Multivariate Elliptical Truncated Moments (Supplementary Material)

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Abstract

In this study, we derived analytic expressions for the elliptical truncated moment generating function (MGF), the zeroth-, first-, and second-order moments of quadratic forms of the multivariate normal, Student’s $t$, and generalized hyperbolic distributions. The resulting formulas were tested in a numerical application to calculate an analytic expression of the expected shortfall of quadratic portfolios with the benefit that moment based sensitivity measures can be derived from the analytic expression. The convergence rate of the analytic expression is fast – one iteration – for small closed integration domains, and slower for open integration domains when compared to the Monte Carlo integration method. The analytic formulas provide a theoretical framework for calculations in robust estimation, robust regression, outlier detection, design of experiments, and stochastic extensions of deterministic elliptical curves results.

In this online appendix we provide a detailed derivation of the proofs.

Keywords: Multivariate truncated moments, Quadratic forms, Elliptical Truncation, Tail moments, Parametric distributions, Elliptical functions

2. Analytic expressions for the MGF of elliptical truncated quadratic forms in multivariate normal (MVN) distributions

Proposition 2.1. Let $X^\top = (X_1, \ldots, X_n)$ have the MVN distribution, with mean vector $\mu_X$ and covariance matrix $\Sigma_X$, $a \in \mathbb{R}$. Define an ellipsoid restriction $C(x, a) = \{x \in \mathbb{R}^n : a \leq (x - \mu_A)^\top A (x - \mu_A)\}$. The truncated MGF of $X$ at the ellipsoid $C(X, a)$ is equal to,

$$\mathbb{E}[\exp(t^\top X)|C(X, a)] = m(t, C(x, a)) = L^{-1} \exp \left( t^\top \mu_X + \frac{1}{2} t^\top \Sigma_X t \right) H_{n; \text{E, b}(a/p)},$$

the truncated zeroth-, first-, and second-order moment of $X$ at the ellipsoid $C(X, a)$ is equal to,

$$\Pr[X|C(X, a)] = m_0(C(X, a)) = L = H_{n; \text{E, b}_0}(a/p),$$

$$\mathbb{E}[X|C(X, a)] = m_1(C(X, a)) = \mathbb{E}\left[ X | a \leq (X - \mu_A)^\top A (X - \mu_A) \right]$$

$$= \mu_X + L^{-1} \sum_{i=0}^{\infty} G_{n+2i}(a/p) c_i |_{a_0}, \quad (1)$$

\textsuperscript{∗}The author wish to thank seminar participants at the Mathematical Finance Days conference 2013, held at the HEC Montreal, and organized by the Institut de Finance Mathématique de Montréal (IFM2), specially to Mario Ghossoub and Alexandre Roch, chair of the derivatives pricing session.

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\[
\mathbb{E} [XX^\top | C(X, a)] = m_2(C(x), a) = \mu_X \mu_X^\top + \Sigma_X + L^{-1} \mu_X \left( \sum_{i=0}^{\infty} G_{n+2i}(a/p)c_i[\partial t, 0] \right)^\top + \\
L^{-1} \left( \sum_{i=0}^{\infty} G_{n+2i}(a/p)c_i[\partial t, 0] \right) \mu_X^\top + L^{-1} \sum_{i=0}^{\infty} G_{n+2i}(a/p)c_i[\partial t, 0], (2)
\]

where,

\[
H_{n,E,b(t)}(s) = \sum_{i=0}^{\infty} c_i G_{n+2i}(s),
\]

where the diagonal matrix \( E = \text{diag}(c_1, \ldots, c_n) \) has the eigenvalues of \( \text{PAP} \) with \( \text{P} \text{P}^\top = \Sigma_X \), and vector \( \mathbf{b}(t) = (b_1(t), \ldots, b_n(t))^\top \) is defined as,

\[
\mathbf{b}(t) = K^{-1} \left( \Sigma_X^{-1/2}(\mu_A - \mu_X) - \Sigma_X^{1/2} \right) \mathbf{c} = K^{-1} \mathbf{c} \mathbf{c}^\top (\mu_A - \mu_X - \Sigma_X t),
\]

