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DOI
10.1017/asb.2015.11

Publication date
2015

Document Version
Author accepted manuscript

Published in
ASTIN Bulletin

Link to publication

Citation for published version (APA):
Boonen, T. J. (2015). Competitive equilibria with distortion risk measures. *ASTIN Bulletin*, *45*(3), 703-728. https://doi.org/10.1017/asb.2015.11

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Download date:27 Jan 2022

Competitive Equilibria with Distortion Risk Measures*

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Appeared in: ASTIN Bulletin 45 (3), 703-728.

October 1, 2015

Abstract

This paper studies optimal risk redistribution between firms, such as banks or insurance companies. The introduction of the Basel II regulation and the Swiss Solvency Test (SST) has increased the use of risk measures to evaluate financial or insurance risk. We consider the case where firms use a distortion risk measure (also called dual utility) to evaluate risk. The paper first characterizes all Pareto optimal redistributions. Thereafter, it characterizes all competitive equilibria. It presents three conditions that are jointly sufficient for existence of a unique equilibrium redistribution. This equilibrium's redistribution and prices are provided in closed form via a representative agent.

JEL classification: D53, D81, G11, G22.

Keywords: competitive equilibria, distortion risk measures, capital asset pricing model.

1 Introduction

In this paper, we study the question how to redistribute risk if firms use a distortion risk measure in order to evaluate risk. There is a relatively large literature that analyzes optimal redistributions of risk, based on the seminal work of Borch (1962) and Wilson (1968). This paper mainly differs in terms of the objective of firms. In this paper, we study optimal risk sharing in the context of distortion risk measures instead of Von Neumann-Morgenstern expected utilities. Distortion risk measures are used to define the preference relations of the firms. De Giorgi and Post (2008) show that the capital asset pricing model with distortion risk measures is empirically better fitting US stock returns.

Distortion risk measures have applications in both actuarial science and finance, being related both to the dual theory of risk (Yaari, 1987) and coherent risk measures (Artzner et al., 1999). Existing literature has investigated the use of distortion risk measures as pricing mechanisms, e.g., Yaari (1987), Chateauneuf et al. (1996), Wang (1996, 2000), and Wang et al. (1997).

*The author thanks Enrico Biffis, Péter Csóka, Anja De Waegenaere, Henk Norde, Hans Schumacher, Mitja Stadje, Dolf Talman, Michel Vellekoop, Bas Werker, Alexander Mürmann, the anonymous referees, participants of SING8, the Netspar Pension Day 2013 and the ARIA annual meeting, and seminar participants of Tilburg University, University of New South Wales, Queensland University of Technology, University of Technology Sydney, Australian National University, University of Amsterdam, University of Waterloo, and Georgia State University for useful comments. Possible errors are my own responsibility.

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Yaari (1987) characterizes dual utility by a modification of the independence axiom in expected utility theory. Instead of requiring independence with respect to probability mixtures of risks, he requires independence with respect to direct mixing the realizations of the risks. The evaluation of a risk is linear in the pay-offs but non-linear in the probabilities. Distortion risk measures coincide with dual utility if firms are risk averse. Distortion risk measures differ in two fundamental ways from expected utility theory. First, they reflect cash-equivalent preferences, implying that cash payments do not affect risk preferences. For example, in insurance this implies that the price of a risk is independent of the initial wealth of the insurer. Second, distortion risk measures attempt to reflect business practices where Expected Shortfall has been gaining practitioner interest. In line with mean-variance preferences, we can formulate a risk preference based on a distortion risk measure by any trade-off between the expectation and a distortion risk measure. This leads to risk-reward preferences as in De Giorgi and Post (2008).

The approach that we propose to optimally redistribute risk is twofold. First, we analyze Pareto optimal redistributions, where all transfers such that all risk is redistributed are allowed. Pareto optimality of redistributions based on risk measures is first studied by Jouini et al. (2008) and it is applied to distortion risk measures by Ludkovski and Rüschendorf (2008) and Ludkovski and Young (2009). Ludkovski and Rüschendorf (2008) focuses on existence of Pareto optima. Jouini et al. (2008) and Ludkovski and Young (2009) characterize all comonotone Pareto optimal risk redistributions as a finite sum of stop-loss contracts on the aggregate risk. We introduce one condition that guarantees that all Pareto optimal risk redistributions are comonotone. Moreover, we derive two conditions that are jointly sufficient to ensure unique Pareto optimal risk redistributions up to side-payments.

Second, we analyze the problem of determining the size of the side-payments. We focus on the competitive equilibria in a well-functioning market where firms act as price-takers. Filipović and Kupper (2008), Dana and Le Van (2010), Dana (2011), and Flåm (2011) analyze existence of equilibria in markets where firms use risk measures. Our focus is on uniqueness of the equilibrium. We provide three jointly sufficient conditions to guarantee that there is a unique equilibrium risk redistribution. The equilibrium prices are derived from the preferences of a representative agent. From this unique equilibrium, we derive a corresponding capital asset pricing model (CAPM) for distortion risk measures.

An important application of the problem described in this paper is insurance. Particularly for the trading of insurance products, tranching of the aggregate risk is empirically observed. Typically, the idiosyncratic part of insurance risk can be hedged via pooling. This creates incentives for insurance in the first place. We show that only the systematic component of insurance risk is priced. Multiple small firms can benefit from this by pooling their risk with other firms. The systematic part of insurance risk cannot be hedged by pooling risk. However, firms can still benefit from trading the systematic part of insurance risk with other firms which face insurance risk in other insurance risk classes. The systematic risk can be shared and, therefore, firms can benefit by redistributing. For instance, the class of longevity risk is an application where there is potential to redistribute risk since death benefit insurers and pension funds have negatively correlated risk exposures (see, e.g., Tsai et al., 2010; Wang et al., 2010).

This paper contributes to the literature on ambiguity. Distortion risk measures are derived via distorted probabilities. Distorted probabilities are typically used to include ambiguity (see, e.g., Chateauneuf et al., 2000; Werner, 2001; Tsanakas and Christofides, 2006; De Castro and Chateuneuf, 2011; Strzalecki and Werner, 2011; Rigotti and Shannon, 2012). With ambiguity, utility of firms is often determined via distorted probabilities and a concave utility function via a Choquet expected utility (Schmeidler, 1989). Chateauneuf et al. (2000), Werner (2001), and

Tsanakas and Christofides (2006) study Pareto optima and equilibria, and assume that the expected utility function is strictly concave. We extend these results by allowing for linear utility functions, i.e., firms are neutral towards risk and averse towards ambiguity (also called Knightian uncertainty). Ambiguity aversion is captured by concavity of the probability distortions (see, e.g., Yaari, 1987; Schmeidler, 1989). In this way, we get a more explicit solution. De Castro and Chateuneuf (2011) analyze the trading volumes under ambiguity. They show that more ambiguity leads to a smaller set of Pareto optimal risk redistributions. If the utility functions are linear and firms use strictly concave probability distortions, we derive in this paper that the set of Pareto optimal risk redistributions is independent of the degree of ambiguity aversion. Strzalecki and Werner (2011) analyze comonotonicity of Pareto optimal risk redistributions in the context of ambiguity. They show that all Pareto optimal risk redistributions are comonotone if firms use strictly convex preferences. If the utility function is linear, however, the preferences are not strictly convex. Moreover, Rigotti and Shannon (2012) show existence of a finite number of competitive equilibria with ambiguity. In our setting, existence is guaranteed as shown by Filipović and Kupper (2008). Our specific setting allows us to analyze uniqueness of the competitive equilibrium.

This paper is set out as follows. Section 2 introduces distortion risk measures and the risk redistribution problem. Section 3 analyzes Pareto optimality. We hereby focus explicitly on uniqueness of Pareto optimal risk redistributions. Section 4 derives the competitive equilibrium prices, as well as conditions such that the corresponding equilibrium risk redistribution is unique. Finally, Section 5 concludes this paper.

2 Distortion risk measures and risk redistribution problems

2.1 Distortion risk measures

Distortion risk measures are developed from research on dual utility by Yaari (1987). Moreover, they are developed as premium principle by Wang (1995). Let Ω a finite state space and $\mathbb{P}$ the physical probability measure on the power set 2^Ω. Moreover, denote $\mathbb{R}^\Omega$ as the space of all real valued stochastic variables on Ω that are realized at a well-defined reference time. These stochastic variables are referred to as risks. A realization of a risk is interpreted as a future loss.

A risk measure is a function $\rho : \mathbb{R}^\Omega \to \mathbb{R}$, i.e., a risk measure maps risks into real numbers. For every risk $Y \in \mathbb{R}^\Omega$, we refer to $\rho(Y)$ as the risk adjusted value of the liabilities. Wang (1995) defines a distortion risk measure by

$$\rho(Y) = \int_0^\infty g^\rho(1 - F_Y(x))\,dx + \int_{-\infty}^0 (g^\rho(1 - F_Y(x)) - 1)\,dx, \text{ for all } Y \in \mathbb{R}^\Omega, \tag{1}$$

for a continuous, concave and increasing distortion function $g^\rho : [0,1] \to [0,1]$ with $g^\rho(0) = 0$ and $g^\rho(1) = 1$, where F_Y is the cumulative density function (CDF) of risk Y. Here, convergence of the integrals is guaranteed by boundedness of the risk Y.

We continue with an alternative representation of distortion risk measures that we use throughout this paper. We get via direct calculations that

$$\rho(Y) = E_{\mathbb{Q}_Y}[Y], \text{ for all } Y \in \mathbb{R}^\Omega, \tag{2}$$

where $\mathbb{Q}_Y : 2^\Omega \to (0,1]$ is the additive mapping such that

$$\mathbb{Q}_Y(\{\omega\}) = g^\rho(\mathbb{P}(Y \geq Y(\{\omega\}))) - g^\rho(\mathbb{P}(Y > Y(\{\omega\}))), \text{ for all } \omega \in \Omega. \tag{3}$$

Since g^ρ is increasing and such that $g^\rho(0) = 0$ and $g^\rho(1) = 1$, it holds that $\mathbb{Q}_Y$ is a probability measure for a given Y. A distortion risk measure evaluated in risk Y is its expectation under the probability measure $\mathbb{Q}_Y$ that assigns higher probabilities to worst-case realizations of the risk Y.

