# Bifurcations of optimal vector fields

Kiseleva, T.; Wagener, F.

**DOI**
10.1287/moor.2014.0655

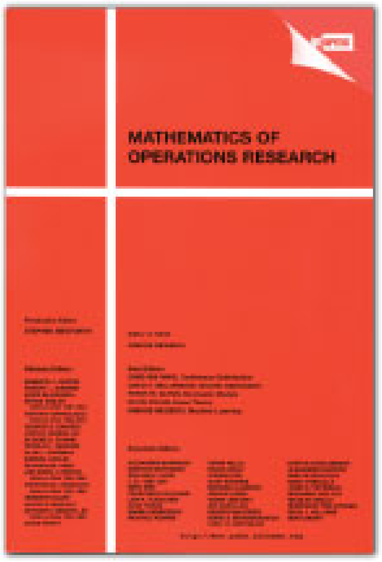
**Publication date**
2015

**Document Version**
Final published version

**Published in**
Mathematics of operations research



**Citation for published version (APA):**
Kiseleva, T., & Wagener, F. (2015). Bifurcations of optimal vector fields. *Mathematics of operations research*, *40*(1), 24-55. https://doi.org/10.1287/moor.2014.0655




# Mathematics of Operations Research



## Bifurcations of Optimal Vector Fields

Tatiana Kiseleva, Florian Wagener

**MATHEMATICS OF OPERATIONS RESEARCH**
Vol. 40, No. 1, February 2015, pp. 24–55
ISSN 0364-765X (print) | ISSN 1526-5471 (online)

http://dx.doi.org/10.1287/moor.2014.0655


# Bifurcations of Optimal Vector Fields

Tatiana Kiseleva
Department of Spatial Economics, VU University Amsterdam, 1081 HV Amsterdam, The Netherlands,
kiseleva.t@gmail.com

Florian Wagener
CeNDEF, Department of Economics and Econometrics, University of Amsterdam, 1018 XE Amsterdam, The Netherlands,
f.o.o.wagener@uva.nl

We study the structure of the solution set of a class of infinite-horizon dynamic programming problems with one-dimensional state spaces, as well as their bifurcations, as problem parameters are varied. The solutions are represented as the integral curves of a multivalued optimal vector field on state space. Generically, there are three types of integral curves: stable points, open intervals that are forward asymptotic to a stable point and backward asymptotic to an unstable point, and half-open intervals that are forward asymptotic to a stable point and backward asymptotic to an indifference point; the latter are initial states to multiple optimal trajectories. We characterize all bifurcations that occur generically in one- and two-parameter families. Most of these are related to global dynamical bifurcations of the state-costate system of the problem.

*Keywords*: infinite horizon problems; multiple optimizers; indifference points; optimal vector fields; bifurcations
*MSC2000 subject classification*: Primary: 37G10, 37N40; secondary: 49N35
*OR/MS subject classification*: Primary: dynamic programming/optimal control (deterministic)
*History*: Received July 15, 2011; revised July 18, 2013. Published online in *Articles in Advance* June 6, 2014.



**1. Introduction.** In the geometric theory of autonomous dynamical systems, the study of structurally stable systems prompted the development of the theory of their bifurcations, which has found many applications. Autonomous dynamical systems arise in optimal control theory as solutions to infinite horizon problems, if the optimal control is expressed in feedback form and substituted back into the dynamic evolution law; such systems will be called "optimal" dynamical systems in the following. The natural question arises whether it is possible to study the bifurcations of an optimal dynamical system in terms of the data of the optimization problem. Results in this direction have been obtained only relatively recently (Wagener [37], Kiseleva and Wagener [23]). The aim of the present work is to develop systematically the bifurcation theory of optimal dynamical systems for one-dimensional state spaces in the continuous time case.

Infinite horizon problems occur most frequently in economic models. Early investigations focused on situations where the optimal dynamics possess a globally attracting steady state (Ramsey [28], Brock and Scheinkman [5]). Since the late 1970s, an increasing number of models were analyzed whose optimal dynamics are not of the globally asymptotically stable type: in growth theory they occur in the context of poverty traps (Dechert and Nishimura [10], Skiba [33]); in fisheries, in the context of overexploitation and even annihilation of fish stocks (Clark [9]). There are environmental models where both industry-promoting but polluting as well as ecologically conservative policies are optimal in the same model, depending on the initial state of the environment (Kiseleva and Wagener [23], Mäler et al. [24], Tahvonen and Salo [34], Wagener [37]); in migration studies, active relocation policies as well as *laissez-faire* policies may occur in the same model (Caulkins et al. [8]); optimal advertising efforts may depend on the initial awareness level of a product (Sethi [30, 32]); the successful containment of epidemics may depend on the initial infection level (Rowthorn and Toxvaerd [29], Sethi [31]); in the control of illicit drug use, high law enforcement as well as low enforcement and treatment of drug users can depend on the initial level of drug abuse (Feichtinger and Tragler [13], Tragler et al. [36]); in R&D policies of firms, the optimal decision between high R&D expenditure investment to develop a technology versus low investment to phase a technology out may depend on the initial technology level (Hinloopen et al. [20]).

In all such models, there is for certain parameter configurations a critical state where both kinds of policies are simultaneously optimal, and where the decision maker is consequently indifferent between them. These points will be called *indifference points* in the following, though they go by many other names as well.[1] Indifference points are related to singularities of the value function of the problem: under conditions that are typically satisfied in many applications (we shall be more precise later on), the value function fails to be continuously differentiable at these points.

[1] For instance Skiba points, Dechert-Nishimura-Skiba points, Dechert-Nishimura-Sethi-Skiba points, regime switching thresholds, Maxwell points, shocks, etc.

The importance of indifference points is perhaps most easily grasped by the analogy to structurally stable one-dimensional continuous time autonomous dynamical systems. Recall that a "structurally stable" system does not change qualitatively if it is perturbed slightly; by restricting to this class, degenerate but a-typical systems are excluded from consideration. For instance, a structurally stable system of the type we are considering has only repelling or attracting steady states, and the qualitative behaviour of the system is known once the location of these steady states is determined. The attracting steady states are the attractors of the system; the basin of attraction of any of these is an open interval, and every repelling steady state is contained in the boundary of two basins of attraction, but is part of neither basin.

If for an autonomous optimal control problem in continuous time with one state dimension the optimal action is known as a function of the state, the optimally controlled state dynamics can be studied. It is characterized qualitatively by the location of the attracting and repelling steady states as well as that of the indifference points. As in the situation of the dynamical systems, all attractors are attracting steady states. But unlike the previous situation, the basins of attraction may contain their boundary points. More precisely, an indifference point is a boundary point to two separate basins of attraction and belongs to both of them, whereas, as before, a repelling steady state is a boundary point to two separate basins while being contained in neither.

Usually, the presence of an indifference point is established numerically for a fixed set of parameter values of the model. To study the dependence of the qualitative properties of the optimal policies on the system parameters, it is possible to do an exhaustive search over all parameter combinations. Such a strategy, although feasible, is however very computing intensive.

A different approach is suggested by the theory of bifurcations of dynamical systems: to identify only those parameter configurations at which the qualitative characteristics of the solutions change. For instance, in Wagener [37] it was shown that indifference points disappear if a heteroclinic bifurcation of the state-costate system occurs. This mechanism, which is a special case of an indifference-attractor bifurcation, one of the bifurcations analyzed below, relates the change of the solution structure of the optimal control problem to a global bifurcation of the state-costate system.

More fundamentally, bifurcation theory identifies and classifies those configurations that are robust under small perturbations of the system or small changes in the system parameters by studying the complementary set of bifurcating configurations. In the present context, this amounts to a bifurcation analysis of optimally controlled or "optimal" vector fields. The purpose of the present article is to develop this theory systematically for optimal control problems with one-dimensional state spaces. Many of the bifurcations involved have already been found in particular systems (see Caulkins et al. [7], Grass [15], Hinloopen et al. [20], Kiseleva and Wagener [23], Wagener [37]).

The main tool of this investigation is a detailed examination, in the neighbourhood of equilibrium points of the state-costate system, of certain critical trajectories and the trajectories on the invariant manifolds of the equilibrium. Their relative positions determine which of them corresponds to an optimal state trajectory. Being close to equilibrium points of the state-costate system, linearization methods can be applied. In this respect, the present approach is more general, as well as more systematic, than that of Wagener [37]; for instance, it allows us also to analyze indifference-attractor bifurcations that are not associated to heteroclinic bifurcations, as occur for instance in Hinloopen et al. [20].

As with the isolated indifference point, parallel to the bifurcation theory of optimal vector fields is a theory of changes in regularity of the associated value functions as problem parameters change. For instance, at an indifference-attractor bifurcation, the value function is $C^1$, but not $C^2$, while to one side of the bifurcation value, it is smooth, whereas at the other side, it is $C^0$ but not $C^1$. In this way, the theory presented below also provides new perspectives on the regularity properties of value functions as a function of problem parameters.

## 2. Illustrative examples.

**2.1. Ramsey saving and poverty traps.** Although they were known before, to most economists indifference points were introduced by Skiba in his article on poverty traps (Skiba [33]). As an example, we consider here a closely related model that shares the main characteristics with Skiba's setup. Skiba considered Ramsey's capital growth model of an economy with a convex-concave production function. We shall consider a concave-convex-concave production function, to avoid having to discuss the boundary of the state space.

That is, we take $x = x(t)$ as a capital stock, $u = u(t)$ as the rate of consumption, and consider the capital dynamics

$$\dot{x} = y(x) - u,$$

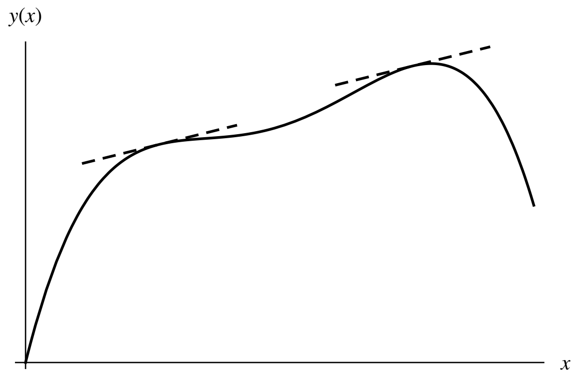



FIGURE 1. Graph of the production function $y(x)$ and "Golden rule" capital levels.

where we shall take the numerical specification $y(x) = 13x - 22x^2 + 16x^3 - 4x^4$ for the production function; this has no special economic significance and it is chosen merely for illustrative purposes (but compare Wagener [38]). The graph of this production function is depicted in Figure 1.

The model requires the planner to maximize the utility functional

$$J = \int_0^\infty g(x, u)e^{-\rho t}\,dt = \int_0^\infty \frac{u^{1-1/\mu}}{1-1/\mu}e^{-\rho t}\,dt,$$

subject to the control restriction

$$0 \le u \le y(x)$$

as well as the dynamic restriction

$$\dot{x} = f(x, u) = y(x) - u.$$

The parameter $\mu > 1$ is the elasticity of intertemporal substitution.

The Hamilton function of this problem is

$$H(x, p) = \max_{u\ge 0}\left(\frac{u^{1-1/\mu}}{1-1/\mu} + p(y(x) - u)\right) = \begin{cases} \dfrac{1}{\mu-1}p^{1-\mu} + py(x) & \text{if } p \ge y(x)^{-1/\mu}; \\ \dfrac{y(x)^{1-1/\mu}}{1-1/\mu} & \text{otherwise.} \end{cases}$$

The maximizer satisfies

$$u = U(x, p) = p^{-\mu}.$$

if $p \ge y(x)^{-1/\mu}$ and $U(x, p) = y(x)$ otherwise.

It can be shown, using methods analogous to those employed in Hinloopen et al. [19], that for all parameter values $\rho > 0$ and $\mu > 1$ the value function $V(x)$ of this problem exists, that $V'(x) \ge 0$ for all $x > 0$, and that it is a viscosity solution of the Hamilton-Jacobi equation

$$\rho V(x) - H(x, V'(x)) = 0,$$

which is continuous and piecewise continuously differentiable. More precisely, it can be shown that the value $V(x)$ may fail to be differentiable, as a function of $x$, in at most one point.

The points where $V$ is differentiable form an open set $D$. At points $x$ from this set, we introduce the optimal feedback rule

$$u^o(x) \stackrel{\text{def}}{=} U(x, V'(x)).$$

If the state-control trajectory pair $(x(t), u(t))$ is such that $x(0) \in D$ and such that it optimizes $J$ subject to both the control and the dynamic restriction, then it can be shown that for all $t > 0$ we have

$$u(t) = u^o(x(t)). \tag{1}$$

Moreover, if the optimal vector field is introduced as

$$f^o(x) = f(x, u^o(x)) = y(x) - u^o(x), \tag{2}$$

then

$$\dot{x}(t) = f^o(x(t))$$

for all $t > 0$.

The necessary conditions for a maximizer imply the existence of a costate trajectory $p = p(t)$, such that

$$\dot{x} = H_p = -p^{-\mu} + y(x), \qquad \dot{p} = \rho p - H_x = p(\rho - y'(x)).$$

If $\bar{x}$ is a steady state level under optimal consumption, then it follows that

$$y'(\bar{x}) = \rho.$$

This is Ramsey's celebrated "golden rule": for concave production functions $y(x)$ the steady state level is independent of the specific form of the utility functional; therefore, under optimal consumption, technology, embodied in the production function, determines the long-run state of an economy.

For nonconcave production functions, as the one illustrated in Figure 1, this is no longer the case. First of all, there can be several golden rule steady states. In Figure 2, we give the optimal costate rule, the value function, and the associated dynamics under optimal consumption for three different values of $\mu$, keeping $\rho$ fixed.

In Figure 2(a) we sketch the optimal feedback rule, the optimal dynamics, as well as the value function for the situation $(\mu, \rho) = (1.1, 0.8)$. The value function has a point of nondifferentiability at $x = \hat{x}$. The points $\bar{x}_1$ and $\bar{x}_2$ are attracting steady state under optimal consumption. That is, all optimal state trajectories $x$ with $0 < x(0) < \hat{x}$ tend to $\bar{x}_1$, whereas those trajectories with $x(0) > \hat{x}$ tend to $\bar{x}_2$. In the terms of the model, if the initial capital stock is too low, that is, lower than the critical level $\hat{x}$, under optimal consumption the economy will end up in the low-capital, low-consumption golden rule state. This state is a "poverty trap" for the economy.

At $\hat{x}$, there are in fact two optimal actions $\hat{u}_L$ and $\hat{u}_R$, which are the left and right limits of $u^o(x)$ as $x$ tends to $\hat{x}$ from the left or from the right, respectively. We conclude that the optimal feedback rule and the optimal vector field are multivalued at $\hat{x}$, with

$$u^o(\hat{x}) = \{\hat{u}_L, \hat{u}_R\} \qquad f^o(\hat{x}) = \{f(\hat{x}, \hat{u}_L), f(\hat{x}, \hat{u}_R)\}.$$

Redefining

$$u^o(x) = \{U(x, V'(x))\} \qquad \text{and} \qquad f^o(x) = \{f(x, U(x, V'(x)))\}$$

for all $x \in D$, relations (1) and (2) are replaced by

$$u(t) \in u^o(x(t)) \qquad \text{and} \qquad \dot{x}(t) \in f^o(x(t))$$

for all optimal state-control pairs $(x(t), u(t))$ and all $t \geq 0$.

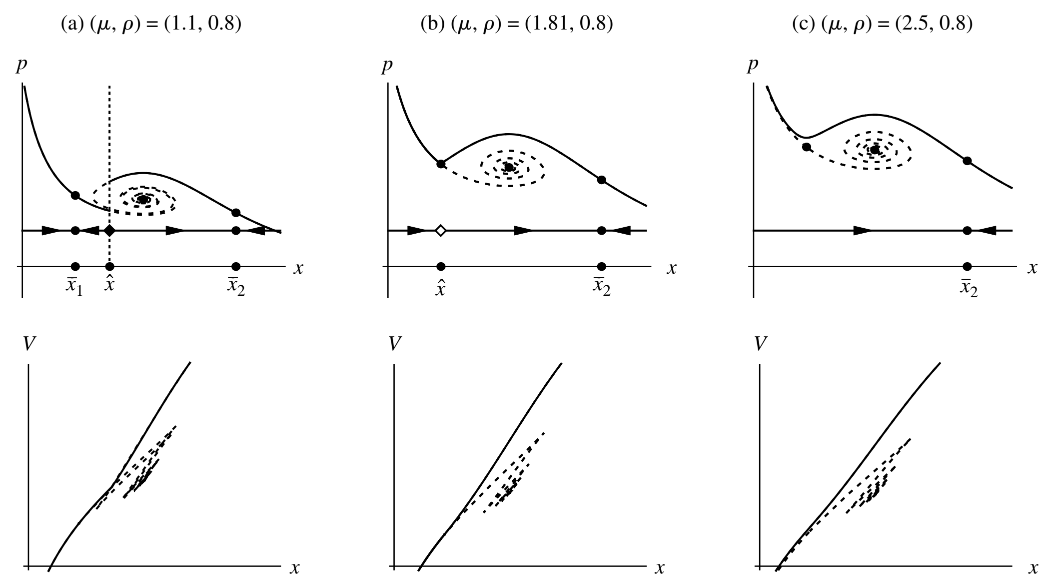



FIGURE 2. State-costate and optimal state dynamics (upper figures) and corresponding value function for Ramsey's problem. In the state-costate diagrams, thick solid curves indicate locally optimal trajectories, thin solid curves critical trajectories; in the value plots, they indicate the respective associated values. If the elasticity of intertemporal substitution $\mu$ is under the critical level $\hat{\mu} \approx 1.81$, initially poor economies are caught in a poverty trap; if it is above $\hat{\mu}$, the trap is avoided. The two situations are separated by an indifference-attractor bifurcation.

The point $\hat{x}$ where the optimal feedback rule is multivalued is an indifference point, as the social planner is indifferent between two alternatives: the total discounted utility $J$ is the same for both trajectories. The occurrence of indifference points in a problem implies that the eventual steady state depends on the initial state of the system, and that we cannot predict the long-time behaviour of the system by mere knowledge of the revenue functional and the state dynamics.

