On cuspidal unipotent representations

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Chapter 6
An extra-special correspondence

We have claimed in last chapter that besides the transportation STMs, there are also STMs which are said to be *extra special*, from the classical affine Hecke algebras of parameter type V (resp. VI) to those Hecke algebras of parameter type III (resp. IV). In order to verify these statements, we shall introduce the extra special algorithm in this chapter, and prove the existence of extra special STMs.

The extra special algorithm yields a bijective correspondence

\[(m, \rho) \leftrightarrow (\delta, \lambda)\]

where \(0 < m \in (\mathbb{Z} \pm 1/4), \delta \in \{0, 1\} \) and \(\rho, \lambda\) partitions (including zero partition). We will denote by \(R\) the collection of all such data \((m, \rho)\). The partitions \(\lambda\) are with all parts odd and distinct.

With the extra special algorithms we can fulfil the classification of the standard STMs, also we can use the extra special correspondence to translate the results concerning on counting the odd cyclotomic polynomials in the case of parameters \(m\) to the case of parameters \(\delta\). More precisely, recall that the parameters \(\delta\) of the \(\mu^{IM}\)-function of the Iwahori-Hecke algebra takes values in \(\{0, 1/2, 1\}\). The case \(\delta_+ = \delta_- = 1/2\) is relative easy to analyse. For the integral parameter cases, we will rely on the extra special bijection, to translate the results on the multiplicities of odd cyclotomic polynomials in the \(q\)-rational part of the formal degree (which is equal to the \(q\)-rational part of the residue of the \(\mu^{IM}_{m_,m_+}\)-function at some residual point) into the multiplicities in the function \(\mu^{IM}_{\delta_-,\delta_+}\) with \(\delta_\pm \in \{0, 1\}\).

This chapter consists of two sections. The first section is devoted to prove the extra special bijection. We will closely follow the description of the extra special
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algorithm as published in [102]. The verification of the existence of extra special
STMs will occupy the second section.

6.1 Description of the extra-special algorithm

The following algorithms are published in the article [102], without proof. Here we
borrow the description of the algorithms there, and explained how to prove that
the two operations denoted by \( \mathcal{E} \) and \( \mathcal{D} \) below are inverse to each other. Meanwhile
we give some instructive examples.

Let \( P_{\text{odd, dist}} \) be the collection of all partitions (including the zero partition) with
odd, distinct parts. Let

\[
R = \{(m, \rho) \mid m \in \mathbb{Z} \pm 1/4, m > 0 \text{ and } \rho \text{ a (possibly zero) partition}\}.
\]

We will define two operations \( \mathcal{E} : P_{\text{odd, dist}} \rightarrow R \) and \( \mathcal{D} : R \rightarrow P_{\text{odd, dist}} \) and prove
they are inverse to each other. We refer to \( \mathcal{E} \) as the extra-special algorithm.

Remark 6.1. We remark that the map \( \mathcal{D} \) is similar, and indeed equivalent to the
algorithm discussed in [24, Section 4.4]. It was also shown in [24] that \( \mathcal{D} \) is an
injective map. We thank Dan Ciubotaru for pointing out the reference [24] to us.

We first recall the \( m \)-tableau. For a real number \( m \) and a partition \( \lambda \), the \( m \)-tableau
(cf. [89], [43]) \( T_m(\lambda) \) of \( \lambda \) is defined to be the tableau of shape \( \lambda \) with its box \( b_{i,j} \)
(where the coordinates \((i, j)\) have the same meaning as for matrix entries) filled
with the nonnegative real number \(|m - i + j|\).

The algorithm \( \mathcal{E} : \lambda \mapsto (m, \rho) \) produces a number \( m \in (\mathbb{Z} \pm 1/4)_+, \) and an
m-tableau, whose shape we call \( \rho \). The steps to produce \( m \) and \( T_m(\rho) \) from
\( \lambda \in P_{\text{odd, dist}} \) are as follows:

1. Arrange the parts of \( \lambda \) as a non-negative integral sequence in decreasing order.
   Define \( j = (\lambda - 1)/2 \), where \( (\lambda - 1)/2 \) means subtracting 1 from all nonzero
   parts of \( \lambda \), and then dividing each part by 2.
   We stress that we do not regard \( j \) as a partition, but as a tuple of nonnegative
   integers, whose length is equal to the number of nonzero parts of \( \lambda \).

2. Let \( \kappa \geq 0 \) be the excess number of parts of the dominant parity type (even or
   odd) of \( j \). Put \( \epsilon = 1 \) if the dominant parity type is odd or if \( \kappa = 0 \) (in which
case we shall call the dominant parity type odd as well), otherwise put \( \epsilon = 0 \).
   Put \( m = \kappa + (2\epsilon - 1)/4 \). This gives us the required number \( m \in (\mathbb{Z} \pm 1/4)_+ \).
(3) Let \( j' = (\gamma_1, \ldots, \gamma_k) \) be the sub-sequence in \( j \) of the \( k \) smallest parts of dominant parity type.

(4) Removing \( j' \) from \( j \) and denote the remaining sub-sequence of \( j \) by \( j'' \). Notice that the parts of \( j'' \) is an even number, and the numbers of odd parts and even parts are equal. Arrange \( j'' \) in \( t \) pairs:
\[
  j'' = \left( (\alpha_1, \beta_1), \ldots, (\alpha_t, \beta_t) \right)
\]
with \( \alpha_1 > \cdots > \alpha_t \) and \( \beta_1 > \cdots > \beta_t \), where for all \( i \), \( \alpha_i \) is of dominant parity type and \( \beta_i \) is of the other parity type.

(5) For every pair \( (\alpha_i, \beta_i) \) we denote by \( T_m(H(\alpha_i, \beta_i)) \) the hook-shaped \( m \)-tableau whose “hand” (the box at the end of its arm) is filled with \((\alpha_i - 1/2)/2\), and whose “foot” has filling \(|\beta_i - 1/2|/2\).

Note that we need to take the absolute value in the latter expression since it might happen that the smallest part of \( j \) is 0. If \( \epsilon = 1 \) then this part 0 of \( j \) will appear as \( \beta_i = 0 \) in the smallest pair \( (\alpha_i, \beta_i) \) of \( j'' \). Also observe that if \( \epsilon = 0 \) then \( \kappa > 0 \), and then this part 0 of \( j \) will appear as the smallest part \( \gamma_\kappa \) of \( j' \). In particular, we always have \((\alpha_i - 1/2)/2 > 0\) for all \( i \).

Let \( T_m(H) \) be the \( m \)-tableau obtained by nesting the hook shaped tableaux \( T_m(H(\alpha_i, \beta_i)) \) in decreasing order. Observe that all hooks \( T_m(H(\alpha_i, \beta_i)) \) contain a box with filling \( m \) (namely the box at the corner) and a box with filling 1/4 (and if \( m = 1/4 \) then these two boxes coincide). We call such hooks \textit{m-hooks}. Hence the leg of an \( m \)-hook has length at least \( \kappa \), since its corner box has filling \( m \) and the box with filling 1/4 is precisely \( \kappa \) boxes below that.

(6) We add horizontal strips \( S_i \) (which may be empty) to \( T_m(H) \) (for \( i = 1, \ldots, \kappa \)). If \( \gamma_i < 2(m - i + 1) + 1/2 \), then the strip \( S_i \) is empty. Otherwise \( S_i \) is the horizontal \((m - i + 1)\)-tableau whose rightmost extremity has filling \(|\gamma_i - 1/2|/2\) (again, we need the absolute value because it might happen that \( \gamma_\kappa = 0 \), namely if \( j \) contains 0 as a part, and if in addition \( \epsilon = 0 \)). These horizontal strips can be added to \( T_m(H) \) in a unique way such that the union is an \( m \)-tableau (so \( S_1 \) is placed at the “armpit” of \( T_m(H) \), and \( S_{i+1} \) is placed just below \( S_i \) for \( i = 1, \ldots, \kappa - 1 \)).