Distortion risk measures are characterized as the risk measures that are coherent and satisfying the two properties *Conditional State Independence* and *Comonotonic Additivity* (see Wang et al., 1997). Coherence is later formally introduced by Artzner et al. (1999). A risk measure $\rho : \mathbb{R}^\Omega \to \mathbb{R}$ is called coherent if and only if it satisfies the four properties *Sub-additivity*, *Monotonicity*, *Positive Homogeneity*, and *Translation Invariance*. The relevance of these properties is widely discussed by Artzner et al. (1999).

Artzner et al. (1999) show that a risk measure ρ is coherent if and only if there exists a set of probability measures Q such that

$$\rho(Y) = \sup \{E_{\mathbb{Q}}[Y] : \mathbb{Q} \in Q\}, \text{ for all } Y \in \mathbb{R}^\Omega. \tag{4}$$

Denote $\mathcal{P}(\Omega)$ as the set of all probability measures on the state space Ω. Based on Denneberg (1994), it holds that a representation of the set Q in (4) for distortion risk measures is given by

$$Q(\rho) = \{\mathbb{Q} \in \mathcal{P}(\Omega) : \mathbb{Q}(A) \leq g^\rho(\mathbb{P}(A)) \text{ for all } A \subset \Omega\}. \tag{5}$$

In the sequel, we discuss the problem to redistribute risk where all firms use distortion risk measures to evaluate risk.

2.2 Risk redistribution problems

Throughout this paper, we fix the set of firms and the discrete state space such that:

- the finite collection of firms is given by $N = \{1, \ldots, n\}$;
- the state space is finite and given by Ω. Without loss of generality, it is assumed that $|\Omega| > 1$;[1]
- the physical probability measure is given by $\mathbb{P} : 2^\Omega \to (0, 1]$.[2] This measure is common knowledge.

Assuming a finite state space will help us later to solve and display the competitive equilibria. Therefore, the reason to assume a finite state space is computational tractability. The techniques we use in this paper are similar to Heath and Ku (2004) and De Giorgi and Post (2008), who also assume a finite state space.

Next, we define risk redistribution problems with distortion risk measures.

Definition 2.1 A *risk redistribution problem with distortion risk measures* is a tuple $(X_i, \rho_i)_{i \in N}$, where

- $X_i \in \mathbb{R}^\Omega$ is the risk held by firm $i \in N$;
- $\rho_i : \mathbb{R}^\Omega \to \mathbb{R}$ is the distortion risk measure that firm $i \in N$ is endowed with. The corresponding distortion function is denoted by g_i.[3]

[1] We denote $|\Omega|$ for the cardinality of the state space Ω.

[2] We assume that $\mathbb{P}(\{\omega\}) > 0$ for each state $\omega \in \Omega$, i.e., the probability that a state occurs is strictly positive. As the state space is finite, this is without loss of generality; states with zero probability can be omitted from the state space.

[3] For notational convenience, we write g_i instead of g^{ρ_i}.

The class of risk redistribution problems with distortion risk measures is denoted by $\mathcal{R}$.

In the sequel, we refer to a risk redistribution problem with distortion risk measures as a risk redistribution problem. There is common knowledge about the risks and risk measures of all firms. Moreover, we define the aggregate risk as $X = \sum_{i \in N} X_i$.

For a risk redistribution problem, we aim to redistribute the aggregate risk X among firms. The objective of a firm is to minimize its risk adjusted value of the liabilities. We allow for all forms of risk redistributions, as long as the aggregate risk is redistributed. The set of *feasible* risk redistributions of a risk redistribution problem $R \in \mathcal{R}$ is given by

$$\mathcal{F}(R) = \left\{ (\widetilde{X}_i)_{i \in N} \in (\mathrm{I\!R}^{\Omega})^N : \sum_{i \in N} \widetilde{X}_i = X \right\}. \tag{6}$$

The set of risk redistributions in (6) allows for, e.g., proportional or stop-loss contracts on the aggregate risk X.

Notation: In the following two sections, we use the following notation. Without loss of generality, we order the state space $\Omega = \{\omega_1, \ldots, \omega_p\}$ such that

$$X(\omega_1) \geq \cdots \geq X(\omega_p). \tag{7}$$

Moreover, we define the set $\Omega_k = \{\omega_1, \ldots, \omega_k\}$ for all $k \in \{1, \ldots, p-1\}$, $\Omega_0 = \emptyset$, and the risk $e_A \in \mathrm{I\!R}^{\Omega}$ for $A \subseteq \Omega$ is given by

$$e_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A, \\ 0 & \text{otherwise,} \end{cases} \tag{8}$$

for all $\omega \in \Omega$.

3 Pareto optimality

3.1 Definition and characterization

In this subsection, we summarize two results in the literature that we need in this paper. Tsanakas and Christofides (2006), Acciaio (2007), Burgert and Rüschendorf (2008), Filipović and Svindland (2008), Jouini et al. (2008), Kiesel and Rüschendorf (2007, 2010), Kaina and Rüschendorf (2009), and Ludkovski and Young (2009) all analyze existence of Pareto optimal risk redistributions in settings where risk is measured by a risk measure. A risk redistribution is called Pareto optimal if there does not exist another feasible redistribution that is weakly better for all firms, and strictly better for at least one firm. The set of *Pareto optimal* risk redistributions of a risk redistribution problem $R \in \mathcal{R}$ is given by

$$\mathcal{PO}(R) = \left\{ (\widetilde{X}_i)_{i \in N} \in \mathcal{F}(R) : \nexists (\hat{X}_i)_{i \in N} \in \mathcal{F}(R) \text{ s.t. } (\rho_i(\hat{X}_i))_{i \in N} \lneqq (\rho_i(\widetilde{X}_i))_{i \in N} \right\}. \tag{9}$$

Similar to Borch (1962) where firms use expected utilities, the set $\mathcal{PO}(R)$ only depends on the risks $X_i, i \in N$, via their sum X.

Next, we introduce side-payments. These are used to characterize Pareto optimal risk redistributions. A risk $Y \in \mathrm{I\!R}^{\Omega}$ is a *side-payment* if there exists a constant $c \in \mathrm{I\!R}$ such that $Y = c \cdot e_{\Omega}$ (i.e., the risk Y is a degenerated stochastic variable). For all $R \in \mathcal{R}$, it holds that

$(\widetilde{X}_i)_{i\in N} \in \mathcal{PO}(R)$ if and only if $(\widetilde{X}_i + c_i e_\Omega)_{i\in N} \in \mathcal{PO}(R)$ for any $c \in \mathbb{R}^N$ such that $\sum_{i\in N} c_i = 0$. This follows straightforwardly from *Translation Invariance* of ρ_i and is also stated by Ludkovski and Young (2009). So, if we find a Pareto optimal risk redistribution $(\widetilde{X}_i)_{i\in N}$, we can construct a set of Pareto optimal risk redistributions by adding zero-sum side-payments $(c_i \cdot e_\Omega)_{i\in N}$ such that $\sum_{i\in N} c_i = 0$ to $(\widetilde{X}_i)_{i\in N}$. If there exists a risk redistribution $(\widetilde{X}_i)_{i\in N}$, such that the set of risk redistributions that arises from adding zero-sum side-payments equals the set of Pareto optimal risk redistributions, we call this risk redistribution $(\widetilde{X}_i)_{i\in N}$ the unique element of $\mathcal{PO}(R)$ up to side-payments.

Definition 3.1 For an $R \in \mathcal{R}$, the risk redistribution $(\widetilde{X}_i)_{i\in N} \in \mathcal{PO}(R)$ is, *up to side-payments*, the unique element of $\mathcal{PO}(R)$ if

$$\mathcal{PO}(R) = \left\{ (\widetilde{X}_i + c_i e_\Omega)_{i\in N} : c \in \mathbb{R}^N, \sum_{i\in N} c_i = 0 \right\}.$$

The term side-payment is inspired by the procedure that first firms pick a Pareto optimal risk redistribution and, thereafter, add or subtract zero-sum side-payments. Uniqueness up to side-payments is first introduced in the context of risk measures by Jouini et al. (2008) in the case where one firm is endowed with a very specific risk measure.

If firms use expected utilities, one obtains every Pareto optimal risk redistribution by maximizing a weighted sum of expected utilities (see Borch, 1962). Then, under mild regularity conditions, every (normalized) weight-vector yields a unique risk redistribution. For distortion risk measures, we also get the Pareto optimal risk redistributions by minimizing the weighted aggregate risk adjusted value of the liabilities, i.e., for every $k \in \mathbb{R}^N_{++}$ we minimize $\sum_{i\in N} k_i \rho_i(\widetilde{X}_i)$ over all $(\widetilde{X}_i)_{i\in N} \in \mathcal{F}(R)$. However, in case of distortion risk measures this minimization problem only has a solution if k equals the ones-vector. If k equals the ones-vector, we get all Pareto optimal risk redistributions as shown by Jouini et al. (2008) and Filipović and Kupper (2008). So, for all $R \in \mathcal{R}$, it holds that $(\widetilde{X}_i)_{i\in N} \in \mathcal{PO}(R)$ if and only if $(\widetilde{X}_i)_{i\in N} \in \mathcal{F}(R)$ and

$$\sum_{i\in N} \rho_i(\widetilde{X}_i) = \min \left\{ \sum_{i\in N} \rho_i(\hat{X}_i) : (\hat{X}_i)_{i\in N} \in \mathcal{F}(R) \right\}. \tag{10}$$

3.2 Special case

The following proposition shows that if there is a firm that is endowed with a smaller distortion function than all other firms, it is Pareto optimal to shift all risk to this firm.

Proposition 3.2 *If $R \in \mathcal{R}$ is such that there exists a firm $i \in N$ such that*

$$g_i(x) \leq g_j(x), \text{ for all } x \in [0,1] \text{ and } j \in N, \tag{11}$$

then for $\widetilde{X}_i = X$ and $\widetilde{X}_j = 0 \cdot e_\Omega$ for all $j \in N\backslash\{i\}$, it holds that $(\widetilde{X}_j)_{j\in N} \in \mathcal{PO}(R)$. Moreover, if

$$g_i(x) < g_j(x), \text{ for all } x \in (0,1) \text{ and } j \in N\backslash\{i\}, \tag{12}$$

then $(\widetilde{X}_j)_{j\in N}$ is, up to side-payments, the unique element of $\mathcal{PO}(R)$.

