{class}}$. Apart from these spectral morphisms, Opdam has given a set of basic STMs between $C_l(m_-, m_+) [q^b]$.
with parameter types \( \mathcal{X} \in \{ I, II, III, IV \} \), as well as for the STMs from \( \mathcal{C}_{III}^{class} \) to \( \mathcal{C}_{class}^{V} \), and from \( \mathcal{C}_{class}^{IV} \) to \( \mathcal{C}_{class}^{VI} \). They are listed below:

\[
\begin{align*}
C_r(m_-, m_+)[q^2] &\sim C_{r+|m_+|} - \frac{1}{2} (m_--\epsilon(m_-), m_+)[q^2] & \text{if } (m_-, m_+) \in \mathcal{V}, m_+ \notin \mathbb{Z} \\
C_r(m_-, m_+)[q^2] &\sim C_{r+2|m_+|-2} (m_- - 2\epsilon(m_-), m_+)[q^2] & \text{if } (m_-, m_+) \in \mathcal{V}, m_+ \in \mathbb{Z} \\
C_r(m_-, m_+)[q] &\sim C_{r+|m_+|} - \frac{1}{2} (m_--\epsilon(m_-), m_+)[q] & \text{if } (m_-, m_+) \in \mathcal{V}^{II} \\
C_r(m_-, m_+)[q] &\sim C_{r+2|m_+|-2} (m_- - 2\epsilon(m_-), m_+)[q] & \text{if } (m_-, m_+) \in \mathcal{V}^{III} \\
C_r(m_-, m_+)[q^2] &\sim C_{2r+\frac{1}{2}a(a+1)+2b(b+1)(\delta_-\delta_+)}[q] & \text{if } (m_-, m_+) \in \mathcal{V}^{IV} \\
C_r(m_-, m_+)[q^2] &\sim C_{2r+\frac{1}{2}a(a+1)+2b(b+1)(\delta_-\delta_+)}[q] & \text{if } (m_-, m_+) \in \mathcal{V}^{VI}.
\end{align*}
\]

**Example 5.2** (cont. Example 5.1). We continue the concrete examples in Example 5.1. For the odd rank group \( G = PU_{18}(k_{nr}) \), we have seen that we can obtain an affine Hecke algebra \( \mathcal{H}(G/\mathcal{P}_{3,3}) = C_3(0,7/2)[q^2] \). Applying the first STM above, we get \( C_3(0,7/2)[q^2] \sim C_6(0,5/2)[q^2] \sim C_8(0,3/2)[q^2] \sim C_9(0,1/2)[q^2] \). We witness that we can reach the minimal object \((5.10)\) in parameter type I.

For the even rank group \( G = PU_{19}(k_{nr}) \), we have \( \mathcal{H}(G/\mathcal{P}_{3,0}) = C_6(3,2,2) \). Applying the second STM above, we have \( C_6(3,2,2)[q^2] \sim C_7(1,2,2)[q^2] \sim C_9(1,2,1)[q^2] \), again we witness that we can reach the minimal object for parameter type II.

The underlying affine morphisms of algebraic tori for these STMs are described as follows. Suppose \( (m_- , m_+) \in \mathcal{V}, n \in \mathbb{Z}_{>0} \). We may assume that \( m_+ \geq 0 \) by virtue of the spectral isomorphisms \( \text{Iso} \). The torus \( T_n(\Xi) := G_m^n(\Xi) \) has \( X_n := X^*(T_n) = \mathbb{Z}^n \) as its character lattice. We consider \( X_n \) as the root lattice of a root system of type \( B_n \).

1. For \( m_\pm \in \mathbb{Z} + 1/2 \), define a homomorphism \( \phi_{m_-, m_+, T} : T_n \rightarrow T_{n+m_+-1/2} \) of algebraic tori over \( \Xi \) by

\[
\phi_{m_-, m_+, T}(t_1, \ldots, t_n) = (t_1, \ldots, t_n, v^b, v^{3b}, \ldots, v^{2b(m_+-1)}).
\]

2. For \( m_+ \in \mathbb{Z}_{>0} \), define a homomorphism \( \psi_{m_-, m_+, T} : T_n \rightarrow T_{n+2m_+-2} \) of algebraic tori over \( \Xi \) by

\[
\psi_{m_-, m_+, T}(t_1, \ldots, t_n) = (t_1, \ldots, t_n, 1, q^{b}, q^{b^2}, q^{b^3}, \ldots, q^{b(m_+-2)}, q^{b(m_+-2)}, q^{b(m_+-1)}).
\]

3. (The extra-sepcial cases \( \mathcal{X} \in \{ \mathcal{V}, \mathcal{VI} \} \)). For \( m_\pm > 0 \), set \( l := 2n + (a/2)(a + 1) + 2b(b+1) \) if \( \mathcal{X} = \mathcal{V} \), or \( l := 2n + (a/2)(a + 1) + 2b^2 - \delta_+ \) if \( \mathcal{X} = \mathcal{VI} \). Let \( \kappa_\pm, \epsilon_\pm \) be defined as in \( (5.1) \). If we let \( l_\pm := \kappa_\pm(\kappa_\pm + \epsilon_\pm - 1/2) \), then \( l = 2n + [l_-] + [l_+] \). For \( m \in \mathbb{Z} \pm \frac{1}{4} \) and \( m > 1 \), let

\[
\sigma_e(m) = (q^\delta, q^{\delta+1}, \ldots, q^{2m-\frac{3}{2}}).
\]

\[75\]
Define the residual points $r_e(m)$ recursively by putting
\[ r_e(1) = r_e(3) := 0, \quad r_e(m) = (\sigma_e(m); r_e(m-1)), \quad m > 1. \]

Finally we define the representing morphism $\xi_{m-, m+, T} : T_n \to T_l$ of the extraspécial STM by
\[ \xi_{m-, m+, T}(t_1, \ldots, t_n) = ( - r_e(m-), v^{-1}t_1, vt_1, \ldots, v^{-1}t_n, vt_n, r_e(m+) ). \quad (5.9) \]

One can verify by direct computation that in the cases $X \in \{ I, II, III, IV \}$, these morphisms indeed define a spectral transfer morphism [103], while the extra-special cases take us more consideration. We will prove in next chapter that $\xi_{m-, m+, T}$ defined in (5.9) above is indeed a spectral transfer morphism, by induction on $m_\pm$, where the recursive definition of $r_e(m)$ and the formula (5.6) will be crucial.

Notice that by the action of $\text{Iso}$, we can interchange $m_-$ and $m_+$, and map $m_\pm$ to $-m_\pm$. Thus, we may, and will, assume that both $m_\pm \geq 0$, and $m_- \leq m_+$. Under this assumption all the morphisms above decrease the value of $m_- + m_+$. The first five STMs are called translation STM. Observe that within the same parameter type, translation STMs commute, and thus we can iterate them (and using $\text{Iso}$) to reach the object which has minimal $m_- + m_+$ value in the respective parameter type. The minimal spectral isogeny classes of these objects are:

\[
\begin{align*}
[C_l(1, 0)[q^2]] & \quad \text{if } X = I, \\
[C_l(1, 1)[q^2]] & \quad \text{if } X = II, \\
[C_l(0, 1)[q]] & \quad \text{if } X = III, \\
[C_l(0, 0)[q]] & \quad \text{if } X = IV.
\end{align*}
\]

As a contrast, the last two STMs will map the source affine Hecke algebra to the minimal object within the same category in one step. They play a special role in the theory of STMs. For instance, the ambient groups of their corresponding parahoric types $(\mathfrak{P}, \sigma)$, are the inner forms $G(u)$, where $u$ corresponds (via Kottwitz’s theorem) to the action of the centre $L^\ast Z$ of the $L$-group in its spin representations. These morphisms are said to be extra-special. We will demonstrate that these two morphisms are indeed STMs.

**Definition 5.3.** Following the terminology in [74, 75], the STMs of $C_{\text{class}}$ generated by the translation and extra-special STMs above, and by the elements of $\text{Iso}$, are said to be standard.

Recall that we assume $G = G(k_{nr})$ is quasi-split over $k$. Let $\mathcal{H}^{\text{IM}}(G)$ be the Iwahori-Hecke algebra of $G$. Since we have known the standard STMs, we can show that there exists an STM $\phi : \mathcal{H}^u \sim \mathcal{H}^{\chi}(G)$ for every classical group $G$ (other than the projective general linear group) which is absolutely simple, $k$-quasi-split and
of adjoint type, and unipotent affine Hecke algebra $\mathcal{H}^u$ of an inner form of $G$. To establish such morphism $\phi$ we shall use the spectral covering morphisms. They have been set up in [75]. Here we summarise the results.

Consider the spectral transfer category $\mathfrak{C}_{es}$ with essential strict STMs as morphisms. Let $\mathfrak{C}(G)$ be the full subcategory of $\mathfrak{C}_{es}$ whose objects are the normalised unipotent affine Hecke algebras attached to the various inner forms $G(u)$ of $G$, where $u$ runs through a complete system of representatives of the classes $[u] \in H^1(k, G)$. Denote by $\mathfrak{H}^u$ the direct sum of all the objects of $\mathfrak{C}(G)$. The group $G^*_{nr}$ of unramified complex characters of $G$ acts naturally on $\mathfrak{H}^u$ which respects the direct summands and their ranks. In particular, $G^*_{nr}$ acts on the unique summand $\mathfrak{H}^{IM}(G)$ of the largest rank of $\mathfrak{H}^u$.

Now we start to describe the standard STMs from the Iwahori-Hecke algebras to the minimal objects.

(I-1) $G = PU_{2n}$. The minimal object is $C_n(0, \frac{1}{2})[q^2]$, normalised as an object in $\mathfrak{C}_I^{class}$. We have a two-to-one spectral covering map $\mathfrak{H}^{IM}(G) \cong B_n(2, 1)[q] \cong C_n(0, \frac{1}{2})[q^2]$, corresponding to an embedding of the latter as an index 2 subalgebra in the former. The group $G^*_{nr}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

(I-2) $G = PU_{2n+1}$. The minimal object is $C_n(\frac{1}{2}, 1)[q^2]$, normalised as an object in $\mathfrak{C}_I^{class}$. We have an isomorphism $\mathfrak{H}^{IM}(G) \cong C_n(\frac{1}{2}, 1)[q^2]$.

(II) $G = SO_{2n+1}$. The minimal object and the Iwahori-Hecke algebra coincide: both are $C_n(\frac{1}{2}, \frac{1}{2})[q]$. In this situation all unipotent affine Hecke algebras are objects of $\mathfrak{C}_II^{class}$.

(III) $G = PCSp_{2n}$. The minimal object is $C_n(0, 1)[q]$, normalised as an object in $\mathfrak{C}_III^{class}$. We have a two-to-one STM $\mathfrak{H}^{IM}(G) \cong C_n(0, 1)[q]$ arising from an embedding of the latter as an index 2 subalgebra in the former. Here, all direct summands $\mathcal{H}$ of $\mathfrak{H}^u$ are either objects of $\mathfrak{C}_III^{class}$ or $\mathfrak{C}_V^{class}$, or there exists a semi-standard covering STM $\mathcal{H} \cong \mathcal{H}'$, arising from an embedding $\mathcal{H}' \subset \mathcal{H}$ of index 2, where $\mathcal{H}' \in \mathfrak{C}_III^{class}$ is an object such that one of the parameters $m_-$ or $m_+$ equals 0.