Observe that for all \( i \) the parity type of \( \gamma_i \) is \( \epsilon \). The smallest possible value of \( \gamma_i \) equals \( \gamma_i = 2(m - i + 1) - 3/2 \), corresponding to \( S_i \) being empty. In particular, the smallest possible value of \( j_\kappa \) is \( j_\kappa = \epsilon \). If all strips \( S_i \) are empty, we have:
\[
  j' = (\epsilon + 2(\kappa - 1), \epsilon + 2(\kappa - 2), \ldots, \epsilon)
\]

Denote the union of the strips \( S_i \) by \( T_m(S) \). Then \( T_m(S) \) is either an \( m \)-tableau or empty.
(7) Finally, $\rho$ is the partition whose Young diagram is the shape formed by the union of the $m$-hooks $H$ and strips $S_i$ in the way indicated above.

As mentioned above, we call the hook-shaped $m$-tableaux $T_m(H(\alpha_i, \beta_i))$ the $m$-hooks of $T_m(\rho)$. Observe that the $m$-hooks of $T_m(\rho)$ are precisely the hooks in $T_m(\rho)$ which are $m$-tableaux, and contain 1/4. Equivalently, a hook of $T_m(\rho)$ which is an $m$-tableau is an $m$-hook if and only if its leg length is at least $\kappa$.

**Example 6.1.** Let $\lambda = (19, 13, 9, 5, 3)$ be an odd distinct partition. We have already ordered the parts of $\lambda$ in decreasing order. Then we have $j = (9, 6, 4, 2, 1)$. We now have 3 even numbers and 2 odd numbers. Thus the dominant parity type is even (hence $\epsilon = 0$) and the excess number $\kappa = 3 - 2 = 1$. We get $m = \kappa + (2\epsilon - 1)/4 = 3/4$.

Now $j' = (2)$. Removing $j'$ from $j$ we obtain $j'' = ((6, 9), (4, 1))$. Therefore we will have 2 hook-shaped $m$-tableaux and 1 horizontal strips. The Young tableau is as follow:

$$
\begin{array}{|c|c|c|}
\hline
3/4 & 7/4 & 11/4 \\
1/4 & 3/4 & 7/4 \\
5/4 & 1/4 & \frac{3}{4} \\
9/4 & & \\
13/4 & & \\
17/4 & & \\
\hline
\end{array}
$$

The box of filling $\frac{3}{4}$ is the only strip (of length 1).

**Example 6.2.** Consider the odd distinct partition $\lambda = (15, 13, 11, 9, 5, 3)$. We find that $j = (7, 6, 5, 4, 2, 1)$, with $\kappa = 0$. Hence $\epsilon = 1$ and $m = \kappa + (2\epsilon - 1)/4 = 1/4$.

Since $j'' = j = ((7, 6), (5, 4), (1, 2))$, and we obtain the following $m$-tableau, consisting of 3 $m$-hooks:

$$
\begin{array}{|c|c|c|c|}
\hline
1/4 & 5/4 & 9/4 & 13/4 \\
3/4 & 1/4 & 5/4 & 9/4 \\
7/4 & 3/4 & 1/4 & \\
11/4 & 7/4 & 3/4 & \\
\hline
\end{array}
$$

There are no strips (even empty strips) appearing in the $m$-tableau above.
Next, we describe the operation $\mathcal{D} : (m, \rho) \mapsto \lambda$, namely how to recover the odd distinct partition $\lambda \in \mathcal{P}_{\text{odd, dist}}$ from $(m, \rho) \in \mathcal{R}$.

(1) Recall that $m \in (\mathbb{Z} \pm 1/4)_+$. Let $\kappa$ be the closest integer to $m$ and write $m = \kappa + (2\epsilon - 1)/4$ with $\epsilon = 0$ or $1$. This uniquely determines a parity type $\epsilon$ and a nonnegative integer $\kappa$. Define $\delta \in \{0, 1\}$ by $\kappa = \delta + 2\mathbb{Z}$. The number $\delta$ is useful later. However, we do not let $\delta$ come into play at present.

(2) The $m$-tableau $T_m(\rho)$ can be written as a disjoint union of nested $T_m(\rho)$-hooks which are themselves $m$-tableaux. If one of these hook shapes is an $m$-hook then all its predecessors are $m$-hooks too (since the condition for such hook to qualify as $m$-hook is that its leg length is at least $\kappa$). Hence $T_m(\rho)$ has a unique decomposition as the union $T_m(H) \cup T_m(S)$ of two $m$-tableaux (both possibly empty) such that $H$ is the largest $m$-tableau contained in $T_m(\rho)$ which is a union of $m$-hooks, and $S$ is the complement of $H$ in $T_m(\rho)$. By the above we see that $S$ is itself an $m$-tableau (or empty) without $m$-hooks, and that $S$ has at most $\kappa$ parts.

(3) We number the shapes of the the nested $m$-hooks in $T_m(\rho)$ in decreasing order as $H_1, \ldots, H_t$. For every $i$, $T_m(H_i)$ defines unique pair of nonnegative integers $(\alpha_i, \beta_i)$ such that $T_m(H_i) = T_m(H(\alpha_i, \beta_i))$. Indeed, if the hand of $T_m(H_i)$ has filling $A_i \in (\mathbb{Z} \pm 1/4)_+$ and its foot has filling $B_i \in (\mathbb{Z} \pm 1/4)_+$, then $\alpha_i$ is the unique positive integer of parity type $\epsilon$ nearest to $2A_i \in (\mathbb{Z} + 1/2)_+$ (which is easily seen to be $2A_i + 1/2$), and $\beta_i$ is the unique non-negative integer of parity type $1 - \epsilon$ nearest to $2B_i \in (\mathbb{Z} + 1/2)_+$. In particular, every such pair $(\alpha_i, \beta_i)$ consists of nonnegative integers with opposite parity type.

(4) Recall that $S$ itself is an $m$-tableaux (or empty) with at most $\kappa$ parts. Let $S_1, \ldots, S_\kappa$ denote the list of rows of $S$ (where some of the rows, or even all of them, may be empty). Let the right-most box of $S_i$ be filled with $C_i \in (\mathbb{Z} \pm 1/4)_+$. If $S_i$ is empty, we define $C_i := |m - i| \in (\mathbb{Z} \pm 1/4)_+$ (this is the filling of the rightmost box of the $(t + i)$-th row of $T_m(\rho)$, provided that $t \neq 0$ (otherwise $S = T_m(\rho)$, in which case this row is empty by assumption)). Define $\gamma_i$, for $i = 1, \ldots, \kappa$, as the unique nonnegative integer of parity type $\epsilon$ nearest to $2C_i \in (\mathbb{Z} \pm 1/2)_+$.

This determines a set of pairs of nonnegative integers (of opposite parity) $(\alpha_i, \beta_i)$, and a set of nonnegative integers $\gamma_j$ uniquely. Observe that these integers are mutually distinct.

(5) Now we form the descending list

\[ j := ((\alpha_1, \beta_1), \ldots, (\alpha_t, \beta_t), \gamma_1, \ldots, \gamma_\kappa). \]

(Observe that the length of $j$ is at least $\kappa$, namely, we have at least the
numbers $\gamma_j$ for $i = 1, \ldots, \kappa$ in our list). Finally we define $\lambda := 2j + 1$. Observe that $\lambda$ has distinct, odd parts, and that $\delta$ (the parity of $\kappa$) is also the parity of the number of parts of $\lambda$.

We can also form the descending list $e$ of the numbers $(\alpha_i - 1/2)/2$, the $(\beta_i - 1/2)/2$ (for $i = 1, \ldots, t$) and the $(\gamma_j - 1/2)/2$ (for $j = 1, \ldots, \kappa$). Observe that the list $e$ may contain at most one negative number as an entry, namely $-1/4$. Then $j = 2e + 1/2$. Also note that the list $|e|$ of absolute values of $e$ is the list consisting of the fillings of arms (the $A_i$) and feet (the $B_i$) of the $m$-hooks $H_i$ of $T_m(\rho)$ (for $i = 1, \ldots, t$), combined with the list of the $C_j$ (for $j = 1, \ldots, \kappa$).

Before showing that $E$ and $D$ are indeed inverse to each other, we shall see some examples of the operation $D$ first.