Proposition 3.2 extends the result of Schmidt (1999), who shows that if (12) holds we have $(\widetilde{X}_j)_{j\in N} \in \mathcal{PO}(R)$. We show that this risk redistribution is, up to side-payments, the unique element of $\mathcal{PO}(R)$. The result of Schmidt (1999) is later generalized by Ludkovski and Young (2008), who determine a subset of all Pareto optimal risk redistributions.

Condition (11) seems restrictive. However, we next provide examples where this condition holds. These examples focus on the case where firms use Expected Shortfall risk measures (see, e.g., Acerbi and Tasche, 2002).

Example 3.3 In this example, we discuss two cases where condition (11) holds. For all $j \in N$, let firm j use the risk measure Expected Shortfall with significance level $\alpha_j \in (0,1]$, which is the distortion risk measure with distortion function $g_j(x) = \min\left\{\frac{x}{\alpha_j}, 1\right\}$, for all $x \in [0,1]$ (Dhaene et al., 2006). One can easily verify that condition (11) is satisfied, where $i \in \operatorname{argmax}\{\alpha_j : j \in N\}$. Hence, according to Proposition 3.2, it is Pareto optimal to redistribute all risk to firm i. Note that it might seem unfair that firm i takes over all risk. Firm i is, however, willing to bear all risk only if the side-payments it receives are sufficiently high. This problem is examined in Section 4.

As a second example, we let every firm $j \in N$ use the risk measure Mean-Expected Shortfall with significance levels $\zeta \in [0,1]$ and $\alpha_j \in (0,1)$, i.e.,

$$\rho^{MES}_{\zeta,\alpha_j}(Y) = \zeta E_{\mathbb{P}}[Y] + (1-\zeta)\rho^{ES}_{\alpha_j}(Y), \text{ for all } Y \in \mathbb{R}^\Omega, \tag{13}$$

where $\rho^{ES}_{\alpha_j}(Y)$ is the Expected Shortfall with significance level $\alpha_j \in (0,1]$. The (ζ, α_j)-Mean-Expected Shortfall is a distortion risk measure with distortion function $g_j(x) = \zeta x + (1-\zeta)\min\left\{\frac{x}{\alpha_j}, 1\right\}$ for all $x \in [0,1]$. One can again verify that condition (11) is satisfied, where $i \in \operatorname{argmax}\{\alpha_j : j \in N\}$. This result is also shown by Asimit et al. (2013) in the context of optimal risk transfers between two divisions within an insurance company, where the objective of the insurer is equal to (10). Generalizing this to multiple divisions is straightforward. More generally, if a firm uses a weighted average of a distortion risk measure and the expectation, its preferences can again be formalized via a distortion risk measure. If every firm $j \in N$ uses

$$\rho_j(Y) = \zeta_j E_{\mathbb{P}}[Y] + (1-\zeta_j)\rho(Y), \text{ for all } Y \in \mathbb{R}^\Omega, \tag{14}$$

for some distortion risk measure ρ and $\zeta_j \in [0,1]$, it is Pareto optimal to shift all risk to a firm $i \in \operatorname{argmax}\{\zeta_j : j \in N\}$. $\triangledown$

3.3 Comonotonicity with the aggregate risk

A risk redistribution $(\widetilde{X}_i)_{i\in N} \in \mathcal{F}(R)$ is comonotone with the aggregate risk if there exists an ordering $(\omega_1, \ldots, \omega_p)$ on the state space Ω such that $\Omega = \{\omega_1, \ldots, \omega_p\}$, and $\widetilde{X}_i(\omega_1) \geq \cdots \geq \widetilde{X}_i(\omega_p)$ for all $i \in N$. So, all stochastic variables in $(\widetilde{X}_i)_{i\in N}$ are comonotone with each other. For distortion risk measures, comonotonicity of Pareto optimal risk redistributions is first studied by Ludkovski and Rüschendorf (2008). Their main result, which is based on Landsberger and Meilijson (1994), states that there exists a Pareto optimal risk redistribution such that all individual posterior risks are comonotone with the aggregate risk X. For a broader class than distortion risk measures, this result is extended by Kiesel and Rüschendorf (2010) by characterizing all Pareto optimal risk redistributions as being μ-comonotone. This is a weaker condition than comonotonicity. We provide a sufficient condition such that all Pareto optimal risk redistributions are comonotone with the aggregate risk X.

We first define a risk measure ρ^*_N that plays a central role in obtaining Pareto optimal risk redistributions.

Definition 3.4 The function $g_N^*: [0,1] \to [0,1]$ of a risk redistribution problem $R \in \mathcal{R}$ is given by $g_N^*(x) = \min\{g_i(x) : i \in N\}$ for all $x \in [0,1]$. Moreover, ρ_N^* is the risk measure as defined in (1) with $g^{\rho_N^*} = g_N^*$.

The function g_N^* is continuous, concave, increasing and such that $g_N^*(0) = 0$ and $g_N^*(1) = 1$. Concavity of the function g_N^* follows from the fact that the minimum of concave functions is concave as well. This leads directly to the following proposition.

Proposition 3.5 *For all $R \in \mathcal{R}$, the risk measure ρ_N^* is a distortion risk measure.*

We show in the sequel of this section that the risk measure ρ_N^* plays a central role in obtaining Pareto optimal risk redistributions.

Next, we provide a closed-form expression of a set of Pareto optimal risk redistributions. To do so, we first define the set of functions $M(R)$ of a risk redistribution problem $R \in \mathcal{R}$ by

$$M(R) = \left\{ m : \{1,\ldots,p-1\} \to N \,\middle|\, m(k) \in \operatorname*{argmin}_{j \in N} g_j\left(\mathbb{P}(\Omega_k)\right) \text{ for all } k \in \{1,\ldots,p-1\} \right\}. \quad (15)$$

A function $m \in M(R)$ assigns to every $k \in \{1,\ldots,p-1\}$ a firm i for which the distortion function g_i is minimal at $\mathbb{P}(\Omega_k)$, i.e., for all $k \in \{1,\ldots,p-1\}$ it holds that $g_{m(k)}(\mathbb{P}(\Omega_k)) = g_N^*(\mathbb{P}(\Omega_k))$. Ludkovski and Young (2009) determine some Pareto optimal risk redistributions as follows. For all $R \in \mathcal{R}$, $m \in M(R)$ and $d \in \mathbb{R}^N$ with $\sum_{i \in N} d_i = X(\omega_p) = \min X$, it holds that $(\widetilde{X}_i)_{i \in N} \in \mathcal{PO}(R)$ where

$$\widetilde{X}_i = \sum_{k=1}^{p-1} [X(\omega_k) - X(\omega_{k+1})] \cdot \mathbb{1}_{m(k)=i} \cdot e_{\Omega_k} + d_i \cdot e_\Omega, \text{ for all } i \in N, \quad (16)$$

and $\mathbb{1}_{m(k)=i} = 1$ if $m(k) = i$ and zero otherwise. Via the risks $d_i \cdot e_\Omega, i \in N$, the functional form (16) allows for all side-payments that ensure $(\widetilde{X}_i)_{i \in N} \in \mathcal{F}(R)$. The size of these side-payments is a central topic in the next section.

In the following proposition, we characterize all Pareto optimal risk redistributions. This result provides the minimum aggregate risk adjusted value of the liabilities in the market after any Pareto optimal risk redistribution.

Proposition 3.6 *For all $R \in \mathcal{R}$, we have $(\widetilde{X}_i)_{i \in N} \in \mathcal{PO}(R)$ if and only if $(\widetilde{X}_i)_{i \in N} \in \mathcal{F}(R)$ and*

$$\sum_{i \in N} \rho_i(\widetilde{X}_i) = \rho_N^*(X). \quad (17)$$

From Proposition 3.6 it follows that the Pareto optimal aggregate risk adjusted value of the liabilities depends on the risk measures $\rho_i, i \in N$, via ρ_N^* only. One can interpret this as that there exists a representative agent whose distortion risk measure equals ρ_N^*. The agent is representative in the sense that its preferences are sufficient to calculate the Pareto optimal aggregate risk adjusted value of the liabilities in the market. In contrast to actuarial equilibrium models with utility functions, the representative agent is a hypothetical firm with a risk measure that is least risk-averse in the market instead of an average (cf. Bühlmann, 1980).

We observe that every risk redistribution of the form (16) is comonotone with the aggregate risk X. If the following condition holds, we show that all Pareto optimal redistributions are comonotone with the aggregate risk.

Condition [SC]: The function g_N^* is strictly concave, i.e.,

$$\lambda g_N^*(x) + (1-\lambda)g_N^*(y) < g_N^*(\lambda x + (1-\lambda)y),$$

for all $\lambda \in (0,1)$ and $x, y \in [0,1]$ such that $x \neq y$.

We next show our main contribution to the literature on Pareto optimal risk redistributions. We will later use this to characterize uniqueness of Pareto optimal risk redistributions up to side-payments.

Proposition 3.7 *If $R \in \mathcal{R}$ is such that condition [SC] holds, all Pareto optimal risk redistributions are comonotone with each the aggregate risk X.*

If the function g_N^* is piecewise linear (e.g., if ρ_N^* equals α-Expected Shortfall), comonotonicity with the aggregate risk X is not guaranteed for Pareto optimal risk redistributions. However, if the distortion functions $g_i, i \in N$ are all strictly concave, the function g_N^* is strictly concave as well. Wirch and Hardy (2001) show that strict concavity of a distortion function is a necessary and sufficient condition for risk measures to strongly preserve second order stochastic dominance. From Proposition 3.7 we get that if $R \in \mathcal{R}$ is such that condition [SC] holds and $k \in \{1, \ldots, p-1\}$ is such that $X(\omega_k) = X(\omega_{k+1})$, it holds for all $(\widetilde{X}_i)_{i \in N} \in \mathcal{PO}(R)$ that $\widetilde{X}_i(\omega_k) = \widetilde{X}_i(\omega_{k+1})$ for all $i \in N$. So, when considering Pareto optimal risk redistributions, the states $\omega_k, \omega_{k+1} \in \Omega$ such that $X(\omega_k) = X(\omega_{k+1})$ are treated in the same manner.