(IV-1) $G = SO_{2n}$. The minimal object is $C_n(0, 0)[q]$, normalised as an object in $\mathfrak{C}_IV^{class}$. We have a non-semi-standard STM $\mathfrak{H}^{IM}(G) \cong C_n(0, 0)[q]$, which is represented by a two-fold covering of tori (recall the explanation in last section). All other direct summands $\mathcal{H}$ of $\mathfrak{H}^u(G)$ are either objects of $\mathfrak{C}_IV^{class}$ with both $m_{\pm}$ even, or objects of $\mathfrak{C}_VI^{class}$ with $\delta_- = \delta_+ = 0$, or there is a semi-standard covering morphism $\mathcal{H} \cong \mathcal{H}'$ arising from an index 2 embedding $\mathcal{H}' \subset \mathcal{H}$ where $\mathcal{H}' \in \mathfrak{C}_VI^{class}$ such that one of the parameters $m_-$ or $m_+$ equals 0.
5.3 The essential uniqueness

Let $G = G(k_{nr})$ be the group of $k_{nr}$ points of a connected, absolutely simple group $G$ of adjoint type, which is defined and quasi-split over $k$, and split over $k_{nr}$. Recall the spectral transfer category $\mathcal{C}_{es}$, with essentially strict STMs as morphisms. Consider the full sub-category $\mathcal{C}_{unip}(G)$ of $\mathcal{C}_{es}$ whose objects are the normalised unipotent affine Hecke algebras associated with the various inner form $G(u)$, where $u \in Z^1(F,G)$ runs through a complete set of representatives of the class $[u] \in H^1(F,G)$. Denote by $\mathcal{H}_{unip}$ the direct sum of all objects in $\mathcal{C}_{unip}(G)$. Note that the Iwahori-Hecke algebra $\mathcal{H}^\text{IM}(G)$ of $G$ is the object of $\mathcal{C}_{unip}(G)$ of maximal rank.

Let $G^*_{nr}$ be the group of unramified complex character of $G$. This group is the Pontryagin dual of $\Omega : G^*_{nr} = \Omega^*$ [28]. Likewise, we can identify $(G^\text{Fus})^*_{nr}$ with the Pontryagin dual $(\Omega^\text{F})^*$ of $\Omega^\text{F}$. Recall that $G$ is assumed to be $k$-quasi-split, hence in terms of the root datum $\mathcal{R}_\text{qs} = (X_\text{qs}, R_0, Y_\text{qs}, R_0^\text{F})$, we have a canonical isomorphism $(G^\text{Fus})^*_{nr} \simeq (X_\text{qs}/\mathbb{Z}R_0)^*$. The group $(\Omega^\text{F})^*$ acts on the set of unipotent characters of $G^\text{Fus}$ by tensoring. On the other hand, this group acts on the Iwahori-Hecke algebra $\mathcal{H}^\text{IM}(G)$ by algebra automorphisms. In general, the group $G^*_{nr}$ naturally acts on $\mathcal{H}_{unip}$, such that direct summands are sent to direct summands, and the ranks are preserved.

Although the correspondence $(a,b) \leftrightarrow (m_-, m_+)$ is not unique, we have the following powerful result.

**Theorem 5.1.** [75, Theorem 3.4] There is a spectral transfer morphism

$$\phi : (\mathcal{H}_{unip}(G), \tau) \rightsquigarrow (\mathcal{H}^\text{IM}(G), \tau^\text{IM})$$

which is $G^*_{nr}$-equivariant, and attains the following essential uniqueness property: If there is another such $G^*_{nr}$-equivariant spectral transfer morphism $\phi'$, then there exists a spectral transfer automorphism $\beta$ of $(\mathcal{H}_{unip}, \tau)$, such that $\phi' = \phi \circ \beta$.

The strategy to prove this theorem is discussed thoroughly in [75]. In next section, we will present the flow of the proof of this theorem. In fact, the proof of the essential uniqueness for those STMs of rank 0, is the basis to establish the general statement, and what we will discuss in next section is mainly the proof of this rank 0 case.
Once the rank 0 essential uniqueness had been set up, we could use the notion of induction of spectral transfer morphisms ([75]) to finish the proof of the general essential uniqueness. In the rest of this section, we shall sketch the idea on this notion.

Given a normalised affine Hecke algebra \( (\mathcal{H}, \tau^d) \), recall that a generic residual coset \( L \) of \( \mathcal{H} \) determines a parabolic root sub-system \( R_P \subset R_0 \). Applying a suitable element \( w \in W_0 \) to \( L \), we can always assume that \( R_P \) is standard, with \( P \) a subset of \( F_0 \), and \( L = \tau^P T_P \) with \( \tau^* \in T_P \). Let \( \mathcal{H}_P \) be the semisimple quotient algebra of \( \mathcal{H} \) whose associated algebraic torus is \( T_P \). Then \( \tau^* \) is a residual point of \( \mathcal{H}_P \). We normalised the trace \( \tau^d_P \) of \( \mathcal{H}_P \) by the rule that \( \tau^d_P(1) = (v - v^{-1})^{\text{rk}(R_P) - \text{rk}(R_0)} d \).

Consider a strict STM \( \phi : (\mathcal{H}', \tau'^d) \rightarrow (\mathcal{H}, \tau^d) \) represented by \( \phi_T : T' \rightarrow R_n \), with \( L = rT_P \) a residual coset. For any inclusion \( P \subset Q \subset F_m \), by [75, Corollary 3.7, 3.8], we have a bijection

\[
\{ R_Q' \subset R'_m : R_Q' \text{ is parabolic} \} \leftrightarrow \{ R_Q \subset R_m : R_Q \text{ contains } R_P, \text{ and is parabolic} \}
\]

induced by the gradient \( D\phi_T \) of \( \phi_T \). After modifying \( \phi_T \) by an appropriate element \( w' \in W(R'_m) \), we can assume that \( R_Q' = (D\phi_T)^{-1}(R_Q) \) is standard and associated with a subset \( Q' \subset F'_m \). By restricting \( \phi_T \) to \( T'_Q \subset T' \) we obtain a map \( (\phi_Q)_T \), which represents spectral transfer morphism \( \phi_Q : \mathcal{H}'_{Q'} \rightarrow \mathcal{H}_Q \). We say that \( \phi_Q \) is the restriction of \( \phi \) to \( \mathcal{H}'_{Q'} \), and \( \phi \) is induced from \( \phi_Q \).

**Remark 5.3.** Every spectral transfer morphism can be induced from a rank 0 spectral transfer morphism. For instance, the STM \( \phi : \mathcal{H}' \rightarrow \mathcal{H} \) above is induced from \( \phi_P : \Xi \rightarrow \mathcal{H}_P \). However, not every rank 0 STM of the form \( \psi : \Xi \rightarrow \mathcal{H}_P \) is the restriction of a STM \( \Psi \) to \( \mathcal{H} \). See [75, 3.1.2].

At this stage we raise a remarkable fact: Let \( G(u) \) be the inner form of \( G \) determined by \( u \in Z^1(k, G) \), and let \( \sigma^u \) be a cuspidal unipotent representation of \( G(u) \). Then up to a non-zero rational constant, the formal degree \( \text{fdeg}(\sigma^u) \) is equal to the formal degree of an Iwahori-spherical unipotent discrete series representation \( \sigma \) of \( G \). Let \( \delta_\sigma \) be the corresponding discrete series representation of \( \mathcal{H}^\text{IM}(G, \mathcal{B}) \) with central character \( W_0 \tau^* \) (here \( \tau^* \) is a generic residual point), then \( W_0 \tau^* \) is uniquely determined by \( \text{fdeg}(\sigma) \), modulo the action of \( G^*_{\text{nr}} \). Consequently, the rank 0 case of the essential uniqueness theorem follows.

Now we call for the notion of induction of STMs we introduce just above. Replace \( G(u) \) by a proper Levi subgroup \( M(u) := C_{G(u)}(S(u))^o \) (so \( S(u) \) is the \( k \)-split part of the connected centre of \( M(u) \)) which carries a cuspidal unipotent representation \( \sigma^u_M \). We have a root datum \( \mathcal{R}_M = (X_M, R_M, Y_M, R'_M) \) of \( M \). By the definition of unipotent representations, we note that \( \sigma^u_M \) factors through \( \sigma^u_M \). Therefore, as in the rank 0 case, one expects that \( \sigma^u_M \) uniquely determines an orbit of cuspidal residual points \( W_{M,0} \tau^* \subset T_M \) of the Iwahori-Hecke algebra \( \mathcal{H}^\text{IM}_M(\mathcal{R}_M) \), up to the
action of $\Omega^*_{\text{M}}$ (The Pontryagin dual of $\Omega_{\text{M}} = X_{\text{M}} / \mathbb{Z} R_{\text{M}}$). Hence up to the action of $\Omega^*_{\text{M}}$, the residual point $\mathfrak{r}^*_{\text{M}}$ is the image of of the representing morphism of the rank 0 STM $\phi_{\text{M}} : \Xi \to \mathcal{H}_{\text{IM}}(M_{\text{ss}})$. Here, $M := C_G(S(u))$ is a k-Levi group of $G$ (so $M$ is quasi-split over $k$, and $M(u)$ is an inner form of $M$), $M_{\text{ss}} = M/S(u)$, and $\mathcal{H}_{\text{IM}}(M_{\text{ss}})$ denotes the Iwahori-Hecke algebra with respect to the Iwahori subgroup $(M \cap B)/(S(u) \cap B)$ of $M_{\text{ss}}$.

The orbit $W_{\text{M,0}} \mathfrak{r}^*_M$ gives rise to a maximal proper subset $J_M$ in the spectral diagram $\Gamma_s(R_M, m_M)$ of $M_{\text{ss}}$. See [75, 3.1.3, 3.1.4] for more details. Because $T^M$ is defined as the subset of $T$ where the dual affine roots in $J_M$ are constant, we deduce that $(\mathfrak{r}^*_M, T^M)$ is uniquely determined by the type of $M_{\text{ss}}$ and $\deg_q(\sigma^*_M)$, up to the action of $W(R_0)$ and of $\Omega^*_{\text{M}}$.

**Remark 5.4.** (i) We point out that in all types of the root system $\Sigma_0$ of $G$, there is a unique way to place a subset of such type $J_M$ into $\Gamma_s(R, m)$ as an excellent subset. See Lusztig [61].

(ii) One can assign an $F_u$-stable parahoric subgroup $\mathcal{P} \subset G$ to the cuspidal unipotent pair $(M(u), \sigma^*_M)$ as explained in Morris [67], such that $\mathcal{P}^F_u \cap M^F_u$ is a maximal parahoric subgroup of $M^F_u$, and the set of affine roots associated with $\mathcal{P}$ possesses a base. This base is a proper subset of $\Gamma_a(R, m)$ which is $\Omega$-invariant. Moreover, $\mathcal{P}^F_u$ admits a cuspidal unipotent representation, say $\delta$. Hence $\mathcal{P}^F_u \cap M^F_u, \delta|_{\mathcal{P}^F_u \cap M^F_u}$ is a maximal unipotent type of $M^F_u$. Also we have $\text{rk}(M) + \text{rk}(\mathcal{P}) = \text{rk}(G)$.

An illustrative example of the above analysis with $G = \text{PGL}_{n+1}$ is given in [75, 3.2.1]. Here we give one example to explain how $J_M$ fits as an excellent subset into the spectral diagram of $H_{\text{IM}}$. More examples can see the full list in Lusztig’s paper [61].