**Example 6.3.** Let us give some examples of the operator $D$:

1. We start with continuing the two Examples 6.1, 6.2.

   In Example 6.1, there are 2 $m$-hooks and 1 strip (of length 1, i.e. a singleton). The “hands” and “feet” of these $m$-hooks and strip give us the descending list
   $$ e = \left( \frac{17}{4}, \frac{11}{4}, \frac{7}{4}, \frac{3}{4}, \frac{1}{4} \right). $$
   Therefore, we can recover the partition $\lambda = 4e + 2 = (19, 13, 9, 5, 3)$. We shall rewrite $\lambda$ as $\lambda = [3, 5, 9, 13, 19]$.

   Likewise, in Example 6.2, after applying the operation $D$, we recover the partition $\lambda = [3, 5, 9, 11, 13, 15]$.

2. Let $m = 1/4$ and $\rho$ be zero. Then $\lambda$ is zero as well. On the other hand, if $m = 3/4$ and $\rho$ is zero, then we have no $m$-hooks, but since $\kappa = 1$ and $\epsilon = 0$, we have one empty strip $S_1$, which yields $C_1 = 1/4$, and $\gamma_1 = 0$. Hence $e = (-1/4)$, and $\lambda = [1]$.

3. Let $m = 15/4$ and $\rho$ be the zero partition. Then $m = 4 - 1/4$ gives $\kappa = 4$ and $\epsilon = 0$. We have no $m$-hooks, and $\kappa = 4$ strips which are all empty. Hence $C_1 = m - 1 = 11/4$, $C_2 = m - 2 = 7/4$, $C_3 = m - 3 = 3/4$ and $C_4 = |m - 4| = 1/4$, and thus $\gamma_1 = 6$, $\gamma_2 = 4$, $\gamma_1 = 2$, $\gamma_1 = 0$. So $e = (11/4, 7/4, 3/4, -1/4)$. We obtain the odd distinct partition $\lambda = [1, 5, 9, 13]$ (let us agree that we may also denote a partition in increasing order, using square brackets as delimiters).

4. For a singleton $m = \left[ \frac{1}{4} \right]$ we have $\kappa = 0, \epsilon = 1$. Hence we have one hook, and no strips (even no empty ones!). We find that $e = (1/4, -1/4)$; thus we
get \( \lambda = [1, 3] \).

5. Consider the following tableau:

\[
\begin{array}{cc}
5/4 & 1/4 \\
\end{array}
\]

Here \( m = 5/4 = 1 + 1/4 \). Therefore \( \kappa = 1 \) and \( \epsilon = 1 \). We have one \( m \)-hook, and one empty strip. Thus \((A_1, B_1) = (5/4, 1/4)\), and \((\alpha_1 - 1/2)/2, (\beta_1 - 1/2)/2) = (5/4, -1/4)\). In addition the empty strip \( S_1 \) yields \( C_1 = |m - 1| = 1/4 \), and thus \( \gamma_1 = 1 \). Thus we form the descending list \( e = (5/4, 1/4, -1/4) \) and recover the odd distinct partition \( \lambda = [1, 3, 7] \).

**Theorem 6.1.** The operations \( E : P_{\text{odd, dist}} \to R \) and \( D : R \to P_{\text{odd, dist}} \) are inverse bijections.

**Proof.** We shall prove that \( D \circ E = \text{Id}_{P_{\text{odd, dist}}} \) and \( E \circ D = \text{Id}_R \). In fact, the proof is quite straightforward.

We first look at \( D \circ E \). Let \( \lambda = (2c_1 + 1, \ldots, 2c_l + 1) \) be an odd distinct partition, with \( c_1 > \cdots > c_l \). Then \( j = (c_1, \ldots, c_l) \). Firstly we consider the case \( \kappa = 0 \), which implies that \( \epsilon = 1, m = 1/4 \) and \( l \) is an even integer. So \( j = j'' \). We arrange the sequence \( j'' \) as

\[
j'' = ((c_1, c_2), \ldots, (c_{l-1}, c_l))
\]

where the \( c_i \) with odd (resp. even) subscripts are odd (resp. even) integers. And we get the \( m \)-tableau from the algorithm as follow:

\[
\begin{array}{cccc}
\frac{1}{4} & \frac{5}{4} & \frac{9}{4} & \ldots & \frac{c_1 - \frac{1}{2}}{2} \\
\frac{3}{4} & \frac{1}{4} & \ldots & \frac{c_l - \frac{1}{2}}{2} \\
\vdots & \vdots \\
\vdots & \frac{|c_{l-1} - \frac{1}{2}|}{2} \\
\vdots \\
\frac{|c_2 - \frac{1}{2}|}{2} \\
\end{array}
\]
There is no strip (even empty strip) in this \( m \)-tableau, because \( \kappa = 0 \). Now we apply the algorithm \( D \) to this \( m \)-tableau. One easily recovers the integers \( c_1, \ldots, c_l \), and hence the partition \( \lambda \).

In case that \( \kappa > 0 \), the analysis is similar to that \( \kappa = 0 \). In fact, if \( \kappa > 0 \), we first compute \( m := \kappa + (2\epsilon - 1)/2 \), and determine the \( m \)-tableau \( T_m(\rho) \). In the tableau below we adopt the notations as in the algorithm. Note that there may be some strips occurring in the \( m \)-tableau. If so, the first strip is an \( m \)-band where the filling of the right-most box is \( |\gamma_1 - 1/2|/2 \). Attention should be addressed to distinct the \( m \)-hooks \( T_m(\alpha_i, \beta_i) \) and the \( m \)-band which is a strip: By definition, the \( m \)-hooks must contain both the boxes of filling \( m \) and of filling \( 1/4 \).

\[
\begin{array}{cccccccccc}
\hline
m & m+1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \alpha_1 - \frac{1}{2} \\
\hline
\vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
\vdots & \vdots & m & \cdots & \cdots & \cdots & \cdots & \alpha_i - \frac{1}{2} \\
1/4 & \cdots & \cdots & m-i+1 & \cdots & \gamma_i - \frac{1}{2} \\
\vdots & \vdots & |\beta_t - \frac{1}{2}|/2 \\
\vdots & \vdots \\
|\beta_1 - \frac{1}{2}| & \hline
\end{array}
\]

Now apply the algorithm \( D \) to the \( m \)-tableau \( T_m(\rho) \) above. We detect that the numbers \( A_i, B_i, C_i \) in the description of the algorithm \( D \) are just the numbers \( \alpha_i, \beta_i, \gamma_i \). This readily shows that the composition \( D \circ E \) is the identity on \( P_{\text{odd,dist}} \).

On the other hand, we want to verify that \( E \circ D = \text{Id}_{\mathbb{R}} \). Given a pair \((m, \rho)\) with \( 0 < m \in \mathbb{Z} \pm 1/4 \), and \( \rho \) a partition, we first determine \( \kappa \) and \( \epsilon \) by writing \( m \) as \( m = \kappa + (2\epsilon - 1)/4 \), with \( \kappa \) the nearest integer to \( m \). Then we identify the \( m \)-hooks and strips in the tableau \( T_m(\rho) \) of shape \( \rho \), and determine the integers \( A_i, B_i, C_i \), and form the sequence

\[
j = ((\alpha_1, \beta_1), \ldots, (\alpha_t, \beta_t), \gamma_1, \ldots, \gamma_k),
\]

where \( \alpha_1 > \cdots > \alpha_t, \beta_1 > \cdots > \beta_t \) and the numbers \( \alpha_i \) are of parity type \( \epsilon \), while the numbers \( \beta_i \) are of parity type \( 1 - \epsilon \).

From this sequence \( j \) we can obtain a partition \( \lambda := 2j + 1 \) with odd and distinct
parts. Note that if instead of \( \beta \), we use the sequence
\[
e = \left( \frac{\alpha_1 - 1/2}{2}, \frac{\beta_1 - 1/2}{2} \right), \ldots, \left( \frac{\alpha_t - 1/2}{2}, \frac{\beta_t - 1/2}{2} \right), \frac{\gamma_1 - 1/2}{2}, \ldots, \frac{\gamma_\kappa - 1/2}{2} \right),
\]
then \( j = 2e + \frac{1}{2}, \) and \( \lambda = 2j + 1 = 4e + 2. \)

Finally, we apply the algorithm \( \mathcal{E} \) to the odd distinct partition \( \lambda \) just obtained. One easily see that we will go back to the original \( m \)-tableau \( T_m(\rho) \) with the same \( m \). This implies that \( \mathcal{E} \circ \mathcal{D} \) is the identify map on \( R \).