3.4 Uniqueness up to side-payments

If there is a unique Pareto optimal risk redistribution up to side-payments, the only question left is to determine the size of the side-payments. From (10), and *Sub-additivity* and *Positive Homogeneity* of the risk measures, it can be shown that the set of Pareto optimal risk redistributions is convex, i.e., for all $(\widetilde{X}_i)_{i \in N}$, $(\hat{X}_i)_{i \in N} \in \mathcal{PO}(R)$, it holds that $(\lambda \widetilde{X}_i + (1-\lambda)\hat{X}_i)_{i \in N} \in \mathcal{PO}(R)$ for all $\lambda \in [0,1]$. Therefore, the set of Pareto optimal risk redistributions can be large even up to side-payments. In this subsection, we identify two joint conditions under which the Pareto optimal risk redistributions are, up to side-payments, unique.

Uniqueness up to side-payments of Pareto optimal risk redistributions would hold if the risk measures are strictly concave (see Filipović and Svindland, 2008; Kiesel and Rüschendorf, 2010). Distortion risk measures are, however, not strictly concave. This can be seen from the properties *Comonotonic Additivity* and *Positive Homogeneity* of distortion risk measures as it holds that $\alpha\rho(X) + (1-\alpha)\rho(Y) = \rho(\alpha X + (1-\alpha)Y)$ for all comonotone $X, Y \in \mathbb{R}^\Omega$ such that $X \neq Y$ and $\alpha \in (0,1)$.

To show uniqueness up to side-payments of Pareto optimal risk redistributions, we first introduce the following condition on $\mathcal{R}$.

Condition [U]: For all $k \in \{1, \ldots, p-1\}$ such that $X(\omega_k) > X(\omega_{k+1})$ there exists a firm $i \in N$ such that for all $m \in M(R)$ it holds that $m(k) = i$.

If condition [SC] holds, we get from Proposition 3.7 that all Pareto optimal risk redistributions are comonotone with the aggregate risk. From this combined with Theorem 2 of Ludkovski and Young (2009) follows the next result.

Theorem 3.8 *If $R \in \mathcal{R}$ is such that condition [SC] holds, there exists a risk redistribution that is, up to side payments, the unique element of $\mathcal{PO}(R)$ if and only if condition [U] holds*

The least risk-averse agent is the analogue to the risk-neutral agent under expected utility. If there exists a globally least risk-averse firm, then this firm bears all the aggregate risk. If there does not exist a globally least risk-averse firm, then it is Pareto optimal that the locally least risk-averse firm bears the local aggregate risk. Under condition [SC], this risk redistribution is unique if there exists a unique least strictly risk-averse firm for those states with local aggregate risk.

Note that $|M(R)| = 1$ is a sufficient condition for condition [U] to hold. Hence, if $|M(R)| = 1$ and condition [SC] holds, it follows from Theorem 3.8 that all Pareto optimal risk redistributions are uniquely determined up to side-payments. One can determine a Pareto optimal risk redistribution via (16). Condition [U] implies that all functions in $M(R)$ differ only for k such that $X(\omega_k) = X(\omega_{k+1})$.

Even if the function g_N^* is not strictly concave, it might be possible to define a strict concave distortion function such that it coincides with the function g_N^* on the relevant subdomain. This subdomain is the following finite collection of probabilities $\{\mathbb{P}(\Omega_k) : k = 1, \ldots, p-1\}$. We illustrate this special case in the following example, where we also illustrate the construction of Pareto optimal risk redistributions.

Example 3.9 Let $N = \{1, 2\}$, $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $\mathbb{P}(\{\omega\}) = \frac{1}{3}$ for all $\omega \in \Omega$, $g_1(x) = \min\left\{1\frac{1}{2}x, 1\right\}$, $g_2(x) = \sqrt{x}$, $X(\omega_1) = 2$, $X(\omega_2) = 1$, $X(\omega_3) = 0$ and $X_1 = X_2 = \frac{1}{2}X$. So, we consider the case where all benefits from risk redistributions arise from the use of different risk measures. Firm 1 uses $\frac{2}{3}$-Expected Shortfall and Firm 2 uses a so-called proportional hazard distortion risk measure with parameter $\frac{1}{2}$. The function g_N^* is given by:

$$g_N^*(x) = \min\{g_i(x) : i \in N\} = \begin{cases} 1\frac{1}{2}x & \text{if } x \leq \frac{4}{9}, \\ \sqrt{x} & \text{otherwise.} \end{cases}$$

The distortion functions g_1, g_2 and g_N^* are displayed in Figure 1. From this figure, we see that there is a unique $i \in \operatorname{argmax}\{g_j(x) : j \in N\}$ that is minimal at $x = \mathbb{P}(\{\omega_1\}) = \frac{1}{3}$ and at $x = \mathbb{P}(\{\omega_1, \omega_2\}) = \frac{2}{3}$. Hence, it holds that $|M(R)| = 1$ and, therefore, condition [U] holds. Moreover, it holds that $m(1) = 1$ and $m(2) = 2$ for $m \in M(R)$. According to (16) with $c_1 = c_2 = 0$, a Pareto optimal risk redistribution is given by $\widetilde{X}_1$ and $\widetilde{X}_2$ such that $\widetilde{X}_1(\omega_1) = 1, \widetilde{X}_1(\omega_2) = 0, \widetilde{X}_1(\omega_3) = 0$, $\widetilde{X}_2(\omega_1) = 1$, $\widetilde{X}_2(\omega_2) = 1$ and $\widetilde{X}_2(\omega_3) = 0$. The construction of this Pareto optimal risk redistribution is shown in Figure 2.

One can readily verify that $\rho_2(\widetilde{X}_2) > \rho_2(X_2)$ and, so, Firm 2 needs to receive a side-payment from Firm 1 in order to be willing to trade. We investigate the size of this side-payment in Section 4.

In this example, the function g_N^* is not strictly concave. However, we can define a strictly concave distortion function that coincides with the function g_N^* on the subdomain $\{\mathbb{P}(\Omega_k) : k = 1, \ldots, p-1\} = \left\{\frac{1}{3}, \frac{2}{3}\right\}$. Since also condition [U] holds, we get from Theorem 3.8 that there is a unique Pareto optimal risk redistribution up to side-payments.

The distortion functions g_1 and g_2 cross only once on the subdomain $(0, 1)$, i.e., $g_1(x) < g_2(x)$ for $x \in (0, x^*)$ and $g_1(x) > g_2(x)$ for $x \in (x^*, 1)$ with $x^* = \frac{4}{9}$. This implies that every Pareto optimal risk redistribution is a stop-loss contract or a deductible on the aggregate risk X and a side-payment; every $(\widetilde{X}_i)_{i \in N} \in \mathcal{PO}(R)$ is such that there exists a $c \in \mathbb{R}$ such that $\widetilde{X}_1 = (X-1)^+ + c \cdot e_\Omega$ and $\widetilde{X}_2 = \min\{X, 1\} - c \cdot e_\Omega$. $\triangledown$

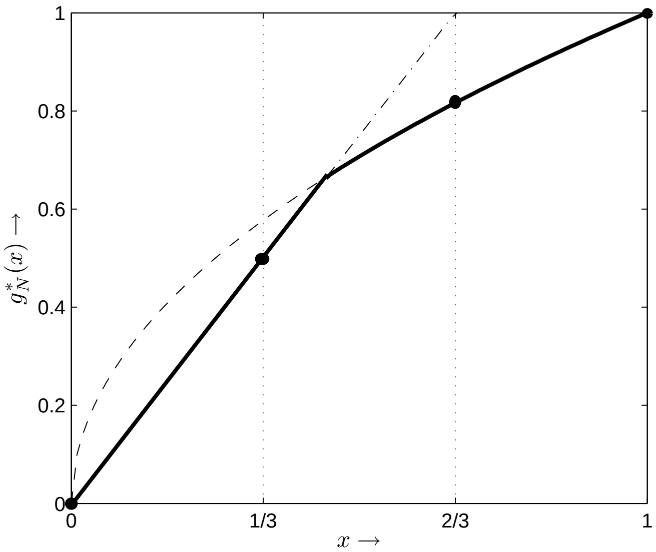

Figure 1: Construction of the function g_N^* via the distortion functions g_1 and g_2 corresponding to Example 3.9. The function g_1 is the dashed-dotted line, g_2 is the dashed line and g_N^* is the solid line.

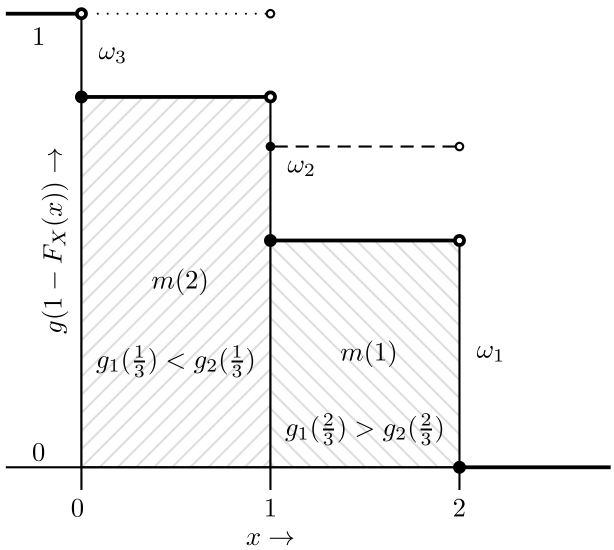

Figure 2: Graphical illustration of the construction of Pareto optimal risk redistributions corresponding to Example 3.9. The figure displays $g(1 - F_X(x))$ for g equal to g_1 (dotted line), g_2 (dashed line) or g_N^* (solid line). If lines coincide, the solid line is shown. From (1) with $\mathbb{P}(X < 0) = 0$ it follows that the area under g_N^* equals $\rho_N^*(X)$. The two shaded blocks are assigned to the firms $m(1) = 1$ and $m(2) = 2$; the risk $\widetilde{X}_1 = e_{\Omega_1}$ is assigned to Firm 1 and the risk $\widetilde{X}_2 = e_{\Omega_2}$ is assigned to Firm 2. The area of the shaded block with $m(i)$ in it represents $\rho_i(\widetilde{X}_i)$ for $i = 1, 2$.