**Example 5.3.** Take $G = \text{PGL}_n$. If $n = md$ is a factorisation of $n$, let $G(u)$ be an inner form of $G$ corresponding to a cocycle $u$ of order $m$. From Example 2.3 we conclude that $G(u) = \text{PGL}_d(D)$, where $D$ is a division algebra over $k$ of rank $m^2$, which contains an unramified extension $k \subset k'$ of degree $m$.

The only cuspidal unipotent representation of $G$ is from its anisotropic inner form $\text{PGL}_1(E) = E^\times/k^\times$, where $E$ is a division algebra over $k$ of rank $n^2$, which contains an unramified extension $k \subset k'$ of degree $n$. This is amount to the case that $d = 1$ and $n = m$.

A maximal $k$-split torus $S \simeq (k^\times)^{d-1}$ defines a Levi subgroup $M(u) = C_{G(u)}(S)$ of $G(u)$ such that $M(u)_{\text{ss}} := M(u)/S$ is isomorphic to $(D^\times/k^\times)^d$. So $J_M$ is of type $\text{A}^{d}_{m-1}$, direct product of $d$ copies of $\text{A}_{m-1}$.

In the spectral diagram of $G$, the root system $R_m^{(1)}$ is of type $\text{A}^{(1)}_{n-1}$. So the underlying diagram of $\Gamma_s(R_m)$ is a cycle of $n$-nodes. We label the nodes by
{0, 1, 2, \ldots, n - 1}, where 0 corresponding to the affine root. Then the subdiagram of the nodes \(\{1, \ldots, m - 1; m + 1, \ldots, 2m - 1; \ldots; (d - 1)m + 1, \ldots, dm - 1\}\) can be identify with \(J_M\). It is clear that such a subset is excellent.

### 5.3.1 Classification of standard STMs

The powerful essential uniqueness Theorem 5.1 has some interesting applications. First of all, we have a classification result of unipotent spectral transfer morphisms. Let us agree to use the shorthand \(\mathcal{C}_{\text{class}}^\chi \cup \mathcal{C}_{\text{class}}^\psi\) for \(\mathcal{C}_{\text{class}}^\chi \cup \mathcal{C}_{\text{class}}^\psi\).

**Proposition 5.2.** [75, Proposition 3.8] All spectral transfer morphism between the objects in each of \(\mathcal{C}_{\text{class}}^I, \mathcal{C}_{\text{class}}^{II}, \mathcal{C}_{\text{class}}^{III}, \mathcal{C}_{\text{class}}^{IV}, \mathcal{C}_{\text{class}}^{V}, \mathcal{C}_{\text{class}}^{VI}\) are generated by the basic translation STMs as well as the extra special STMs, and the group \(\text{Iso}\) of spectral isomorphisms. Further, the basic translation STMs commute with each other.

We now turn to the cuspidal residual points as the images of rank 0 STMs. Recall our assumptions on \(G\). However, here we no more assume that \(G\) is \(k\)-quasi-split. Instead, let \(G_{qs}\) be the \(k\)-quasi-split group in the same inner class of \(G\), and \(G = G_{qs}(u)\) is the inner form determined by the cocycle \(u\). We have seen that for each minimal objects \(H_{\text{min}} = H(\mathcal{R}, m_{\text{min}}^\chi)\) of the category \(\mathcal{C}_{\text{class}}^\chi\), there exists a spectral covering map

\[H_{\text{IM}} \to H_{\text{min}}.\]

The rank 0 object of \(\mathcal{C}_{\text{class}}^\chi, \chi \in \{I, II, III, IV, V, VI\}\) is \(\mathcal{H}_0^\chi := (\Xi, d^0 = \text{fdeg}_q(\sigma))\) from the definition.

**Proposition 5.3.** [102, Proposition 3.5, Corollary 3.6] Assume the existence of standard STMs to \(\mathcal{H}_{\text{min}}^\chi\). If \(W_0 \tau \in \text{Res}(\mathcal{R}, m_{\text{min}}^\chi)\) satisfies that

\[
(p_{\text{min}}^\chi)^{(\{\tau\}} = \text{fdeg}_q(\sigma)
\]

for some maximal unipotent type \((\mathcal{P}, \sigma)\) of \(G\), where \(\mathcal{P}\) is \(F_u\)-stable, then up to the action of \(\text{Aut}_{es}(\mathcal{H}_{\text{min}}^\chi)\), \(W_0 \tau\) determines a rank 0 STM

\[\phi : \mathcal{H}_0^\psi \to \mathcal{H}_{\text{min}}^\chi,\]

where \(\psi = \chi\), or \(\psi = V\) and \(\chi = III\), or \(\psi = VI\) and \(\chi = V\).

### 5.4 Proof of the essential uniqueness

As promise in last section, we will explain the flow of the proof for Theorem 5.1 at the rank 0 case (also called \textit{cuspidal} case in [75]). First of all, notice that for
split exceptional groups, this essential uniqueness holds by Reeder’s result [81], and we will show in the last chapter that it also holds for non-split but quasi-split exceptional groups. Therefore, we will concentrate on classical groups.

For the classical groups, the case of $G = \text{PGL}_n$ is relatively easy and has been discussed completely in [75, 3.2.1], and hence we can restrict ourselves to the cuspidal case for the groups $G = \text{PU}_n, \text{SO}_{2n+1}, \text{PCSp}_{2n}, \text{P}(\text{CO}_{2n}^0)$ and $\text{P}((\text{CO}_{2n+2}^-)^0)$, as claimed in [75, 3.2.6].

In fact, verifying the essential uniqueness in the cuspidal case for these five types of classical groups is a major topic of [102]. The authors of [102] reduce the proof of theorem 5.1 to Proposition 5.3 stated above (which is in fact two propositions in [102]). It is again a case-by-case check to prove Proposition 5.3. To avoid duplication, here we shall only discuss the idea of the proof and state the key results presented in [102].

The guiding rule is to use the constrain equality (5.12), combining with the fact that the formal degrees of cuspidal unipotent representations of classical $G$ are reciprocal of products of even cyclotomic polynomials $\Phi_{2j}(q), j \in \mathbb{Z}_{>0}$. The formal degrees are given as in Section 3.2.3. Any cuspidal unipotent pair $(p_{a,b}, \sigma)$ gives us two non-negative integers $a, b$, which corresponds, via the rule (5.5), to a pair of “spectral parameters” $(m_-, m_+)$. Thus we can determine the type $\mathcal{X} \in \{\text{I}, \text{II}, \text{III}, \text{IV}, \text{V}, \text{VI}\}$ of the Hecke algebra $C_r(m_-, m_+)[q^b]$. We then use, either (composition of) translation STMs, or extra special STMs, from this Hecke algebra to the minimal object in the corresponding $\mathcal{C}_\text{class}^\mathcal{X}$ to prove Proposition 5.3.

### 5.4.1 Reduction to linear residual points

We first recall the $m$-tableau. For a real number $m$ and a partition $\lambda$, the $m$-tableau (cf. [89, 43]) $T_m(\lambda)$ of $\lambda$ is defined to be the tableau of shape $\lambda$ with its box $b_{i,j}$ (where the coordinates $(i, j)$ have the same meaning as for matrix entries) filled with the nonnegative real number $|m - i + j|$. This number is called the filling of the box $b_{i,j}$.

In our situation we have $m \in \mathbb{Z}/4$. Let $\rho$ be any non-zero partition. To $(m, \rho)$ we associate the $W_0(B_n)$-orbit of content vectors $W_0(B_n)\xi(m, \rho) \in \mathbb{Q}^n$. This orbit is the collection of vectors $\xi(m, \rho)$ such that the list of absolute values of coordinates of $\xi(m, \rho)$ coincides with the list of box fillings of $T_m(\rho)$ (counted with multiplicity). Here the root system

$$B_n = \{\pm x_i : 1 \leq i \leq n\} \cup \{\pm x_i \pm x_j : 1 \leq i \neq j \leq n\}$$

is realised as the set of linear functionals on $\mathbb{Q}^n$. 

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Proposition 5.4. The followings are equivalent.

1. There is a content vector \( \xi(m, \rho) \) which is an \( m \)-linear residual point for the root system \( B_n \), in the sense of [43].

2. The orbit \( W_0(B_n)\xi(m, \rho) \) is the central character of a discrete series character of the graded affine Hecke algebra \( \overline{H}_m(B_n) \) of the root system \( B_n \), with parameter 1 for the roots \( \pm x_i \pm x_j \) and \( m \) for the roots \( \pm x_i \).

If any of these conditions holds, the \( m \)-splitting \( S_m(\rho) \) of \( T_m(\rho) \) is well defined (cf. [89, Definition 5.16, Lemma 5.17]), and we say that \( \rho \) is \( m \)-regular.

For the proof, see [76] and [71].

Let \( q \in \mathbb{R}_{>0} \) and \( b \in \mathbb{Z}_{>0} \), and \( k = \log(q) \). Given a pair \((m, \rho)\) as above, we denote by

\[
\overline{r}_{(b;m;\rho)} := \exp(b \, k \xi(m, \rho)) = (q^{b \, c_1}, \ldots, q^{b \, c_n})
\]

the generic residual point, where \( \xi(m, \rho) = (c_1, \ldots, c_n) \) is the \( m \)-linear residual point whose coordinates are the fillings of the boxes of \( T_m(\rho) \). (Here we can order the coordinates arbitrarily since the Weyl group action permutes the order of the coordinates.) We form a set of vectors \( W_0(B_n)\overline{r}_{(b;m;\rho)}(q) \) with \( \overline{r}_{(b;m;\rho)}(q) = (q^{b \, c_1}, \ldots, q^{b \, c_n}) \in \mathbb{R}_{>0}^n \). For simplicity, we often suppress the parameters \( m \) and \( b \) from the notation, and simply write \( \overline{r}_{\rho} \) instead of \( \overline{r}_{(b;m;\rho)}(q) \) or even of \( \overline{r}_{(b;m;\rho)}(q) \) if there is little danger of confusion.

The residual point \( W_0 \overline{r} \in \text{Res}(\mathbb{R}, m_{\min}) \) obeying the equality (5.12), is the central character of a discrete series of \( C_{n}(m_-, m_+)\). From [76], we know that the central character of a discrete series character of \( C_{n}(m_-, m_+)\) can always be expressed as the \( W_0(B_n) \)-orbit of a residual point of the form

\[
\overline{r}_{(\rho_-, \rho_+)} := (- \overline{r}_{(b;m_-;\rho_-)}(q), \overline{r}_{(b;m_+;\rho_+)}(q)),
\]

where \( \rho_{\pm} \) denotes an \( m_{\pm} \)-regular partition of \( n_{\pm} \), with \( n_- + n_+ = n \). We will always assume (without loss of generality) that \( m > 0 \), that \( b \in \mathbb{N} \), \( m_\pm \in \mathbb{Q}_+ \) are such that \( b(m_+ + m_-) \in \mathbb{Z} \) and \( b(m_+ - m_-) \in \mathbb{Z} \). This assumption is satisfied for unipotent affine Hecke algebras, and guarantees that all factors in the \( \mu \) function are of the form \( (1 \pm q^n) \) with \( n \in \mathbb{Z} \).