Value function, optimal feedback rule, and optimal vector field are sketched in Figure 2(c) for the situation $(\mu, \rho) = (2.5, 0.8)$. It turns out that here there is no indifference point, and all trajectories eventually end up in the high-capital, high-consumption state $\bar{x}_2$.

It is now of interest to determine the critical level $\hat{\mu}$ of the intertemporal elasticity of substitution, such that under optimal consumption, an economy with low initial capital level $x_0$ is caught in the poverty trap if $\mu < \hat{\mu}$, while it reaches the high-consumption state if $\mu > \hat{\mu}$.

The two situations differ in the qualitative characteristics of the optimal dynamics. The intertemporal elasticity of substitution $\mu$ measures the willingness of the planner to exchange present consumption for higher future consumption: if this willingness is low, more precisely, if $\mu < \hat{\mu}$, there are two attracting steady states, separated by the threshold $\hat{x}$; only if the initial capital levels of the economy is above $\hat{x}$, it will converge to the high-consumption state $\bar{x}_2$. If on the other hand the elasticity is sufficiently high, i.e., if $\mu > \hat{\mu}$, the economy ends up in the high-consumption state independently of the initial capital level.

That is, if the intertemporal elasticity of substitution is greater than the critical value $\hat{\mu}$, it is optimal for the economy to save itself out of the poverty trap.

At $\mu = \hat{\mu}$, the capital dynamics, given optimal decisions, changes qualitatively; or, in the terminology of dynamical system theory that will be used extensively in the following, the family of optimal vector fields parametrised by $\mu$ bifurcates at $\hat{\mu}$. Figure 2(b) illustrates the bifurcating situation, which occurs at $(\mu, \rho) \approx (1.81, 0.8)$. It turns out that there is only one generic mechanism, the *indifference-attractor bifurcation*, through which an indifference point and an attracting steady state of an optimal vector field can disappear, and this is illustrated by Figures 2(a)–2(c).

**2.2. Pollution management: rapid or gradual?** Ecosystems under stress, like the food web of a lake polluted by the runoff of artificial fertilizer used on the surrounding fields, may feature tipping points. These are stress levels at which the system state changes catastrophically (in the sense of Thom [35] and Zeeman [41]) to a different regime. A decision problem of a manager that has to weight the benefits generated by economic activity, like farming, and those generated by the lake has been formulated by Mäler et al. [24]. The system dynamics are

$$\dot{x} = u + \varphi(x), \qquad \varphi(x) = -bx + \frac{x^2}{x^2+1},$$

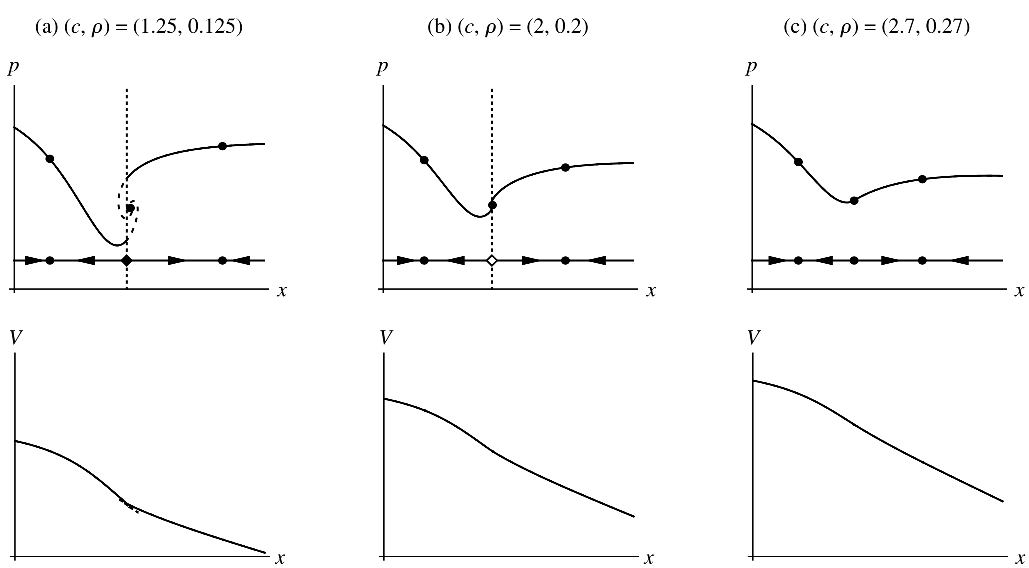



Figure 3. State-costate and optimal state dynamics (upper figures) and corresponding value function for the lake problem. Legend as in Figure 2. The state changes rapidly under optimal loading around the threshold in panel 3(a) and slowly in panel 3(c). These situations are separated by an indifference-repeller bifurcation (panel 3(b)).

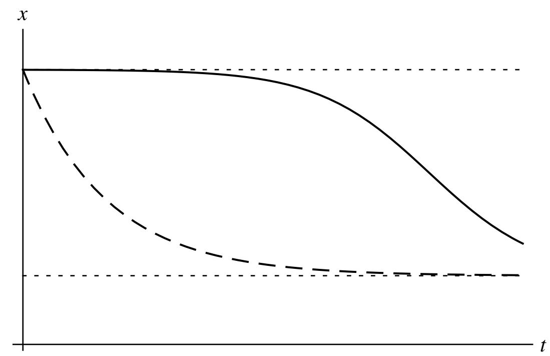



FIGURE 4. Difference between dynamics at an indifference point and a repeller. State evolutions starting near an indifference point (broken) and at a repeller (solid).

where $x$ measures the pollutant concentration in a lake, $u$ is the pollutant loading rate, and $\varphi$ the reaction function of the lake, which for small values of the sedimentation rate $b$ features a tipping point. A social planner is assumed to optimize

$$J = \int_0^\infty (\log u - cx^2) e^{-\rho t} \, dt,$$

subject to the control restriction $u > 0$. Here $\log u$ models the benefit derived from the economic activity and $-cx^2$ the benefit loss from the presence of pollutants in the lake. The parameter $c$ models the economic weight that is assigned to a clean ecosystem.

On the surface, there is a trade-off between increasing the parameter $\rho$ and increasing $c$; after an increase in $\rho$, more emphasis is placed on the short term, and the short-run gains from economic activity win over the long-run losses from pollution. An increase in $c$ compensates for this by giving more weight to pollution.

But when $\rho$ and $c$ are increased simultaneously, the qualitative nature of the dynamics under optimal loading changes. At $(c, \rho) \approx (2, 0.2)$, the optimal dynamics change in an *indifference-repeller bifurcation*, as the indifference point changes into a repeller, see Figure 3.

This entails an important quantitative difference for the state trajectories under optimal control, illustrated in Figure 4: as a repeller is an equilibrium state of the vector field, trajectories starting close to it tend to remain for a long time in its vicinity, before diverging away. Typically, at an indifference point $x = 0$, the state dynamics under optimal control is of the form $\dot{x} = 1$ if $x \geq 0$ and $\dot{x} = -1$ if $x \leq 0$, whereas at a repeller $x = 0$ it takes the form $\dot{x} = x$. If $x(0) = \delta > 0$ is the initial position of a trajectory $x(t)$, at the repeller the time $T_r$ needed for $x$ to double its distance from the repeller equals $T_r = \log 2$, whereas at the indifference point the "doubling" time $T_i$ equals $T_i = \delta$; we see that $T_i / T_r \to 0$ as $\delta \to 0$: the trajectory takes much more time to get away from the repeller than it does for getting away from the indifference point.

## 3. Setting.

**3.1. Definitions.** Let $X \subset \mathbb{R}$ be a closed interval and $U \subset \mathbb{R}^r$ a closed convex set with nonempty interior. Let $\rho > 0$ be a positive constant and $f: X \times U \to \mathbb{R}$, $g: X \times U \to \mathbb{R}$ be infinitely differentiable, or *smooth*, in the interior of $X \times U$, and such that all derivatives can be extended continuously to a neighbourhood of $X \times U$.

For an interval $I \subset \mathbb{R}$, denote the lower and upper boundary point, if they exist, by $x_l$ and $x_u$, and set $\partial I = \{x_l, x_u\}$. Then the *outward pointing "vector"* $\nu(x)$ is defined as

$$\nu(x_l) = -1, \qquad \nu(x_u) = 1. \tag{3}$$

The following is the first in a string of assumptions, which are assumed to be fulfilled for the rest of the article.

ASSUMPTION 3.1. *The vectors $f$ are assumed to be always inward pointing at $\partial X$; that is*

$$\nu(x) f(x, u) \leq 0 \tag{4}$$

*for all $x \in \partial X$ and all $u \in U$.*

Definition 3.1. An (admissible) *programme with initial state* $\xi$ is a pair $(\xi, u)$, where the *initial state* $\xi$ is a point in $X$, and the *control* is a function $u$: $[0, \infty) \to U$ that is locally Lebesgue integrable and essentially bounded; that is, bounded on a subset of a full measure of $[0, \infty)$.

A given programme $(\xi, u)$ determines an associated absolutely continuous *state trajectory* as the solution to the ordinary differential equation

$$\dot{x} = f(x, u), \tag{5}$$

for almost all $t \in [0, \infty)$, with initial value

$$x(0) = \xi. \tag{6}$$

State trajectories associated to admissible programmes are also called admissible.

The set of all admissible programmes with initial point $\xi$ is denoted by $\mathcal{A}_\xi$; the set of all programmes is denoted by $\mathcal{A} = \cup_\xi \mathcal{A}_\xi$.

Remark 3.1. Every locally Lebesgue integrable and essentially bounded function $u$ gives rise to a state trajectory $x$, as condition (4) insures that no such trajectory can leave $X$. A programme is therefore determined by a control and an initial state; a programme in $\mathcal{A}_\xi$ is determined solely by a control.

Remark 3.2. Note that any $x \in X$ such that $0 \notin f(x, U)$ is a threshold value that can be crossed at most once by any state trajectory.

Let $\rho > 0$ be a positive number. The *value* of a programme $(\xi, u)$ is the value, whenever it is defined, of the *objective functional*

$$J(\xi, u) = \int_0^\infty g(x(t), u(t)) \mathrm{e}^{-\rho t} \, \mathrm{d}t, \tag{7}$$

where $x$ is the state trajectory associated to the programme. Note that $J$ might be finite only for a subset of $\mathcal{A}$. As we are not interested in the existence problem by itself, but rather in the structure of the set of maximizers and its dependence on the problem parameters, we make the following assumption.

Assumption 3.2. *For all admissible controls, the integral (7) either exists and is finite, or it tends to $-\infty$. Moreover, for every initial state $\xi \in X$, the objective functional takes a finite maximum on $\mathcal{A}_\xi$.*

A 5-tuple $\mathcal{P} = (X, U, f, g, \rho)$, where $\rho > 0$, that satisfies Assumptions 3.1, 3.2 and 3.4, which will be formulated further below, determines the (family of) *infinite horizon problems* to find for every $\xi \in X$ a control $u^*$ such that $J(\xi, u) \le J(\xi, u^*)$ for all admissible controls $u$. A solution to this problem is a *maximizer* or an *optimal control*. An individual infinite horizon problem with initial state $\xi$ is denoted by $\mathcal{P}_\xi$.

Remark 3.3. If the control $u^*$ is such that $J$ takes its maximum at $(\xi, u^*)$, then this is called an optimal control, and $(\xi, u^*)$ an optimal programme. The associated state trajectory is also called optimal.

If $\tau \ge 0$ and $v$: $[0, \infty) \to \mathbb{R}^n$ a given function, let its $\tau$-time translate $v_\tau$ be given as

$$v_\tau(t) = v(\tau + t).$$

Optimal trajectories enjoy the following time invariance property, commonly known as the dynamic optimization principle, whose proof is straightforward.

Theorem 3.1. *Let $\mathcal{P}$ be a given infinite horizon problem. If the programme $(\xi, u)$ is optimal in $\mathcal{A}_\xi$, and if $x$ is its associated state trajectory, then for any $\tau \ge 0$, the programme $(x(\tau), u_\tau)$ is optimal in $\mathcal{A}_{x(\tau)}$.*

**3.2. The Hamilton-Jacobi equation for the value function.** A central notion in optimal control is the Hamilton-Jacobi differential equation; this is a nonlinear implicit first order partial differential equation with real characteristics, which is typical for hyperbolic equations like the wave equation or the Schrödinger equation. As always when a hyperbolic structure is present, solutions can be described either by rays or by waves. In the present context the former are the optimal trajectories, whereas the latter are the level sets of the optimal value function.

Let $\mathcal{P} = (X, U, f, g, \rho)$ be a given infinite horizon problem. Introduce

$$P(x, p, u) = g(x, u) + pf(x, u).$$

The following nondegeneracy assumption allows solving for interior optimal controls using the implicit function theorem.

Assumption 3.3. *For all $(x, p, u) \in X \times \mathbb{R} \times U$, the inequality*

$$\frac{\partial^2 P}{\partial u^2}(x, p, u) < 0 \tag{8}$$

*holds true.*

The (reduced) Hamilton function of the optimization problem $\mathcal{P}$ is specified as

$$H(x, p) = \max_{u \in U}\{g(x, u) + pf(x, u)\}.$$

Assumption 3.3 implies that this maximum is taken at a unique point

$$u = \kappa(x, p), \tag{9}$$

where $\kappa$ depends smoothly on its arguments if its value is in the interior of $U$.

The value of the infinite horizon problem $\mathcal{P}_\xi$ is defined as

$$V(\xi) = \max_{u \in \mathcal{A}_\xi} J(\xi, u).$$

The maximum exists by Assumption 3.2. If $V$ is continuously differentiable, it is a solution of the infinite horizon Hamilton-Jacobi equation

$$\rho V(x) - H(x, V'(x)) = 0; \tag{10}$$

cf. Carathéodory [6]. If the value function $V$ is not differentiable, it can still satisfy Equation (10) in the viscosity sense.

Definition 3.2. The function $V$ is a *viscosity subsolution* of the equation

$$G(x, V(x), V'(x)) = 0$$

at a point $\hat{x}$, if

$$G(x, \varphi(x), \varphi'(x)) \leq 0$$

for all differentiable functions $\varphi$ such that $\varphi(\hat{x}) = V(\hat{x})$ and $V(x) \leq \varphi(x)$ for all $x$ in some neighbourhood of $\hat{x}$.

The function $V$ is a *viscosity supersolution*, if

$$G(x, \varphi(x), \varphi'(x)) \geq 0$$

for all differentiable $\varphi$ such that $\varphi(\hat{x}) = V(\hat{x})$ and $V(x) \geq \varphi(x)$ in a neighbourhood of $\hat{x}$.

The function $V$ is a *viscosity solution* at $\hat{x}$, if it is both a viscosity supersolution and a viscosity subsolution there. It is a viscosity solution on $X$, if it is a viscosity solution at every point $x \in X$.

As announced earlier, all infinite horizon problems considered here are assumed to have value functions that solve the associated Hamilton-Jacobi equation in the viscosity sense.

Assumption 3.4. *The value function of $\mathcal{P}$ is continuous and the unique viscosity solution of the Hamilton-Jacobi Equation* (10) *associated to $\mathcal{P}$.*

If $u$ maximizes $J$, if $x$ is its associated state trajectory, then for almost all $t$ such that $V$ is differentiable at $x(t)$, we have $u(t) = \kappa(x(t), V'(x(t)))$.

**3.3. The (reduced) canonical vector field.** Dually to the Hamilton-Jacobi approach, optimal programmes can be represented in terms of state-costate trajectories $z(t) = (x(t), p(t))$. Precisely, if $x$ is an optimal state trajectory, there exists a costate trajectory $p$ such that $(x, p)$ satisfies the reduced canonical equations

$$\dot{x} = F_1(x, p) = H_p(x, p), \qquad \dot{p} = F_2(x, p) = \rho p - H_x(x, p), \tag{11}$$

which define the reduced canonical vector field $F = (F_1, F_2)$. Moreover, the state-costate pair $(x, p)$ satisfies the initial condition

$$x(0) = \xi \tag{12}$$

and the terminal "transversality" condition

$$\lim_{t \to \infty} \mathrm{e}^{-\rho t} p(\tilde{x} - x) \leq 0 \tag{13}$$

for all admissible state trajectories $\tilde{x}$. Trajectories of the reduced canonical vector field $F$ are called *extremal*. Extremal trajectories that additionally satisfy the boundary conditions (12) and (13) are called *critical*. The contents of the first order conditions of optimal control theory can then be expressed by stating that noncritical trajectories cannot be maximizers.

**3.4. Properties of the reduced canonical vector field.** Assumption (3.3) implies that the strong Legendre-Clebsch condition

$$H_{pp}(x, p) > 0 \tag{14}$$

holds for all $(x, p)$, which are such that $\kappa(x, p)$ is in the interior of $U$. Such a point $(x, p)$ is called *interior*. One of the implications of this condition is that eigenspaces of interior equilibria of the reduced canonical vector field are never vertical. More precisely, the following lemma holds.

LEMMA 3.1. *If $z = (x, p)$ is interior, all eigenvectors $v$ of $DF(z)$ can be written in the form $v = (1, w)$.*

PROOF. The lemma is implied by the statement that if $H_{pp}(z) \neq 0$, then $(0, 1)$ cannot be an eigenvector of

$$DF(z) = \begin{pmatrix} H_{px}(z) & H_{pp}(z) \\ -H_{xx}(z) & \rho - H_{px}(z) \end{pmatrix}.$$

This is easily verified. □

LEMMA 3.2. *If $z$ is interior and if $v_1 = (1, w_1)$ and $v_2 = (1, w_2)$ are two eigenvectors of $DF(z)$ with $\lambda_1 < \lambda_2$, then $w_1 < w_2$.*

PROOF. The first component of the vector equation $DF(z)v_i = \lambda_i v_i$ reads as

$$H_{px}(z) + H_{pp}(z) w_i = \lambda_i.$$

As $H_{pp}(z) > 0$, the lemma follows. □

The value of the objective $J$ over an extremal trajectory can be computed by evaluating the Hamilton function at the initial point (e.g., Skiba [33], Wagener [37]).