4 Competitive equilibria

4.1 Uniqueness of the competitive equilibrium

In the previous section, we analyzed Pareto optimality of risk redistributions. Under the conditions [SC] and [U] in Theorem 3.8, there exists a unique Pareto optimal risk redistribution up to side-payments. In this section, we identify some Pareto optimal risk redistributions as the *competitive equilibria*. Competitive equilibria in insurance markets are studied by, e.g., Aase (1993, 2010) for the case where firms use expected utility functions. Under three regularity conditions on the utility functions, Aase (1993, 2010) proves existence and uniqueness of the equilibrium. This result is inspired by Borch (1962), who shows this for special cases. For mean-variance investors, the competitive equilibrium corresponds with the classical CAPM equilibrium price of risk as derived by Sharpe (1964). Dana (2011) finds existence of a representative agent in the market where firms use *strictly* concave risk measures. If firms use distortion risk measures, we show in this subsection some conditions under which the market prices are unique.

Chateauneuf et al. (2000) and Tsanakas and Christofides (2006) analyze equilibria in the case in which firms evaluate risk via $\rho_i(u_i(X_i))$,[4] where u_i is a strictly concave utility function and ρ_i a risk measure. Here, the role of risk measures is to include ambiguity aversion via max-min ambiguity-averse preferences. They assume strict concavity of the utility function as a sufficient condition to have uniqueness of the competitive equilibrium. In this subsection, we relax the assumption of a strictly concave utility function and obtain the competitive equilibria and corresponding capital asset pricing model.

Let there be a complete market. This implies existence of state prices, i.e., prices for the Arrow-Debreu assets $e_{\{\omega\}}$ with $\omega \in \Omega$. The pricing formula is given by $\pi(\hat{p}, Y) = \sum_{k=1}^{p} \hat{p}_k Y(\omega_k)$ for some price vector $\hat{p} \in \mathbb{R}_+^\Omega$. We assume that the risk-free rate is zero, i.e., $\pi(\hat{p}, e_\Omega) = 1$. A competitive equilibrium is a vector of prices $\hat{p} \in \mathbb{R}_+^\Omega$ and a risk redistribution $(\hat{X}_i)_{i\in N} \in (\mathbb{R}^\Omega)^N$ such that given the prices, each firm $i \in N$ individually minimizes $\rho_i(\hat{X}_i)$ under a budget constraint, i.e., $\hat{X}_i$ solves

$$\min_{\widetilde{X}_i \in \mathbb{R}^\Omega} \rho_i(\widetilde{X}_i), \tag{18}$$

$$\text{s.t.} \quad \pi(\hat{p}, \widetilde{X}_i) \geq \pi(\hat{p}, X_i), \tag{19}$$

and the price vector $\hat{p}$ satisfies $\pi(\hat{p}, e_\Omega) = 1$ and induces market-clearing by equating aggregate supply and demand, i.e., $(\hat{X}_i)_{i\in N} \in \mathcal{F}(R)$. Competitive equilibria rely on the assumption that there is a competitive environment, where individual transactions have no influence on the prices. So, the number of firms needs to be sufficiently large.

The following lemma follows directly from Theorem 3.2 of Filipović and Kupper (2008). This result is an adjustment of the First Fundamental Welfare Theorem in case firms use distortion risk measures. Filipović and Kupper (2008) show this for the class of monetary utility functions, which include distortion risk measures as a subclass.

Lemma 4.1 *For all risk redistribution problems in $\mathcal{R}$, there exists a competitive equilibrium. Moreover, every equilibrium risk redistribution is Pareto optimal.*

Next, we focus on characterizing uniqueness of the competitive equilibrium. We introduce the following condition, which assumes that there are no states in Ω in which the realization of the

[4]For a wider class of risk measures ρ_i, this is also called Choquet-, max-min- or rank-dependent expected utility.

aggregate risk X is the same.

Condition [SO]: $X(\omega_1) > \cdots > X(\omega_p)$.

The following theorem shows that the conditions [SC] and [SO] are jointly sufficient to guarantee uniqueness of the equilibrium prices.

Theorem 4.2 *If $R \in \mathcal{R}$ is such that condition [SC] holds, it holds that*

- *an equilibrium price vector is given by*

$$\hat{p}_k = g_N^*(\mathbb{P}(\Omega_k)) - g_N^*(\mathbb{P}(\Omega_{k-1})), \text{ for all } k \in \{1, \dots, p\}; \tag{20}$$

- *the equilibrium price vector $\hat{p}$ is unique if and only if condition [SO] holds.*

If the conditions [SC] and [SO] hold, we have that $\hat{p}$ is unique, and such that $\hat{p} = \mathbb{Q}_X \in Q(\rho_N^*)$, where $\mathbb{Q}_X$ is as in (3) with $g^\rho = g_N^*$, and $Q(\rho_N^*)$ is defined in (5).[5]. If condition [SC] does not hold, we can show that the vector $\hat{p}$ in (20) is still the unique equilibrium price vector if condition [SO] holds. However, the reversed statement does not necessarily hold true. If condition [SO] does not hold, we get that the equilibrium risk redistribution is still unique if conditions [SC] and [U] hold. However, the prices are not unique (see Theorem 4.2).

We show in the following proposition that under the conditions [SC] and [SO], the risk adjusted value of the liabilities of every equilibrium risk redistribution equals the equilibrium price of the prior risk of the firms.

Proposition 4.3 *If $R \in \mathcal{R}$ is such that conditions [SC] and [SO] hold, every equilibrium risk redistribution $(\hat{X}_i)_{i \in N}$ is such that*

$$\rho_i(\hat{X}_i) = E_{\mathbb{Q}_X}[X_i] = \sum_{k=1}^{p} [g_N^*(\mathbb{P}(\Omega_k)) - g_N^*(\mathbb{P}(\Omega_{k-1}))] X_i(\omega_k), \tag{21}$$

for all $i \in N$.

Theorem 4.2 states the unique equilibrium prices under conditions [SC] and [SO]. Moreover, Theorem 3.8 states that under the additional condition [U] these unique equilibrium prices correspond with a unique risk redistribution. If firms use expected utilities, Aase (1993) shows that under two regularity conditions on utility functions only, there is existence of the equilibria. Existence of the equilibria in the risk redistribution problem is guaranteed. If firms use expected utilities, there is a third regularity condition necessary to ensure uniqueness of the equilibrium risk redistribution. All three regularity conditions in Aase (1993) are imposed on the utility functions only. From Theorem 3.8 and Theorem 4.2, we get the following main result of this paper.

Theorem 4.4 *If $R \in \mathcal{R}$ is such that conditions [SC], [SO] and [U] hold, there is a unique equilibrium risk redistribution.*

[5]Note that $\mathbb{Q}_X$ is a probability measure whereas $\hat{p}$ is a vector. Here, we mean that $\mathbb{Q}_X(\{\omega_k\}) = \hat{p}_k$ for all $k \in \{1, \dots, p\}$. In the sequel, we interpret $\hat{p}$ as a probability measure as well.

In contrary to the conditions of Aase (1993) for expected utilities, the two conditions [SO] and [U] also depend on the aggregate risk.

The condition [SC] implies that the results in Theorem 4.2, Proposition 4.3 and Theorem 4.4 do not apply to Expected Shortfall. Suppose that condition [SC] does not hold. Ludkovski and Rüschendorf (2008) show that there exists a Pareto optimal risk redistribution that is comonotone with the aggregate risk. Then, we derive the following result in the same way as for Theorem 4.2.

Proposition 4.5 *If $R \in \mathcal{R}$ is such that conditions [SO] and [U] hold, then there exists a unique equilibrium $(\widetilde{X}_i)_{i\in N}$ that is comonotone with the aggregate risk.*

Note that many authors explicitly assume that risk redistributions are comonotone with the aggregate risk (see, e.g., Ludkovski and Young, 2009; Asimit et al., 2013).

In this paper, we assume that there is a finite space. This allows us to solve the competitive equilibria in Theorem 4.2 via linear programming. Generalizing the characterization of the uniqueness of competitive equilibria to an infinite state space would be an interesting topic for future research. Whereas the condition [U] has a straightforward translation to the class of continuous risks, the condition [SO] has not.

Next, we return to the problem in Example 3.9 and compute the equilibrium risk redistribution for this case.

Example 4.6 In this example, we provide the equilibrium risk redistribution problem in Example 3.9. We get that $\mathbb{Q}_X = \left(\frac{1}{2}, \sqrt{\frac{2}{3}} - \frac{1}{2}, 1 - \sqrt{\frac{2}{3}}\right)$. This leads to

$$\rho_i(\hat{X}_i) = E_{\mathbb{Q}_X}[X_i] = \frac{1}{\sqrt{6}} + \frac{1}{4} \approx 0.66, \tag{22}$$

for all $i \in N$, where $\hat{X}_i$ is the corresponding risk redistribution. This implies that for the risk redistribution $(\widetilde{X}_i)_{i\in N}$ provided in Example 3.9, there is a side-payment of size $\frac{1}{\sqrt{6}} - \frac{1}{4} \approx 0.16$ made by Firm 1 to Firm 2. Then, the risk redistribution $\hat{X}_1$ and $\hat{X}_2$ is given by $\hat{X}_1(\omega_1) \approx 1.16$, $\hat{X}_1(\omega_2) \approx 0.16$, $\hat{X}_1(\omega_3) \approx 0.16$, $\hat{X}_2(\omega_1) \approx 0.84$, $\hat{X}_2(\omega_2) \approx 0.84$, and $\hat{X}_2(\omega_3) \approx -0.16$. $\triangledown$

Next, we proceed with a more complex example, in which we illustrate two issues with uniqueness of the equilibrium.

Example 4.7 Let $N = \{1, 2, 3\}$, $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$, $\mathbb{P}(\{\omega\}) = \frac{1}{5}$ for all $\omega \in \Omega$, $g_1(x) = \min\left\{1\frac{1}{2}x, 1\right\}$, $g_2(x) = \sqrt{x}$, $g_3(x) = \min\{1\frac{1}{2}x, \frac{1}{2} + \frac{1}{2}x\}$ $X(\omega_1) = 4$, $X(\omega_2) = 1$, $X(\omega_3) = 0$, $X(\omega_4) = -1$, $X(\omega_5) = -4$ and $X_1 = X_2 = X$ and $X_3 = -X$. Firm 3 uses the $(\frac{1}{2}, \frac{1}{2})-$Mean-Expected Shortfall risk measure. This example resembles a situation where firms face a systematic risk-factor, but have different exposure to it.