If \( m \not\in \{0, 1/2, 1, \ldots, n-1\} \), all partitions \( \rho \vdash n \) are \( m \)-regular (see [43, 89]). This implies that the set of \( W_0 \)-orbits of the \( m \)-linear residual points in this situation is in canonical bijection to the set of all partitions \( \rho \) of \( n \). This is the case for example for type V and type VI classical unipotent affine Hecke algebras.

However, for \( m \in (\mathbb{Z}/2)_{\geq 0} \) (both \( m_\pm \) are in this set, for all classical unipotent affine Hecke algebras of type I to IV) not all partitions \( \rho \vdash n \) are generic. In order to parametrise the set of \( W_0 \)-orbits of the \( m \)-linear residual points in these cases,
Slooten [89] devised a general notion of a generalised “distinguished unipotent class”. For \( m \in \mathbb{Z}_{\geq 0} \), the generalised distinguished unipotent classes are partitions \( \lambda \) of \( 2n + m^2 \) with distinct odd parts, and with length at least \( m \); for \( m \in (\mathbb{Z} + 1/2)_{\geq 0} \) they are partitions \( \lambda \) of \( 2n + m^2 - 1/4 = 2n + \lfloor m^2 \rfloor \), with distinct even parts, of length at least \( m - 1/2 = \lfloor m \rfloor \). If \( \lambda \vdash 2n + \lfloor m^2 \rfloor \) is a distinguished unipotent \( m \)-class, then there exists a corresponding non-empty set of partitions \( \rho \vdash n \) whose sets of content vectors \( \xi(m, \rho) \) are equal, and form a \( W_0(B_n) \)-orbit of \( m \)-linear residual points whose jump sequence ([43, 89]) is equal to \((\lambda - 1)/2\). This characterises Slooten’s parametrisation alluded to above.

In view of the above explanation, given a distinguished unipotent \( m \)-class \( \lambda \vdash 2n + \lfloor m^2 \rfloor \) and a corresponding partition \( \rho \vdash n \), we will often write \( \vec{r}_{(m, \lambda)} \) instead of \( \vec{r}_{(m, \rho)} \) by abuse of notation.

### 5.4.2 The multiplicities of cyclotomic polynomials

Next, we shall develop a formula to count the multiplicities of cyclotomic polynomials in the residue of the \( \mu \)-function. Recall that \( \mathbf{M} \subset \mathbf{K}^\times \) is a free abelian group. Let \( \mathbf{M}_0 \subset \mathbf{M} \) be the subgroup generated by \( \mathbb{Q}^\times \) and the (linearly independent) set \( \{ \Phi_n \mid n \in \mathbb{N} \} \) of cyclotomic polynomials \( \Phi_n := \Phi_n(q) \). By Möbius inversion, \( \mathbf{M}_0 \) can also be spanned by \( \mathbb{Q}^\times \) and the linearly independent set \( \{ q^n - 1 \mid n \in \mathbb{N} \} \). For each element \( f \in \mathbf{M}_0 \), we can thus express \( f \) as some rational constant times the product of cyclotomic polynomials \( \Phi_n \), or as a product of rational constant, and powers of elements of \( \{ q^n - 1 \mid n \geq 1 \} \):

\[
f = r_1 \prod_n \Phi_n^{\text{cycl}_{f}(n/2)} = r_2 \prod_n (q^n - 1)^{\text{mult}_{f}(n/2)}, r_1, r_2 \in \mathbb{Q}^\times.
\]

By this way we have defined two functions \( \text{cycl}_f, \text{mult}_f : (\mathbb{Z}/2)_+ \rightarrow \mathbb{Z} \). Note that these two functions are of finite support in \((\mathbb{Z}/2)_+\). (We apologise to the readers for this convention to divide the argument of cycl and mult by 2, but this turns out to be convenient in the context of this thesis.)

**Remark 5.5.** When we speak of the multiplicity of a cyclotomic polynomial \( \Phi_n \) in \( f \in \mathbf{M}_0 \), we simply mean the exponent \( \text{cycl}_f(n/2) \) of \( \Phi_n \) as an irreducible factor of \( f \). Thus “\( f \) does not contain \( \Phi_n \) as a factor” is amount to say that \( \text{cycl}_f(n/2) = 0 \), while “\( f \) contains (or has) cyclotomic polynomial” is an equivalent saying of “there exists some \( j \in \mathbb{Z}_{>0} \) such that \( \text{cycl}_f(j/2) \neq 0 \). Notice that we do not require \( \text{cycl}_f(j/2) > 0 \) here! Therefore, if we say “\( f \) contains (or has) \( \Phi_j \) (\( j \in \mathbb{Z}_{>0} \)) as an irreducible factor”, we mean \( \text{cycl}_f(j/2) \neq 0 \).

If \( k \in (\mathbb{Z} + 1/2)_+ \), since \( q^{2k} - 1 = \prod_{j|2k} \Phi_j \), we have the following relation:

\[
\text{cycl}(k) = \sum_{d \geq 1} \text{mult}(dk).
\]

(5.14)
If there is no danger of confusion, we often omit the subscript $f$ in \( \text{mult}_f \) and \( \text{cycl}_f \).

Our normalisation of the trace

\[
\tau_{m,m_+}(1) = \frac{d_{m,m_+}^{0}}{(q^b - q^{-b})^n}
\]

of a rank \( n \) unipotent affine Hecke algebra \( (\mathcal{H}^{\mathfrak{m}}_{m,m_+}, \tau_{m,m_+}) \) of type I to VI contains only one odd cyclotomic polynomial with nonzero multiplicity, namely \( \Phi_1 \) with multiplicity \( -n \). As a consequence we have the following obvious but important observation: Modulo even cyclotomic polynomial factors, rational constants and powers of \( q \), we have a factorisation:

\[
f := \mu_{m_+,m_+}^{n}(\vec{r}_{(\rho_-,\rho_+)}{+}) \cong \mu_{m_+,m_+}^{n}(\vec{r}_{0,\rho_-}{+}) \mu_{m_-,m_+}^{n}(\vec{r}_{0,\rho_+}{+}) := f_- f_+ \quad (5.15)
\]

Clearly, if the multiplicities of all odd cyclotomic polynomials in both factors \( f_{\pm} \) on the right hand side are zero, then the same thing is true for \( f \) on the left hand side. Remarkably, for classical unipotent affine Hecke algebras of type I to VI, the converse is also true:

**Proposition 5.5.** Let \( \vec{r}_{(\rho_-,\rho_+)} \) be a residual point for a classical unipotent affine Hecke algebra \( (\mathcal{H}^{\mathfrak{m}}_{m,m_+}, \tau_{m,m_+}) \) of type I to VI. If the support \( \text{Supp}(\text{cycl}_f_{\pm}) \) of \( \text{cycl}_f_{\pm} \) is not contained in \( \mathbb{Z} \) for at least one of \( f_- \) or \( f_+ \), let \( p_{\pm} + 1/2 \in \text{Supp}(\text{cycl}_f_{\pm}) \cap (\mathbb{Z} + 1/2) \) denote the maximal element. In this case we have:

\[
\text{cycl}_f_{\pm}(p_{\pm} + 1/2) > 0,
\]

In particular, the support of \( \text{cycl}_f \) is not contained in \( \mathbb{Z} \) in this situation either, and if \( p + 1/2 \in \text{Supp}(\text{cycl}_f) \cap (\mathbb{Z} + 1/2) \) is the maximal element, then \( \text{cycl}_f(p + 1/2) > 0 \).

Consequently, if the left hand side of (5.15) has no odd cyclotomic polynomial factor, then the same is true for the two factors on the right hand side of (5.15).

We also point out the obvious fact that:

\[
\mu_{m_+,m_+}^{n}(\vec{r}_{0,\rho_+}{+}) \cong \mu_{m_+,m_+}^{n}(\vec{r}_{0,\rho_-}{+}) \quad (5.16)
\]

modulo even cyclotomic factors, for any choice of \( m_{\pm} \in \mathbb{Z}/4 \) such that the pair \((m_{\pm}^0,m_{\pm})\) also belongs to a type I to IV, whose base parameter \( b' \) of \((m_{\pm}^0,m_{\pm})\) is the same as for \((m_{\pm},m_{\pm})\) (thus \( b' = b \)). We often choose \( m_{\pm} \) as small as possible. For example, for the types V and VI we choose \( m_{\pm} = 1/4 \), so that the expressions to analyse are both of the form \( \mu_{1/4,m}^{n}(\vec{r}_{0,\rho}) \) with appropriate values for \( m \) and \( \rho \).

From here onward we may thus concentrate on the individual factors of the right hand side of (5.15), provided that we prove this positivity assertion on odd cyclotomic multiplicities for these factors, of course. We will omit the subscripts \( \pm \)
While we are focusing on these individual factors, we will also ignore powers of $q$ and rational constants since we are only interested in the $q$-rational factors (thus we can replace all factors of the form $(1 - q^a)$ by $(1 - q^{|a|})$).

### 5.4.3 Reduction by using standard STMs

We have already discussed the existence of the standard STMs ([75], [103]). In this section we shall use the standard STMs to illustrate how to reduce the problem of counting multiplicities of cyclotomic polynomials to some elementary cases. Furthermore, using the so-called extra-special STM (whose existence will be verified in next chapter), we can translate this problem in terms of partitions. At the end we will see that the fact that there is no odd cyclotomic polynomials in the formal degrees (of cuspidal unipotent representations of classical groups) puts a strong constrain on the corresponding partition and the associated Young tableau.

A standard STM (of types I to VI)

\[ \Psi : \mathcal{H}_l^{m_-m_+} \to \mathcal{H}_l^{\delta_-, \delta_+} \]

between two classical affine Hecke algebras gives rise to a corresponding morphism on the spectra of the centres (i.e. the associated algebraic tori) of the corresponding Hecke algebras

\[ \Psi_Z : \text{Spec } \mathcal{H}_l^{m_-m_+} \to \text{Spec } \mathcal{H}_l^{\delta_-, \delta_+} \]

sending orbits of residual points to orbits of residual points. More precisely, if \( W_{1,0} \to (\lambda_-, \lambda_+) \) is an orbit of residual points of \( \mathcal{H}_l^{m_-m_+} \) with

\[ \to (\lambda_-, \lambda_+) = (-\to (b_1, m_-; m_-), \to (b_1, m_+; m_+)), \]

then there exists a unique orbit of residual points \( W_{2,0} \to (\lambda_-, \lambda_+) \) of \( \mathcal{H}_l^{\delta_-, \delta_+} \) satisfying

\[ \Psi_Z(W_{1,0} \to (\lambda_-, \lambda_+)) = W_{2,0} \to (\lambda_-, \lambda_+), \]

where we may choose \( \to (\lambda_-, \lambda_+) \) in the standard form:

\[ \to (\lambda_-, \lambda_+) = (-\to (b_2, \delta_-; \delta_-), \to (b_2, \delta_+; \delta_+)). \]

Moreover, the main property of STMs ([74, Theorem 3.11]) implies the equality of residues:

\[ \mu_{\delta_-, \delta_+}^{\{\to (\lambda_-, \lambda_+)\}}(\to (\lambda_-, \lambda_+)) = c \mu_{m_-m_+}^{\{\to (\rho_-, \rho_+)\}}(\to (\rho_-, \rho_+)) \]  

(5.17)

for some constant \( c \in \mathbb{Q}^\times \). Thus \( \Psi_Z \) essentially preserves the formal degree ([74, Theorem 3.11]). Hence we can analyse the $q$-rational parts of the formal degrees of \( \mathcal{H}_l^{\delta_-, \delta_+} \) on either side.