with \( K \) a matrix with the eigenvectors of the orthogonal decomposition of \( \Sigma_X^{1/2} \mathbf{A} \Sigma_X^{1/2} \), vector \( \mathbf{b}_0 = \{ b_{1,0}, \ldots, b_{n,0} \} \) is equal to the vector \( \mathbf{b}(t) \) evaluated at \( t = 0 \), the coefficients \( c_i \) are defined through a recursive equation,

\[
c_0 = \exp \left( -\frac{1}{2} \mathbf{b}(t)^\top \mathbf{b}(t) \right) \prod_{j=1}^{n} (p/e_j)^{1/2},
\]

\[
c_i = (2i)^{-1} \sum_{k=0}^{i-1} d_{i-k} c_k, \forall i \geq 1,
\]

and coefficients are equal,

\[
d_i = \sum_{j=1}^{n} (1 - p/e_j)^i + p \sum_{j=1}^{n} \left( b_j(t)^2/e_j \right) (1 - p/e_j)^{i-1},
\]

for \( i \in \{1, \ldots, n\} \), coefficients \( c_{i,0} \) are equal to \( c_i \) substituting \( \mathbf{b}(t) \) by \( \mathbf{b}_0 \),

\[
c_{i,0} \equiv \left[ c_i \right]_{t=0},
\]

\( G_{n+2i}(s) = 1 - F_{n+2i}(s) \), \( F_{n+2i}(s) \) is the distribution of a central chi-squared with \( n + 2i \) degrees of freedom, and the term \( c_i[\partial t, 0] \) refers to a vector of the partial derivatives of the coefficient \( c_i \), where the component \( j \)-th is \( \left[ \frac{\partial c_i}{\partial t} \right]_{t=0} \), then,

\[
\left[ \frac{\partial c_i}{\partial t} \right]_{t=0} = \frac{\partial^2 c_i}{\partial t^2} \bigg|_{t=0},
\]

\[
\left[ \frac{\partial c_i}{\partial t} \right]_{t=0}
\]

\[1\]The central chi-squared cumulative density function with \( \nu \) degrees of freedom is,

\[
F_{\nu}(x) = \frac{\gamma(\nu/2, x/2)}{\Gamma(\nu/2)},
\]

where \( \Gamma(x) \) is the gamma function, \( \gamma(x, y) \) is the lower-incomplete gamma function,

\[
\Gamma(x) = \int_0^\infty t^{x-1} \exp(-t) dt,
\]

\[
\gamma(x, y) = \int_0^y t^{x-1} \exp(-t) dt.
\]

2
Proof. The density of $X$ is,

$$
\phi_n(x, \mu_X, \Sigma_X) = (2\pi)^{-n/2}|\Sigma_X|^{-1/2} \exp \left( -\frac{1}{2}(x - \mu_X)^\top \Sigma_X^{-1}(x - \mu_X) \right).
$$

(7)

To calculate (1) and (2), we use the moment generating function approach of Tallis [5]. Define the abbreviated integral operator as,

$$
\int_{a_1}^\infty \cdots \int_{a_n}^\infty (\cdot)dx_1 \cdots dx_n = \int_{a_n}^\infty (\cdot)dx,
$$

(6)

Using the definition in (7), the truncated MGF of $X$ can be calculated as,

$$
E[\exp(t^\top X)|a \leq (X - \mu_A)^\top A(X - \mu_A)| = m(t,C(x,a))
$$

$$
= L^{-1}(2\pi)^{-n/2}|\Sigma_X|^{-1/2} \int_{C(x,a)} \exp \left( -\frac{1}{2}(x - \mu_X)^\top \Sigma_X^{-1}(x - \mu_X) + t^\top x \right)dx,
$$

(8)

where,

$$
L = (2\pi)^{-n/2}|\Sigma|^{-1/2} \int_{C(x,a)} \exp \left( -\frac{1}{2}(x - \mu_X)^\top \Sigma_X^{-1}(x - \mu_X) \right)dx.
$$