We thus have proved that the two operations \( \mathcal{E} \) and \( \mathcal{D} \) are inverse to each other.

\[\Box\]

### 6.2 The extra-special spectral transfer morphisms

In this section, we shall prove by direct computation, the existence of extra special STMs. The term extra special means the parameter type \( \mathcal{X} \in \{V, VI\} \). Recall that every pair of extra special parameters \((m_-, m_+)\) with \( 0 < m_+ \in (\mathbb{Z} \pm 1/4) \) determines a pair \((a, b)\) with \( a, b \in \mathbb{Z}_{\geq 0} \). For two non-negative integers \( r \leq l \), we define a morphism \( \xi^r_{T, m_-, m_+} : T_r \to T_l \) as follows. Here

\[
l := \begin{cases} 
2r + \frac{1}{2}a(a + 1) + 2b(b + 1) & \text{if } \mathcal{X} = V, \\
2r + \frac{1}{2}a(a + 1) + 2b^2 - \delta_+ & \text{if } \mathcal{X} = VI. 
\end{cases}
\quad (6.1)
\]

Write \( l_\pm = \kappa_\pm(2\kappa_\pm + 2\epsilon_\pm - 1)/2 \), then \( l = 2r + [l_-] + [l_+] \). Firstly, for \( m \in \mathbb{Z} \pm 1/4 \) and \( m > 0 \), we define residual points \( r_e(m) \) recursively by putting \( r_e(\frac{1}{4}) = r_e(\frac{3}{4}) := \emptyset \) and

\[
r_e(m) = (\sigma_e(m); r_e(m - 1))
\]

for \( m > 1 \), with

\[
\sigma_e(m) = (q^\delta, q^{\delta+1}, \ldots, q^{2m-\frac{3}{2}}), m > 1.
\]

Then we define the representing morphism of the extra-special STM by

\[
\xi^r_{T, m_-, m_+}(t_1, \ldots, t_r) := (v^{-1}t_1, vt_1, \ldots, v^{-1}t_r, vt_r, -r_e(m_-), r_e(m_+)). \quad (6.2)
\]

**Proposition 6.1.** The morphism \( \xi^r_{T, m_-, m_+} \) defines a spectral transfer morphism called extra-special

\[
\xi^r_{m_-, m_+} : \mathcal{H}_{m_-, m_+}^r \rightsquigarrow \mathcal{H}_{\delta_-, \delta_+}^l
\]

with \( l \) defined above.

Please recall that \( \mathcal{H}_{m_-, m_+}^r \) is in base \( q^2 \), but \( \mathcal{H}_{\delta_-, \delta_+}^l \) is in base \( q \). Before prove this statement it is good to see some illustrating examples.

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Example 6.4. Suppose \( m_{\pm} = 1/4 \). Then \( \delta_{\pm} = 0 \), and \( a = b = 0 \), and \( r_e(1/4) = \emptyset \). If moreover the rank \( r = 0 \) then we have nothing to verify because \( d_{1/4,1/4} \) reduces to 1, as does \( d_{0,0} \), and there is no root contributing the \( \mu \)-functions so that both \( \mu \)-functions reduce to 1 as well.

Now assume \( r > 0 \). Since \( m_{\pm} = 1/4 \), we note that this is of parameter type VI with \( a = b = 0 \). Hence \( l = 2r \), so that \( d_{1/4,1/4} = (v^2 - v^{-2})^{-r} \) (recall that we have base \( q^2 \) here) and \( d_{0,0} = (v - v^{-1})^{-2r} \). The \( \mu \)-function \( \mu_{0,0}^{2r} \) has only contributions from the type D roots:

\[
\mu_{0,0}^{2r} = (v - v^{-1})^{-2r} \prod_{1 \leq i < j \leq 2r} \frac{(1 - t_i t_j^{-1})^2 (1 - t_i t_j)^2}{(1 - q t_i t_j^{-1})(1 - q^{-1} t_i t_j)(1 - q t_i^{-1} t_j)(1 - q^{-1} t_i^{-1} t_j)}
\]

We compute the pull-back \( \xi^* (\mu_{0,0}^{2r}(L)) \), so we need to substitute \( (t_1, t_2, \ldots, t_{2r-1}, t_{2r}) \) by \( (v s_1, v^{-1} s_1, v s_2, v^{-1} s_2, \ldots, v s_r, v^{-1} s_r) \) in \( \mu_{0,0}^{2r} \), after regularising the expression along the image \( L \) of \( \xi \). Concerning the regularisation, observe that the parabolic subsystem of roots which is constant on the image is

\[
\{(t_1 t_2^{-1})^{\pm 1}, (t_3 t_4^{-1})^{\pm 1}, \ldots, (t_{2r-1} t_{2r}^{-1})^{\pm 1}\},
\]

of type \( A_1^r \). After dropping the singular factors, the remaining “constant factors” yield (including the normalisation factor \( (v - v^{-1})^{-2r} \)), up to a power of \( v \) which is irrelevant for us (as explained before): \( (v^2 - v^{-2})^{-r} \), which is indeed the normalisation factor \( d_{1/4,1/4} \).

For the non constant part, by Weyl group invariance, it is enough to check the contribution one type B root in each \( W_0 \)-orbit. First consider \( s_1 s_2 \). The roots whose pull back along \( \xi \) equals a nonzero power of \( s_1 s_2 \) times a power of \( v \) are \( t_1 t_3, t_2, t_3, t_1 t_4, t_2 t_4 \) (together with their opposites). These yield \( q s_1 s_2, s_1 s_2, s_1 s_2 \) and \( q^{-1} s_1 s_2 \) respectively, so these root give us the factor (after some cancellations, and up to powers of \( v \) and other characters of \( T^r \) (which are irrelevant anyhow)):

\[
\frac{(1 - s_1 s_2)^2}{(1 - q^{-2} s_1 s_2)(1 - q^2 s_1 s_2)}
\]

which shows that the pull back of \( \mu_{0,0}^{2r} \) yields a factor which is a type D \( \mu \) function with base \( q^2 \), as desired for \( \mu_{1/4,1/4}^{r} \).

The type D roots which pull back to a power of \( v \) times a nonzero power of \( s_1 \) (a type \( A_1 \) root) are only \( t_1 t_2 \) (and its opposite). Its pull back is \( s_1^2 \). This gives a factor in \( \xi^* (\mu_{0,0}^{2r}(L)) \) of the form:

\[
\frac{(1 - s_1)^2 (1 + s_1)^2}{(1 + v^{-1} s_1)(1 + v s_1)(1 - v^{-1} s_1)(1 - v s_1)}
\]
This second fraction is exactly the factor in $\mu_{1/4,1/4}^r$ (with base $q^2$) for this last remaining type of root in the type B root system. In other words, the condition (T3) is satisfied. Hence we have verified that $\xi_{T,1/4,1/4}^r$ represents an extra-special STM.

Similar computations show easily (using the same map $\xi_T^r$ as in Example 6.4, but different $\mu$-functions on the target torus) the following Lemma, which will serve as an induction base:

**Lemma 6.1.** Proposition 6.1 is true for $(m_-, m_+) = (1/4, 1/4), (3/4, 1/4), (1/4, 3/4)$ and $(3/4, 3/4)$.

The following example is more complicated, since it includes the first induction step:

**Example 6.5.** Take $m_- = 1/4, m_+ = 5/4 = 1 + 1/4$. Then this parameter is of type V and $a = 1, b = 0$. Also we have $\kappa_- = \delta_- = 0, \kappa_+ = \delta_+ = 1$ and $\varepsilon_\pm = 1$ thus $l = 2r + 1$. Suppose the rank $r > 0$, plus with $r_c(1/4) = 0, r_c(5/4) = (q)$. We see that, with $\xi = \xi_{T,1/4,5/4}^r$,

$$\xi(s_1, \ldots, s_r) = (v^{-1}t_s, vt_s, \ldots, v^{-1}s_r, vs_r, q) = (t_1, t_2, \ldots, t_{2r-1}, t_{2r}, t_{2r+1}).$$

This time we have one more coordinate in the residual point (compare to last example). And $\mu_{0,1}^{2r+1} = (v - v^{-1})^{-(2r+1)}\mu_{0,1}^{D,2r+1}\mu_{0,1}^{A,2r+1}$ is a product of the normalisation factor $(v - v^{-1})^{-(2r+1)}$, the factor $\mu_{0,1}^{D,2r+1} = \mu_{0,0}^{2r+1}$ (the $\mu$-function associated to type D roots only) and the $\mu$-function $\mu_{0,1}^{A,2r+1}$ associated the type $A_1^{2r+1}$ roots $t_i^{\pm 1}$ of the type B root system:

$$\mu_{0,1}^{A,2r+1} = \prod_{k=1}^{l} \frac{(1 - t_k)^2}{(1 - qt_k)(1 - q^{-1}t_k)}. \quad (6.6)$$

The function $\mu_{1/4,5/4}^r$ has a normalisation factor $d_{1/4,5/4} = (v^2 - v^{-2})^{-r}(v^2 + v^{-2})^{-1}$ (cf. [75, equations (35),(36)]).