Since we have enough standard STMs to map all unipotent Hecke algebras to a minimal object of \( \mathcal{E}_{\text{class}} \), we may conclude that:
Proposition 5.6. In order to prove Proposition 5.5 it suffices to prove this Proposition for the minimal objects of \((5.10)\).

This is our first step of reduction.

We now consider the concrete effects of various standard STMs on the orbit of residual points. For a standard translation STM \(\Psi\) [75, Section 3.2.4], we can use Slooten’s parametrisation of such orbits in terms of “unipotent partitions” to describe the action of \(\Psi_Z\) on Weyl group orbits of residual points. Namely, if \((\lambda_-, \lambda_+)\) is a pair of unipotent partitions of type I, II, III or IV with parameters \((m_-, m_+)\), and \(\Psi\) translates this parameter pair to a pair \((m'_-, m'_+)\), then \((\lambda_-, \lambda_+)\) is also a pair of unipotent partitions for the pair \((m'_-, m'_+)\), and

\[
\Psi_Z(W_0(-\tilde{r}^\mathfrak{b}_{(m_-,;\lambda_-)}, \tilde{r}^\mathfrak{b}_{(m_+;\lambda_+)}) = W_0'(-\tilde{r}^\mathfrak{b}_{(m'_-;\lambda_-)}, \tilde{r}^\mathfrak{b}_{(m'_+;\lambda_+)})
\]

Such standard translation STMs exist whenever \(m_\pm \in \mathbb{Z}/2\), \(m'_\pm\) lies between \(m_\pm\) and 0, \(m_\pm - m'_\pm \in \mathbb{Z}\) if \(m_\pm \in \mathbb{Z} + 1/2\), and \(m_\pm - m'_\pm \in 2\mathbb{Z}\) if \(m_\pm \in \mathbb{Z}\).

Besides these translation STMs, we have the spectral isomorphisms. Recall the group \(\text{Iso}\) of spectral isomorphisms is generated by two isomorphisms \(\eta\), which interchanges \(m_+\) and \(m_-\), and \(\eta_+\), which sends \(m_+\) to \(-m_+\) and leaves \(m_-\) invariant. These two transformations have the following effect on the orbits of residue points in their standard presentations:

\[
\eta_Z(W_0(-\tilde{r}^\mathfrak{b}_{(m_-,;\lambda_-)}, \tilde{r}^\mathfrak{b}_{(m_+;\lambda_+)}) = W_0(-\tilde{r}^\mathfrak{b}_{(m_+;\lambda_+)}, \tilde{r}^\mathfrak{b}_{(m_-;\lambda_-)})
\]

and

\[
\eta_{+,Z}(W_0(-\tilde{r}^\mathfrak{b}_{(m_-,;\lambda_-)}, \tilde{r}^\mathfrak{b}_{(m_+;\lambda_+)}) = W_0(-\tilde{r}^\mathfrak{b}_{(m_-,;\lambda_-)}, \tilde{r}^\mathfrak{b}_{(m_-;\lambda_+}')})
\]

(where \(\lambda_+\) is the conjugate of \(\lambda_+\)).

Last but not least, there are the extra special standard STMs, whose action on the residual points involves the bijections described by the extra special algorithms (see next chapter). In fact, let \(R\) be the collection of pairs \((m, \rho)\) where \(0 < m \in \mathbb{Z} + 1/4\) and \(\rho\) a partition, and denote by \(P_{\text{odd, dist}}\) the set of all partitions \(\lambda\) whose parts are all odd and distinct. The extra special algorithm provides a bijection \((m, \rho) \leftrightarrow (\delta, \lambda)\), where \((m, \rho) \in R\) and \(\lambda \in P_{\text{odd, dist}}\), and moreover \(\delta \in \{0, 1\}\) is completely determined by \(m\).

Let \((m_-, m_+)\) be a pair of parameters of type V or VI. There exists a unique extra special standard STM \(\Psi^e : \mathcal{H}^\mathfrak{g}_{(m_-, m_+)} \twoheadrightarrow \mathcal{H}^\mathfrak{g}_{(\delta_, \lambda_+)}\). Consider two pairs \((m_-, \rho_-), (m_+, \rho_+)\) in \(R\) with \(\rho_\pm\) partitions of \(n_\pm\), where \(n = n_- + n_+\). Let

\[
(m_\pm, \rho_\pm) \leftrightarrow (\delta_\pm, \lambda_\pm)
\]

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according to the extra special algorithm. Then

\[ \Psi^e_Z(W_{1,0}(-\overline{\eta}(2,m_-;\rho_-), \overline{\eta}(2,m_+;\rho_+))) = W_{2,0}(-\overline{\eta}(1,\delta_-;\lambda_-), \overline{\eta}(1,\delta_+;\lambda_+)) \]

It is a very important fact that every pair \(((\delta_-, \lambda_-), (\delta_+, \lambda_+))\) arises in the image of a suitable extra special STM.

We recall that if \(m \in \mathbb{Z}_{\geq 0}\) (i.e. types I, III, IV), a minimal object has parameters \((\delta_-, \delta_+\)) in the set \(\{(0, 1/2), (1/2, 1), (0, 1), (0, 0), (1, 1)\}\), hence at least one of \(\delta_\pm\) is in \(\{0, 1\}\). Say \(\delta = \delta_+ \in \{0, 1\}\), and let \(\lambda \vdash 2n + \delta\) be a partition with odd, distinct parts. Let

\[(m, \rho) \leftrightarrow (\delta, \lambda)\]

correspond each other through the extra special algorithm, where the left pair is expressed as a Young tableau \(T_m(\rho)\) and \(\lambda\) is an odd distinct partition whose number of parts is congruent to \(\delta\) modulo 2. As mentioned above, all residue points of the form \(\overline{\eta}(b=1,\delta;\lambda)\) are the image under an appropriate extra special STM \(\Psi : \mathcal{H}_n^{(1/4,m)} \rightarrow \mathcal{H}_n^{(0,\delta)}\) of a residue point of the form \(\overline{\eta}(2,m_+;\rho_+).\) Our second step of reduction is, instead of analysing formal degrees for the parameters \((0, \delta) \in \mathcal{V}\) (with \(b = 1\)), we can analyse the situation for the generic parameters \((1/4, m) \in \mathcal{V}\) (with \(b = 2\)). This turns out to be an important simplification.

By the above reductions, to prove Proposition 5.5, it suffices to establish the following:

**Proposition 5.7.** Denote \(R^*\) the subset of \(R\) consisting of elements \((m, \rho)\) such that \(\rho \vdash n\) a partition (possibly zero) and such that \(g(m, \rho) := \mu_{1/4,m}^{n(\overline{\eta}(0,\rho))}(\overline{\eta}_{0,\rho})\) has odd cyclotomic polynomial factors. For arbitrary \((m, \rho) \in R^*, \) set

\[ p_{\text{max}} := \max\{p \in \mathbb{Z}_{>0} : \text{cycl}_{g(m,\rho)}(p + 1/2) \neq 0\}, \]

then \(\text{cycl}_{g(m,\rho)}(p_{\text{max}} + 1/2) > 0.\)

The same statement is true for all 1/2-unipotent classes \(\lambda \vdash 2n\) (i.e. \(\lambda \vdash 2n\) has even, distinct parts), and factors of the form \(\Phi_{2p+1}\) of the \(\mu_{1/2,1/2}^{n(\overline{\eta}(0,\lambda))}\).

The proof of Proposition 5.7 (and hence of Proposition 5.5) is presented in [102]. In the following sections we will record the results.

### 5.4.4 The case \(\delta = 1/2\)

This is the case where the parameter \(m \in \mathbb{Z} + 1/2\), i.e. of type II. So the group is \(G = \text{SO}_{2n+1}\). The minimal object in this case is the unipotent affine Hecke algebra \(\mathcal{H}^n_{(1/2,1/2)} = C_n(1/2, 1/2)[q].\)
This relative easy case can be regarded as an application of the notion of residual point and generalised unipotent classes. If $\lambda \vdash 2n(n \in \mathbb{Z}_{>0})$ is a partition with even and distinct parts, we write $(\lambda - 1)/2$ for the sequence obtaining from subtracting every nonzero parts of $\lambda$ by 1, and dividing by 2. Thus every coordinate of $(\lambda - 1)/2$ is in $(\mathbb{Z} + 1/2)_+$. Let $\rho \vdash n$ be the partition such that the content vector $\xi(1/2, \rho)$ of the $1/2$-tableau $T_{1/2}(\rho)$ has a jump sequence equal to $(\lambda - 1)/2$.

Define a function $h : (\mathbb{Z} + 1/2)_+ \to \mathbb{Z}_{>0}$ by the rule that for each $k \in (\mathbb{Z} + 1/2)_+$, $h(k)$ is equal to the number of boxes in $T_{1/2}(\rho)$ with filling $k$. From [43, Section 4] we see that $h$ describes a linear residual point for $B_n$ and $m = 1/2$ if and only if the following two conditions hold:

(A) If $p$ is the largest element in the support of $h$, then $h(p) = 1$.

(B) For all $x \geq 1/2$, we have $h(x + 1) \in \{h(x), h(x) - 1\}$.

Let $\vec{r}$ be the corresponding residual point for $\mathcal{H}^n_{(1/2,1/2)}$. From this characterisation we can prove that in order that the rational function $\mu_{1/2,1/2}(\vec{r})$ contains no odd cyclotomic polynomials, it is sufficient and necessary that $h$ is support on $[0,p]$ and $h(x) = p + 1 - x$. If $h$ is not of this form, then the highest odd cyclotomic polynomial $\Phi_{2j+1}$ appears in the numerator of $\mu_{1/2,1/2}(\vec{r})$. From this result we deduce the following result.

**Proposition 5.8** ($\delta = 1/2$). [102, Corollary 5.5] There exists a positive residual point $\vec{r}$ such that $\mu_{1/2,1/2}(\vec{r})$ contains no odd cyclotomic polynomials if, and only if $n = i(i + 1)/2$ for some $i \in \mathbb{Z}_{>0}$. In this case we have a partition $\lambda = [2i,2i-2,2i-4,\ldots,2] \vdash 2n = i(i + 1)$, and $\vec{r}$ is of the form $\vec{r}_{(b=1,1/2,\lambda)}$. This residual point determines a cuspidal STM $C_0(1/2,i+1/2)[q] \sim C_n(1/2,1/2)[q]$.

Define $a,b \in \mathbb{Z}_{>0}$ by $\{a,b\} = \{i,i+1\}$. Then $\mu_{1/2,1/2}(\vec{r}) = d_b^0(q)d_b^1(q)$. Let $\pi$ be the corresponding cuspidal unipotent representation. If $i \equiv 0,3(\mod 4)$, then $\pi$ is a representation of $SO_{i(i+1)+1}(k)$. If $i \equiv 1,2(\mod 4)$, then $\pi$ is a representation of the non-trivial inner form of $SO_{i(i+1)+1}(k)$.