THEOREM 3.2. *Let $\mathcal{P}$ be an infinite horizon problem, let $z = (x, p)$ be a trajectory of its reduced canonical vector field with initial point $(\xi, \eta)$, and let $u(t) = \kappa(z(t))$ be the associated control function. If $\lim_{t\to\infty} H(z(t))e^{-\rho t} = 0$, then*

$$J(\xi, u) = \frac{1}{\rho} H(\xi, \eta).$$

The condition $He^{-\rho t} \to 0$ is known in economics as the "no Ponzi" condition; cf. also Michel [25].

In the situation that the state space $X$ is one-dimensional, the reduced canonical vector field $F$ defines a dynamical system in the plane. As it has the special property that

$$\operatorname{div} F = \rho > 0, \tag{15}$$

it follows that the phase flow of $F$ expands all regions at rate $\rho > 0$. In particular, there are no invariant regions with positive surface measure. This has immediate implications for the geometry of the possible phase curves of the system: it is impossible that any phase curve is closed, or that it would form a homoclinic loop, or that several phase curves would form a heteroclinic loop, as in all of these situations there is an invariant region in phase space with positive surface measure.

Moreover, the possible bifurcations of the dynamical system defined by $F$ are restricted as well: both Hopf and homoclinic bifurcations are impossible, as their occurrence implies that for some parameter values there are invariant regions with positive measure. Bifurcations like the saddle-node bifurcation or the heteroclinic bifurcation are not ruled out. It is however important to note that not all bifurcations of the reduced canonical vector field necessarily involve the optimal trajectories.

**3.5. Indifference points and the optimal vector field.** A central difference between "ordinary" vector fields and optimal vector fields that result from finding maximizers of an objective functional of the form (7) is the occurrence of indifference points for optimal vector fields.

DEFINITION 3.3. Let $\mathcal{P}$ be an infinite horizon problem. If $\xi \in X$ is such that there are two optimal controls $u_1, u_2 \in \mathcal{A}_\xi$, such that the associated state trajectories $x_1$ and $x_2$ satisfy $x_1(t) \neq x_2(t)$ for some $t \in [0, \infty)$, then $\xi$ is called an *indifference point*. The totality of indifference points form the *indifference set*; its complement in $X$ is the *domain of uniqueness*.

The existence of situations where the maximization problem has several solutions motivates the following definitions. Recall that if $S$ is a set, the *power set* $2^S$ of $S$ is the set of all subsets of $S$.

Definition 3.4. The *optimal costate rule* of a problem $\mathcal{P}$ is the set valued map $p^o$: $X \to 2^{\mathbb{R}}$ with the property that if $\eta \in p^o(\xi)$, then the trajectory $z = (x, p)$ of the reduced canonical vector field with initial value

$$z(0) = (\xi, \eta)$$

gives rise to an optimal control

$$u(t) = \kappa(z(t)).$$

Associated to the optimal costate rule are the *optimal feedback rule*

$$u^o(x) = \kappa(x, p^o(x)),$$

and the *optimal* (*state*) *vector field*

$$f^o(x) = H_p(x, p^o(x)) = f(x, u^o(x)),$$

which are both set valued as well.

Remark 3.4. In regions of differentiability of the value function $V$, the optimal costate rule is given as

$$p^o(x) = \{V'(x)\}.$$

Remark 3.5. A curve $x$: $[0, \infty) \to X$ is a trajectory of an optimal vector field if

$$\dot{x}(t) \in f^o(x(t))$$

for all $t \geq 0$. Such a trajectory is optimal.

Remark 3.6. At an indifference point $\hat{x}$, the set $f^o(\hat{x})$ contains two elements.

Remark 3.7. The optimal vector field is closely related to the "regular synthesis" of an optimal control problem (Boltyanskii [4]), which is a single-valued vector field on state space whose trajectories are optimal. A regular synthesis is a selection from the optimal vector field; the latter preserves more information about the optimal trajectories, at the price of not being single valued.

The following theorem is true in all dimensions, but it has special significance in the one-dimensional context.

Theorem 3.3. *Let $x$ be an optimal state trajectory of an infinite horizon problem $\mathcal{P}$. The sets $p^o(x(t))$ and $f^o(x(t))$ are single valued for all $t > 0$.*

Proof. See Fleming and Soner [14], p. 44, corollary I.10.1. □

An immediate corollary of this theorem is the following result.

Theorem 3.4. *Let $\mathcal{P}$ be an infinite horizon problem with one-dimensional state space. Then the indifference points of $\mathcal{P}$ are isolated.*

Proof. Let $\xi$ denote the indifference point. If one of the two orbits originating there is constant, its initial costate $\eta_1$ is a critical point of the strictly convex function $p \mapsto H(\xi, p)$, and hence a global minimizer of this function. As the second orbit has a different initial costate $\eta_2$, the value of $H$, and hence, by Theorem 3.2, the value of $J$ for that orbit is strictly larger. This rules out constant orbits originating at $\xi$.

If the two orbits originating at $\xi$ are both increasing or both decreasing, one has to be a time-translated version of the other. As they have the same initial point, they coincide.

The only situation that remains is that one trajectory is increasing and the other decreasing, and that their union covers a neighbourhood of $\xi$. By Theorem 3.3, there cannot be another indifference point located in this neighbourhood. □

Dynamical systems on a one-dimensional state space that are defined by a vector field have two kinds of generic singularities: hyperbolic attractors and hyperbolic repellers. Both are equilibria, and the knowledge of these special points is sufficient to reconstruct the flow of the system qualitatively. Additionally to these, an optimal vector field may have indifference points as well. Again, the knowledge of the equilibria and the indifference points suffices to reconstruct the qualitative features of behaviour of the optimal state trajectories.

In one-dimensional problems an indifference point is an initial point of two trajectories that have necessarily different long-run behaviour. This is not necessarily true for problems with higher dimensional state spaces, or for discrete time problems (see, e.g., Moghayer and Wagener [26]).

**4. Bifurcations of optimal vector fields.** The analysis of bifurcations of a parameterized family of optimal vector fields is performed in terms of the reduced canonical vector field, but it is worthwhile to point out that the latter is an auxiliary construct.

The optimal vector field defines a continuous time evolution on the state space. When the state space is one-dimensional, the evolution has certain special properties: trajectories sweep out intervals that are bounded by optimal attractors and optimal repellers or indifference points. At a bifurcation, the qualitative structure of the set of all trajectories changes. For instance, in a saddle-node bifurcation, an attractor and a repeller coalesce and disappear, together with the trajectory that joins them. Analogously, in an indifference-attractor bifurcation, an indifference point and an attractor coalesce and disappear, again together with the trajectory joining them. It is clearly impossible that a repeller and an indifference point coalesce, as they cannot be joined by a trajectory: there is necessarily another singularity contained in the interval of which they are the boundary points. There is however a third possible bifurcation scenario: a repeller may change into an indifference point. This also modifies the structure of the set of trajectories, for the equilibrium trajectory that remains in the repelling state for all time has no equivalent in the situation with the indifference point.

The indifference-attractor bifurcation and the different kinds of indifference-repeller bifurcations that we have just sketched out have no counterpart in the theory of dynamical systems: they are typical for dynamic optimization problems. Instances of indifference-attractor bifurcations have been analyzed in Wagener [37, 39], where they were linked to heteroclinic bifurcations of the reduced canonical vector field.

The present paper generalizes these results by providing a semilocal analysis around the singularities of the reduced canonical vector field that are involved in the bifurcation in question. Indifference-attractor bifurcations that do not involve heteroclinic connections (as occurring in, e.g., Hinloopen et al. [20]) are captured by this more general approach.

**4.1. Notions from dynamical system theory.** We need several notions from the geometrical theory of dynamical systems. The solution flows of two vector fields are considered to be qualitatively the same, if the first is a continuous distortion of the second. Precisely, two vector fields are *topologically conjugate*, if all trajectories of the first can be mapped homeomorphically onto trajectories of the second; that is, by a continuous invertible transformation whose inverse is continuous as well.

Let $F$ be a vector field that is defined on an open subset of some Euclidean space. An equilibrium $\bar{z}$ of $F$ is called *hyperbolic*, if no eigenvalue of $DF(\bar{z})$ is situated on the imaginary axis. The sum of the generalized eigenspaces associated to the hyperbolic eigenvalues is the hyperbolic eigenspace $E^h$, which can be written as the direct sum of the stable and unstable eigenspaces $E^s$ and $E^u$, associated to the stable and unstable eigenvalues, respectively. The sum of the eigenspaces associated to the eigenvalues on the imaginary axis is the neutral eigenspace $E^c$. The center-unstable and center-stable eigenspaces $E^{cu}$ and $E^{cs}$ are the direct sums $E^c \oplus E^u$ and $E^c \oplus E^s$, respectively.

The center manifold theorem (see Hirsch et al. [21]), ensures the existence of manifolds that are invariant under the flow of the dynamical system and that are tangent to the stable and unstable eigenspaces. The simplest instance of this is a saddle steady state in the plane, where the linearization of the vector field around the steady state has a stable eigenvalue $\lambda^s$ and an unstable eigenvalue $\lambda^u$ with $\lambda^s < 0 < \lambda^u$, and associated eigenspaces $E^s$ and $E^u$. Then the center manifold states that there are two invariant curves, the stable manifold $W^s$ and the unstable manifold $W^u$, that are tangent to the corresponding eigenspaces at the saddle. All trajectories on $W^s$ tend toward the saddle as time goes to infinity; all trajectories on $W^u$ tend toward the saddle as time goes to minus infinity. In the two-dimensional situation, these manifolds are also known as separatrices of the solution flow. The center manifold theorem generalizes this situation.

Theorem 4.1 (Center Manifold Theorem). *Let $F$ be a $C^k$ vector field on $\mathbb{R}^m$, $k \geq 2$, and let $F(\bar{z}) = 0$. Let $E^u$, $E^s$, $E^c$, $E^{cu}$ and $E^{cs}$ denote the generalized eigenspaces of $DF(\bar{z})$ introduced above. Then there are $C^k$ manifolds $W^s$ and $W^u$ tangent to $E^s$ and $E^u$ at $\bar{z}$, and $C^{k-1}$ invariant manifold $W^c$, $W^{cu}$ and $W^{cs}$ tangent to $E^c$, $E^{cu}$ and $E^{cs}$, respectively, at $\bar{z}$. These manifolds are all invariant under the flow of $F$; the manifolds $W^s$ and $W^u$ are unique, whereas $W^c$, $W^{cu}$ and $W^{cs}$ need not be.*

Invariant manifolds can be used to choose convenient coordinates around an equilibrium point of a vector field. For instance, let $F(0) = 0$, let $E^1$ and $E^2$ be two linear subspaces, both invariant under $DF(0)$, such that

$$E^1 \oplus E^2 = \mathbb{R}^m,$$

and let $W^1$ and $W^2$ be two invariant manifolds that are locally around 0 parameterized as the graphs of functions

$$w^1 \colon E^1 \to E^2, \qquad w^2 \colon E^2 \to E^1,$$

satisfying $Dw^1(0) = 0$, $Dw^2(0) = 0$. For a sufficiently small neighbourhood $N$ of 0 and for $(z_1, z_2) \in U \subset E^1 \times E^2$, define the coordinate transformation

$$(\zeta_1, \zeta_2) = (z_2 - w^1(z_1), z_1 - w^2(z_2)). \tag{16}$$

In the new coordinates, the vector field has the form

$$F(\zeta) = \begin{pmatrix} A_1\zeta_1 + \zeta_1\varphi_1(\zeta) \\ A_2\zeta_2 + \zeta_2\varphi_2(\zeta) \end{pmatrix}, \tag{17}$$

where $\varphi_i(\zeta) \to 0$ as $\zeta \to 0$.

For a hyperbolic equilibrium of a vector field on the plane, a much stronger result is available, the $C^1$ linearization theorem of Hartman (cf. Hartman [17, 18]; Palis and Takens [27]).

Theorem 4.2 (Hartman's $C^1$ Linearization Theorem). *Let $F$: $\mathbb{R}^2 \to \mathbb{R}^2$ be a $C^2$ vector field in the plane, and let $z = 0$ be a hyperbolic equilibrium of $F$. Then there is a neighbourhood $N$ of 0 and coordinates $\zeta$ on $N$, such that*

$$F(\zeta) = DF(0)\zeta$$

*in these coordinates.*

We illustrate this theorem by using it to introduce the notion of strong stable and strong unstable manifold of a planar vector field (see Homburg et al. [22] for a general definition). Let $F$ be a $C^2$ vector field in the plane, such that $z = 0$ is an equilibrium of $F$ with eigenvalues $0 < \lambda^u < \lambda^{uu}$. The eigenvalue $\lambda^{uu}$ is called the strong stable eigenvalue. Then the *strong stable manifold* $W^{uu}$ is a $C^1$ curve that is invariant under $F$ and tangent to $E^{uu}$ at $z = 0$. To show that such a curve exists, we invoke Hartman's theorem.

We may assume that $DF(0) = \left(\begin{smallmatrix} \lambda^u & 0 \\ 0 & \lambda^{uu} \end{smallmatrix}\right)$ is already diagonal; otherwise we perform a linear coordinate transformation to achieve this. Let $\varphi$: $\mathbb{R}^2 \to \mathbb{R}^2$ be the $C^1$-linearizing coordinate transformation. Then $\varphi$ is a (local) $C^1$ diffeomorphism: there is a neighbourhood of $z = 0$ such that $\varphi$ is invertible and such that its inverse is $C^1$ as well. In the present situation $\varphi(z) = z + O(|z|^2)$, as the linear part of the vector field is not changed by the transformation. In the new coordinates, the flow of $F$ takes the form

$$\dot{\zeta}_1 = \lambda^u\zeta_1, \qquad \dot{\zeta}_2 = \lambda^{uu}\zeta_2,$$

and the line $E^{uu}$ spanned by the vector $(0, 1)$ is invariant. Then the curve $\varphi^{-1}(E^{uu})$ is $C^1$, invariant under the original vector field $f$, and tangent to $E^{uu}$ at $z = 0$.

As is well known, the natural (but abstract) description of state-costate space $X \times \mathbb{R}$ is its interpretation as the cotangent bundle $T^*X$, which is a vector bundle with base $X$ and natural projection $\pi$: $T^*X \to X$. In the present context, using the full formalism is excessive; but the mathematical structure is there and plays an important role, as an optimal costate rule is a (multivalued) section of $T^*X$. To replace the notion of "fibres of $T^*X$," we introduce the notion of "vertical leaves."

Definition 4.1. Let $\pi$: $X \times \mathbb{R} \to X$ be the projection $\pi(x, p) = x$. The *vertical leaf* $L_x \subset X \times \mathbb{R}$ through the point $x$ is the set

$$L_x = \pi^{-1}(x) = \{x\} \times \mathbb{R}. \tag{18}$$

Theorem 4.3. *The invariant manifolds $W^s$ and $W^u$ of an interior saddle $e$ of the reduced canonical vector field $F$ can be represented as the graphs $p = w^s(x)$ and $p = w^u(x)$ of smooth functions $w^s$ and $w^u$.*

Proof. At $e$, the invariant manifolds $W^s$ and $W^u$ are tangent to the stable and unstable eigenspaces $E^s$ and $E^u$, respectively; these intersect all vertical leaves $L_x$ transversally by Lemma 3.1. Therefore, if $z(s) = (x(s), p(s))$ is a smooth parameterization of one of these manifolds with $z(0) = e$, then $x'(0) \neq 0$. Denote by $s = s(x)$ the inverse of $x = x(s)$. Consequently

$$w(x) = p(s(x))$$

is well defined and smooth in a neighbourhood of $x_e$. □

We need two notions from the geometrical theory of dynamical systems. The first is the unfolding of an object $X$: a family $X_\mu$, continuously or differentiably depending on a parameter $\mu$, *unfolds* $X$ if $X_0 = X$.

The second is the notion of "vertical vector field": if the vector field $F_\mu$ defines a family of dynamical systems

$$\dot{x} = F_\mu(x),$$

then equivalently we can interpret the parameter $\mu$ as a state variable, and consider the single *vertical vector field* $(F_\mu, 0)$ defining

$$\dot{x} = F_\mu(x), \qquad \dot{\mu} = 0. \tag{19}$$

Note that the name vertical vector field is more natural if the parameter is written first, as we then obtain $(0, F_\mu)$.

**4.2. Optimal vector fields with isolated singularities.** We classify the behaviour of an optimal vector field $f^o$ around a given point $x_0$. If there is a neighbourhood $N$ of $x_0$ such that $f^o(x) = \{f(x)\}$ for all $x \in N$, then $f^o$ is *univalued around* $x_0$ and *represented by* $f$. If moreover $f(x) \neq 0$ for all $x \in N$, then $x_0$ is a *regular* point; if $f(x_0) = 0$ and $f'(x_0) < 0$, then $x_0$ is a (hyperbolic) attractor; if $f(x_0) = 0$ and $f'(x_0) > 0$, a (hyperbolic) repeller. If the set $f^o(x_0)$ contains two elements, then $x_0$ is an indifference point. Points $x_0$ that fall, for a given $f^o$, into one of these categories are *stable* or *nonbifurcating*; all other points are *bifurcating*. An optimal vector field containing only stable points is a *stable vector field*.

We have the following result about the regularity of the value function that corresponds to a stable optimal vector field.