The distortion function g_N^* is not strictly concave on the relevant subdomain. Moreover, the condition [U] does not hold. We get for all $m \in M(R)$ that $m(3) = m(4) = 2$. From this and (16), we get that it is Pareto optimal if Firm 2 bears the risk $4 \cdot e_\Omega + \min\{X, 0\} = e_{\Omega_3} + 3 \cdot e_{\Omega_4}$. Moreover, we get that $m(1), m(2) \in \{1, 3\}$ for all $m \in M(R)$. From (16) and the fact that the set of Pareto optimal risk redistributions is convex, we get that any comonotone risk redistribution of $(X)^+ = 3 \cdot e_{\Omega_1} + e_{\Omega_2}$ to the firms in $\{1, 3\}$ is Pareto optimal. However, since the distortion function g_N^* is linear on the sub-domain $\{0, \mathbb{P}(\Omega_1), \mathbb{P}(\Omega_2)\}$, we get that every non-negative risk redistribution of $(X)^+$ to the firms in $\{1,3\}$ is Pareto optimal. Hence, a Pareto optimal risk redistribution is not necessarily comonotone with the aggregate risk.

Now adjust the problem such that the distortion function of Firm 3 is strictly higher on the sub-domain $(0,1)$ than before, i.e., $\hat{g}_3(x) > g_3(x)$ for all $x \in (0,1)$, and let the rest remain the same. Then, condition [U] is satisfied. We get that the Pareto optimal risk redistributions are not unique up to side-payments (note that condition [SC] is still not satisfied). The only risk redistribution that is comonotone with the aggregate risk is given by $\widetilde{X}_1 = (X)^+$, $\widetilde{X}_2 = \min\{X, 0\}$ and $X_3 = 0 \cdot e_\Omega$. The unique equilibrium prices are given by $\mathbb{Q}_X = (0.3, 0.3, \sqrt{0.6} - 0.6, \sqrt{0.8} - \sqrt{0.6}, 1 - \sqrt{0.8})$. From Proposition 4.5, we get that the unique equilibrium risk redistribution that is comonotone with the aggregate risk, denoted by $(\hat{X}_i)_{i \in N}$, is such that $\rho_1(\hat{X}_1) = E_{\mathbb{Q}_X}[X_1] \approx 0.96$, $\rho_2(\hat{X}_2) \approx 0.96$, and $\rho_3(\hat{X}_3) = E_{\mathbb{Q}_X}[X_3] - \approx -0.96$. This leads to $\hat{X}_1 \approx (X)^+ - 0.54 \cdot e_\Omega$, $\hat{X}_2 = \min\{X, 0\} + 1.5 \cdot e_\Omega$, and $\hat{X}_3 = -0.96 \cdot e_\Omega$.

Note that the equilibrium does not depend on the distortion function $\hat{g}_3$ as long as we have $\hat{g}_3(x) > g_N^*(x)$ for all $x \in \{\mathbb{P}(\Omega_k) : k = 1, \ldots, p-1\}$. Moreover, the risk adjusted value of the liability corresponding to the equilibrium is lower for Firm 3 due to that the risk of this firm is anti-comonotone with the aggregate risk, and hence useful for diversification. The fact that the distortion function $\hat{g}_3$ is never minimal on the interval $(0,1)$ does not influence this value. $\triangledown$

Next, we discuss some comparative statics of the equilibrium. Assume that the conditions [SC], [SO] and [U] hold. Let $m \in M(R)$ and pick a $k \in \{1, \ldots, p-1\}$ and a firm $i \in N$. Now, suppose that we increase the distortion function g_i only in $g_i(\mathbb{P}(\Omega_k))$ such that it is still concave on the subdomain $\{0\} \cup \{\mathbb{P}(\Omega_k) : k = 1, \ldots, p\}$, *ceteris paribus*. Denote the new distortion function for firm i by $\widetilde{g}_i$. This leads possibly to a different function $\widetilde{m} \in M(\widetilde{R})$. We get the following three possible effects:

- if $m(k) = i$ and $\widetilde{m}(k) \neq i$, the risk $[X(\omega_k) - X(\omega_{k+1})] \cdot e_{\Omega_k}$ is now borne by firm $\widetilde{m}(k)$ in equilibrium. Firm $m(k)$ charges, however, a higher price for bearing this risk, i.e., the equilibrium price for the state ω_k increases with the amount $g_{\widetilde{m}(k)}(\mathbb{P}(\Omega_k)) - g_i(\mathbb{P}(\Omega_k))$. The equilibrium price for state ω_{k+1} decreases with this amount. The aggregate risk adjusted value of the liabilities in the market increases, but the sign of the individual effect on the risk adjusted value of the liabilities for firm i is unknown;

- if $m(k) = i$ and $\widetilde{m}(k) = i$, then the risk $[X(\omega_k) - X(\omega_{k+1})] \cdot e_{\Omega_k}$ is still borne by firm i in equilibrium. Firm i charges, however, a higher price for bearing this risk, i.e., the equilibrium price for the state ω_k increases with the amount $\widetilde{g}_i(\mathbb{P}(\Omega_k) - g_i(\mathbb{P}(\Omega_k))$. The equilibrium price for the state ω_{k-1} decreases with this amount. The aggregate risk adjusted value of the liabilities in the market increases, but the sign of the individual effect on the risk adjusted value of the liabilities for firm i is unknown;

- if $m(k) \neq i$, the equilibrium prices and risk redistributions are not changed.

4.2 Capital asset pricing model

In this subsection, we derive a capital asset pricing model under the conditions [SC], [SO] and [U]. One can interpret $\hat{p} = \mathbb{Q}_X$ in Proposition 4.3 as a *risk neutral* probability measure for obtaining the price of all X_i. The Radon-Nikodym derivative is given by

$$\frac{d\mathbb{Q}_X}{d\mathbb{P}}(\{\omega_k\}) = \frac{g_N^*(\mathbb{P}(\Omega_k)) - g_N^*(\mathbb{P}(\Omega_{k-1}))}{\mathbb{P}(\{\omega_k\})}, \tag{23}$$

for all $k \in \{1, \dots, p\}$, which extends the Radon-Nikodym derivative used by Tsanakas (2004) and De Giorgi and Post (2008) for heterogeneous risk measures. So, the equilibrium prices are such that $\pi(\hat{p}, X_i) = E_{\mathbb{P}}\left[X_i \frac{d\mathbb{Q}_X}{d\mathbb{P}}\right]$ for all $i \in N$. This leads to

$$\pi(\hat{p}, X_i) = E_{\mathbb{P}}[X_i] + cov\left(X_i, \frac{d\mathbb{Q}_X}{d\mathbb{P}}\right), \text{ for all } i \in N, \tag{24}$$

where $\hat{p}$ is the unique equilibrium price vector. So, if the risk X_i is independent of the aggregate X, firm i only gets a risk where its risk adjusted value of the liabilities equals the expectation of its prior risk X_i. The stochastic variable $\frac{d\mathbb{Q}_X}{d\mathbb{P}}$ is comonotone with the risk X due to concavity of the function g_N^*. Therefore, only co-movements with the market risk X are priced. Also, the equilibrium price depends on the aggregate risk X via the ordering on the state space Ω such that $X(\omega_1) \geq \dots \geq X(\omega_p)$ only.

If the equilibrium prices are unique, it follows from (24) that for all $R \in \mathcal{R}$ such that conditions [SC] and [SO] hold, we have

$$E_{\mathbb{P}}[RR_i] - 1 = \beta_i \left(E_{\mathbb{P}}[RR_m] - 1\right), \tag{25}$$

where $i \in N$, $RR_i = \frac{X_i}{\pi(\hat{p}, X_i)}$ and $RR_m = \frac{X}{\pi(\hat{p}, X)}$ and

$$\beta_i = \frac{cov\left(RR_i, \frac{d\mathbb{Q}_X}{d\mathbb{P}}\right)}{cov\left(RR_m, \frac{d\mathbb{Q}_X}{d\mathbb{P}}\right)}. \tag{26}$$

The factor β_i in (25) is a market beta in a representation of the CAPM-model with distortion risk measures. Note that the risk-free rate is assumed to be zero.

De Giorgi and Post (2008) empirically test the CAPM model for the case where all firms use the same distortion risk measure using US stock returns and find a better fit than the CAPM model with mean-variance investors. If we would test the equilibrium prices with distortion risk measures, we would assume a functional form of the representative distortion function g_N^*. De Giorgi and Post (2008) assume a functional form of the distortion function that all firms use. Hence, to test our model is analogue to the test of De Giorgi and Post (2008). Hence, our model with distortion risk measures has a better fit than the CAPM model with mean-variance investors as well.

5 Conclusion

This paper studies optimal risk redistribution between firms. In contrast to previous literature on preferences given by distortion functions, we study uniqueness of the competitive equilibrium. We provide three conditions that are jointly sufficient to have a unique equilibrium. Two out of the three conditions that we propose do depend on the aggregate risk in the market. This is in contrast to the literature on uniqueness of the equilibrium with expected utilities, where sufficient conditions do typically depend on the utility functions only (Aase, 1993).

The equilibrium prices follow from the preferences of a hypothetical representative agent. In contrast to when firms use expected utilities, this representative agent resembles a least risk-averse agent in the market instead of an average risk-averse agent (cf. Bühlmann, 1980).

We derive the results in this paper for concave distortion functions. This assumption allows us to study risk redistributions that are comonotone with the aggregate risk. Whereas the

Expected Shortfall is a popular used distortion risk measure, this assumption excludes the use of Value-at-Risk. Studying competitive equilibria with Value-at-Risk is a promising topic for future research. Another topic for future research is to examine competitive equilibria in the presence of agents that minimize distortion risk measures, and agents that maximize expected utilities. Whereas distortion risk measures might be induced by regulation for firms, the preferences of individual consumers are often modeled via expected utilities.

A Proofs

Proof of Proposition 3.2 The first part of the proof follows from Ludkovski and Young (2009).