### 5.4.5 The case $\delta \in \{0,1\}$

This case includes the parameter type $\mathcal{X} \in \{\text{III, IV, V, VI}\}$, i.e. either $m_{\pm} \in \mathbb{Z}$ (type III, IV) or $m_{\pm} \in (\mathbb{Z} \pm 1/4)_+$ (type V, VI, the extra-special cases). Since we can apply the extra-special STMs to connect these two situations, we just need to deal with the extra-special cases, i.e. type V, VI. In fact, consider an extra-special STM $\Phi : \mathcal{H}^n_{1/4,m} \sim \mathcal{H}^l_{0,\delta}$. The $W_0$-orbits of generic $\Xi$-values residual points of $\mathcal{H}^n_{1/4,m}$ are naturally in one-to-one correspondence with partitions $\rho \vdash r$. This correspondence is realised by defining $\vec{r}_\rho := \exp(2k\zeta)$, where $k = \log(q)$, and the coordinates of the linear residual point $\zeta$ are the fillings of the $m$-tableau $T_m(\rho)$.
Applying the extra-special algorithm to \((m, \rho)\) we obtain \((\delta, \lambda)\) where \(\lambda\) is an odd distinct partition. This partition \(\lambda\) gives rise to a residual point \( \vec{r}_\lambda \) of \(\mathcal{H}_{0,\delta}^l\) by the following recipe: We can recover a partition \(\Lambda\) from \(\lambda\), such that the \(\delta\)-extremities of \(T_\delta(\Lambda)\) are all distinct, and equal to \((\lambda - 1)/2\). Let \(\xi\) be the linear residual point whose coordinates are the fillings of \(T_\delta(\Lambda)\). Then \(\vec{r}_\lambda = \exp(k\xi)\) is the desired residual point of \(\mathcal{H}_{0,\delta}^l\). We remark that such \(\Lambda\) always exists, and \(W_0 \vec{r}_\lambda\) does not depend on the choice of \(\Lambda\).

We proceed to define some useful notations on the \(m\)-tableaux \(T_m(\rho)\) (with \(m \in (\mathbb{Z} \pm 1/4)_+,\) and \(\rho\) some partition of some nonnegative integer \(r\)). The entries (i.e. fillings) of the upper-left, upper-right and lower-left cornered boxes of \(T_m(\rho)\) will be denoted by \(m, p_+, p_-\) respectively. Let \(a_m \in \{0, 1\}\) be defined by \(m - a_m \in \mathbb{Z}\). The entry of the last box below to \(p_+\) is denoted by \(r_+\), and the entry of the last box horizontal to the right of \(p_-\) will be called \(r_-\). Immediately we can observe that \(p_+ - m \in \mathbb{Z}_{>0}\). The \(a_m\)-diagonal inside the \(m\)-tableau indicates the change of congruence class modulo \(\mathbb{Z}\). In other words, the difference between two entries of \(T_m(\rho)\) on different sides of the \(a_m\)-diagonal is not an integer. Also notice that all entries of \(T_m(\rho)\) are in the same congruence class modulo \(\mathbb{Z}\) if and only \(p_- - m \in \mathbb{Z}\).

Below is an example of an \(m\)-tableau with \(m = 5/4, p_+ = r_+ = 21/4, p_- = 7/4, a_m = 1/4\) and \(r_- = 3/4\),

\[
\begin{array}{cccccc}
5/4 & 9/4 & 13/4 & 17/4 & 21/4 \\
1/4 & 5/4 & 13/4 \\
3/4 & 1/4 & 5/4 \\
7/4 & 3/4 \\
\end{array}
\]

By the reductions above, we only need to consider the multiplicities of odd cyclotomic polynomials occurring in the residue of the regularised \(\mu\)-function \(\mu_{1/4,m}^{r, \{\vec{r}_{0,\rho}\}}\) at the residue point \(\vec{r}_{0,\rho}\). We write \(\mu_{1/4,m}^r\) as a rational function on the algebraic torus as follow:

\[
\mu_{1/4,m}^r \equiv (1 - q)^{-r} \prod_{1 \leq i < j \leq n} \frac{(1 - t_i t_j)^2 (1 - t_i^{-1} t_j)^2}{(1 - q^2 t_i t_j)(1 - q^{-2} t_i t_j)(1 - q^2 t_i^{-1} t_j)(1 - q^{-2} t_i^{-1} t_j)} \times \prod_{z=1}^n \frac{(1 - t_z^2)^2}{(1 - q^{2m t_z})(1 - q^{-2m t_z})} \quad (\text{mod even cyclotomic polynomials}).
\]

Here \(n\) is the rank of the algebraic group \(G\). Under the action of a suitable Weyl group element, all coordinates of \(\vec{r}_\rho := \vec{r}_{0,\rho}\) are of the form \(v^{4x} = q^{2x}\) where \(x \in (\mathbb{Z} \pm 1/4)_+\).
Our objective is to find residual points such that, this residue (here we denote it by \( f(m, \rho) \)) does not contain any odd cyclotomic polynomial. In the disguise of partitions and Young tableaux, the requirement that \( \text{cycl}_f = 0 \) is translated to the restriction to the shapes of the \( m \)-tableaux. After thorough consideration, the following characterisation has been proved in [102].

**Theorem 5.2.** [102, Thm 5.12] In order that \( \text{cycl}_{f(m, \rho)}(k) = 0 \) for any \( k \in (\mathbb{Z} + 1/2)_+ \) with \( k \geq (p_+ + p_-)/4 \), the pair \((m, \rho)\) must be one of the following candidates:

(a) \( m \) is arbitrary and \( \rho \) is zero.

(b) \( m = 1/4 \) and \( T_m(\rho) \) is a square diagram, so that \( r_- = r_+ = 1/4 \).

(c) \( m = 3/4 \), and \( T_m(\rho) \) is a rectangular diagram, and \( r_- = r_+ = 1/4 \). In this case we can write \( p_+ = n + 3/4, \ p_- = n + 1/4 \) for some \( n \in \mathbb{Z}_{\geq 0} \).

(d) \( m = 5/4 \), and \( T_m(\rho) \) is a rectangular diagram, and \( r_- = r_+ = 3/4 \). In this case we can write \( p_+ = 2n + 5/4, \ p_- = 2n + 3/4 \) for some \( n \in \mathbb{Z}_{\geq 0} \).

(e) \( m = 7/4 \), and \( T_m(\rho) \) is a rectangular diagram, and \( r_- = r_+ = 1/4 \). In this case we can write \( p_+ = 2n + 7/4, \ p_- = 2n + 1/4 \) for some \( n \in \mathbb{Z}_{\geq 0} \).

If \((m, \rho)\) does not belong to this list, then the set

\[
\{ k \in \mathbb{Z} + 1/2, \ k \geq (p_+ + p_-)/4 : \text{cycl}_{f(m, \rho)}(k) \neq 0 \}
\]

is not empty. The largest element \( k_{\text{max}} \) of this set satisfies \( \text{cycl}_{f(m, \rho)}(k_{\text{max}}) > 0 \).

Now we can translate the result of Theorem 5.2 to the corresponding partitions \( \lambda \) (and hence obtain the residual points of \( \mathcal{H}^l_{0, \delta} \)) by applying the extra-special algorithm. We record the relating result.

**Theorem 5.3.** [102, Corollary 5.13] Suppose that \( \delta \in \{0, 1\} \). Every positive linear residual point \( \vec{\tau}^* \) such that the residue \( \mu^{n, \{\vec{\tau}^*\}}_{0, \delta} \) contains no odd cyclotomic factors, is of the form \( \vec{\tau}^* = \vec{\tau}^*_\lambda \), where \( \lambda \vdash \delta + 2n \) is a partition with odd, distinct parts. The pair \((\delta, \lambda)\) must belong to one of the following cases:

(a) Choose \( m \in (\mathbb{Z} + 1/4)_+ \) and define \( \kappa \in \mathbb{Z}_{\geq 0} \) and \( \epsilon \in \{0, 1\} \) by writing \( m = \kappa + (2\epsilon - 1)/4 \). Define \( \lambda = [1 + 2\epsilon, 5 + 2\epsilon, \ldots, 4(\kappa - 1) + 1 + 2\epsilon] \), and \( \delta \in \{0, 1\} \) by \( \kappa \equiv \delta \mod 2 \). Define \( n \) by \( 2n + \delta = \kappa(k + 2\epsilon - 1) \). Then \( \vec{\tau}^*_\lambda \) represents a cuspidal (extra-special) unipotent STM \( \mathcal{H}^{l}_{1/4, m} \hookrightarrow \mathcal{H}^{n}_{0, \delta} \). In particular, modulo powers of \( q \) and rational constants, we have \( \mu^{n, \{\vec{\tau}^*\}}_{0, \delta} = d^{0}_{1/4, m} \) (with \( d^{0}_{1/4, m} \) as in (5.6)), and for all these cases \( \mu^{n, \{\vec{\tau}^*\}}_{0, 0} \) indeed has no odd cyclotomic factors.

(b) For \( i \in \mathbb{Z}_{\geq 0} \), put \( \lambda = [1, 3, \ldots, 4i + 1, 4i + 3] \vdash 2n \) with \( n = 2(i + 1)^2 \), and put \( \delta = 0 \). Then \( \vec{\tau}^* = \vec{\tau}^*_\lambda \) represents a cuspidal unipotent STM \( \mathcal{H}^{0, 4i + 2}_{0, 2i + 2} \hookrightarrow \mathcal{H}^{0, 0}_{0, 0} \), and \( \mu^{n, \{\vec{\tau}^*\}}_{0, 0} = (d^{a}_{0}(q))^2 \) with \( a = i + 1 \). In particular, \( \mu^{n, \{\vec{\tau}^*\}}_{0, 0} \) indeed no odd
cycloptomic factors for all these cases. (Note: We consider the empty diagram with \( \delta = 0 \) as belonging to case (a).)

(c) For \( i \in \mathbb{Z}_{\geq 0} \), put \( \lambda = [1, 3, \ldots, 4i + 3, 4i + 5] \vdash 2n + 1 \) with \( n = 2(i + 1)(i + 2) \), and put \( \delta = 1 \). Then \( \overline{\tau} = \overline{\tau}_\lambda \) represents a cuspidal unipotent STM \( \mathcal{H}_{0,2i+3} \sim \mathcal{H}_{0,1}^{n} \), and \( \mu_{0,1}^{n,\{\overline{\tau}\}} = (d_0^B(q))^2 \) with \( b = i + 1 \). In particular, for all these cases \( \mu_{0,1}^{n,\{\overline{\tau}\}} \) indeed has no odd cyclotomic factors.

(d) Let \( i \in \mathbb{Z}_{\geq 0} \), put \( n = (i+1)^2 - 1, \lambda = [3, 5, 7, \ldots, 8i + 5, 8i + 7] \vdash 2n + 1 \) and \( \delta = 1 \). In this case, modulo powers of \( v \) and rational constant, we have

\[
\mu_{0,1}^{n,\{\overline{\tau}\}} = \frac{(1 - q^{4(r+1)})}{(1 - q)} \mu_{0,0}^{n+1,\{\overline{\tau}'\}} = \frac{(1 - q^{4(r+1)})}{(1 - q)} (d_a^D(q))^2
\]

where \( \overline{\tau}' = \overline{\tau}'_{\lambda'} \) with \( \lambda' = [1, 3, 5, \ldots, 8i + 7] \) (then \( \overline{\tau}' = (1, \overline{\tau}') \)), and \( a = 2i + 2 \). In particular, \( \mu_{0,1}^{n,\{\overline{\tau}\}} \) has no odd cyclotomic polynomials as factors if and only if \( i = 2^s - 1 \) for some \( s \geq 0 \).