Concerning the normalisation factor, observe that the parabolic subsystem of roots which is constant on the image $L$ of $\xi$ is $\{(t_1t_2^{-1})^{t_1}, (t_3t_4^{-1})^{t_1}, \ldots, (t_{2r-1}t_{2r})^{t_1}, t_{2r+1}^{t_1}\}$ now, of type $A_1^{r+1}$. Due to the extra roots $t_{2r+1}$ in this list, and the extra factor $(v - v^{-1})^{-1}$ in the normalisation factor of $\mu_{0,1}^{2r+1}$, compared to the previous example, the pull back $\xi^*(\mu_{0,1}^{2r+1})^{(L)}$ has the extra normalisation factors:

$$(v - v^{-1})^{-1}(v - v^{-1})(v^2 + v^{-2})^{-1} = (v^2 + v^{-2})^{-1}, \text{ as required.}$$

Concerning the analysis of type D-roots: This is the same as in the previous example. This is correct, due to the observation that the type D root factors of $\mu_{1/4,5/4}^r$ and $\mu_{1/4,1/4}^r$ are the same.
Finally consider the factors in \( \xi^*(\mu_{0,1}^{2r+1})^L \) involving powers of \( v \) times powers of \( s_1 \), thus factors of the form \((1 \pm v^n s_1^m)\). The roots which are pulled back to a character of the form \( v^n s_1^m \) with \( m \) nonzero are: \( t_1, t_2, t_1 t_2, t_1 t_2 r+1, t_1 t_2 r+1, t_2 t_2 r+1, t_2 t_2 r+1 \) and their opposites.

The first two roots of these are of type \( A_1^{2r+1} \), and restrict to \( vs_1 \) and \( v^{-1}s_1 \) respectively. In view of (6.6), these yields a factor of \( \xi^*(\mu_{0,1}^{2r+1})^L \) as follows:

\[
\frac{(1 - vs_1)^2(1 - v^{-1}s_1)^2}{(1 - v^{-3}s_1)(1 - v^{-1}s_1)(1 - v^3s_1)} = \frac{(1 - vs_1)(1 - v^{-1}s_1)}{(1 - v^{-3}s_1)(1 - v^3s_1)} \tag{6.7}
\]

The remaining possibilities are type D, and give us \( s_1^2, v^3s_1, v^{-1}s_1, vs_1, v^{-3}s_1 \). This produces:

\[
\frac{(1 + s_1)^2(1 - s_1)^2(1 - v^{-3}s_1)^2(1 - v^{-1}s_1)^2(1 - vs_1)^2(1 - v^3s_1)^2}{(1 + vs_1)(1 + v^{-1}s_1)(1 - v^{-5}s_1)(1 - v^{-3}s_1)(1 - v^{-1}s_1)^3(1 - vs_1)^3(1 - v^3s_1)(1 - v^5s_1)}
\]

which taken together yields a factor of \( \xi^*(\mu_{0,1}^{2r+1})^L \) of the form:

\[
\frac{(1 + s_1)^2(1 - s_1)^2}{(1 + vs_1)(1 + v^{-1}s_1)(1 - v^{-5}s_1)(1 - v^{-3}s_1)}
\]

which indeed is the factor coming from the type \( A_1^r \) root \( s_1 \) in \( \mu_{1/4,5/4}^r \).

This verifies that the morphism \( \xi \) defined above represents an STM as indicated.

Proof of Proposition 6.1. Recall that \( m_\pm \in \mathbb{Z} \pm \frac{1}{4} \). Using spectral isomorphisms \( \text{Iso} = \langle \eta_+, \eta \rangle \) we can assume that \( m_\pm > 0 \). So we can write \( m_- = \kappa_- + (2\epsilon_- - 1)/4 \) with \( \kappa_- \in \mathbb{Z}_{\geq 0} \) and \( \epsilon_- \in \{0, 1\} \). Let \( \delta_- \) be defined by \( \kappa_- - \delta_- \in 2\mathbb{Z} \). In other words, \( \delta_- = 0 \) (resp. 1) if \( \kappa_- \) is even (resp. odd). Similarly we have \( m_+ = \kappa_+ + (2\epsilon_+ - 1)/4 \) and \( \delta_+ \).

We will apply an inductive argument on \( m_- + m_+ \), where the induction base is provided by Lemma 6.1. Let us first assume that \( m_+ \geq m_- > 0 \) and that \( m_+ > 1 \).

By induction we may now assume that \( \xi^r_{m_-, m_- + 1} \) represents an STM.

Note that \( \delta_\pm, \epsilon_\pm \in \{0, 1\} \). We define \( \delta_\pm^c, \epsilon_\pm^c \) by the rules that \( \delta_\pm + \delta_\pm^c = 1, \epsilon_\pm + \epsilon_\pm^c = 1 \). Also we define \( A(m) = 2m - \frac{3}{2} \) and write \( A_\pm = A(m_\pm) = 2(\kappa_- - 1) + \epsilon_\pm \in \mathbb{Z}_{\geq -1} \).

Observe that \( A_+ \geq 1 \) by our assumptions.

It is a good place to introduce more notations. The \( \mu \)-function associated to the source normalised Hecke algebra \( \mathcal{H}^{r}_{m_-, m_+, r^d} \) of the alleged STM \( \xi = \xi^r_{m_-, m_+} \) with parameters \( m_\pm \) will be denoted by \( \mu^r_{m_-, m_+, d} \), where we often omit the rank \( r \) if there should be no confusion. When \( d = d^r_{m_-, m_+} \) we will simply write \( \mu^r_{m_-, m_+} \).
The $\mu$-function of the target is denoted by $\mu_{\delta_-,\delta_+}^l$, with $l$ given as in (6.1). Recall that we have, up to irrelevant factors, $\mu_{\delta_-,\delta_+}^l = d_{\delta_-,\delta_+}^l \mu_{A_-,\delta_+}^l$ with

$$d_{\delta_-,\delta_+}^l = (v - v^{-1})^{-l}(v + v^{-1})^{-\delta_+}$$  \hspace{1cm}  (6.8)

(the second factor arises in [75, Proposition 2.5] from the fact that the reductive quotient $\mathbb{F}$ of a minimal $F$-stable parahoric in the case $(\delta_-, \delta_+) = (1, 1)$ is an $\mathbb{F}_q$-torus of split rank $l$ whose maximal $\mathbb{F}_q$-anisotropic subtorus has $q+1$ rational points over $\mathbb{F}_q$).

$$\mu_{A_-,\delta_+}^l = \prod_{i=1}^l \frac{(1 - t_i t_j^{-1})^2}{(1 + q\delta - t_i)(1 + q^{-\delta} - t_i)(1 - q^\delta + t_i)(1 - q^{-\delta} + t_i)}$$

$$\mu_{A_-,\delta_+}^l = \prod_{1 \leq i < j \leq l} \frac{(1 - t_i t_j^{-1})^2}{(1 - q t_i t_j)(1 - q^{-1} t_i t_j)}$$  \hspace{1cm}  (6.9)