Next, let $R \in \mathcal{R}$ be such that there exists a firm $i \in N$ for which (12) holds. One can easily verify that from (2)-(3) it follows that

$$\rho_i(Y) \leq \rho_j(Y), \text{ for all } Y \in \mathbb{R}^\Omega \text{ and } j \in N. \tag{27}$$

Suppose that there exists an $(\hat{X}_j)_{j\in N} \in \mathcal{PO}(R)$ such that for at least one firm $j \neq i$ its risk $\hat{X}_j$ is not a side-payment. For every risk $Y \in \mathbb{R}^\Omega$ that is not a side-payment, it holds that

$$\begin{aligned}\rho_i(Y) &= \sum_{k=1}^{p-1} g_i(\mathbb{P}(\{\omega_1,\ldots,\omega_k\}))[Y(\omega_k) - Y(\omega_{k+1})] + Y(\omega_p) &(28)\\ &< \sum_{k=1}^{p-1} g_j(\mathbb{P}(\{\omega_1,\ldots,\omega_k\}))[Y(\omega_k) - Y(\omega_{k+1})] + Y(\omega_p) &(29)\\ &= \rho_j(Y), &(30)\end{aligned}$$

for all $j \neq i$, where $Y(\omega_1) \geq \cdots \geq Y(\omega_p)$. Here, (28) follows from direct calculations with (1), (29) follows from (12) and that there exists a $k \in \{1,\ldots,p-1\}$ such that $Y(\omega_k) - Y(\omega_{k+1}) > 0$ and $\mathbb{P}(\{\omega_1,\ldots,\omega_k\}) \in (0,1)$. From this it follows directly that

$$\begin{aligned}\rho_i(X) &\leq \sum_{j\in N} \rho_i(\hat{X}_j) &(31)\\ &< \sum_{j\in N} \rho_j(\hat{X}_j), &(32)\end{aligned}$$

where (31) follows from *Sub-additivity* of ρ_i, and (32) follows from (27) and (28)-(30). Combining (31)-(32) with $\rho_j(0\cdot e_\Omega) = 0$ for all $j \in N\backslash\{i\}$ yields a contradiction with (10). Hence, for all $(\hat{X}_j)_{j\in N} \in \mathcal{PO}(R)$ and $j \neq i$ it follows that $\hat{X}_j$ is a side-payment. This concludes the second part of the proof. □

Proof of Proposition 3.6 Let $R \in \mathcal{R}$. From (10) we get that it is sufficient to show that $\sum_{i\in N} \rho_i(\widetilde{X}_i) = \rho^*_N(X)$ for an $(\widetilde{X}_i)_{i\in N} \in \mathcal{PO}(R)$. Pick $(\widetilde{X}_i)_{i\in N} \in \mathcal{PO}(R)$ as in (16) with $m \in M(R)$ and $d \in \mathbb{R}^N$ such that $\sum_{i\in N} d_i = X(\omega_p)$. The risk $\widetilde{X}_i$ is constructed such that

$$\begin{aligned}\widetilde{X}_i(\omega_k) - \widetilde{X}_i(\omega_{k+1}) &= \sum_{\ell=k}^{p-1}[X(\omega_\ell) - X(\omega_{\ell+1})]\cdot \mathbb{1}_{m(\ell)=i} + d_i &(33)\\ &\quad - \left(\sum_{\ell=k+1}^{p-1}[X(\omega_\ell) - X(\omega_{\ell+1})]\cdot \mathbb{1}_{m(\ell)=i} + d_i\right)\\ &= [X(\omega_k) - X(\omega_{k+1})]\cdot \mathbb{1}_{m(k)=i}, &(34)\end{aligned}$$

for all $k \in \{1,\dots,p-1\}$ and $i \in N$ and, moreover, $\sum_{i\in N} \widetilde{X}_i(\omega_p) = \sum_{i\in N} d_i = X(\omega_p)$. So, it holds that $\widetilde{X}_i(\omega_1) \geq \cdots \geq \widetilde{X}_i(\omega_p)$ for all $i \in N$. From this and (33)-(34) it follows that

$$\begin{aligned}
\sum_{i\in N} \rho_i(\widetilde{X}_i) &= \sum_{i\in N}\left[\sum_{k=1}^{p-1} g_i(\mathbb{P}(\Omega_k))[\widetilde{X}_i(\omega_k) - \widetilde{X}_i(\omega_{k+1})] + \widetilde{X}_i(\omega_p)\right] \\
&= \sum_{i\in N}\left[\sum_{k=1}^{p-1} g_i(\mathbb{P}(\Omega_k))[X(\omega_k) - X(\omega_{k+1})]\cdot \mathbb{1}_{m(k)=i} + d_i\right] \\
&= \sum_{k=1}^{p-1}\sum_{i\in N} g_i(\mathbb{P}(\Omega_k))[X(\omega_k) - X(\omega_{k+1})]\cdot \mathbb{1}_{m(k)=i} + \sum_{i\in N} d_i \\
&= \sum_{k=1}^{p-1}[X(\omega_k) - X(\omega_{k+1})]\sum_{i\in N} g_i(\mathbb{P}(\Omega_k))\cdot \mathbb{1}_{m(k)=i} + X(\omega_p) \\
&= \sum_{k=1}^{p-1}[X(\omega_k) - X(\omega_{k+1})]\min\{g_i(\mathbb{P}(\Omega_k)) : i \in N\} + X(\omega_p) \\
&= \sum_{k=1}^{p-1}[X(\omega_k) - X(\omega_{k+1})]g_N^*(\mathbb{P}(\Omega_k)) + X(\omega_p) \\
&= \rho_N^*(X).
\end{aligned}$$

This concludes the proof. □

Proof of Proposition 3.7 Let $R \in \mathcal{R}$ be such that condition [SC] holds and suppose that $(\widetilde{X}_j)_{j\in N} \in \mathcal{PO}(R)$ is such that there exist $i \in N$ and $k \in \{1,\dots,p-1\}$ where $\widetilde{X}_i(\omega_k) < \widetilde{X}_i(\omega_{k+1})$. Recall from (2) and (3) that $\rho_N^*(X) = E_{\mathbb{Q}_X}[X]$, where $\mathbb{Q}_X$ is the additive probability measure such that $\mathbb{Q}_X(\{\omega_\ell\}) = g_N^*(\mathbb{P}(\Omega_\ell)) - g_N^*(\mathbb{P}(\Omega_{\ell-1}))$ for all $\ell \in \{1,\dots,p\}$. One can verify that, due to concavity of the function g_N^*, it holds that $\mathbb{Q}_X \in Q(\rho_N^*) \subseteq Q(\rho_j)$ for all $j \in N$ and, so,

$$\rho_j(\widetilde{X}_j) \geq E_{\mathbb{Q}_X}[\widetilde{X}_j], \text{ for all } j \in N. \tag{35}$$

Next, we show that

$$\rho_i(\widetilde{X}_i) > E_{\mathbb{Q}_X}[\widetilde{X}_i]. \tag{36}$$

Since $Q(\rho_N^*) \subseteq Q(\rho_i)$, it follows that $\rho_i(\widetilde{X}_i) \geq \rho_N^*(\widetilde{X}_i)$ and, so, it is sufficient to show $\rho_N^*(\widetilde{X}_i) > E_{\mathbb{Q}_X}[\widetilde{X}_i]$. We will show that

$$\begin{aligned}
&[g_N^*(\mathbb{P}(\Omega_{k-1}\cup\{\omega_{k+1}\})) - g_N^*(\mathbb{P}(\Omega_{k-1}))]\widetilde{X}_i(\omega_{k+1}) \\
&\quad + [g_N^*(\mathbb{P}(\Omega_{k+1})) - g_N^*(\mathbb{P}(\Omega_{k-1}\cup\{\omega_{k+1}\}))]\widetilde{X}_i(\omega_k) \\
&> [g_N^*(\mathbb{P}(\Omega_k)) - g_N^*(\mathbb{P}(\Omega_{k-1}))]\widetilde{X}_i(\omega_k) + [g_N^*(\mathbb{P}(\Omega_{k+1})) - g_N^*(\mathbb{P}(\Omega_k))]\widetilde{X}_i(\omega_{k+1}).
\end{aligned} \tag{37}$$

Equivalently, (37) can be written as

$$\begin{aligned}
&\widetilde{X}_i(\omega_{k+1})[g_N^*(\mathbb{P}(\Omega_{k-1}\cup\{\omega_{k+1}\})) - g_N^*(\mathbb{P}(\Omega_{k-1})) - g_N^*(\mathbb{P}(\Omega_{k+1})) + g_N^*(\mathbb{P}(\Omega_k))] \\
&\quad > \widetilde{X}_i(\omega_k)[g_N^*(\mathbb{P}(\Omega_{k-1}\cup\{\omega_{k+1}\})) - g_N^*(\mathbb{P}(\Omega_{k-1})) - g_N^*(\mathbb{P}(\Omega_{k+1})) + g_N^*(\mathbb{P}(\Omega_k))].
\end{aligned} \tag{38}$$

Since the function g_N^* is strictly concave and $\mathbb{P}(\{\omega\}) > 0$ for all $\omega \in \Omega$, it follows that

$$g_N^*(\mathbb{P}(\Omega_{k-1} \cup \{\omega_{k+1}\})) - g_N^*(\mathbb{P}(\Omega_{k-1})) - g_N^*(\mathbb{P}(\Omega_{k+1})) + g_N^*(\mathbb{P}(\Omega_k)) > 0.$$

From this it follows that (38) holds and, so, (37) holds.