(e) Let \( i \in \mathbb{Z}_{\geq 0} \), put \( n = (i+1)^2 + 1, \delta = 0 \), and \( \lambda = [1, 3, 5, \ldots, 8i + 5, 8i + 9] \vdash 2n \). Then \( \overline{\tau}_\lambda = (\overline{\tau}'_{\lambda'}, q^{4(i+1)}) \), where \( \lambda' = [1, 3, 5, \ldots, 8i + 7] \). Since \( \overline{\tau}'_{\lambda'} \) represents a standard translation morphism, we see that \( \overline{\tau} \) is the image under the standard translation STM \( \mathcal{H}_{1,m}^{\delta} \sim \mathcal{H}_{0,0}^{n} \) of the residual point \( t = (q^{4(r+1)}) \), where \( m = 4(i+1) \). Consequently, modulo powers of \( v \) and rational constants we have:

\[
\mu_{0,0}^{n,\{\overline{\tau}\}} = \frac{(1 - q^{4(r+1)})}{(1 + q^{4(r+1)})(1 - q)} (d_a^D(q))^2
\]

with \( a = 2i + 2 \). In particular, \( \mu_{0,0}^{n,\{\overline{\tau}\}} \) has no odd cyclotomic polynomials as factors if and only if \( i = 2^s - 1 \) for some \( s \geq 0 \).

Together with the results of Theorem 5.2 the above results imply that in all cases, if a residue of the form \( \mu_{0,\delta}^{n,\{\overline{\tau}\}} \) or of the form \( \mu_{1/4,m}^{r,\{\overline{\tau}\}} \) contains odd cyclotomic polynomials with nonzero multiplicity, then the highest odd cyclotomic factor which has nonzero multiplicity in the residue in fact has positive multiplicity.

5.5 Consequence of the essential uniqueness

We introduce the notation \( \mathcal{I}(G, u) \) for the collection of \( G \)-conjugacy classes of all maximal unipotent types \( s = (u, \mathcal{P}, \sigma) \), where \( \mathcal{P} \in G(u) \) is a maximal \( F_u \)-stable parahoric, \( \sigma \) is a cuspidal unipotent representation of the reductive quotient \( \overline{\mathcal{P}} \).
We make a convention that we will identify equivalence classes of irreducible smooth representations of $G(k)$ with their corresponding irreducible admissible characters. Let $\mathcal{U}^0(G(k))$ be the set of all cuspidal unipotent characters of $G(k)$, and $\mathcal{U}^0(G(k), u, \mathcal{P}, \sigma)$ be the subset consists of irreducible cuspidal unipotent characters $\chi_{\pi}$ such that $\pi$ appears in $c\text{-Ind}_{G_u}^{G_k} \sigma$. Recall the $s$-spherical Hecke algebra $\mathcal{H}_s := \text{End}_{G_u}(\mathcal{U}^0(G, u, \mathcal{P}, \sigma))$ is isomorphic to the group algebra $\mathbb{C}[\Omega^F(\mathcal{P})]$. We thus obtain a bijection between $\mathcal{U}^0(G, u, \mathcal{P}, \sigma)$ and $(\Omega^F(\mathcal{P}))^*$.

Define $\overline{\Lambda}^{dulp,cud}(G, [u])$ to be the set of $G^\vee$-conjugacy classes of discrete unramified Langlands parameters $\lambda$, such that there is a cuspidal unipotent representation $\pi$ of $G(u)$ such that $\text{fdeg}_q(\pi) = \gamma_q(\lambda)$. Put

$$\overline{\Lambda}^{dulp,cud}(G) = \bigcup_{[u] \in H^1(F, G)} \overline{\Lambda}^{dulp,cud}(G, [u]).$$

Recall that ([71, Appendix]) every discrete unramified Langlands parameter $\lambda$ of $G$ is determined by the images $\vartheta s := \lambda(\text{Frob} \times (\frac{1}{0} 1))$ and $c := \lambda(1 \times (\frac{q^{1/2}}{0} 0 \frac{q^{-1/2}}{0} ))$ in $L^G := \langle \vartheta \rangle \ltimes G^\vee$. And $\mathcal{P} := sc$ is a residual point of the Iwahori-Hecke algebra of $G$. In this way we set up a bijection between equivalence classes of discrete unramified Langlands parameter of $G$, and the $W_0$-orbits of residual points of $\mathcal{H}^{\text{IM}}(G)$.

The requirement that $\mathcal{P} = sc$ is a residual point implies that $c$ is the image of a distinguished nilpotent element in the centraliser $\mathfrak{g}_s$ of $s$ in the Lie algebra $\text{Lie}(G^\vee) = \mathfrak{g}$. This forces this centraliser $\mathfrak{g}_s$ to be semisimple. By [79], this is amount to say that $s$ is the image of a vertex $v_s$ in the closure of the dual alcove $\mathcal{C}^\vee$ under the exponential map defined there. The vertex $v_s$ corresponds to a node in the Kac diagram [79, Section 3.4], which is the “twisted” version of our spectral diagram.\footnote{Namely, if we reverse the arrows in the Kac diagram by multiply the “short roots” by 2 or 3, then we get a diagram which is the underlying affine Dynkin diagram of the spectral diagram. See also the full list of Kac diagrams with parameters in Kac’s book [51, §4.8]. We also observe that for each split groups, the Kac diagram coincides with the spectral diagram.}

Hence we can also conclude that $s$ comes from a node in the Kac diagram, or in the spectral diagram, while the orbit $[\lambda]$ of the Langlands parameter $\lambda$ corresponds to an $\Omega_{Y_{qs}}$-orbit in the spectral diagram $\Gamma_s$ of $G$, where $\mathcal{R}_{qs} = (X_{qs}, R_0, Y_{qs}, R'_0)$ is the root datum associated with $\mathcal{H}^{\text{IM}}(G_{qs})$, and $\Omega_{Y_{qs}} := Y_{qs}/\mathbb{Z}R'_0$. We also recall that $\Omega^F := X_{qs}/\mathbb{Z}R_0$.

Let $\overline{\Lambda}^{dulp}(G)$ be the set of $G^\vee$-conjugacy classes of discrete unramified Langlands parameters of $G$. Recall the group $G(k)_{nr}$ of unramified complex characters of $G(k)$ (which is the Pontryagin dual of $\Omega^F$) acts on $\overline{\Lambda}^{dulp}(G)$, and we have $\overline{\Lambda}^{dulp}(G_{sc}) = (\Omega^F)^* \Lambda^{dulp}(G)$, where $G_{sc}$ denotes the simply connected of $G$. This applies particularly to the case where $G$ is of adjoint type.
We are ready to state the main result of this thesis as follow.

**Theorem 5.4.** [102, Theorem 1.1] Let \( G_{qs} \) be the quasi-split form of \( G \), and suppose that \( G \) is an inner form determined by \([u] \in H^1(F, G)\).

(i) For each cuspidal unipotent representation \( \pi \) of \( G \), say \( \pi \) is inducing from the maximal unipotent type \((u, \mathcal{P}, \sigma)\), there exists a unique \((\Omega^F)^*\)-obrit \([\lambda_{sc}] = (\Omega^F)[\lambda] \in \Lambda^{dulp(G_{sc})}\), such that

\[
\text{fdeg}_q(\pi) = \gamma_q(\lambda_{sc}). \tag{5.20}
\]

(ii) For every \( u \in Z^1(F, G) \), we have the following canonical surjection

\[
\varepsilon : \mathcal{I}(G, u) \to \overline{\Lambda^{dulp.cad}(G_{sc})}, \tag{5.21}
\]

whose fibre above \([\lambda_{sc}]\) are those conjugacy classes \([([\mathcal{P}, \sigma])\) satisfying that (5.20) for each cuspidal unipotent representation \( \pi \in \mathcal{U}^0(G, u, \mathcal{P}, \sigma) \) for a uniquely determined orbit of maximal \( F_u \)-stable parahoric subgroups of \( G(u) \).

(iii) In situation of (ii), the group \((\Omega^F/\Omega^F(\mathcal{P}))^*\) is contained in \((\Omega^F)^*_{[\lambda]}\), the isotropy group of \([\lambda]\) in \((\Omega^F)^*\), where \([\lambda] \in \Lambda^{dulp(G)}\) maps to \([\lambda_{sc}] \in \Lambda^{dulp(G_{sc})}\). (It does not matter which \([\lambda]\) above \([\lambda_{sc}]\) we choose, since \([\lambda_{sc}] = (\Omega^F)^*[\lambda]\).)

(iv) Recall we have identified \([\lambda]\) with a residual orbit \( W_0 \overset{\tau}{\to} \) with \( \tau = sc \), and \( s \) corresponds to a node \( v_s \) in the Kac diagram. (In Lusztig’s notation: \( J = I - \{v_s\} \).) Let \( n_s \) be the coefficient of the node \( v_s \) in the Kac diagram (as list in [79]). Let \( \phi(\cdot) \) be the Euler totient function.

Continue the situation of (ii). Define \( a := |(\Omega^F)^*[\lambda]|, \ a' := |(\Omega^F(\mathcal{P}))^*|, \ b := \phi(n_s)|(\Omega^F)^*[\lambda]: (\Omega^F/\Omega^F(\mathcal{P}))^*|, \) and \( b' \) to be the cardinality of the set

\[
\{[\mathcal{P}, \sigma'] \in \mathcal{I}(G, u) : \text{deg}(\sigma') = \text{deg}(\sigma)\}.
\]

In other words, \( b' \) is equal to the size of fibre of \( \varepsilon \) described in (ii) in which \((\mathcal{P}, \sigma)\) lies. Then \( a, b, a', b' \) are equal to Lusztig’s \( a, b, a', b' \) in [61] provided that \( G = G_{ad} \). We have \( ab = a'b' \) for all cases. In particular, \( b \) is equal to the number of cuspidal local systems supported by \([\lambda]\) on which \( LZ \) acts by \([u]\) via Kottwitz’s isomorphism.

**Remark 5.6.** Observe that to prove this theorem, it suffices to treat the case that \( G = G_{ad} \). Notice that for classical groups, in non-extra-special cases we always have \( b' = 1 \). However, \( b' \) can be bigger than 1 in extra-special case, as we will see in the example in next section. For classical groups, the coefficient \( n_s \) is either 1 or 2, so \( \phi(n_s) = 1 \).
5.6 Check with classical groups

The joint-work [102] by the author and his supervisor gives a full proof for the case of classical groups of Theorem 5.4. It is a case by case check. We will sketch the idea to prove Theorem 5.4 (iii), (iv) presented in [102], and then illustrate the conclusions by an example.

We start by explain the relation of discrete unramified local Langlands parameters with residual points. By Borel’s theorem, the semisimple conjugacy classes in $\vartheta G^\vee$ is in natural bijection with the categorial quotient $W_0\backslash T$, which can be interpreted as the totality of the central characters of $\mathcal{H}^{\text{IM}}(G)$ supporting discrete series characters. Recall the root system $R_m$ of the Iwahori-Hecke algebra $\mathcal{H}^{\text{IM}}(G)$. Let $\Delta := \text{wt}(R_m^\vee)/\mathbb{Z}R_m^\vee$ be the group of special diagram automorphism ([61]). It has a subgroup $\Omega_Y := Y_{qs}/\mathbb{Z}R_m^\vee$. And there is a short exact sequence

$$1 \longrightarrow \Omega_Y \longrightarrow \Delta \longrightarrow (\Omega^F)^* \longrightarrow 1.$$ 

In our present situation, the quasi-split $G$ is of adjoint type $G_{ad}$, hence $\Omega_Y$ is trivial. We thus draw our first conclusion: If $G = G_{ad}$, then $(\Omega^F_{ad})^* = \Delta = \text{wt}(R_m^\vee)/\mathbb{Z}R_m^\vee$.