On the other hand $\mu_{m_-,m_+}^r = d_{m_-,m_+}^r \mu_{D_r}^r(q^2) \mu_{m_-,m_+}^A$ with

$$d_{m_-,m_+}^r = (v - v^{-1})^{-r} d_{m_-,m_+}^0$$

and $d_{m_-,m_+}^0$ the normalisation factor given by (5.6) (as in [75, equation (36)]), where $\mu_{D,r}^r(q^2)$ is similar as $\mu_{D,l}^l$, only with rank $r$ instead of $l$ and in base $q^2$ instead of $q$, and finally:

$$\mu_{A_-,m_+}^A = \prod_{i=1}^r \frac{(1 - s_i^2)^2}{(1 + v^{4m - s_i})(1 + v^{-4m - s_i})(1 - v^{4m + s_i})(1 - v^{-4m + s_i})}$$  \hspace{1cm}  (6.10)

Recall the morphism $\xi := \xi_{T,m_-,m_+}^r : T_r \to T_l$ is defined by

$$\xi_{T,m_-,m_+}^r(s_1, \ldots, s_r) = (-r_e(m_-), v^{-1}s_1, vs_1, \ldots, v^{-1}s_r, vs_r, r_e(m_+))$$  \hspace{1cm}  (6.11)

where $r_e(\frac{1}{2}) = r_e(3 \frac{1}{2}) = 0$, $r_e(m) = (\sigma_e(m), r_e(m - 1))$ and

$$\sigma_e(m) = (q^\delta, q^{\delta+1}, \ldots, q^{2m - \frac{3}{2}}).$$

We denote $\mathcal{T}_0 := (-r_e(m_-), r_e(m_+))$. Note that if we re-order the coordinates of $\xi(s_1, \ldots, s_r)$ (or of $\mathcal{T}_0$), or invert them, then the result lies in the same $W_{2,0}$-orbit. The $\mu$-function invariant for $W_{2,0}$, hence for such modifications.
Let us first consider the rank 0 case \( r = 0 \) (this is the main challenge, as we will see). We need to verify the condition (T3), assuming that \( m_+ \geq m_- > 0 \) and \( m_+ > 1 \), and (by the induction hypothesis) that \( s_{T,m_-m_+}^0 \) represents an STM.

Write \( l_0 = [l_-] + [l_+] \) for the rank of the target Hecke algebra if we use the parameters \((m_-, m_+)\), and \( l'_0 = [l_-] + [l'_+] \) if we use the parameters \((m'_-, m'_+)\) := \((m_-, m_+ - 1)\). Observe that \( \delta'_- = \delta_-. \) and \( \delta'_+ = \delta^c_+ \). By our normalisations of the Hecke algebras it suffices to show the following identity for the ratio of residues:

\[
C_{m_-m_+} := \frac{\mu^{D,l_0,\{\tau_0\}}_{\delta_- \delta_+} (\tau_0) \mu^A_{\delta_- \delta_+} (\tau_0)}{\mu^{D,l'_0,\{\tau'_0\}}_{\delta_- \delta_+} (\tau'_0) \mu^A_{\delta_- \delta_+} (\tau'_0)} = \frac{d^0_{\delta_- \delta_+} (\mu^0_{\delta_- \delta_+}) (\tau_0)}{d^0_{\delta_- \delta_+} (\mu^0_{\delta_- \delta_+}) (\tau'_0)}.
\]

(6.12)

The latter equals, up to powers of \( v \) and rational constants,

\[
A(m_+) := \frac{d^0_{\delta_- \delta_+} (\mu^0_{\delta_- \delta_+}) (\tau_0)}{d^0_{\delta_- \delta_+} (\mu^0_{\delta_- \delta_+}) (\tau'_0)} = (v - v^{-1})^{A_+ + \delta^c_+} (v + v^{-1})^{(\delta_+ - \delta_-)} \frac{d^0_{m_-m_+}}{d^0_{m_-m_+ - 1}}.
\]

(the second equality is easy to check, using that \( l_0 - l'_0 = A_+ + \delta^c_+ = 2m_+ - \frac{1}{2} - \delta_+ \)).

In the equations below we will simplify notations by omitting the references to the rank (if the arguments are given, the rank equals the number of coordinates of the argument so the explicit references to the ranks are superfluous), and we will simply write “reg” to indicate that we need to omit the factors that are identically 0 after evaluation at the argument. Finally, an expression like \( \mu^{D,\text{reg}}(\tau_1; \tau_2) \) means that we only consider the product in the numerator and the denominator of those type D-roots \( t_i^{\pm 1}e_j^{\pm 1} \) for which \( t_i \) is a coordinate of \( \tau_1 \) and \( t_j \) is a coordinate of \( \tau_2 \), and only those factors which are not identically 0. Since \( \tau_0 = (\tau^*_0, \sigma_e(m_+)) \), we see that \( C_{m_-m_+} \) is equal to

\[
\frac{\mu^A_{\delta_- \delta_+} (\tau'_0)}{\mu^A_{\delta_- \delta_+} (\tau'_0)} \frac{\mu^A_{\delta_- \delta_+} (\sigma_e(m_+)) \times \mu^{D,\text{reg}} (\sigma_e(m_+)) \mu^{D,\text{reg}} (-r_e(m_-); \sigma_e(m_+)) \mu^{D,\text{reg}} (\sigma_e(m_+); r_e(m_+ - 1))}{\mu^{D,\text{reg}} (-r_e(m_-); \sigma_e(m_+)) \mu^{D,\text{reg}} (\sigma_e(m_+); r_e(m_+ - 1))}.
\]

(6.13)

It is easy to see that

\[
\frac{\mu^A_{\delta_- \delta_+} (\tau'_0)}{\mu^A_{\delta_- \delta_+} (\tau'_0)} = \prod_{u_i \in \tau_e(m_-)} \frac{(1 + \gamma^{\delta^c_+} u_i)(1 + \gamma^{-\delta^c_+} u_i)}{(1 + \gamma^{\delta^c_+} u_i)(1 + \gamma^{-\delta^c_+} u_i)} \prod_{w_j \in \tau_e(m_+ - 1)} \frac{(1 - \gamma^{\delta^c_+} w_j)(1 - \gamma^{-\delta^c_+} w_j)}{(1 - \gamma^{\delta^c_+} w_j)(1 - \gamma^{-\delta^c_+} w_j)}
\]

(6.14)
while

\[
\mu_{\delta_-,\delta_+}^{A,\text{reg}}(\sigma_e(m_+)) = \prod_{t_k \in \sigma_e(m_+)} \frac{(1 - t_k^2)^2}{(1 + q^{\delta-t_k})(1 + q^{-\delta-t_k})(1 - q^{\delta+t_k})(1 - q^{-\delta+t_k})}
\]

\[
\begin{cases}
1 & (\delta_-, \delta_+) = (0, 0) \\
\frac{1+q^2m_+ - 3/2}{(1+q)(1+q^{2m_+ - 1/2})} & (\delta_-, \delta_+) = (1, 0) \\
\frac{1-q^{2m_+ - 3/2}}{(1-q)(1-q^{2m_+ - 1/2})} & (\delta_-, \delta_+) = (0, 1) \\
\frac{(1+q^{2m_+ - 3/2})(1-q^2)}{1+q^{2m_+ - 1/2}} & (\delta_-, \delta_+) = (1, 1)
\end{cases}
\]

\[
= \left(1 + \frac{q^{A_+}}{1 + q^{A_+ + 1}}\right)^\delta_-(1 - \frac{q^{A_+}}{1 - q^{A_+ + 1}})^\delta_+ (1 + q)^(-1)^\delta_+ \delta_-(1 - q)^\delta_+
\]

(6.15)

We denote this last expression by \((P1) = \mu_{\delta_-,\delta_+}^{A,\text{reg}}(\sigma_e(m_+))\).