Theorem 4.4. *Let $\mathcal{P} = (X, U, f, g, \rho)$ be a given infinite horizon problem with value function $V$, let $N \subset X$ be an open interval, and let $f^o$ be the optimal vector field of $\mathcal{P}$. Assume that $(x, V'(x))$ is interior for all $x$.*

(i) *If $f^o$ has an indifference point in $N$, then $V$ is Lipschitz continuous in $N$.*

(ii) *If $N$ contains only regular points, or at most one hyperbolic attractor of $f^o$, then $V$ is smooth in $N$.*

(iii) *Let $N$ contain only regular points and a single hyperbolic repeller $x_e$ of $f^o$. Set $e = (x_e, V'(x_e))$ and let $F$ be the reduced canonical vector field (11); then $F(e) = 0$ and the eigenvalues $\lambda_1, \lambda_2$ of $DF(e)$ are real; choose the indices such that $\lambda_1 \leq \lambda_2$.*

(a) *If $\lambda_1 < 0$, then $V$ is smooth in $N$.*

(b) *If $\lambda_1 > 0$, then $V \in C^{1+\beta-\varepsilon}(N)$ for every $\varepsilon > 0$, where $\beta = \lambda_2/\lambda_1$.*

Remark 4.1. If $s > 0$, a function $v$ is said to be $C^s$ Hölder regular, notation $v \in C^s$, if $v \in C^k$, with $k$ the largest integer smaller than or equal to $s$, and if $v^{(k)}$ satisfies a Hölder condition with Hölder exponent $s - k$.

Proof. Consider first the situation that all points in $N$ are regular, and that $f^o$ is represented by $f$ there. By regularity, we have for $x \in N$ that

$$H_p(x, V'(x)) = f(x) \neq 0,$$

and consequently the Hamilton-Jacobi Equation (10) can be solved for $V'$ on $N$, yielding

$$V' = \Phi(x, V),$$

where $\Phi$ is smooth. It follows that the solution $V$ of this explicit ordinary differential equation is smooth as well.

Let then $\xi$ be an indifference point. It follows from Theorem 3.4 that there is a neighbourhood $N$ of $\xi$, not containing $\xi$, such that $p^o$ is represented by the smooth function $p = V'$ on $N$, and such that $p$ has a jump discontinuity at $\xi$. This implies that the function $V$ is Lipschitz.

Consider now the situation that $N$ contains additionally to regular points a hyperbolic attracting point $x_e$. Set $p(x) = V'(x)$ and $e = (x_e, p(x_e))$; the graph of $p$ in the state-costate space is traced out by solution curves of the reduced canonical vector field $F$. In particular

$$\dot{x} = f(x) = H_p(x, p(x)),$$

and as $x_e$ is an attracting point of $f$, the point $e$ is an equilibrium of $F$ with a nonempty stable manifold. As the product of the eigenvalues of $DF(e)$ equals $\rho > 0$, the point $e$ is a saddle. But the stable manifold of a saddle of a smooth vector field is smooth (Hirsch et al. [21]). Moreover, it can be represented by the graph of a smooth function (Theorem 4.3). As the stable manifold coincides with the graph of $p$, it follows that $p$, and hence $V$, is smooth as well.

Finally, we consider the situation that $N$ contains a single hyperbolic repeller $x_e$ as the only nonregular point. This is by far the most laborious case. If the eigenvalues of $DF(e)$ satisfy $\lambda_1 < 0 < \lambda_2$, then $e$ is a saddle point, and the graph of $p(x) = V'(x)$ is necessarily contained in the unstable manifold of $e$, which is smooth and can be represented as the graph of a smooth function.

Using the smooth transformation (16), we may assume that around $e$ the state-costate system takes the form

$$\dot{y}_1 = \lambda_1 y_1 + y_1 \varphi_1(y_1, y_2), \qquad \dot{y}_2 = \lambda_2 y_2 + y_2 \varphi_2(y_1, y_2), \tag{20}$$

with $|\varphi_i(y)| \le C|y|$; the equilibrium $e$ has been transported to the origin $y=0$ by the coordinate change. Let $y_2 = p(y_1)$ be the graph of $p$ in these coordinates: it consists of three trajectories of (20), the origin and two curves $y^1$ and $y^2$ that are both asymptotic to the origin as $t \to -\infty$.

The invariance condition $\dot{y}_2 = p'(y_1)\dot{y}_1$, after making the substitutions $y_1 = y$ and $y_2 = p(y)$, is equivalent to

$$(\lambda_1 y + y\varphi_1(y,p))p' = \lambda_2 p + p\varphi_2(y,p). \tag{21}$$

This is an implicit differential equation for $p$, with singularity locus $y=0$. We claim that its solutions are of the form $y^{\lambda_2/\lambda_1}q(y)$, where $q$ is continuous except for $y=0$, where it has possibly a jump discontinuity.

Dividing Equation (21) by $\lambda_1 p$ yields for $y \neq 0$

$$\left(1+\frac{\varphi_1}{\lambda_1}\right)\frac{yp'}{p} = \frac{\lambda_2}{\lambda_1} + \frac{\varphi_2}{\lambda_1}.$$

This is equivalent to

$$\frac{yp'}{p} = \beta + \psi(y,p),$$

where $\beta = \lambda_2/\lambda_1$, and where $\psi = (\varphi_2 - \beta\varphi_1)/(1+\varphi_1)$ is a smooth function such that $\psi(0,0)=0$. Write

$$\psi(y,p) = y\psi_1(y,p) + p\psi_2(y,p),$$

using Hadamard's lemma, with $\psi_1$ and $\psi_2$ smooth functions, and introduce a new dependent variable $q(y)$ by $p = y^\beta q$. The equation for $q$ then reads as

$$q' = \frac{\psi}{y}q = (\psi_1 + y^{\beta-1}q\psi_2)q.$$

This is an explicit differential equation for $q$, whose right-hand side is smooth if $y \neq 0$ and continuous at $y=0$. The curves $z_1$ and $z_2$ therefore correspond to bounded and continuous solutions $q_1$ and $q_2$ of this equation, which are defined for $y \le 0$ and $y \ge 0$, respectively. We conclude that $p$ is given as

$$p(y) = \begin{cases} y^\beta q_1(y) & \text{if } y<0, \\ y^\beta q_2(y) & \text{if } y>0, \\ 0 & \text{if } y=0. \end{cases}$$

As $p$ is $C^{\beta-\varepsilon}$ Hölder regular, for every $\varepsilon > 0$, this proves the final claim of the theorem. □

**4.3. Codimension one bifurcations.** We now come to the main part of the article: in this subsection, we discuss the codimension one bifurcations of optimal vector fields; in the next, bifurcations of higher codimension are treated.

A codimension-one bifurcation is a bifurcation that cannot be avoided in a one-parameter family of systems. There are three such bifurcations: indifference repeller, indifference attractor, and saddle node. It turns out that there are two configurations of the state-costate system that can give rise to indifference-repeller bifurcations of the optimal vector field; they are referred to as type 1 and type 2, respectively.

A general remark on notation: the codimension of a bifurcation will be denoted by a subscript, whereas the type is indicated, if necessary, by additional information in brackets. For instance, the abbreviation $\mathrm{IR}_1(2)$ denotes a codimension one indifference repeller bifurcation of type 2.

#### 4.3.1. $\mathrm{IR}_1$(1) bifurcation.

Definition 4.2. An interior point $e=(x_e, p_e)$ is a (codimension one) *indifference repeller singularity of type* 1, notation $\mathrm{IR}_1(1)$, of an optimization problem $\mathcal{P}$ with reduced canonical vector field $F$, if for all sufficiently small compact interval neighbourhoods $N$ of $x_e$ in $X$ there exists a function $p_0$: $N \to \mathbb{R}$, such that the following conditions hold.

(i) The point $e$ is an equilibrium of $F$ such that the eigenvalues $\lambda^u$, $\lambda^{uu}$ of $DF(e)$ satisfy $0 < \lambda^u < \lambda^{uu}$.

(ii) We have $p_e = p_0(x_e)$ and $p_0$ represents $p^o$ on $N$, that is,

$$p^o(x) = \{p_0(x)\} \quad \text{for all } x \in N.$$

(iii) Let $W^{uu}$ denote the strong unstable manifold of $F$ at $e$, parameterized as the graph of $w\colon N \to \mathbb{R}$. Also, let $\nu(x)$ be the outward pointing vector of $N$. There is exactly one $\bar{x} \in \partial N$ such that

$$p_0(\bar{x}) = w(\bar{x}), \tag{22}$$

whereas for $x \in \partial N$ and $x \neq \bar{x}$, we have that

$$\nu(x)(p_0(x) - w(x)) < 0. \tag{23}$$

The definition is illustrated in Figure 6(b).

Theorem 4.5. *Consider a family $\mathcal{P}_\mu$ of optimization problems, depending on a parameter $\mu \in \mathbb{R}^q$, with value functions $V_\mu$ and reduced canonical vector fields $F_\mu$, such that $\mathcal{P}_0$ has an $IR_1(1)$ singularity $e_0$ with associated costate function $p_0\colon N \to \mathbb{R}$. Assume that there is a neighbourhood $\Gamma \subset \mathbb{R}^q$ of 0 such that the following conditions hold.*

(i) *If $e_\mu$ is the family of equilibria of $F_\mu$ unfolding $e_0$, then the eigenvalues of $DF_\mu(e_\mu)$ satisfy $0 < \lambda^u_\mu < \lambda^{uu}_\mu$ for all $\mu \in \Gamma$.*

(ii) *Let the strongly unstable manifold $W^{uu}_\mu$ of $e_\mu$ be parameterized as the graph $p = w_\mu(x)$ of a differentiable function $w_\mu\colon N \to \mathbb{R}$. Let $\bar{x}$ be such that $p^o(\bar{x}) = \{w_0(\bar{x})\}$. There is a function $d_\mu\colon \partial N \to \mathbb{R}$, differentiable in $\mu$, such that*

$$p^o_\mu(x) = \{d_\mu(x)\}$$

*for all $x \in \partial N$ and all $\mu \in \Gamma$. The function*

$$\alpha(\mu) = \nu(\bar{x})(d_\mu(\bar{x}) - w_\mu(\bar{x})),$$

*is defined on $\Gamma$ and satisfies*

$$\alpha(0) = 0 \qquad \text{and} \qquad D\alpha(0) \neq 0.$$

*Then the optimal vector field $f^o_\mu$ restricted to $N$ is for $\alpha(\mu) < 0$ topologically conjugate to*

$$Y(x) = \{x\},$$

*whereas for $\alpha(\mu) > 0$ it is conjugate to*

$$Y(x) = \begin{cases} \{-1\}, & x < 0, \\ \{-1, 1\}, & x = 0, \\ \{1\}, & x > 0. \end{cases}$$

*Moreover, if $\alpha(\mu) < 0$, then for every $\varepsilon > 0$, the value function $V_\mu \in C^{1+\beta-\varepsilon}(N)$, with $\beta = \lambda^{uu}/\lambda^u > 1$; if $\alpha(\mu) = 0$, then $V_\mu \in C^1$, and $V'_\mu$ is Lipschitz continuous on $N$; finally, if $\alpha(\mu) > 0$, then $V_\mu$ itself is Lipschitz continuous on $N$.*

Remark 4.2. The existence of a point $\bar{x}$ with the property stated in the theorem is guaranteed by the definition of $IR_1(1)$ singularity.

Remark 4.3. The equality $\alpha(0) = 0$ is a direct consequence of condition (22).

The bifurcation diagram is given in Figure 5. Figure 6 illustrates the contents of the theorem: shown is a neighbourhood of a repelling equilibrium of the reduced canonical vector field in $(x, p)$-coordinates. Thin dashed lines indicate linear unstable eigenspaces of the equilibrium; the strongly unstable eigenspace corresponds to the line with the largest gradient. Approaching the equilibrium are two trajectories of the reduced canonical vector field. The solid part of these curves are optimal.

Also shown in the first row of Figure 6 is the optimal state dynamics, which has for $\alpha(\mu) \leq 0$ a repeller, indicated by a black dot, and for $\alpha(\mu) > 0$ an indifference point, indicated by a black diamond and a vertical dotted line. The graphs in the second row give the values of $J$ that are associated to the orbits of the reduced canonical vector field. The solid part of these curves correspond to the value function.

At the bifurcation, the relative position of the optimal trajectories and the strongly unstable manifold changes: for $\alpha(\mu) < 0$ the backward extension of the optimal trajectories are tangent to $E^u$ at either side of the equilibrium. This ensures that the equilibrium itself corresponds to an optimal repeller. For $\alpha(\mu) > 0$, the backward extensions are tangent to $E^u$ at the same side of the equilibrium. One of them necessarily intersects the line $x = x_e$, which implies that $e$ cannot be an optimal trajectory.

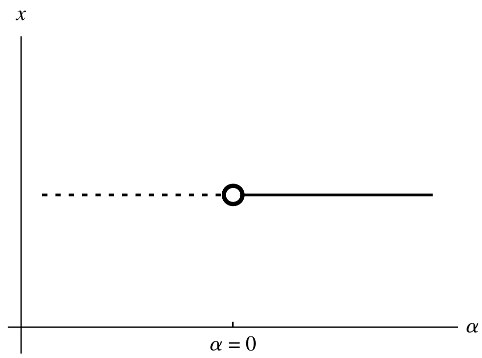



Figure 5. Bifurcation diagram of the indifference-repeller bifurcations $\mathrm{IR}_1(1)$ and $\mathrm{IR}_1(2)$. A repelling equilibrium that exists for $\mu < 0$ (dotted line) changes to an indifference point for $\mu > 0$ (solid line).

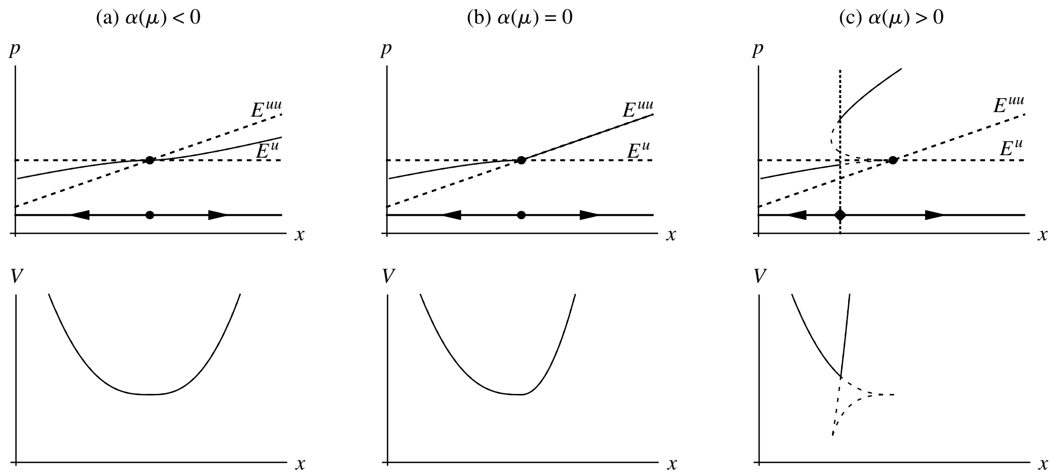



Figure 6. Before, at, and after the indifference-repeller bifurcation point. Upper row: trajectories of the state-costate system and the corresponding optimal state dynamics. Lower row: value functions. Note that for $\alpha < 0$, the value function is nondifferentiable at the indifference point; for $\alpha = 0$, it is differentiable, but not twice differentiable; for $\alpha > 0$, the degree of differentiability of the value function depends on the ratio of the eigenvalues.

Proof. Let $E^{uu} = \mathbb{R}v^{uu}$ and $E^u = \mathbb{R}v^u$ be the eigenspaces spanned by the eigenvectors $v^{uu} = (1, w^{uu})$ and $v^u = (1, w^u)$ of $DF(e)$ corresponding to the eigenvalues $\lambda^{uu}$ and $\lambda^u$, respectively. Note that $w^{uu} > w^u$ as a consequence of Lemma 3.2.

Hartman's linearization theorem then implies that there is a neighbourhood of $e$ and a $C^1$ diffeomorphism $\zeta = \zeta(z)$, with inverse

$$z = z(\zeta) = (x(\xi, \eta), p(\xi, \eta)),$$

mapping $e$ to 0, such that the linear map $D\zeta(0)$ maps $v^{uu}$ to $(1, 0)$ and $v^u$ to $(0, -1)$ and such that in these coordinates the vector field $F$ takes the form

$$\dot{\zeta} = \begin{pmatrix} \lambda^{uu} & 0 \\ 0 & \lambda^u \end{pmatrix} \zeta. \tag{24}$$

As a consequence of these choices, the map $\zeta$ as well as its inverse are orientation preserving; in particular

$$\det Dz > 0.$$

Moreover

$$x_\xi^0 = x_\xi(0, 0) = \pi_1 Dz(0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \pi_1 v^{uu} = 1 > 0,$$

$$x_\eta^0 = x_\eta(0, 0) = \pi_1 Dz(0) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\pi_1 v^u = -1 < 0,$$

where $\pi_1$: $\mathbb{R}^2 \to \mathbb{R}$ denotes the projection on the first component, and where $x_\xi$ denotes partial derivation with respect to $\xi$ etc. By continuity, there is a neighbourhood of the origin such that

$$x_\xi > 0 \qquad \text{and} \qquad x_\eta < 0 \tag{25}$$

on this neighbourhood.

Let $\bar{x}_i$, $i = 1, 2$ be such that

$$N = [\bar{x}_1, \bar{x}_2], \tag{26}$$

and set

$$\bar{z}_i = (\bar{x}_i, \bar{p}_i) = (\bar{x}_i, p(\bar{x}_i, \mu)) \tag{27}$$

and $\zeta_i = \zeta(z_i)$.

Trajectories starting at $\bar{z}_1$ and $\bar{z}_2$ are optimal. Moreover, their time-forward parts leave the region $N \times \mathbb{R}$ eventually. Any optimal trajectory within this region is either equal to the equilibrium $e$, or it leaves this region and necessarily passes through one of the points $\bar{z}_1$ or $\bar{z}_2$. Let $z_1$ and $z_2$ be therefore the trajectories of the reduced canonical vector field that are such that $z_i(0) = \bar{z}_i$ for $i = 1, 2$. The equilibrium $e$ and the time-backward parts of $z_1$ and $z_2$ are the only candidates for optimal trajectories in the region $N \times \mathbb{R}$.