Let $y = x - \mu_X - \Sigma_X t$, then (8) becomes,

$$
m(t,C(x,a)) = L^{-1}(2\pi)^{-n/2}|\Sigma|^{-1/2} \int_{C(y,a)} \exp \left( t^\top \mu_X + \frac{1}{2}t^\top \Sigma_X t \right)\int_{C(y,a)} \exp \left( -\frac{1}{2}y^\top \Sigma_X^{-1}y \right)dy,
$$

(9)

where, $C(y,a) = \{y \in \mathbb{R}^n : a \leq (y - (\mu_A - \mu_X - \Sigma_X t))^\top A(y - (\mu_A - \mu_X - \Sigma_X t))\}$

The distinction in (9) is from the MVN, with ellipsoid restriction $C(y,a)$. Applying an orthogonal decomposition $E = K^\top P^\top A^\top P K$, where $PP^\top = \Sigma_X$, and $K$ the orthogonal matrix with the eigenvectors of $P^\top A P$, setting $b(t) = (b_1(t), \ldots, b_n(t))^\top$ such that,

$$
b(t) = K^{-1}P^{-1}(\mu_A - \mu_X - \Sigma_X t),
$$

and defining $z = K^{-1}P^{-1}y$, the distribution (9) is transformed in,

$$
m(t,C(x,a)) = L^{-1}(2\pi)^{-n/2} \int_{C(z,a)} \exp \left( t^\top \mu_X + \frac{1}{2}t^\top \Sigma_X t \right)\int_{C(z,a)} \exp \left( -\frac{1}{2}z^\top z \right)dz,
$$

(10)

where,

$$
C(z,a) = \{z \in \mathbb{R}^n : a \leq (z - b(t))^\top E(z - b(t))\},
$$

with $E$ a diagonal matrix with the eigenvalues of $PP^\top$, and diagonal components $e_i, 1 \leq i \leq n$. From Ruben [3], the distribution in (10) can be expressed as a series expansion of central chi-squared random variables,

$$
m(t,C(x,a)) = L^{-1} \exp \left( t^\top \mu_X + \frac{1}{2}t^\top \Sigma_X t \right) \sum_{i=0}^\infty G_{n+2i}(a/p)c_i,
$$

$$
= L^{-1} \exp \left( t^\top \mu_X + \frac{1}{2}t^\top \Sigma_X t \right) H_n E_b(t)(a/p),
$$

(11)

where $p$ is an arbitrary positive constant.\(^2\) Set $t = 0$ in (11) and define,

$$
b_0 = |b(t)|_{t=0} = (b_1(0), \ldots, b_n(0)) = K^{-1}P^{-1}(\mu_A - \mu_X),
$$

\(^2\)In Ruben [3] an upper bound for $p$ is derived, $p < \min(e_i), 1 \leq i \leq n$. In Genz and Bretz [2], they referenced an algorithm of Sheil and O’Muircheartaigh [4] where a series of values for $p$ are tested, finding that $p = 29/32 \min(e_i)$ had an optimal balance between the speed of the algorithm and the convergence.
we derive the value of \( L \), the zeroth-order moment,

\[
L = H_{n,E,b_0}(a/p),
\]

where,

\[
H_{n,E,b_0}(a/p) = \sum_{i=0}^{\infty} G_{n+2i}(a/p)c_{i,0},
\]

and \( c_{i,0}, d_{i,0} \) are defined by (6), and (3), (4), and (5) substituting \( b(t) \) by \( b_0 \). First-order moments (1) are calculated deriving the moment generating function,

\[
\begin{align*}
E \left[ X^a \right] & = m_1(C(x,a)) = \frac{\partial m(t)}{\partial t} \bigg|_{t=0} \\
& = \left[ \frac{\partial}{\partial t} \left( \exp \left( t^T \mu_X + \frac{1}{2} t^T \Sigma_X t \right) \frac{H_{n,E,b(t)}(a/p)}{H_{n,E,b_0}(a/p)} \right) \right]_{t=0} \\
& = \mu_X + \frac{1}{H_{n,E,b_0}(a/p)} \left[ \frac{\partial H_{n,E,b(t)}(a/p)}{\partial t} \right]_{t=0} \\
& = \mu_X + \frac{1}{H_{n,E,b_0}(a/p)} \sum_{i=0}^{\infty} G_{n+2i}(a/p) \frac{\partial c_i}{\partial t} \bigg|_{t=0}.
\end{align*}
\]