Next we consider \(\mu_{D,\text{reg}}^{D,-r_e(m_-);\sigma_e(m_+)}\) and \(\mu_{D,\text{reg}}^{D,\text{reg}}(\sigma_e(m_+); r_e(m_+ - 1))\).

\[
\mu_{D,\text{reg}}^{D,-r_e(m_-);\sigma_e(m_+)} := \prod_{t_i \in \sigma_e(m_+), t_j \in r_e(m_-)} \frac{(1 - t_i t_j^{-1})^2 (1 - t_i t_j)^2}{(1 - q t_i t_j^{-1})(1 - q^{-1} t_i t_j)(1 - q t_i t_j)(1 - q^{-1} t_i t_j)}
\]

\[
\mu_{D,\text{reg}}^{D,\text{reg}}(\sigma_e(m_+), r_e(m_+ - 1)) = \prod_{t_i \in \sigma_e(m_+), t_j \in r_e(m_+ - 1)} \frac{(1 - q^{\delta_+ t_i^{-1}})(1 - q^{A_+ t_i^{-1}})(1 - q^{\delta_+ t_j})(1 - q^{A_+ t_j})}{(1 - q^{\delta_+ - 1 t_i^{-1}})(1 - q^{A_+ - 1 t_i^{-1}})(1 - q^{\delta_+ - 1 t_j})(1 - q^{A_+ - 1 t_j})}
\]

(6.16)

where \(A_+ = 2m_+ - 3/2\). We can likewise obtain

\[
\mu_{D,\text{reg}}^{D,\text{reg}}(\sigma_e(m_+), r_e(m_+ - 1)) = \prod_{t_i \in \sigma_e(m_+), t_j \in r_e(m_+ - 1)} \frac{(1 - t_i t_j^{-1})^2 (1 - t_i t_j)^2}{(1 - q t_i t_j^{-1})(1 - q^{-1} t_i t_j)(1 - q t_i t_j)(1 - q^{-1} t_i t_j)}
\]

\[
\mu_{D,\text{reg}}^{D,\text{reg}}(\sigma_e(m_+), r_e(m_+ - 1)) = \prod_{t_i \in \sigma_e(m_+), t_j \in r_e(m_+ - 1)} \frac{(1 - q^{\delta_+ t_i^{-1}})(1 - q^{A_+ t_i^{-1}})(1 - q^{\delta_+ t_j})(1 - q^{A_+ t_j})}{(1 - q^{\delta_+ - 1 t_i^{-1}})(1 - q^{A_+ - 1 t_i^{-1}})(1 - q^{\delta_+ - 1 t_j})(1 - q^{A_+ - 1 t_j})}
\]

(6.17)

Observe that up to some power of \(v\), we can cancel some factors from (6.14), (6.16) and (6.17) and obtain

\[
\frac{\mu_{\delta_-,\delta_+}^{\delta_+,\delta_-}(\mathcal{R}_0')}{\mu_{\delta_-,\delta_+}^{\delta_+,\delta_-}(\mathcal{R}_0')} \times \mu(D, q, -r_e(m_-), \sigma_e(m_+)) \times \mu(D, q, \sigma_e(m_+), r_e(m_+ - 1))
\]

\[
= \prod_{t_j \in r_e(m_-)} \frac{(1 + q^{A_+ t_j^{-1}})(1 + q^{A_+ t_j})}{(1 + q^{A_+ + 1 t_j^{-1}})(1 + q^{A_+ + 1 t_j})} \prod_{t_j \in r_e(m_+ - 1)} \frac{(1 - q^{A_+ t_j^{-1}})(1 - q^{A_+ t_j})}{(1 - q^{A_+ + 1 t_j^{-1}})(1 - q^{A_+ + 1 t_j})}
\]

(6.18)
Chapter 6

Denote

\[(P2) = \prod_{t_j \in r_e(m_-)} \frac{(1 + q^{A+}t_j^{-1})(1 + q^{A+}t_j)}{(1 + q^{A+}+1t_j^{-1})(1 + q^{A+}+1t_j)}\]

and

\[(P3) = \prod_{t_j \in r_e(m_+ - 1)} \frac{(1 - q^{A+}t_j^{-1})(1 - q^{A+}t_j)}{(1 - q^{A+}+1t_j^{-1})(1 - q^{A+}+1t_j)}\]

They can be further simplified. Observe that

\[r_e(m_-) = (\sigma_e(m_-), r_e(m_- - 1)) = (\sigma_e(m_-), \sigma_e(m_- - 1)), \ldots, \sigma_e\left(\frac{7 - 2\epsilon_-}{4}\right)\]

The number of \(\sigma_e\)'s in \(r_e(m_-)\) is \(\kappa_-\). Recall that for \(g \in \mathbb{Z}_{>0}\) we defined:

\[\sigma_e(g + \frac{2\epsilon_- - 1}{4}) = (q^g, q^{g+1}, \ldots, q^{2(g-1)+\epsilon_-})\]

where \(\bar{g} = 0\) if \(g\) is even and \(\bar{g} = 1\) if \(g\) is odd. Therefore we can write \(P2\) as

\[(P2) = \prod_{t_j \in r_e(m_-)} \frac{(1 + q^{A+}t_j^{-1})(1 + q^{A+}t_j)}{(1 + q^{A+}+1t_j^{-1})(1 + q^{A+}+1t_j)}
= \prod_{g=2-\epsilon_-}^{\kappa_-} \prod_{t_j \in \sigma_e(g+\frac{2\epsilon_- - 1}{4})} \frac{(1 + q^{A+}t_j^{-1})(1 + q^{A+}t_j)}{(1 + q^{A+}+1t_j^{-1})(1 + q^{A+}+1t_j)}\]

\[(6.19)\]

\[= \prod_{g=2-\epsilon_-}^{\kappa_-} \frac{(1 + q^{A+}-t^{2g-2+\epsilon_-})(1 + q^{A+}+\bar{g})}{(1 + q^{A+}+1-\bar{g})(1 + q^{A+}+(2g-1+\epsilon_-))}\]

\[= \prod_{g=2-\epsilon_-}^{\kappa_-} \frac{(1 + q^{A+}-t^{2g-2+\epsilon_-})}{(1 + q^{A+}+(2g-1+\epsilon_-))} \times \frac{(1 + q^{A+}+\bar{g})}{(1 + q^{A+}+1-\bar{g})}\]

Notice that \(\delta_-\) indicates the parity of \(\kappa_-\), so

\[\prod_{g=2-\epsilon_-}^{\kappa_-} \frac{(1 + q^{A+}+\bar{g})}{(1 + q^{A+}+1-\bar{g})} = \begin{cases} 
1+q^{A+} & \text{if } \epsilon_- \neq \delta_- \\
1+q^{A+}+1 & \text{if } \epsilon_- = \delta_- = 0 \\
1+q^{A+}+1 & \text{if } \epsilon_- = \delta_- = 1 
\end{cases}\]

So \((P2)\) is equal to

\[\frac{(1 + q^{A+}-t^{2+\epsilon_-})}{(1 + q^{A+}+1-\epsilon_-)} \epsilon_-=\epsilon_+^{\epsilon_-}\delta_- \frac{(1 + q^{A+}-4+\epsilon_-)}{(1 + q^{A+}+3-\epsilon_-)} \cdots \frac{(1 + q^{A+}-A_-)}{(1 + q^{A+}+A_-+1)}\]

\[(6.20)\]
where \( A_- = 2m_- - 3/2 \). (Recall our assumptions \( m_+ \geq m_- > 0 \) and \( m_+ > 1 \), hence \( A_+ \geq A_- \geq -1 \) and \( A_+ \geq 1 \). Observe that \( (P2) = 1 \) if \( 0 < m_- < 1 \).

Similarly we can compute that \((P3)\) is equal to

\[
\left( \frac{1 - q^{A_+ + \epsilon_+}}{1 - q^{A_+ + 1 - \epsilon_+}} \right)^{\epsilon_+ \delta_+ + \epsilon_+ \delta_-} \frac{(1 - q^2)(1 - q^4) \cdots (1 - q^{A_+ - 2 + \epsilon_+})}{(1 - q^{2A_+ - 1})(1 - q^{2A_+ - 3}) \cdots (1 - q^{A_+ + 3 - \epsilon_-})}. \tag{6.21}
\]

The final term to compute is \( \mu_{D, \text{reg}}(\sigma_e(m_+)) = \mu_{D_0, \text{reg}}(\sigma_e(m_+)) \mu_{D \neq 0, \text{reg}}(\sigma_e(m_+)) \). Here \( D_0 \) refers to the type D-roots which have coordinate sum equal to zero (this is a maximal proper parabolic root subsystem, irreducible of type A), and \( D \neq 0 \) refers to the remaining roots of type D (this is not a root subsystem).