From (37), we get that $E_{\mathbb{Q}'}[\widetilde{X}_i] > E_{\mathbb{Q}_X}[\widetilde{X}_i]$ where

$$\mathbb{Q}'(\omega) = \begin{cases} g_N^*(\mathbb{P}(\Omega_{k+1})) - g_N^*(\mathbb{P}(\Omega_{k-1} \cup \{\omega_{k+1}\})) & \text{if } \omega = \omega_k, \\ g_N^*(\mathbb{P}(\Omega_{k-1} \cup \{\omega_{k+1})) - g_N^*(\mathbb{P}(\Omega_{k-1})) & \text{if } \omega = \omega_{k+1}, \\ \mathbb{Q}_X(\omega) & \text{otherwise.} \end{cases}$$

From $\mathbb{Q}' \in Q(\rho_N^*)$ it follows that $\rho_N^*(\widetilde{X}_i) \geq E_{\mathbb{Q}'}[\widetilde{X}_i]$. So, we have shown that (36) holds. To conclude, it follows that

$$\begin{aligned} \sum_{j \in N} \rho_j(\widetilde{X}_j) &\geq \sum_{j \in N \setminus \{i\}} E_{\mathbb{Q}_X}[\widetilde{X}_j] + \rho_i(\widetilde{X}_i) && (39) \\ &> \sum_{j \in N} E_{\mathbb{Q}_X}[\widetilde{X}_j] && (40) \\ &= E_{\mathbb{Q}_X}[X] && (41) \\ &= \rho_N^*(X), && (42) \end{aligned}$$

where (39) follows from (35), (40) follows from (36), and (42) follows from (2) and (3). From Proposition 3.6 we get that a risk redistribution $(\hat{X}_j)_{j \in N} \in \mathcal{F}(R)$ is Pareto optimal only if $\sum_{j \in N} \rho_j(\hat{X}_j) = \rho_N^*(X)$. So, it follows that $(\widetilde{X}_j)_{j \in N}$ is not Pareto optimal, which is a contradiction. Hence, we have $\widetilde{X}_j(\omega_1) \geq \cdots \geq \widetilde{X}_j(\omega_p)$ for all $j \in N$ and $(\widetilde{X}_j)_{j \in N} \in \mathcal{PO}(R)$. This concludes the proof. $\square$

Proof of Theorem 4.2 Let $R \in \mathcal{R}$ be such that condition [SC] holds. First, we show the second result, i.e., that the equilibrium price vector $\hat{p}$ is unique if and only if condition [SO] holds. Next, we show the "$\Leftarrow$" ("if") part of the proof. Let condition [SO] hold. According to Lemma 4.1, there exists an equilibrium. Pick an equilibrium $(\hat{p}, (\hat{X}_i)_{i \in N})$. Lemma 4.1 states that every equilibrium risk redistribution is Pareto optimal and, so, we have $(\hat{X}_i)_{i \in N} \in \mathcal{PO}(R)$. From this, strict concavity of the function g_N^* and Proposition 3.7 it follows that all risks $\hat{X}_i, i \in N$ are comonotone with the aggregate risk, i.e., $\hat{X}_i(\omega_1) \geq \cdots \geq \hat{X}_i(\omega_p)$ for all $i \in N$. Hence, the objective function for firm $i \in N$ in (18) can be written as

$$\rho_i(\widetilde{X}_i) = \sum_{k=1}^{p} [g_i(\mathbb{P}(\Omega_k)) - g_i(\mathbb{P}(\Omega_{k-1}))] \widetilde{X}_i(\omega_k), \tag{43}$$

which is minimized over all $\widetilde{X}_i \in \mathbb{R}^\Omega$ such that $\widetilde{X}_i(\omega_1) \geq \cdots \geq \widetilde{X}_i(\omega_p)$ and $\pi(\hat{p}, \widetilde{X}_i) \geq \pi(\hat{p}, X_i)$. A minimum is obtained in $\widetilde{X}_i = \hat{X}_i$.

Since the constraints and the objective function in (43) are all affine, we get that the equilibrium risk redistribution $(\hat{X}_i)_{i \in N}$ satisfies the Kuhn-Tucker conditions. The Kuhn-Tucker conditions are obtained by the first-order conditions of the following function with respect to $\widetilde{X}_i(\omega_k)$:

$$\begin{aligned} &\sum_{k=1}^{p} \Big([g_i(\mathbb{P}(\Omega_k)) - g_i(\mathbb{P}(\Omega_{k-1}))] \widetilde{X}_i(\omega_k) - \lambda_i \hat{p}_k [\widetilde{X}_i(\omega_k) - X_i(\omega_k)] \Big) \\ &\qquad - \sum_{k=1}^{p-1} \gamma_{i,k} \left[\widetilde{X}_i(\omega_k) - \widetilde{X}_i(\omega_{k+1}) \right] \end{aligned} \tag{44}$$

in $\widetilde{X}_i(\omega_k) = \hat{X}_i(\omega_k)$, for all $k \in \{1,\ldots,p\}$ and $i \in N$, where $\lambda_i \geq 0$ and $\gamma_{i,k} \geq 0$ are the Kuhn-Tucker multipliers of the constraints $\pi(\hat{p}, \widetilde{X}_i) \geq \pi(\hat{p}, X_i)$ and $\widetilde{X}_i(\omega_k) \geq \widetilde{X}_i(\omega_{k+1})$, respectively. These Kuhn-Tucker conditions are then given by

$$g_i(\mathbb{P}(\Omega_k)) - g_i(\mathbb{P}(\Omega_{k-1})) = \begin{cases} \lambda_i\hat{p}_k + \gamma_{i,k} & \text{if } k = 1, \\ \lambda_i\hat{p}_k + \gamma_{i,k} - \gamma_{i,k-1} & \text{if } k = 2,\ldots,p-1, \\ \lambda_i\hat{p}_k - \gamma_{i,k-1} & \text{if } k = p, \end{cases} \tag{45}$$

for all $k \in \{1,\ldots,p\}$ and $i \in N$, which hold under the constraints $\lambda_i[\pi(\hat{p}, \hat{X}_i) - \pi(\hat{p}, X_i)] = 0, \lambda_i \geq 0, \gamma_{i,k}\left[\hat{X}_i(\omega_k) - \hat{X}_i(\omega_{k+1})\right] = 0$ and $\gamma_{i,k} \geq 0$ for all $k \in \{1,\ldots,p-1\}$ and $i \in N$. Since $g_i(0) = 0$ and $g_i(1) = 1$, it holds that

$$\sum_{k=1}^{p}[g_i(\mathbb{P}(\Omega_k)) - g_i(\mathbb{P}(\Omega_{k-1}))] = 1, \tag{46}$$

and, moreover, it holds that

$$\sum_{k=1}^{p}\hat{p}_k = 1, \tag{47}$$

since $\pi(\hat{p}, e_\Omega) = 1$ and

$$\gamma_{i,1} + \sum_{k=2}^{p-1}(\gamma_{i,k} - \gamma_{i,k-1}) - \gamma_{i,p-1} = 0. \tag{48}$$

From (45), (46), (47) and (48) it follows that $\lambda_i = 1$ for all $i \in N$ and, so, we can write (45) as:

$$g_i(\mathbb{P}(\Omega_k)) - g_i(\mathbb{P}(\Omega_{k-1})) = \begin{cases} \hat{p}_k + \gamma_{i,k} & \text{if } k = 1, \\ \hat{p}_k + \gamma_{i,k} - \gamma_{i,k-1} & \text{if } k = 2,\ldots,p-1, \\ \hat{p}_k - \gamma_{i,k-1} & \text{if } k = p, \end{cases} \tag{49}$$

for all $k \in \{1,\ldots,p\}$ and $i \in N$. Since $X(\omega_1) > X(\omega_2)$, it holds that there exists at least one firm $i_0 \in N$ such that $\gamma_{i_0,1} = 0$. From this and $\gamma_{j,1} \geq 0$ for all $j \in N$ it follows that

$$\hat{p}_1 = g_{i_0}(\mathbb{P}(\Omega_1)) = g_N^*(\mathbb{P}(\Omega_1)) \text{ and } \gamma_{i,1} = g_i(\mathbb{P}(\Omega_1)) - g_N^*(\mathbb{P}(\Omega_1)), \text{ for all } i \in N. \tag{50}$$

If $p > 2$, it follows from (49) and (50) that for $k = 2$ we get

$$g_i(\mathbb{P}(\Omega_2)) - g_N^*(\mathbb{P}(\Omega_1)) = \hat{p}_2 + \gamma_{i,2}, \text{ for all } i \in N, \tag{51}$$

and, so, we get

$$\hat{p}_2 = g_N^*(\mathbb{P}(\Omega_2)) - g_N^*(\mathbb{P}(\Omega_1)) \text{ and } \gamma_{i,2} = g_i(\mathbb{P}(\Omega_2)) - g_N^*(\mathbb{P}(\Omega_2)), \tag{52}$$

for all $i \in N$. Continuing this procedure for all $k \in \{1,\ldots,p\}$, we obtain by induction the unique equilibrium price vector $\hat{p}$. This concludes the first part of the proof.

Next, we show the "$\Rightarrow$" ("only if") part of the proof. Let there exists a unique equilibrium price vector $\hat{p}$. Suppose condition [SO] does not hold. Then, there exists a $k \in \{1,\ldots,p-1\}$ such that $X(\omega_k) = X(\omega_{k+1})$. Construct the price vectors as in (20) for every ordering on the state space Ω such that $X(\omega_1) \geq \cdots \geq X(\omega_p)$. These vectors are all equilibrium prices as they satisfy the conditions in (45) for some Kuhn-Tucker multipliers. Due to strict concavity of the function g_N^* (condition [SC]), these price vectors are not identical. This is a contradiction with the assumption that there is a unique equilibrium price vector.

The first result follows from the fact that $\hat{p}$, as defined in (20), satisfies (49). This concludes the proof. $\square$

Proof of Proposition 4.3 Let $R \in \mathcal{R}$ be such that conditions [SC] and [SO] hold. From Theorem 4.2 it follows that $\hat{p} = \mathbb{Q}_X \in Q(\rho_N^*)$, where $\mathbb{Q}_X(\{\omega_k\}) = g_N^*(\mathbb{P}(\Omega_k)) - g_N^*(\mathbb{P}(\Omega_{k-1}))$ for all $k \in \{1, \ldots, p\}$. So, from (5) it follows that $\hat{p} \in \bigcap_{j \in N} Q(\rho_j) \subseteq Q(\rho_i)$ for all $i \in N$. From this and (4) it follows that $\pi(\hat{p}, Y) \leq \rho_i(Y)$ for all $i \in N$ and all $Y \in \mathbb{R}^\Omega$. So, any risk $\hat{X}_i \in \mathbb{R}^\Omega$ such that $\rho_i(\hat{X}_i) = \pi(\hat{p}, \hat{X}_i) = \pi(\hat{p}, X_i) = E_{\hat{p}}[X_i]$ minimizes the system (18)-(19). $\square$

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