We only consider the situation that $\bar{x} = \bar{x}_2$, the proof in the other case being analogous. First, we claim that $N$ can be chosen sufficiently small that the backward part of $z_1$ intersects all vertical leaves $L_x$ transversally. In linearizing coordinates, a leaf $L_x$ takes the form

$$L_x\colon\ x(\xi, \eta) = x.$$

Equation (23) implies that $\bar{\eta}_1 > 0$. Again in linearizing coordinates, the trajectory $z_1$ can be computed from (24):

$$z_1\colon\ (\xi, \eta) = (\bar{\xi}_1 e^{\lambda^{uu} t}, \bar{\eta}_1 e^{\lambda^u t}).$$

Eliminating $t$ from these equations, the trajectory is represented as the graph of a function $v_1$, which is defined for $\xi < 0$ as

$$z_1\colon\ \eta = v_1(\xi) = c_1 |\xi|^{1/\beta},$$

where $\beta = \lambda^{uu}/\lambda^u > 1$ and $c_1 = \bar{\eta}_1 |\bar{\xi}_1|^{-1/\beta} > 0$. This trajectory intersects the leaf $L_x$ transversally if $x(\xi, v_1(\xi)) = x$ and

$$0 \neq \frac{d}{d\xi} x(\xi, v_1(\xi)) = x_\xi - \frac{c_1 x_\eta}{\beta} |\xi|^{(1-\beta)/\beta}.$$

By Equation (25) both terms of this sum are positive in some neighbourhood of the origin, demonstrating the claim.

Next, we claim two facts about $z_2$. Firstly, for $\alpha \le 0$ the backward part of $z_2$ also intersects all vertical leaves transversally; secondly, if $\alpha > 0$ is small, the first tangency of the backward part of $z_2$ to the vertical direction is to a leaf that is intersected by the backward part of $z_1$ as well.

As above, in linearizing coordinates the trajectory $z_2$ can be represented by the graph of a function $v_2$, defined for $\xi > 0$, and taking the form

$$z_2\colon\ \eta = v_2(\xi) = c_2 \xi^{1/\beta},$$

where $\beta$ is as above and $c_2 = \bar{\eta}_2 \bar{\xi}_2^{-1/\beta}$. In particular, by definition of $\alpha = \alpha(\mu)$, we have that $c_2 = 0$ if $\alpha = 0$ and $(c_2)_\alpha > 0$. Consider the transversality condition

$$0 \neq \frac{d}{d\xi} x(\xi, v_2(\xi)) = x_\xi + x_\eta \frac{c_2}{\beta} \xi^{(1-\beta)/\beta}.$$

If $\alpha \le 0$, both terms are nonnegative in a neighbourhood of the origin. This proves the first claim about $z_2$.

For the second claim, note that the curve $z_2$ is tangent to a leaf $L_x$ if

$$0 = \frac{d}{d\xi} x(\xi, v_2(\xi)) = x_\xi^0 + x_\eta^0 \frac{c_2}{\beta} \xi^{(1-\beta)/\beta} + \chi(\xi, v_2(\xi)), \tag{28}$$

where $\chi(\xi, \eta) = O(|\xi| + |\eta|)$. To solve this equation, introduce

$$\theta = \left( \frac{-x_\eta^0 c_2}{x_\xi^0 \beta} \right)^{\beta/(\beta-1)} \tag{29}$$

and make the *Ansatz* $\xi = b^\beta \theta$. Equation (28) then takes the form

$$1 - b^{1-\beta} + \theta\tilde{\chi}(b^\beta, b, \theta) = 0, \tag{30}$$

with $\tilde{\chi}$ a differentiable function. It follows from the implicit function theorem that this equation has a unique root

$$b_* = 1 + \delta(\theta), \tag{31}$$

with $\delta(0) = 0$. We find that $z_2$ is tangent to the leaf $L_{x_*}$, where

$$x_* = x(b_*^\beta \theta, v_2(b_*^\beta \theta)).$$

To complete the proof of the second claim, we have to show that $x_* < x_e$. Write

$$x_* - x_e = x_\xi^0 b_*^\beta \theta + x_\eta^0 c_2 b_* \theta^{1/\beta} + \psi(b_*^\beta \theta, c_2 b_* \theta^{1/\beta}),$$

where $\psi(\xi, \eta) = O(\xi^2 + \eta^2)$. Use (29) to write $c_2$ in terms of $\theta$ and eliminate $b_*$ using (31) to arrive at

$$x_* - x_e = x_\xi^0 \theta(1+\delta)((1+\delta)^{\beta-1} - \beta + O(\theta)).$$

For small positive values of $\theta$, this expression is negative as $\beta > 1$. We conclude that the leaf $L_{x_*}$ is also intersected by $z_1$, proving the second claim about $z_2$.

We have shown that for $\alpha(\mu) \leq 0$, the continuous curve formed by the union of the trajectories $z_1$, $z_2$ and $e$ intersects each leaf $L_x$ exactly once, and defines therefore a continuous function $x \mapsto p(x, \mu)$, which necessarily represents the optimal costate map $p_\mu^o$.

For $\alpha(\mu) > 0$, the trajectory $z_1$ intersects the leaf $L_{x_*}$ at $z_{1*}$, while $z_2$ is tangent to $L_{x_*}$ at $z_{2*}$. As $\dot{x} = H_p = 0$ at $z_{2*}$ and $H$ is strictly convex in $p$, it follows that

$$H(z_{2*}) < H(z_{1*}).$$

The segment of $z_2$ connecting $\bar{z}_2$ to $z_{2*}$ intersects all leaves $L_x$ with $x_* < x < \bar{x}$ transversally, in particular $L_{x_e}$. Call the intersection point $z_{2e}$. Again by convexity of $H$ in $p$, it follows that

$$H(z_{2e}) > H(e) = \lim_{t \to -\infty} H(z_1(t)).$$

Consequently there is a state $\hat{x} \in (x_*, x_e)$ and intersections $\hat{z}_1$ and $\hat{z}_2$ of $z_1$ and $z_2$ with $L_{\hat{x}}$, respectively, such that

$$H(\hat{z}_1) = H(\hat{z}_2).$$

Theorem 3.2 then implies that $\hat{x}$ is an indifference point.

It remains to prove the claims about the differentiability of the value function. For the nonbifurcating situations $\alpha(\mu) > 0$ and $\alpha(\mu) < 0$, they follow directly from Theorem 4.4. If $\alpha(\mu) = 0$, the optimal costate $p_\mu^o(x)$ is univalued for all $x$ and is represented by a continuous function $p(x)$ everywhere, whose graph is tangent to $E^{uu}$ as $x \downarrow x_e$ and tangent to $E^u$ as $x \uparrow x_e$. The function $p(x) = V'(x)$ is therefore Lipschitz continuous. This proves the final claim of the theorem.

**4.3.2. $\mathrm{IR}_1(2)$ bifurcation.** An indifference-repeller singularity of type 2 occurs in certain situations when the dynamics of the repeller is a Jordan node. Specifically, consider the situation that the vector field $F$ on $\mathbb{R}^2$ has an equilibrium $e = (x_e, p_e)$, that its linearization $DF(e)$ has two equal positive eigenvalues $\lambda_1 = \lambda_2 = \lambda > 0$, and such that its proper unstable eigenspace $E^{pu}$ is only one-dimensional. By the Hartman theorem, there is a $C^1$ curve $W^{pu}$, the *proper unstable invariant manifold*, which is the image of $E^{pu}$ in general coordinates; trajectories $z(t)$ in $W^{pu}$ are characterized by the requirement that

$$\limsup_{t \to -\infty} \|z(t) - e\| \mathrm{e}^{-\lambda t} < \infty.$$

Definition 4.3. An interior point $e = (x_e, p_e)$ is a (codimension one) *indifference repeller singularity of type* 2, notation $\mathrm{IR}_1(2)$, of an optimization problem $\mathcal{P}$ with reduced canonical vector field $F$, if for all sufficiently small compact interval neighbourhoods $N$ of $x_e$, there exists a function $p_0$: $N \to \mathbb{R}$, such that the following conditions hold.

(i) The point $e$ is an equilibrium of $F$ such that the eigenvalues $\lambda_1, \lambda_2$ of $DF(e)$ satisfy $\lambda_1 = \lambda_2 = \rho/2$.
(ii) We have $p_0(x_e) = p_e$ and $p_0$ represents $p^o$ on $N$.

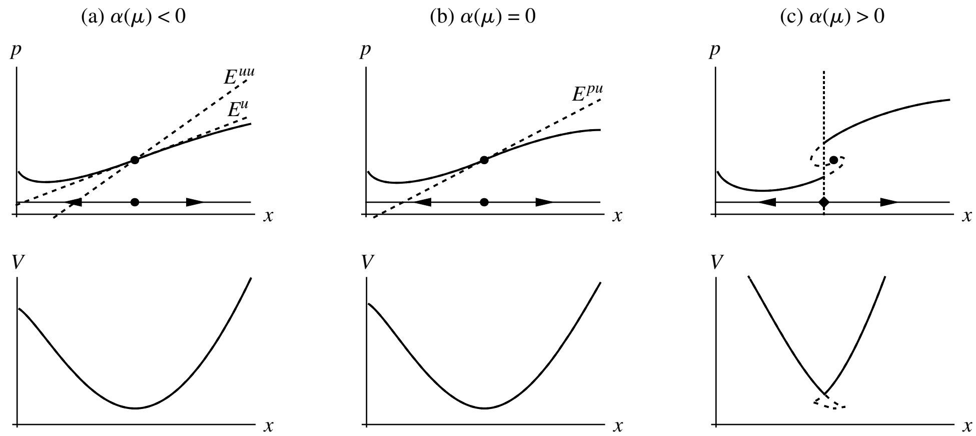



FIGURE 7. Before, at, and after the type 2 indifference-repeller bifurcation point.

(iii) Let $W^{uu}$ denote the strong unstable manifold of $F$ at $e$, parameterized as the graph of the function $w: N \to \mathbb{R}$. Also, let $\nu(x)$ be the outward pointing vector of $N$. For all $x \in \partial N$, we have that

$$\nu(x)(w(x) - p_0(x)) > 0. \tag{32}$$

This singularity also gives rise to an indifference repeller bifurcation. Figure 7 illustrates the bifurcation mechanism, which is different from the $\mathrm{IR}_1(1)$ bifurcation. At bifurcation, the equilibrium of the reduced canonical vector field is a Jordan node. When the eigenvalues move off the real axis, the equilibrium turns into a focus, implying the existence of an indifference point. On the other hand, when the eigenvalues remain on the real axis but separate from each other, two independent eigenspaces $E^{uu}$ and $E^{u}$ are generated. Condition (32) then ensures the existence of an optimal repeller.

THEOREM 4.6. *Consider a family of optimization problems $\mathcal{P}_\mu$, depending on a parameter $\mu \in \mathbb{R}^q$, with value functions $V_\mu$ and reduced canonical vector fields $F_\mu$, such that $\mathcal{P}_0$ has an $IR_1(2)$ singularity $e_0$. Assume that there is a neighbourhood $\Gamma \subset \mathbb{R}^q$ of 0 such that the following conditions hold.*

(i) *If $e_\mu$ is the family of equilibria of $F_\mu$ that unfolds $e_0$, let $D(\mu)$ and $T(\mu)$ denote the trace and the determinant of $DF_\mu(e_\mu)$, and let the function $\alpha: \Gamma \to \mathbb{R}$ be given as*

$$\alpha(\mu) = D(\mu) - \frac{T(\mu)^2}{4}.$$

*Then $\alpha(0) = 0$ and*

$$D\alpha(0) \neq 0.$$

(ii) *There is a family of functions $d_\mu: \partial N \to \mathbb{R}$, depending differentiably on $\mu \in \Gamma$, such that*

$$p^o_\mu(x) = \{d_\mu(x)\}$$

*for all $x \in \partial N$ and all $\mu \in \Gamma$.*
*Then the optimal vector field $f^o$ restricted to $N$ is for $\alpha(\mu) < 0$ topologically conjugate to*

$$Y(x) = \{x\},$$

*whereas for $\alpha(\mu) > 0$ it is conjugate to*

$$Y(x) = \begin{cases} \{-1\} & x < 0, \\ \{-1, 1\} & x = 0, \\ \{1\} & x > 0. \end{cases}$$

*Moreover, if $\alpha(\mu) < 0$, then for every $\varepsilon > 0$, the value function $V_\mu \in C^{1+\beta-\varepsilon}(N)$, with*

$$\beta = \frac{T(\mu) + 2\sqrt{-\alpha(\mu)}}{T(\mu) - 2\sqrt{-\alpha(\mu)}};$$

*if $\alpha(\mu) = 0$, then $V_\mu \in C^2(N)$; finally, if $\alpha(\mu) > 0$, then $V_\mu$ is Lipschitz continuous on $N$.*

Proof. There is a linear map $C_0$ such that

$$C_0^{-1} DF_0(e) C_0 = \begin{pmatrix} \rho/2 & 1 \\ 0 & \rho/2 \end{pmatrix}.$$

Arnold's matrix unfolding theorem (Arnold [1]) then implies that there are functions $\alpha = \alpha(\mu)$ and $\beta = \beta(\mu)$, and a family of maps $C_\mu$ unfolding $C_0$, all smoothly depending on $\mu$ and such that

$$DF_\mu(e_\mu) = C_\mu \begin{pmatrix} \rho/2+\beta & 1 \\ \alpha & \rho/2+\beta \end{pmatrix} C_\mu^{-1}.$$

As $F_\mu$ is a reduced canonical vector field, it follows moreover that $\operatorname{trace} DF_\mu = \rho$ for all $\mu$, and hence that $\beta(\mu) = 0$ everywhere.

Introducing $A_\alpha = C_\mu^{-1} DF_\mu(e_\mu) C_\mu$, the eigenvalues of $DF_\mu(e_\mu)$ and consequently also those of $A_\alpha$ take the form

$$\lambda_1 = \frac{\rho}{2} - \sqrt{-\alpha}, \qquad \lambda_2 = \frac{\rho}{2} + \sqrt{-\alpha};$$

the corresponding eigenvectors of $A_\alpha$ are

$$v_1 = (1, -\sqrt{-\alpha}), \qquad v_2 = (1, \sqrt{-\alpha}).$$

For $\alpha < 0$ these eigenvectors have the same ordering as the corresponding eigenvectors of $DF_\mu(e_\mu)$; cf. Lemma 3.2. It follows that the matrix $C_\mu$ is necessarily orientation preserving for $\alpha(\mu) < 0$ and, by continuity, for all other values of $\mu$.

Define $\bar{x}_i$, $\bar{p}_i$, and $\bar{z}_i$ as in (26) and (27).

For $\alpha(\mu) < 0$, there is a family $W_\mu^{uu}$ of strong unstable manifolds, depending continuously on $\mu$, and parameterized as the graph of a family of $C^1$ functions $w_\mu$ around $x_e$. In particular, if $N$ and $\Gamma$ are sufficiently small

$$\nu(x)(w_\mu(x) - p_\mu(x)) > 0$$

for all $x \in \partial N$ and $\mu \in \Gamma$. By continuity, the backward trajectories through $\bar{z}_1$ and $\bar{z}_2$ intersect all vertical leaves $L_x$ transversally and are tangent to the weak unstable direction $E^u$ at $x_e$. But this implies that they form, together with the equilibrium $e$, the graph of a $C^1$ function $p_\mu$ that is defined on $N$, and for which

$$p_\mu^o(x) = \{p_\mu(x)\}.$$

For $\alpha(\mu) = 0$, the optimal costate map $p^o$ is represented by the function $p_0$ on $N$. Necessarily, the graph of $p_0$ is formed by two trajectories of $F_0$ as well as the equilibrium point $e$. These trajectories intersect the vertical leaves $L_x$ (cf. (18)) transversally and they are tangent to $E^{pu}$ at $x_e$; equivalently, the graph of $p_0$ is tangent to the proper unstable manifold $W^{pu}$ at $e$. Condition (32) then implies $p_0(x) > w(x)$ if $\bar{x}_1 < x < x_e$ and $p_0(x) < w(x)$ if $x_e < x < \bar{x}_2$.

When $\alpha(\mu) > 0$, the eigenvalues are complex, and the backward trajectories $z_1$ and $z_2$ emanating from $\bar{z}_1$ and $\bar{z}_2$, respectively, spiral toward $e$ as $t \to -\infty$. Let $t_*$ be the largest $t \le 0$ such that $\dot{x}_2(t) = 0$. Then necessarily

$$x_* = x_2(t_*) < x_e.$$

The trajectory $z_2$, restricted to $[t_*, 0]$, can be parameterized as the graph of a continuous function $p_2$: $[x_*, \bar{x}_2] \to \mathbb{R}$. In the same way, if $t^*$ is the largest value of $t \le 0$ such that $\dot{x}_1(t) = 0$, then $z_1$ restricted to $[t^*, 0]$ can be parameterized as the graph of the function $p_1$: $[\bar{x}_1, x^*] \to \mathbb{R}$, where

$$x^* = x_1(t^*) > x_e.$$

Moreover, as $H$ is strictly convex and $H_p(x_*, p_2(x_*)) = 0$, it follows that

$$H(x_*, p_2(x_*)) < H(x_*, p_1(x_*));$$

likewise

$$H(x^*, p_2(x^*)) > H(x^*, p_1(x^*)).$$

By continuity, there is a point $\hat{x} \in [x_*, x^*]$ such that

$$H(\hat{x}, p_1(\hat{x})) = H(\hat{x}, p_2(\hat{x})).$$

By Theorem 3.2, this is an indifference point.

The statements about the differentiability of $V_\mu$ follow for $\alpha(\mu) \neq 0$ from Theorem 4.4 combined with, for $\alpha(\mu) < 0$, the equality

$$\beta = \frac{\lambda_2}{\lambda_1} = \frac{\rho + 2\sqrt{-\alpha}}{\rho - \sqrt{-\alpha}}.$$

If $\alpha(\mu) = 0$, the graph of the function $p_0$ representing $p^o$ is tangent to $E^{pu}$ at $e$; consequently $p_0$ is $C^1$, and $V_0$ is $C^2$. □

**4.3.3. IA$_1$ bifurcation.** The indifference-attractor bifurcation, to which we turn next, was first analyzed in a situation where the reduced canonical vector field $F$ features two saddle equilibria (Wagener [37]). For the critical value of the bifurcation parameter, a heteroclinic connection between the two appears: that is, the unstable manifold of one saddle coincides with the stable manifold of the other. The present formulation in terms of value functions and optimal costate rules is more general, as it also captures the situation of a reduced canonical vector field having only a single saddle, whose unstable manifold, at bifurcation, gives rise to an optimal trajectory tending to infinity (Hinloopen et al. [20]).