Recall that \( \sigma_e(m_+) = (q^{\delta_+}, q^{\delta_+ + 1}, \ldots, q^{2m_+ - 3/2}) \). We can easily compute that

\[
\mu_{D_0, \text{reg}}(\sigma_e(m_+)) = \prod_{1 \leq i < j \leq A_+ - \delta_+} \frac{(1 - q^{j - i})^2}{(1 - q^{j - i + 1})(1 - q^{j - i - 1})} = \frac{(1 - q^{A_+ + 1 - \delta_+})}{(1 - q^{A_+ + 1 - \delta_+})}.
\tag{6.22}
\]

by considering the multiplicities of the range of \( j - i \in \{1, 2, \ldots, A_+ - \delta_+ - 1\} \). And the same idea applies to computing \( \mu_{D \neq 0, \text{reg}}(\sigma_e(m_+)) \):

\[
\mu_{D \neq 0, \text{reg}}(\sigma_e(m_+)) = \prod_{1 \leq i < j \leq A_+ - \delta_+} \frac{(1 - q^{j + 2(\delta_+ - 1)})}{(1 - q^{j + 2\delta_+ - 1})(1 - q^{j + 2\delta_+ - 3})} = \begin{cases} 
(1 - q^{1 + 2\delta_+})(1 - q^{3 + 2\delta_+}) \cdots (1 - q^{4 + \delta_+ - 1}) & \text{if } \epsilon_+ = \delta_+ \\
(1 - q^{1 + 2\delta_+})(1 - q^{2 + 2\delta_+}) \cdots (1 - q^{4 + \delta_+ - 2}) & \text{if } \epsilon_+ \neq \delta_+.
\end{cases}
\tag{6.23}
\]

Here, if \( \delta_+ = 0 \) then the denominator starts with \( (1 - q^2) \).

We denote \( \mu_{D, \text{reg}}(\sigma_e(m_+)) \), \( \mu_{D_0, \text{reg}}(\sigma_e(m_+)) \), \( \mu_{D \neq 0, \text{reg}}(\sigma_e(m_+)) \) respectively by \((P4)\), \((P4a)\) and \((P4d)\).

Now we multiply \((P1), (P2), (P3)\) and \((P4)\). The 4 parameters \( \epsilon_+, \delta_+, \epsilon_-, \delta_- \) take values in \( \{0, 1\} \) independently. So basically we need to consider 16 cases (The parity of \( A_+ \) is the same as \( \epsilon_+ \)). But we spot a simplification when taking the
and hence no matter the values of \( \epsilon_+ \) or not. Here in the third and the fourth equations we insert in both the numerators and denominators the factors in square brackets to produce the factors \( 1 + q^\ast (\ast = 1, \ldots , A_+) \) in the denominator.

We proceed to combine with (P4a) and (P1) to obtain

\[
(1 - q^{A_+ + \epsilon_+}) (1 - q^{A_+ + 1 - \epsilon_+}) (1 - q^2) (1 - q^4) \cdots (1 - q^{A_+ - 2 + \epsilon_+}) (1 - q^{2A_+ - 1}) (1 - q^{2A_+ - 3}) \cdots (1 - q^{A_+ + 3 - \epsilon_-})
\]

\[
\times (1 - q^2) (1 - q^3) \cdots (1 - q^{A_+ + 1 - \epsilon_+}) (1 - q^{A_+ + 1}) \epsilon_+ \delta_+ \times (1 + q^{A_+}) \times (1 + q^{A_+ + 1}) \frac{\delta_- (-1)^{\delta_+}}{(1 + q)(1 + q^2) \cdots (1 + q^{A_+})}
\]

\[
\times (1 + q)(1 + q^2) \cdots (1 + q^{A_+}) (1 - q^{A_+ + 1}) \frac{\delta_-}{(1 + q)(1 + q^2) \cdots (1 + q^{A_+})}
\]

\[
= (1 + q)^{A_+ + 1 - \delta_+} \times \left( \frac{1 - q^{A_+}}{1 - q^{A_+ + 1}} \right)^{\delta_+} \times \left( \frac{1 + q^{A_+}}{1 + q^{A_+ + 1}} \right)^{\delta_-} \frac{(1 + q)^{\delta_- (-1)^{\delta_+}} (1 - q)^{\delta_+}}{(1 + q)(1 + q^2) \cdots (1 + q^{A_+})}
\]

no matter the value of \( \delta_+ \).

Finally we multiply with (P2). Note first that

\[
\left( \frac{1 + q^{A_+ + \epsilon_-}}{1 + q^{A_+ + 1 - \epsilon_-}} \right)^{\epsilon_- \delta_- + \epsilon_\delta_- + \epsilon_\delta_-} \left( \frac{1 + q^{A_+}}{1 + q^{A_+ + 1}} \right)^{\delta_-} = \frac{1 + q^{A_+ + \epsilon_-}}{1 + q^{A_+ + 1}}
\]

and hence no matter the values of \( \epsilon_- \) and \( \delta_- \), the total product \( C_{m_-, m_+ - 1} \) is equal
Based on the result of rank 0 case, to prove (6.25), we just need to verify that

\[
\frac{\mu_{\delta_+,R_0\backslash R_L}(\xi'(s),\sigma_e(m_+))}{\mu_{\delta_+,R_0'\backslash R_L'}(\xi'(s))} = \prod_{i=1}^{r} \frac{(1 - q^{-2(m_+-1)s_i})(1 - q^{2m_+})}{(1 - q^{-2m_+})}\frac{1 - q^{2m_+}}{(1 - q^{-2m_+})}. \tag{6.27}
\]

It is enough to consider the contribution to the factors of the right hand side involving \(s_1\). For this we need to consider the contribution in the numerator of the left hand side of the set of type D-roots \(t_i^{\pm 1}t_j^{\pm 1}\) such that \(t_i = v^{-1}s_1\) or \(vs_1\), and \(t_j\) is a coordinate of \(\sigma_e(m_+)\). In addition, we need to consider the contribution in the
numerator and denominator of the left hand side of the roots \( t_1 = v^{-1}s_1 \), \( t_2 = vs_1 \) and their opposites. Therefore, in this computation we may assume that \( r = 1 \).

The contribution of the type \( A_r^1 \) roots \( t_1 = v^{-1}s_1 \) and \( t_2 = vs_1 \) is:

\[
\frac{\mu^2_{\delta_-,\delta_+}(-s_1,s_1)}{\mu^2_{\delta_-,\delta_+}(-s_1,s_1)} = \frac{(1 - q^{\delta_+ - 1/2}s_1)(1 - q^{\delta_+ + 1/2}s_1)(1 - q^{-\delta_+ - 1/2}s_1)(1 - q^{-\delta_+ + 1/2}s_1)}{(1 - q^{\delta+ - 1/2}s_1)(1 - q^{\delta+ + 1/2}s_1)(1 - q^{-\delta+ - 1/2}s_1)(1 - q^{-\delta+ + 1/2}s_1)} = \left(\frac{(1 - q^{3/2}s_1)(1 - q^{-3/2}s_1)}{(1 - q^{1/2}s_1)(1 - q^{-1/2}s_1)}\right)^{(-1)^t_+}.
\]

Next, consider the type D-roots \( t_1^\pm_1 \) and \( t_2^\pm_1 \) in the numerator, with \( t_1 = v^{-1}s_1 \), \( t_2 = vs_1 \), and \( t_j \in \sigma_e(m+) \). These yield:

\[
\prod_{s=\delta_+}^{2m_+ - 3/2} \frac{(1 - q^{-s+1/2}s_1)(1 - q^{-s-1/2}s_1)(1 - q^{s+1/2}s_1)(1 - q^{s-1/2}s_1)}{(1 - q^{-s+3/2}s_1)(1 - q^{-s-3/2}s_1)(1 - q^{s+3/2}s_1)(1 - q^{s-3/2}s_1)} = \frac{(1 - q^{2m_+ - 2}s_1)(1 - q^{-2m_+ + 2}s_1)(1 - q^{-\delta_+ - 1/2}s_1)(1 - q^{-\delta_+ + 1/2}s_1)}{(1 - q^{-\delta_+ + 3/2}s_1)(1 - q^{\delta_+ - 3/2}s_1)(1 - q^{2m_+ s_1})(1 - q^{2m_+ s_1})}.
\]

Multiplying the two expressions quickly leads to \((6.27)\). Using induction on \( m_- + m_+ \) (as in the rank 0-case), and the induction basis Lemma 6.1, finishes the proof that \( \xi^r_{T,m_-,m_+} \) represents an extra-special spectral transfer morphism.

\( \square \)