One can also interpret (cf. [76]) the collection of central characters of $\mathcal{H}^{\text{IM}}(G)$ which support a discrete series character as the collection of pairs $(\Omega_Y s(e), W_{s(e)}\xi)$, where the first component denotes an $\Omega_Y$-orbit of vertices $s(e)$ of the spectral diagram (can be viewed as a vertex of the Kac diagram in our present circumstance), and the second component is an orbit of linear residual points $W_{s(e)}\xi$ of the graded affine Hecke algebra $\mathcal{H}_e$ associated with $\Omega_Y s(e)$. Further, there is a canonical bijection between the latter orbits $W_{s(e)}\xi$ of linear residual points, and the collection of distinguished nilpotent orbits $N \subset g^*$, where $s(e) \in T^0$ is of finite order.

An element $\gamma \in (\Omega^F_{ad})^*$ fixes $\lambda = \lambda_{ad}$ if and only if $\gamma$ fixes $s(e)$ and the action of $\gamma$ on $g^*$ fixes $N$ (equivalently, $\gamma$ should fix $W_{s(e)}\xi$). Fortunately, we have the important fact that the orbits $W_{s(e)}\xi$ of a graded affine Hecke algebra $\mathcal{H}_e$ associated with an irreducible based root system $(R, F)$ is invariant under all diagram automorphisms of $(R, F)$. Therefore, we draw the third conclusion that: Let $W_0 \mathcal{F}$ corresponds $(e, W_{s(e)}\xi)$ as above. If $W_0 \mathcal{F}$ defines a standard STM, then the isotropy group $\Delta_{\mathcal{F}}$ of $W_0 \mathcal{F}$ in $\Delta$ is equal to the isotropy subgroup $\Delta_e$ of the vertex $e$ in the Kac diagram.

On the arithmetic side, we have a similar result: The isotropy subgroup of $(\mathcal{P}, \sigma)$ in $\Omega^F_{ad}$ is the stabiliser subgroup $\Omega^F_{ad}(\mathcal{P})$ of $\mathcal{P}$. This is a consequence of the fact that any unipotent pair $(\mathcal{P}, \sigma)$ is inert for twisting by automorphisms of $\mathcal{F}$. 

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These conclusions we just derived, as well as the uniqueness property of $W_0 \vec{P}$ satisfying the constrain equation

$$(\mu_{\text{min}})_q((\vec{P})) := (\mu_{\delta^-\delta^+})_q((\vec{e}^-e^+)) = \text{fdeg}_q(\sigma) \quad (5.22)$$

are the tools to prove the Theorem 5.4. We explain this theorem by the example of the projective conformal symplectic group $\text{PCSp}_{2n}$. For the discussion of other groups, we refer to [102, Section 6].

### 5.6.1 Example: The group $\text{PCSp}_{2n}$

In this case the group $\Omega_{\text{ad}} = C_2 := \langle \rho \rangle$. The Frobenius automorphism acts on the affine Dynkin diagram of $\text{PCSp}_{2n}$ as a diagram automorphism which we will denote by $\vartheta$. Under this action we have $\Omega/(1-\vartheta)\Omega \simeq \langle \rho \rangle$. We thus have to consider the non-trivial inner form $u = \rho$. Thus, we separate the two cases (i) $u = 1$ and (ii) $u = \rho$.

For every $u \in \Omega/(1-\vartheta)\Omega$, the inner twist $F_u = \text{Ad}(u) \circ F$ of the Frobenius element acts on the building of $G$ via a diagram automorphism $\vartheta u$. Let $(u, P_{a,b}, \sigma)$ be a maximal $F_u$-stable unipotent type of $G(u)$. The conjugacy class of the $F_u$-stable parahoric subgroup $P_{a,b}$ is determined by two integers $a, b \in \mathbb{Z}_{\geq 0}$ (subject to conditions which will be explicit in each case below). In the diagram of an apartment of the building of $G$, the subdiagram corresponding to these conjugacy classes is a maximal proper subdiagram. We box this subdiagram, so the boxed subdiagram is the root diagram of a finite group of Lie type supporting a cuspidal unipotent characters. In other words, the integers $a, b$ should satisfy the rank condition in Section 3.2.3.

(i) We tabulate the result corresponding to $u = 1$ firstly. This case is of parameter type $X = \text{III}$. The Frobenius action $\vartheta$ is trivial (the split case), and $n \geq 2$. (See also [61, 7.48, 7.49, 7.50].) In this case, we have $b = b' = 1$ (always), and $a = a' = 1$ if $m_- \neq 0$, and $a = a' = 2$ if $m_- = 0$.

(ii) If $u = \rho$ is non-trivial, we write $G(\rho)$ for the inner form of $G = \text{PCSp}_{2n}$. This is the extra-special case of parameter type $X = \text{V}$. The element $\rho$ acts on the affine Dynkin diagram of $G$, introducing some type $^2A$ factors in the maximal $F_u$-stable parahoric subgroup $P$, which admits a cuspidal unipotent representation $\sigma$ such that the $q$-rational part of its formal degree is given by $\text{fdeg}_q(\sigma) = d_a^{(2A)}(q^2)d_b^B(q^2)$. The group $P_{a,b}$ has a factor of type $^2A_{(a/2)(a+1)}(q^2)$, and a factor of type $C_{b(b+1)}(q^2)$ (by which we mean the restriction of scalars group defined over $\mathbb{F}_q$, whose group of $\mathbb{F}_q$-rational points is equal to $\text{Sp}_{2b^2+2b}(\mathbb{F}_q)$). This factors appears in the subdiagram of $P_{a,b}$ in the affine Dynkin diagram of $G$ as two isomorphic components, which are interchanged by $\rho$.)

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Table 5.1: $G = \text{PCSp}_{2n}$, split

| $\{2a+1, 2b+1\}$ | $\{m_+ + m_-, |m_+ - m_-|\}$ |
|---------------------|----------------------------------|
| $a, b, m_\pm \in \mathbb{Z}_{>0}, a \geq b, m_+ - m_- \notin 2\mathbb{Z}$ | $\lambda \leftrightarrow (\lambda_-, \lambda_+) \leftrightarrow (m_-, m_+)$ |

### \text{Id}_{q}(\sigma) = d_a^{B}(q)d_b^{B}(q)

$P_{a,b} = D_a = B_{b(b+1)}$

<table>
<thead>
<tr>
<th>$J = D_{m_+^2/2} \times B_{(m_+^2 - 1)/2}$</th>
<th>$J = D_{m_+^2/2} \times B_{(m_+^2 - 1)/2}$ if $a - b &gt; 0$ is odd</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J = D_{m_+^2/2} \times B_{(m_+^2 - 1)/2}$ if $a + b$ is even</td>
<td>$J = B_{(m_+^2 - 1)/2}$ if $a = b$</td>
</tr>
</tbody>
</table>

### $\Omega^{F}_{ad}(P_{a,b})$

| $\begin{cases} 
1 & \text{if } a \neq b \\
C_2 & \text{if } a = b 
\end{cases}$ |

### $\Delta_{\varphi} = \begin{cases} 
C_2 & \text{if } m_- \neq 0 \\
1 & \text{if } m_- = 0 
\end{cases}$

$a = b \iff m_- = 0$

The natural numbers $a, b$ determines $(m_-, m_+)$ by the relation $\{1/2 + a, 1/2b\} = \{m_- + m_+, |m_- - m_+|\}$, and we obtain a residual point $\mathfrak{r}^{\#}_{(\lambda_-, \lambda_+)}$ by computing the partitions $\lambda_\pm$ (whose length $\ell(\lambda_\pm) := \ell_\pm$ is equal to $\kappa_\pm$).

The isotropy group $\Omega^{F}(P_{a,b})$ is always equal to $C_2 = \langle \rho \rangle$. So we have two cuspidal characters. On the geometric side, we have $\Delta_{\varphi} = C_2$ unless $m_- = 1/4$. But $m_- = 1/4$ is equivalent to that $\lambda_-$ is the zero partition.

**Definition 5.4.** We use the notation $2^{(2a+b)}$ to express an extra special 2-group of order $2^{2a+b}$, with $2^{2a+b-1}$ representations of dimension 1, and $2^{b-1}$ irreducible representations of dimension $2^a$.

We learn from [75, Proposition 13.4] that if $\lambda_-$ is not a zero partition, the centraliser group $A_{\lambda_{ad}}$ is a group of type $2^{(\ell_- + \ell_+ - 3) + 2}$. Therefore, if $\lambda_- \neq 0$, there are **two** representations of $A_{\lambda_{ad}}$ of dimension $2^{(\ell_- + \ell_+ - 3)/2}$ whose restriction on $LZ \simeq (\Omega/(1-\theta)\Omega)^*$ is $\rho$ times the identity, and these two “extra special” cuspidal local systems correspond to the **two** cuspidal characters.

Now consider the case that $\lambda_-$ is a zero partition. We then have $\Delta_{\varphi} = 1$, and the group $A_{\lambda_{ad}}$ is a group of type $2^{(\ell_+ - 1) + 1}$, with **one** representation of dimension $2^{(\ell_+ - 1)/2}$ which is corresponds to a cuspidal local system. And we have two Langlands parameters $\lambda_{ad}$ and $\lambda'_{ad}$ over $\lambda_{sc}$ (thus the $(\Omega^\theta)^*$-orbits of $\lambda_{ad}$ and of $\lambda'_{ad}$ coincide), corresponding to the two cuspidal characters.

Let us list the informations for the inner form $G(\rho)$ in two tables.

We read off from the table for $u = \rho$ that we always have $a = a' = 2$ since $\Omega^{F} = \Omega^{F}(P) = C_2$, and $b = b' = 1$ if $m_- = 1/4$, while $b = b' = 2$ if $m_- \neq 1/4$. So $ab = a'b'$ in any case.
Table 5.2: The inner form $G(\rho)$ of $G = \text{PCSp}_{2n}$

<table>
<thead>
<tr>
<th>Expression</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>${1/2 + a, 2b + 1} = {m_+ + m_-,</td>
<td>m_+ - m_-</td>
</tr>
<tr>
<td>$\lambda \leftarrow (\lambda_-, \lambda_+) \leftarrow (m_-, m_+)$</td>
<td></td>
</tr>
<tr>
<td>$\text{fdeg}_q(\sigma) = d_a^{2A}(q^2)d_b^{B}(q^2)$</td>
<td>$m_- = 1/4 \leftrightarrow \lambda_- = 0$</td>
</tr>
<tr>
<td>$\mathcal{P}<em>{a,b} = 2A</em>{(a/2)(a+1)-1} \times C_{b(b+1)} \times C_{b(b+1)}$</td>
<td>For the subdiagram $J$, see [61, 7.51, 7.52, 7.53]</td>
</tr>
<tr>
<td>$\Omega_{ad}(\mathcal{P}_{a,b}) = C_2 = \langle \rho \rangle$</td>
<td>$\Delta_\tau = \begin{cases} 1 &amp; \text{if } m_- = 1/4 \ C_2 &amp; \text{else} \end{cases}$</td>
</tr>
</tbody>
</table>

We conclude that in both (classical and extra-special) cases, Theorem 5.4 holds. This completes the proof for $G = \text{PCSp}_{2n}$. 