Definition 4.4. An infinite horizon problem $\mathcal{P}$ with reduced canonical vector field $F$ has a (codimension one) *indifference attractor singularity* $e = (x_e, p_e)$, notation IA$_1$, if for all sufficiently small compact interval neighbourhoods $N$ of $x_e$, there exists a function $p_0$: $N \to \mathbb{R}$ such that the following conditions hold.

(i) The point $e$ is an equilibrium of $F$ such that the eigenvalues $\lambda^s, \lambda^u$ of $DF(e)$ satisfy $\lambda^s < 0 < \lambda^u$.

(ii) We have $p_0(x_e) = p_e$ and $p_0$ represents $p^o$ on $N$.

(iii) Let $W^s$ and $W^u$ denote, respectively, the stable and the unstable manifold of $F$ at $e$, parameterized as the graphs of $w^s, w^u$: $N \to \mathbb{R}$, respectively. If $\partial N = \{\bar{x}_1, \bar{x}_2\}$, then

$$p_0(\bar{x}_1) = w^s(\bar{x}_1), \qquad p_0(\bar{x}_2) = w^u(\bar{x}_2). \tag{33}$$

Remark 4.4. This definition does not require the points $\bar{x}_1$ and $\bar{x}_2$ to be ordered in a certain way.

Theorem 4.7. *Consider a family of infinite horizon problems $\mathcal{P}_\mu$, depending on a parameter $\mu \in \mathbb{R}^q$, such that $\mathcal{P}_0$ has an IA$_1$ singularity $e_0$ with associated representing costate function $p_0$: $N \to \mathbb{R}$, and let $\bar{x}_1$ and $\bar{x}_2$ be as in (33). Assume that there is a neighbourhood $\Gamma \subset \mathbb{R}^q$ of 0 such that the following conditions hold.*

(i) *If $e_\mu$ is the family of equilibria of $F$ unfolding $e_0$, the eigenvalues of $DF_\mu(e_\mu)$ satisfy $\lambda^s_\mu < 0 < \lambda^u_\mu$ for all $\mu \in \Gamma$.*

(ii) *There is a function $d_\mu$: $\partial N \to \mathbb{R}$, differentiable in $\mu$, such that*

$$p^o(x) = \{d_\mu(x)\}$$

*for all $x \in \partial N$ and all $\mu \in \Gamma$.*

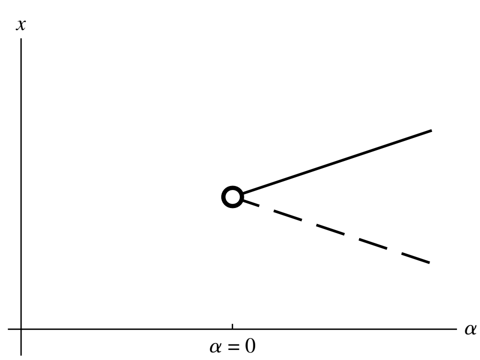



Figure 8. Bifurcation diagram of the indifference-attractor bifurcation IA$_1$. No singularity exists for $\mu < 0$, while an attractor (dashed line) and an indifference point (solid line) have come into being for $\mu > 0$ (solid line).

(iii) *Let the stable and the unstable manifolds $W^s_\mu$ and $W^u_\mu$ of $e_\mu$ be parameterized as graphs $p = w^s_\mu(x)$ and $p = w^u_\mu(x)$ of differentiable functions $w^s_\mu, w^u_\mu\colon N \to \mathbb{R}$, depending differentiably on $\mu$. The function*

$$\alpha(\mu) = \nu(\bar{x}_2)(w^u_m u(\bar{x}_2) - d_\mu(\bar{x}_2)),$$

*for which $\alpha(0) = 0$ by (33), is defined on $\Gamma$ and satisfies*

$$D\alpha(0) \neq 0.$$

(iv) *For all $\mu \in \Gamma$ such that $\alpha(\mu) > 0$, the equality*

$$d_\mu(\bar{x}_1) = w^s_\mu(\bar{x}_1)$$

*holds.*

*Then the optimal vector field $f^o$ restricted to $N$ is for $\alpha(\mu) < 0$ topologically conjugate to*

$$Y(x) = \{1\}$$

*whereas for $\alpha(\mu) > 0$ it is conjugate to*

$$Y(x) = \begin{cases} \{-x\}, & x < 1, \\ \{-1, 1\}, & x = 1, \\ \{1\}, & x > 1. \end{cases}$$

*The value function $V_\mu$ is smooth for $\alpha(\mu) < 0$, $C^1$ with Lipschitz continuous derivative for $\alpha(\mu) = 0$, and Lipschitz continuous for $\alpha(\mu) > 0$.*

The bifurcation diagram is given in Figure 8, and the theorem is illustrated in Figure 9. As for the $\mathrm{IR}_1(1)$ bifurcation, at the bifurcation the relative position of the optimal trajectories and the "most unstable" invariant manifold changes.

Proof. The idea of the proof runs as follows. If $\alpha(\mu) < 0$, the trajectory through $\bar{z}_1$ intersects the line $x = x_e$. Theorem 3.2, combined with the fact that $p \mapsto H(x_e, p)$ takes its minimum at $p_e$, then implies that the constant trajectory $e$ cannot be optimal at all in this case. For $\alpha(\mu) > 0$, the backward part of the optimal trajectory through the point $\bar{z}_2 = (\bar{x}_2, d_\mu(\bar{x}_2))$ has a vertical tangent at a certain point. Past this point, the trajectory cannot be optimal, even locally. It follows that $x_e$ is locally optimal.

To execute this programme, it is again convenient to work with Hartman's linearization theorem. Restricted to a neighbourhood of the saddle, in linearizing coordinates the vector field $F_\mu$ takes the form

$$\dot{\zeta} = \begin{pmatrix} \lambda^u & 0 \\ 0 & \lambda^s \end{pmatrix} \zeta.$$

The coordinates are chosen such that the coordinate transformation is orientation preserving; moreover, the direction of the axes is chosen such that

$$x_\xi > 0, \qquad x_\eta < 0. \tag{34}$$

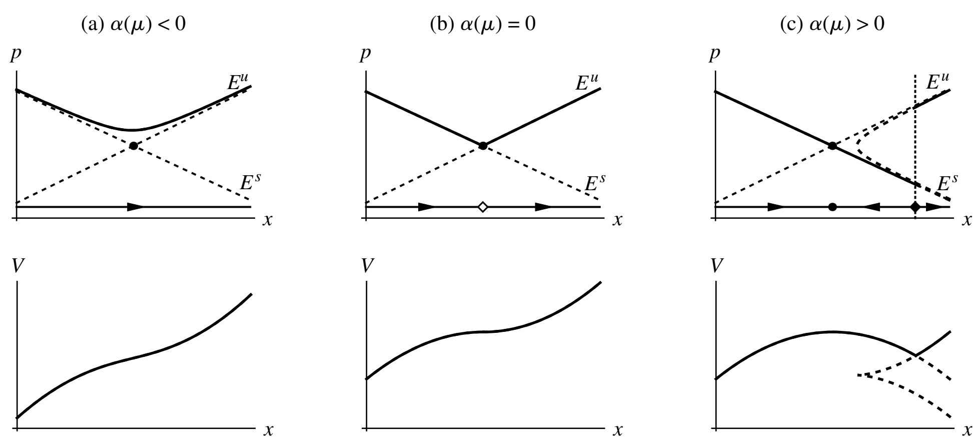



Figure 9. Before, at, and after the indifference-attractor bifurcation point.

Note that the unstable and stable manifolds are in these coordinates equal to the horizontal and vertical coordinate axes, respectively.

As in the proof of Theorem 4.5, set $\bar{x}_i$, $\bar{p}_i$ and $\bar{z}_i$ as in (26) and (27).

Assume that $\bar{x}$ of point 2 of Definition 4.4 satisfies $\bar{x} = \bar{x}_2$; the opposite situation can be handled analogously. If $\bar{\xi}_2$ and $\bar{\eta}_2$ are defined as

$$x(\bar{\xi}_2, \bar{\eta}_2) = \bar{x}, \qquad p(\bar{\xi}_2, \bar{\eta}_2) = w(\bar{x}) + \alpha,$$

then it follows as in the proof of Theorem 4.5 that

$$(\bar{\eta}_2)_\alpha < 0$$

and $\bar{\eta}_2 = 0$ if $\alpha = 0$.

The trajectory $z_2$ of the reduced canonical vector field through $\bar{z}_2$ takes in linearizing coordinates the form

$$z_2\colon\ (\xi, \eta) = (\bar{\xi}_2 \mathrm{e}^{\lambda^u t}, \bar{\eta}_2 \mathrm{e}^{\lambda^s t}).$$

It can also be represented by the graph of a function $v_2$, which for $\xi > 0$ is defined as

$$z_2\colon\ \eta = v_2(\xi) = c_2 \xi^{-\beta},$$

with $\beta = -\lambda^s/\lambda^u > 0$ and $c_2 = \bar{\eta}_2 \bar{\xi}_2^{\beta}$. The curve $z_2$ is tangent to a vertical leaf if

$$0 = \frac{\mathrm{d}}{\mathrm{d}\xi} x(\xi, v_2(\xi)) = x_\xi - x_\eta \beta c_2 \xi^{-\beta-1}.$$

If $\alpha < 0$, then $\bar{\eta}_2 > 0$ and it follows from (34) that both terms on the right-hand side are positive. Hence $z_2$ intersects all vertical leaves transversally, and therefore defines a $C^1$ function $x \mapsto p_\mu(x)$, which then necessarily satisfies

$$p_\mu^o(x) = \{p_\mu(x)\}$$

for all $x \in N$.

Consider now the situation that $\alpha > 0$ and, consequently, that $\bar{\eta}_2 < 0$ and $c_2 < 0$. Let $\delta > 0$ be a positive constant, such that all points $(\xi, \eta)$ with $|\xi| < \delta$ and $|\eta| < \delta$ are contained in the range $R$ of the linearizing transformation $\zeta$. Define $\xi_\delta > 0$ by the requirement that $v_2(\xi_\delta) = -\delta$, that is

$$\xi_\delta = \left(\frac{-c_2}{\delta}\right)^{1/\beta}.$$

If necessary after choosing the domain of definition of the linearizing transformation smaller, we may assume that condition (34) is strengthened to the following: there are constants $c, C > 0$ such that $c < x_\xi < C$ and $-C < x_\eta < -c$ in $R$. Then

$$\left.\frac{\mathrm{d}x}{\mathrm{d}\xi}\right|_{\xi=\xi_\delta} = x_\xi + x_\eta \beta \frac{-v_2(\xi_\delta)}{\xi_\delta} < C - c\beta\delta \frac{1}{\xi_\delta},$$

and this is negative if $\alpha$, and hence $\xi_\delta$, is sufficiently close to zero. Likewise, we have

$$\left.\frac{\mathrm{d}x}{\mathrm{d}\xi}\right|_{\xi=\delta} = x_\xi - x_\eta \beta c_2 \delta^{-\beta-1} > c - \frac{C\beta}{\delta^{\beta+1}} c_2,$$

and this is positive if $\alpha$ is close to zero. Consequently, there exists $\xi = \xi_*(\alpha)$ such that $\mathrm{d}x/\mathrm{d}\xi = 0$ if $\xi = \xi_*$ and such that $\mathrm{d}x/\mathrm{d}\xi > 0$ if $\xi > \xi_*$. Set $x_* = x(\xi_*, v_2(\xi_*))$.

We have shown that the trajectory $z_2$ is tangent to $L_{x_*}$ and intersects transversally the leaves $L_x$ with $x_* < x < \bar{x}_2$. It therefore parametrises the graph of a function $p = p_2(x)$. As $\dot{x} = H_p$ along trajectories, we have

$$H_p(x_*, p_2(x_*)) = 0.$$

Let $p_1\colon N \to \mathbb{R}$ be such that its graph parameterizes the stable manifold $W^s$ of $s$. Strict convexity of $p \mapsto H(x_*, p)$ implies

$$H(x_*, p_2(x_*)) < H(x_*, p_1(x_*)). \tag{35}$$

Define functions $V_1$ on $N$ and $V_2$ on $[x_*, \bar{x}_2]$ by

$$V_i(x) = \frac{H(x, p_i(x))}{\rho}, \quad i = 1, 2.$$

Then Equation (35) is equivalent to

$$V_2(x_*) < V_1(x_*).$$

To establish the opposite inequality for some $x^* \in [x_*, \bar{x}_2]$, consider the situation for $\alpha(\mu) = 0$, when $\bar{z}_2 \in W^u$. Then $V_2$ is defined for all $x_e < x < \bar{x}_2$. Moreover,

$$\lim_{x \downarrow x_e} V_2(x) = V_1(x).$$

Note that since $V_i'(x) = p_i(x)$ and

$$p_2(x) > p_1(x)$$

for all $x_e < x < \bar{x}_2$, it follows that

$$V_2(x) - V_1(x) = \int_{x_e}^{x} (p_2(\sigma) - p_1(\sigma))\, \mathrm{d}\sigma > 0$$

for all $x > x_e$. This implies in particular that

$$H(x, p_2(x)) > H(x, p_1(x)) \tag{36}$$

for all $x > x_e$, if $\alpha(\mu) = 0$.

Fix $x^* \in (x_*, \bar{x}_2)$. Then for $\alpha(\mu) < 0$ sufficiently close to 0, Equation (36) implies, by continuity, that

$$H(x^*, p_2(x^*)) > H(x^*, p_1(x^*)). \tag{37}$$

As a consequence of (35) and (37), there is $\hat{x} \in (x_*, x^*)$ such that

$$H(\hat{x}, p_1(\hat{x})) = H(\hat{x}, p_2(\hat{x})).$$

By Theorem 3.2, it follows that $\hat{x}$ is an indifference point.

The statements about the differentiability of the value functions follow as before from Theorem 4.4 and the fact that $V_0' = p_0$ is Lipschitz continuous. $\square$

**4.3.4. The saddle-node bifurcation.** The saddle-node bifurcation of dynamical systems has a natural counterpart as a bifurcation of optimal vector fields.

Recall that a family of vector fields $f_\mu: \mathbb{R}^m \to \mathbb{R}^m$ can be viewed as a single vector field $g: \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$ by writing

$$\begin{pmatrix} \dot{x} \\ \dot{\mu} \end{pmatrix} = g(x, \mu) = \begin{pmatrix} f_\mu(x) \\ 0 \end{pmatrix}.$$

Consider the situation that for $\mu = 0$ the point $\bar{z}$ is an equilibrium of $f_0$, and that $Df_0(\bar{z})$ has a single eigenvalue 0. Then $Dg(\bar{z}, 0)$ has two eigenvalues zero and an associated two-dimensional eigenspace $E^c$. The center manifold theorem applied to $g$ implies that there is a differentiable invariant manifold $W^c$ of $g$ that is tangent to $E^c$ at $(\bar{z}, 0)$. The manifold $W^c$ can be viewed as a parameterized family of invariant manifolds $W^c_\mu$, which are defined for $\mu$ taking values in a full neighbourhood of $\mu = 0$. Note that center manifolds need not be unique.

Definition 4.5. An interior point $e = (x_e, p_e)$ is a (codimension one) *saddle-node singularity*, notation $\mathrm{SN}_1$, of an optimization problem $\mathcal{P}$ with reduced canonical vector field $F$, if for all sufficiently small compact interval neighbourhoods $N$ of $x_e$ in $X$ there exists a function $p_0: N \to \mathbb{R}$, such that the following conditions hold.

(i) The point $e$ is an equilibrium of $F$ such that the eigenvalues $\lambda_1$, $\lambda_2$ of $DF(e)$ satisfy $\lambda_1 = 0$, $\lambda_2 = \rho$.
(ii) The function $p_0$ represents $p^o$; moreover, the graph of $p_0$ is a center manifold $W^c$ of $F$ at $e$.
(iii) Write $F(x, p) = (F_1(x, p), F_2(x, p))$. The restriction

$$F^c(x) = F_1(x_e + x, p_0(x_e + x))$$

of $F$ to $W^c$ satisfies

$$F^c(0) = 0, \qquad (F^c)'(0) = 0, \tag{38}$$

and

$$(F^c)''(0) \neq 0. \tag{39}$$

THEOREM 4.8. *Consider a family $\mathcal{P}_\mu$ of optimization problems, depending on a parameter $\mu \in \mathbb{R}^q$, with value functions $V_\mu$ and reduced canonical vector fields $F_\mu$, such that $\mathcal{P}_0$ has a $SN_1$ singularity with associated costate function $p_0$: $N \to \mathbb{R}$. Assume that there is a neighbourhood $\Gamma \subset \mathbb{R}^q$ of 0 such that the following conditions hold.*

(i) *There is a family of costate functions $p_\mu$: $N \to \mathbb{R}$, depending differentiably on $\mu$ and unfolding $p_0$, such that for every $\mu \in \Gamma$ the function $p_\mu$ represents $p^o_\mu$.*

(ii) *The manifold $W^c = \{(x, p, \mu): p = p_\mu(x)\}$ is a center manifold of the vertical vector field $(F, 0)$ at $(e, 0)$.*

(iii) *Let $F^c_\mu$ be given as*

$$F^c_\mu(x) = (F_\mu)_1(x_e + x, p_\mu(x_e + x)).$$

*The function*

$$\alpha(\mu) = F^c_\mu(0)$$

*satisfies*

$$D\alpha(0) \neq 0.$$

*Then the optimal vector field $f^o_\mu$ restricted to $N$ is for $\mu \in \Gamma$ topologically conjugate to*

$$Y_\mu(x) = \{\alpha(\mu) - \sigma x^2\},$$

*where $\sigma$ is given as*

$$\sigma = -(F^c_0)''(0).$$

*If $\sigma\alpha(\mu) \leq 0$, the value function $V_\mu$ is smooth; if $\sigma\alpha(\mu) > 0$, then $V_\mu \in C^\beta$ with*

$$\beta = \frac{C}{|\alpha(\mu)|^{1/2}} + O(1),$$

*where $C > 0$ is a fixed constant.*

PROOF. This is a direct consequence from the usual saddle-node bifurcation theorem. In particular, the equilibria of $F^c_\mu$ are of the form

$$x_i = (-1)^{i+1}\sqrt{\alpha}\sigma + O(\sqrt{\alpha}), \quad i = 1, 2$$

and the eigenvalues $\lambda^{(i)}_1$, $\lambda^{(i)}_2$ of $F$ at $(x_i, p_\mu(x_i))$ satisfy

$$\lambda^{(i)}_1 = (-1)^i 2\sqrt{\sigma\alpha} + O(\alpha)$$

and

$$\lambda^{(i)}_1 + \lambda^{(i)}_2 = \rho,$$

as a consequence of (15). The statement now follows from Theorem 4.4 and the relation

$$\beta^{(2)} = \frac{\lambda^{(2)}_2}{\lambda^{(2)}_1} = \frac{\rho - \lambda^{(2)}_1}{\lambda^{(2)}_1} = \frac{\rho}{2\sqrt{\sigma}}\alpha^{-1/2} + O(1). \quad \square$$

**4.4. Codimension two bifurcations.** Most codimension two situations are straightforward extensions of the corresponding codimension one bifurcations. The results in this subsection are therefore stated more briefly and less formally. An exception is made for the indifference-saddle-node bifurcation.

### 4.4.1. A model case: The $\mathrm{IR}_2(1, 1)$ bifurcation.

DEFINITION 4.6. An interior point $e = (x_e, p_e)$ is a (codimension two) *indifference repeller singularity of type* $(1, 1)$, notation $\mathrm{IR}_2(1, 1)$, of an optimization problem $\mathcal{P}$ with reduced canonical vector field $F$, if conditions (i) and (ii) of Definition 4.2 hold, while condition (iii) is replaced by the requirement that

$$p_0(x) = w(x)$$

for all $x \in \partial N$.

THEOREM 4.9. *Consider a family $\mathcal{P}_\mu$ of optimization problems, depending on a parameter $\mu \in \mathbb{R}^q$, that has for $\mu = 0$ an $IR_2(1, 1)$ singularity. Assume that there is a neighbourhood $\Gamma \subset \mathbb{R}^q$ of 0 such that the following conditions hold.*

(i) *If $e_\mu$ is the family of equilibria of $F_\mu$ unfolding $e_0$, then the eigenvalues of $DF_\mu(e_\mu)$ satisfy $0 < \lambda^u_\mu < \lambda^{uu}_\mu$ for all $\mu \in \Gamma$.*

(ii) *There is a function* $d_\mu\colon \partial N \to \mathbb{R}$, *differentiable in* $\mu$, *such that*

$$p^o_\mu(x) = \{d_\mu(x)\}$$

*for all* $x \in \partial N$ *and all* $\mu \in \Gamma$.

(iii) *Let the strongly unstable manifold* $W^{uu}_\mu$ *of* $e_\mu$ *be parameterized as the graph* $p = w_\mu(x)$ *of a differentiable function* $w_\mu\colon N \to \mathbb{R}$, *and let* $\partial N = \{x_1, x_2\}$. *The function*

$$\alpha(\mu) = (\nu(x_1)(d_\mu(x_1) - w_\mu(x_1)), \nu(x_2)(d_\mu(x_2) - w_\mu(x_2))),$$

*is defined on* $\Gamma$ *and satisfies*

$$\alpha(0) = (0, 0) \qquad \textit{and} \qquad \operatorname{ran} D\alpha(0) = 2.$$

*Then the optimal vector field* $f^o$ *restricted to* $N$ *is topologically conjugate to*

$$Y(x) = x$$

*if* $\alpha_1(\mu) \le 0$ *and* $\alpha_2(\mu) \le 0$, *whereas it is conjugate to*

$$Y(x) = \begin{cases} -1 & x < 0, \\ \{-1, 1\} & x = 0, \\ 1 & x > 0, \end{cases}$$

*if* $\alpha_1(\mu) > 0$ *or* $\alpha_2(\mu) > 0$. *In particular, the curves* $\alpha_1(\mu) = 0$, $\alpha_2(\mu) < 0$ *and* $\alpha_2(\mu) = 0$, $\alpha_1(\mu) < 0$ *are codimension one indifference-repeller bifurcation curves of type* 1.

The proof is a simple modification of the proof of the codimension one case and is therefore omitted.

**4.4.2. Other indifference-repeller and indifference-attractor bifurcations.** Looking at the definition of the $\mathrm{IR}_1(2)$ bifurcation, it is clear that bifurcations of higher codimension are obtained when condition (32) is violated at a boundary point. If this happens at one of the boundary points, a codimension two situation is obtained where an $\mathrm{IR}_1(1)$ and an $\mathrm{IR}_1(2)$ curve meet in a $\mathrm{IR}_2(1,2)$ point. If it happens at both boundary points, a codimension three situation arises, denoted $\mathrm{IR}_3$, where two $\mathrm{IR}_1(1)$ and a $\mathrm{IR}_1(2)$ surface meet. To avoid unnecessary repetitions, the exact definitions for these bifurcations are not formulated; they can all be modeled on Definition 4.6 and Theorem 4.9. Their bifurcation diagrams are given in Figures 10(b) and 11.

Likewise, a codimension two bifurcation is obtained if condition (33) is replaced by

$$p(x_1) = w^u(x_1), \qquad p(x_2) = w^u(x_2). \tag{40}$$

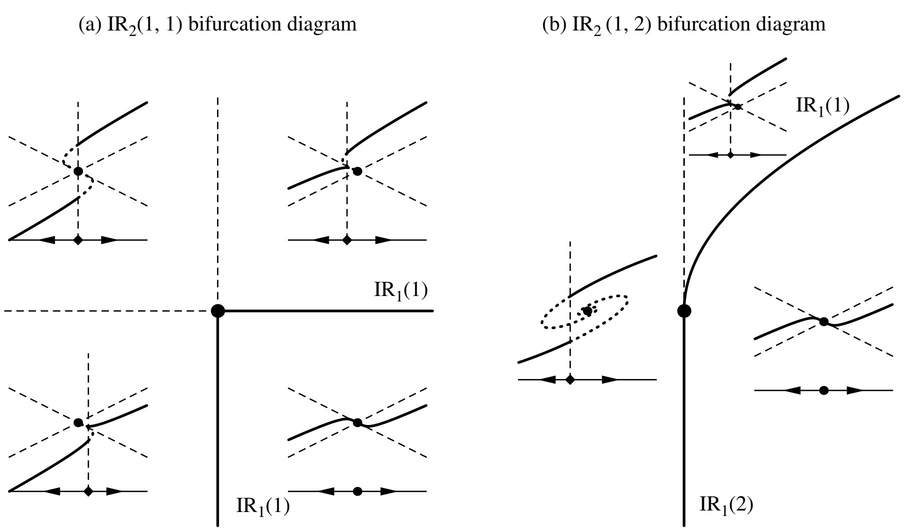



FIGURE 10. Bifurcation diagrams of indifference-repeller bifurcations of codimension two. Shown is a two-dimensional parameter plane. Drawn lines correspond to bifurcations of the optimal vector field, dashed lines to changes of the reduced canonical vector field that do not correspond to such bifurcations. The insets show the state-costate dynamics in the respective parameter regions.

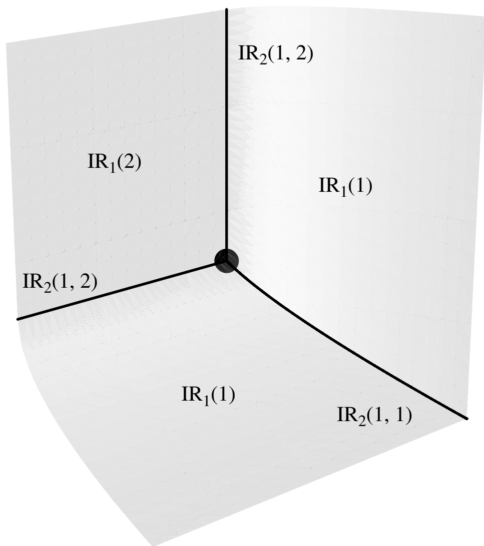



FIGURE 11. IR$_3$ bifurcation diagram.

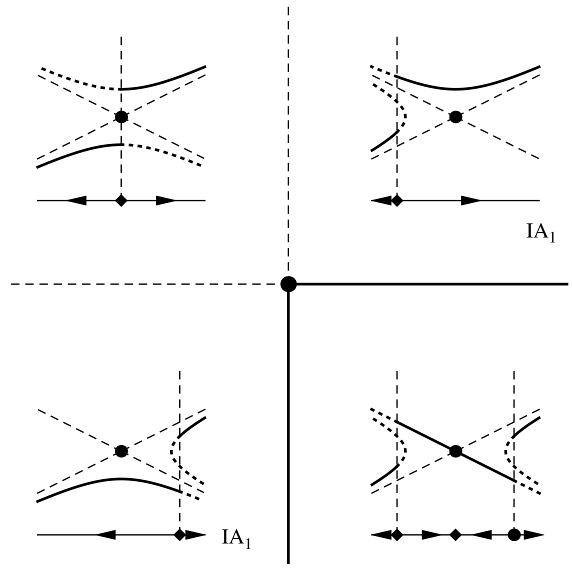



FIGURE 12. IA$_2$ bifurcation diagram.

This is a two-sided or double indifference attractor bifurcation, denoted IA$_2$. Its bifurcation diagram is given in Figure 12.

The relation of the IA$_2$ bifurcation of optimal vector fields to the cusp bifurcation of ordinary vector fields is the same as the IA$_1$ bifurcation to the saddle-node bifurcation, in the following sense.

Let $\mathcal{P}_\alpha$ be a family of optimisation problems, depending on a parameters $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$, and let $c(\mu) = (\alpha_1(\mu), \alpha_2(\mu))$ be a one-dimensional curve in $\mathbb{R}^2$. Then $\tilde{\mathcal{P}}_\mu = \mathcal{P}_{\alpha(\mu)}$ is another family of optimization problems.

Assume that the family $\mathcal{P}_\alpha$ has a IA$_2$ bifurcation point at $\alpha = 0$, and assume further that the curve $c$ passes through this bifurcation point as depicted in Figure 13(b). Then the one-parameter family $\tilde{\mathcal{P}}_\mu$ has for $\mu < 0$ a single indifference point, which for $\mu = 0$ splits into two indifference points, generating an additional attractor: the one-parameter bifurcation diagram is given in Figure 13(a).

We propose to call this bifurcation a "trident" bifurcation (of type 1), T$_1$(1), as it arises exactly analogously from the IA$_2$ bifucation as the "pitchfork" bifurcation arises from the cusp bifurcation. Like the pitchfork bifurcation, the trident bifurcation is not stable under small perturbations: if $c$ is perturbed to $\tilde{c}$, the trident bifurcation breaks into a nonbifurcating family of indifference points and an IA$_1$ bifurcation.

Moreover, as the pitchfork bifurcation is stable in the class of dynamical systems that are symmetric with respect to a reflection (or $\mathbb{Z}_2$) symmetry around the bifurcating singularity, also the trident bifurcation is stable in the class of optimization problems that are symmetric with respect to a reflection symmetry around the bifurcating singularity.

Just as there are two types of indifference-repeller bifurcations, there are expected to be two types of trident bifurcations; it appears that the bifurcation documented in Figure 3 of Caulkins et al. [7] would be a trident

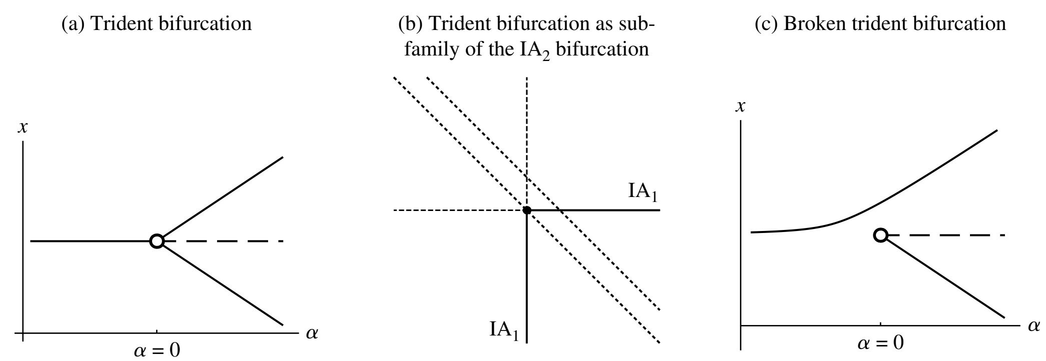



FIGURE 13. Trident bifurcation as a special subfamily of the IA$_2$ bifurcation.

bifurcation where the central singularity of the state-costate system at bifurcation is a Jordan node, just as in the situation of the $\mathrm{IR}_1(2)$ bifurcation; we would call this a trident bifurcation of type two, to be denoted $\mathrm{T}_1(2)$.

**4.4.3. Degenerate saddle-node bifurcations.** Finally, the bifurcations that are direct analogues of the corresponding bifurcations of dynamical systems, that is, the degenerate saddle-node bifurcations like the cusp ($\mathrm{SN}_2$), the swallowtail ($\mathrm{SN}_3$) etc. can be treated similarly to the saddle-node itself.



**4.4.4. The indifference-saddle-node bifurcation.** The indifference-attractor and both types of indifference-repeller bifurcations involve hyperbolic equilibria of the reduced canonical vector field; in contrast, the saddle-node bifurcation involves a nonhyperbolic equilibrium, but the graph of the optimal costate rule coincides with the center manifold of the nonhyperbolic equilibrium at bifurcation. The final bifurcation to be considered is the indifference-saddle-node bifurcation, where at bifurcation the graph of the optimal costate rule coincides with the unstable manifold of the nonhyperbolic equilibrium instead.

Definition 4.7. An interior point $e = (x_e, p_e)$ is a (codimension two) *indifference-saddle-node singularity*, notation $\mathrm{ISN}_2$, of an optimization problem $\mathcal{P}$ with reduced canonical vector field $F$, if for all sufficiently small compact interval neighbourhoods $N$ of $X$ there exists a function $p_0$: $N \to \mathbb{R}$, such that the following conditions hold.

(i) The point $e$ is an equilibrium of $F$, such that the eigenvalues $\lambda_1$, $\lambda_2$ of $DF(e)$ satisfy $\lambda_1 = 0$, $\lambda_2 = \rho$.

(ii) We have $p_e = p_0(x_e)$ and $p_0$ represents $p^o$ on $N$.

(iii) Let $W^u$ denote the unstable manifold of $F$ at $e$, parameterized as the graph of a function $w^u$: $N \to \mathbb{R}$. There is exactly one $\bar{x} \in \partial N$ such that

$$p_0(\bar{x}) = w^u(\bar{x}). \tag{41}$$

(iv) There is a center manifold $W^c$ of $F$ at $e$, parameterized as the graph of $w^c$: $N \to \mathbb{R}$, such that for $x \in \partial N \backslash \{\bar{x}\}$, we have that

$$p(x) = w^c(x). \tag{42}$$

(v) The restriction

$$F^c(x) = F_1(x_e + x, w^c(x_e + x))$$

of $F$ to $W^c$ satisfies

$$F^c(0) = 0, \qquad (F^c)'(0) = 0,$$

and

$$(F^c)''(0) \neq 0.$$

Theorem 4.10. *Consider a family $\mathcal{P}_\mu$ of optimization problems, depending on a parameter $\mu \in \mathbb{R}^q$, with reduced vector fields $F_\mu$, such that $\mathcal{P}_0$ has an $\mathrm{ISN}_2$ singularity with associated costate function $p_0$: $N \to \mathbb{R}$. Assume that there is a neighbourhood $\Gamma \subset \mathbb{R}^q$ of $0$ such that the following conditions hold.*

(i) *There is a family of functions $d_\mu$: $\partial N \to \mathbb{R}$, differentiable in $\mu$, such that*

$$p_\mu^o(x) = \{p_\mu(x)\}$$

*for all $x \in \partial N$ and $\mu \in \Gamma$, and such that*

$$\alpha_2(\mu) = d_\mu(\bar{x}) - d_0(\bar{x})$$

*satisfies*

$$D\alpha_2(0) \neq 0.$$

(ii) *There is a family of center manifolds $W_\mu^c$, parameterized as the graphs of functions $w_\mu^c$: $N \to \mathbb{R}$, such that*

$$d_\mu(x) = w_\mu^c(x)$$

*if $x \in \partial N \backslash \{\bar{x}\}$.*

(iii) *Let $F_\mu^c$ be the restriction*

$$F_\mu^c(x) = (F_\mu)_1(x_e + x, w_\mu^c(x_e + x))$$

*of $F$ to $W_\mu^c$. Then the function*

$$\alpha_1(\mu) = F_\mu^c(0)$$

*satisfies*

$$D\alpha_1(0) \neq 0.$$

(iv) *Let $\alpha(\mu) = (\alpha_1(\mu), \alpha_2(\mu))$.Then* $\operatorname{ran} D\alpha(0) = 2$.

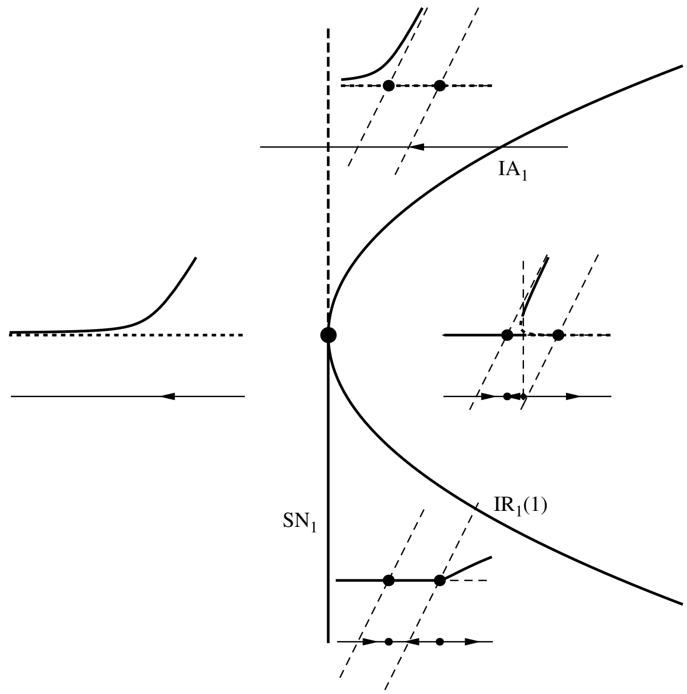



FIGURE 14. ISN$_2$ bifurcation diagram.

*Then there is a differentiable function $C(\alpha_2)$ such that $B(0) = B'(0) = 0$ and $B''(0) \neq 0$, and such that the problem has an indifference-attractor bifurcation if*

$$\alpha_1 = B(\alpha_2), \qquad \alpha_2 > 0,$$

*an indifference-repeller bifurcation if*

$$\alpha_1 = B(\alpha_2), \qquad \alpha_2 < 0,$$

*and a saddle-node bifurcation curve if*

$$\alpha_1 = 0, \qquad \alpha_2 < 0.$$

The bifuration diagram of this bifurcation is given in Figure 14.

PROOF. We only treat the situation that $(F_0^c)''(0) > 0$, the other being entirely analogous.

By an orientation preserving transformation (cf. (17)) we put the system in coordinates $\zeta = (\xi, \eta)$ such that the center manifold $W_\mu^c$ corresponds to $\eta = 0$ for all $\mu$ close to $\mu = 0$, and for $\mu = 0$ the unstable manifold $W^u$ corresponds to $\xi = 0$. In these coordinates, the vertical system, that is the system augmented by the parameter equation $\dot{\mu} = 0$ (see (19)), takes the form

$$\dot{\xi} = \alpha_1(\mu) + f_0(\xi, \mu) + \eta f_1(\zeta, \mu), \tag{43}$$

$$\dot{\eta} = \rho\eta + \eta g_1(\zeta, \mu), \tag{44}$$

$$\dot{\mu} = 0. \tag{45}$$

The fact that $\mathcal{P}_0$ has an indifference-saddle-node singularity implies that $f_0(\xi, \mu) = c(\mu)\xi^2 + O(\xi^3)$. Moreover, we have that $(F_0^c)''(0) > 0$ implies $c(0) > 0$, and $D\alpha_1(0) \neq 0$. Consequently a saddle-node bifurcation of the family of reduced canonical vector fields occurs at $(\xi, \eta) = (0, 0)$ for $\alpha_1(\mu) = 0$. In this bifurcation, a family of hyperbolic saddles and one of hyperbolic unstable equilibria is generated.

The saddle equilibria have associated to them unique unstable invariant manifolds $W_\mu^u$; the unstable equilibria have associated to them strongly unstable manifolds $W_\mu^{uu}$, which are also unique. An indifference-attractor bifurcation occurs if $(x, p_\mu(x)) \in W_\mu^u$; an indifference-repeller bifurcation of type 1 occurs if $(x, p_\mu(x)) \in W_\mu^{uu}$. The main thing to prove is that the manifolds $W_\mu^u$ and $W_\mu^{uu}$ can be parameterized as graphs of differentiable functions

$$w_\mu^u, w_\mu^{uu}: N \to \mathbb{R},$$

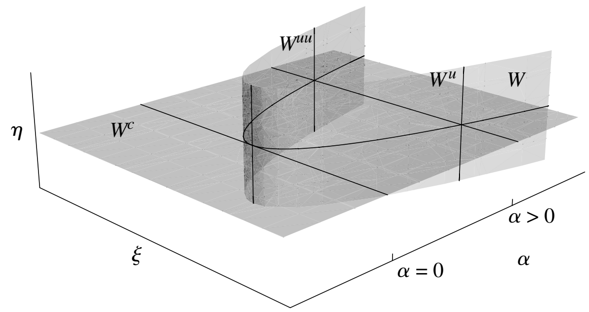



FIGURE 15. The manifold $W$. Its intersection with the plane $\eta = 0$ is given by the manifold $\alpha_1(\mu) + f_0(\xi, 0) = 0$.

and that these depend differentiably on $\mu$ if $\alpha_1(\mu) \neq 0$. This is not automatic, for the function $w^u$ and $w^{uu}$ will not be differentiable as functions of $\mu$, having necessarily at $\alpha_1(\mu) = 0$ a singularity of the order $\sqrt{\alpha_1(\mu)}$.

We however claim that the closure of the invariant set

$$W = \bigcup_{\mu} (W^u_\mu \cup W^{uu}_\mu) \tag{46}$$

forms a differentiable manifold. From Figure 15, it seems likely that $W$ can be described as the level set

$$W\colon \alpha_1(\mu) = -f_0(\xi, \mu) + \eta w(\xi, \eta, \mu),$$

where $w$ is a function yet to be determined. The condition that $W$ is invariant under the flow of (43)–(45) leads to a first order partial differential equation for the function $w$; this equation is singular for $\eta = 0$.

To solve this equation using the method of characteristics, introduce $w = w(t)$ as an independent variable by setting

$$\eta w = \alpha_1 + f_0(\xi, \mu).$$

Deriving with respect to time and using Equations (43)–(45) yields

$$\eta \dot{w} = -\dot{\eta} w - \frac{\partial f_0}{\partial \xi} \dot{\xi} = -w(\rho + g_1)\eta + \frac{\partial f_0}{\partial \xi}(w + f_1)\eta.$$

Dividing out $\eta$ formally, an equation for $\dot{w}$ is obtained. Together with Equations (43)–(45), the following system is obtained:

$$\dot{\xi} = \eta w + \eta f_1, \qquad \dot{w} = -\rho w - w g_1 - \frac{\partial f_0}{\partial \xi}(w + f_1),$$
$$\dot{\eta} = \rho \eta + \eta g_1, \qquad \dot{\mu} = 0.$$

Linearizing the new system at $(\xi, \eta, w, \mu) = (0, 0, 0, 0)$ yields

$$\begin{pmatrix} \dot{\xi} \\ \dot{\eta} \\ \dot{w} \\ \dot{\mu} \end{pmatrix} = \begin{pmatrix} 0 & & & \\ & \rho & & \\ & & -\rho & \\ & & & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ w \\ \mu \end{pmatrix}.$$

Again invoking the center manifold theorem, we find that there is an invariant center-unstable manifold $W^{cu}$ that is tangent to the center-unstable eigenspace $E^{cu} = \{w = 0\}$. Let this manifold be parameterized, in a neighbourhood of the origin, as

$$W^{cu}\colon w = w^{cu}(\xi, \eta, \mu).$$

Then $w^{cu}$ is the function we have been looking for.

A final note on $W$: as for $\mu = 0$ the unstable manifold $W^u$ is tangent to $\alpha_1 = 0$ at $\xi = 0$, the function $w$ in

$$W\colon \alpha_1(\mu) = -f_0(\xi, \mu) + \eta w(\xi, \eta, \mu), \tag{47}$$

has to satisfy $w = \xi^2 \tilde{w}$.

Indifference-attractor or indifference repeller bifurcations occur if $(\bar{x}, d_\mu(\bar{x})) \in W$. The equations

$$x = \bar{x}, \qquad p = d_0(\bar{x}) + \alpha_2$$

take, in $(\xi, \eta)$-coordinates, the form

$$\xi = c_1\alpha_2 + O(\varepsilon^2 + \alpha_2^2), \qquad \eta = c_{21}\alpha_2 + c_{22}\varepsilon + O(\varepsilon^2 + \alpha_2^2).$$

Note however that if $\mu = 0$, then $W$ is given by $\xi = 0$. Moreover, by assumption $d_0(\bar{x}) \in W$; therefore the equations actually read as

$$\xi = \alpha_2(c_1 + O(\varepsilon + \alpha_2)), \qquad \eta = c_{21}\alpha_2 + c_{22}\varepsilon + O(\varepsilon^2 + \alpha_2^2).$$

Substitution in Equation (47) yields the indifference-attractor and indifference-repeller bifurcation curves

$$\alpha_1 = c(\mu)c_1^2\alpha_2^2 + O(\alpha_2^3).$$

Taken together with the saddle-node curve

$$\alpha_1 = 0,$$

this yields the bifurcation diagram. Finally, note that if $d_\mu(\bar{x}) > w_\mu^{uu}(\bar{x})$, the saddle-node bifurcation does not correspond to a bifurcation of the optimal vector field. □

**4.4.5. DISN$_3$ bifurcation.** It is possible that a "double" ISN singularity, denoted DISN$_3$, occurs if conditions (41) and (42) of Definition 4.7 are replaced by the condition that

$$p_0(x) = w^u(x)$$

for all $x \in \partial N$. This is clearly a codimension three situation.

**5. Conclusions.** The present article analyses the simplest bifurcations of infinite horizon optimization problems. Clearly, this theory will have to be extended to incorporate bifurcations of problems with higher dimensional state spaces. But a quick heuristic argument shows that matters will be more complicated already in systems with two state dimensions, with more nonstandard phenomena occurring: by nonstandard, we mean "not occurring in dynamical systems." For example indifference points in systems with planar state spaces are replaced by a complex of indifference curves, which divide the state space in basins of attraction. A topological change in the structure of these basins then corresponds to a bifurcation of the optimal vector field.

Bifurcations of optimal vector fields are linked to perestroikas of shocks, that is, indifference points (Arnold [2], Bogaevsky [3]). This link is not straightforward, as that theory deals with singularities of solutions of the time-dependent Hamilton-Jacobi equation, whereas we deal with families of time-independent Hamilton-Jacobi equations as well as the implied dynamics under optimal control. For instance, consider the perestroikas of shocks given in the second row of Figure 1 of Bogaevsky [3]. It shows how the structure of shocks in solutions of time-dependent Hamilton-Jacobi equations evolve as time changes. In the bifurcation diagrams of the bifurcations of optimal vector fields, shocks (that is, indifference points) of solutions evolve as a parameter changes. It is no coincidence that we obtain a subset of the perestroikas. The perestroika $A_3$ of Figure 1 of Bogaevsky [3] occurs stably in the IR$_1$ and IA$_1$ bifurcations; the perestroika $A_1^3$ of the same figure occurs in the "trident" bifurcation T$_1$, which is not stable for generic systems, but which is stable in a $\mathbb{Z}_2$-symmetric context.

The link between bifurcations of optimal vector fields and perestroikas of shocks will become more readily apparent for higher dimensional state spaces. Topological changes in the relative basins of attraction of planar attractors are expected to feature some of the perestroikas in the second row of Figure 2 in Bogaevsky [3]; this is a clear topic for future research. Moreover, in higher dimensional dynamical systems, bifurcations that are nonlocal in state space may occur, something that is impossible in the one-dimensional situation.

Another research area suggested by the present article is the bifurcation theory of optimal control problems in discrete time and of stochastic control problems. For initial work in the former direction see Moghayer and Wagener [26]; the latter problem has been taken up in Grass et al. [16], which is based on stochastic bifurcation theory (Wagenmakers et al. [40], Diks and Wagener [11, 12]). Also in these fields much work is still to be done.

**Acknowledgments.** We are grateful for helpful remarks from Christophe Deissenberg, Dieter Grass and Saeed Mohammadian Moghayer. Critical remarks of Alex Vladimirsky helpful to improve an earlier draft substantially. Financial support by the Netherlands Organisation for Scientific Research (NWO) under a MaGW-Vidi grant is gratefully acknowledged.

## References

[1] Arnold VI (1988) *Geometrical Methods in the Theory of Ordinary Differential Equations* (Springer, New York).
[2] Arnold VI (1990) *Singularities of Caustics and Wave Fronts* (Kluwer, Dordrecht, The Netherlands).
[3] Bogaevsky A (2005) Perestroikas of shocks in optimal control. *J. Math. Sci.* 126(4):1229–1242.
[4] Boltyanskii VG (1966) Sufficient conditions for optimality and the justification of the dynamic programming method. *SIAM J. Control Optim.* 4:326–361.
[5] Brock WA, Scheinkman J (1976) Global asymptotic stability of optimal control systems with applications to the theory of economic growth. *J. Econom. Theory* 12(1):164–190.
[6] Carathéodory C (1935) *Variationsrechnung und partielle Differentialgleichungen erster Ordnung* (B.G. Teubner, Berlin).
[7] Caulkins JP, Feichtinger G, Grass D, Tragler G (2007) Bifurcating DNS thresholds in a model of organizational bridge building. *J. Optim. Theory Appl.* 133(1):19–35.
[8] Caulkins JP, Feichtinger G, Johnson M, Tragler G, Yegorov Y (2005) Skiba thresholds in a model of controlled migration. *J. Econom. Behav. Organ.* 57(4):490–508.
[9] Clark CW (1976) *Mathematical Bioeconomics: The Optimal Management of Renewable Resources* (Wiley-Interscience, New York).
[10] Dechert WD, Nishimura K (1983) A complete characterization of optimal growth paths in an aggregated model with a non-concave production function. *J. Econom. Theory* 31(2):332–354.
[11] Diks CGH, Wagener FOO (2008) A bifurcation theory for a class of discrete time Markovian stochastic systems. *Physica D* 237(24):3297–3306.
[12] Diks CGH, Wagener FOO (2011) Phenomenological and ratio bifurcations of a class of discrete time stochastic processes. *Indagationes Mathematicae* 22(3–4):207–221.
[13] Feichtinger G, Tragler G (2002) Skiba thresholds in optimal control of illicit drug use. Zaccour G, ed. *Optimal Control and Differential Games: Essays in Honor of Steffen Jørgensen*, Chap. 1 (Kluwer, Dordrecht, The Netherlands), 3–22.
[14] Fleming WH, Soner HM (2006) *Controlled Markov Processes and Viscosity Solutions* (Springer, New York).
[15] Grass D (2012) Numerical computation of the optimal vector field: Exemplified by a fishery model. *J. Econom. Dynam. Control* 36(10):1626–1658.
[16] Grass D, Kiseleva T, Wagener FOO (2013) Small-noise asymptotics of Hamilton-Jacobi-Bellman equations and bifurcations of stochastic optimal control problems. Preprint.
[17] Hartman P (1960) On local homeomorphisms of Euclidean spaces. *Boletín de la Sociedad Matemática Mexicana* 5(2):220–241.
[18] Hartman P (1964) *Ordinary Differential Equations* (John Wiley & Sons, New York).
[19] Hinloopen J, Smrkolj G, Wagener FOO (2013) In defense of trusts: R&D cooperation in global perspective. Technical report, Tinbergen Institute, Amsterdam.
[20] Hinloopen J, Smrkolj G, Wagener FOO (2013) From mind to market: A global, dynamic analysis of R&D. *J. Econom. Dynam. Control* 37(12):2729–2754.
[21] Hirsch MW, Pugh CC, Shub M (1977) *Invariant Manifolds* (Springer, Berlin).
[22] Homburg AJ, Osinga HM, Vegter G (1995) On the computation of invariant manifolds of fixed points. *Zeitschrift für angewandte Mathematik und Physik* 46(2):171–187.
[23] Kiseleva T, Wagener FOO (2010) Bifurcations of optimal vector fields in the shallow lake model. *J. Econom. Dynam. Control* 34(5):825–843.
[24] Mäler KG, Xepapadeas A, de Zeeuw A (2003) The economics of shallow lakes. *Environment. Resource Econom.* 26:603–624.
[25] Michel P (1982) On the transversality condition in infinite horizon optimal problems. *Econometrica* 50(4):975–985.
[26] Moghayer SM, Wagener FOO (2009) Genesis of indifference thresholds and infinitely many indifference points in discrete time. CeNDEF Working Paper 09-14, University of Amsterdam,
[27] Palis J, Takens F (1993) *Hyperbolicity and Sensitive Chaotic Dynamics at Homoclinic Bifurcations* (Cambridge University Press, Cambridge, UK).
[28] Ramsey FP (1928) A mathematical theory of saving. *Econom. J.* 38(152):543–559.
[29] Rowthorn R, Toxvaerd F (2012) The optimal control of infectious diseases via prevention and treatment. CEPR Discussion paper no. DP8925, Centre for Economic Policy Research, London.
[30] Sethi SP (1977) Nearest feasible paths in optimal control problems: Theory, examples, and counterexamples. *J. Optim. Theory Appl.* 23(4):563–579.
[31] Sethi SP (1978) Optimal quarantine programmes for controlling an epidemic spread. *J. Oper. Res. Soc.* 29(3):265–268.
[32] Sethi SP (1979) Optimal advertising policy with the contagion model. *J. Optim. Theory Appl.* 29(4):615–627.
[33] Skiba AK (1978) Optimal growth with a convex-concave production function. *Econometrica* 46(3):527–539.
[34] Tahvonen O, Salo S (1996) Nonconvexities in optimal pollution accumulation. *J. Environment. Econom. Management* 31(2):160–177.
[35] Thom R (1975) *Structural Stability and Morphogenesis. An Outline of a General Theory of Models* (W. A. Benjamin, Reading, MA).
[36] Tragler G, Caulkins JP, Feichtinger G (2001) Optimal dynamic allocation of treatment and enforcement in illicit drug control. *Oper. Res.* 49(3):352–362.
[37] Wagener FOO (2003) Skiba points and heteroclinic bifurcations, with applications to the shallow lake system. *J. Econom. Dynam. Control* 27(9):1533–1561.
[38] Wagener FOO (2005) Structural analysis of optimal investment for firms with non-concave revenue. *J. Econom. Behav. Organ.* 57(4):474–489.
[39] Wagener FOO (2006) Skiba points for small discount rates. *J. Optim. Theory Appl.* 128(2):261–277.
[40] Wagenmakers E-J, Molenaar PCM, Grasman RPPP, Hartelman PAI, van der Maas HLJ (2005) Transformation invariant stochastic catastrophe theory. *Physica D* 211(3–4):263–276.
[41] Zeeman EC (1976) Catastrophe theory. *Sci. Amer.* 234(4):65–70, 75–83.