The partial derivative of coefficients is expressed as a recursive equation. Define,

\[
\begin{align*}
c_{i,[a,t,0]} & \equiv \left[ \frac{\partial c_i}{\partial t} \right]_{t=0} \\
c_{i,0} & \equiv \left[ c_i \right]_{t=0}.
\end{align*}
\]

The term \( c_{i,[a,t,0]} \) refers to a vector where the component \( j \)-th is \( \left[ \frac{\partial m}{\partial t} \right]_{t=0} \). We derive,

\[
\begin{align*}
\frac{\partial}{\partial t_k} b_j \bigg|_{t=0} & = -K_{j,:}^{-1} \Sigma X_{(:,k)}, \\
\frac{\partial}{\partial t_k} b_j^2 \bigg|_{t=0} & = -2b_{j,0} K_{j,:}^{-1} \Sigma X_{(:,k)},
\end{align*}
\]

where \( K_{i,:}^{-1} \) is the row \( i \)-th of the inverse of the eigenvector matrix \( K^{-1} \). Then,

\[
\begin{align*}
c_{0,[a,0]} & = \exp \left( -\frac{1}{2} b_0^T b_0 \right) \left( b_0^T K^{-1} \Sigma X \right)^T \prod_{j=1}^{n} (p/e_j)^{1/2}, \\
c_{i,[a,0]} & = (2i)^{-1} \sum_{k=0}^{i-1} d_{i-k,0}[a,t,0]c_{k,0} + \sum_{k=0}^{i-1} d_{i-k,0}c_{k,[a,0]} \right), i \geq 1, \quad (12) \\
d_{i,[a,0]} & = -2ip \left( \lambda \odot K^{-1} \right)^{-1} \Sigma X, \]

where \( \odot \) is the element to element matrix multiplication, \( \lambda(i) = \{ \lambda_1(i), \ldots, \lambda_n(i) \} \), \( \lambda_j(i) = \frac{(1-p/e_j)^{1/2}}{e_j} b_{j,0} \),

\( c_{0,[a,0]} \) is a vector of dimension \( n \) that has as component \( j \)-th,

\[
\exp \left( -\frac{1}{2} b_0^T b_0 \right) \left( b_0^T K^{-1} \Sigma X_{(:,j)} \right)^T \prod_{i=1}^{n} (p/e_i)^{1/2},
\]
and $d_{i,[0,0]}$ is a vector of dimension $n$ that has as component $j$th,

$$-2ip(\lambda \odot K^{-1})P^{-1}\Sigma_{X(j,:)}$$

with $\Sigma_{X(j,:)}$ and $P_{j,:}$ the $j$-th column of $\Sigma_X$ and $P$.

Second-order moments in (2) are calculated deriving $m(t)$ once more,

$$E[XX^T | a \leq (X - \mu)^T \Sigma_X^{-1}(X - \mu)] = m_2(C(x,a)) = \frac{\partial^2 m(t)}{\partial t \partial t} \bigg|_{t=0}$$

$$= \left[ \frac{\partial^2}{\partial t \partial t} \left( \exp \left( t^T \mu_X + \frac{1}{2} t^T \Sigma_X t \right) \frac{H_n.E.b(a/p)}{H_n.E.b(a/p)}(a/p) \right) \right]_{t=0} + \frac{\partial H_n.E.b(a/p)}{\partial t} \bigg|_{t=0}$$

$$= \mu_X^T \Sigma_X + \mu_X^T \frac{1}{H_n.E.b(a/p)} \left[ \frac{\partial H_n.E.b(a/p)}{\partial t} \bigg|_{t=0} \right] + \frac{1}{H_n.E.b(a/p)} \left[ \frac{\partial^2 H_n.E.b(a/p)}{\partial t^2} \bigg|_{t=0} \right]$$

$$= \mu_X^T \Sigma_X + \mu_X \left( \frac{\sum_{i=0}^{\infty} G_{n+2i}(a/p)c_i[\alpha,0]}{H_n.E.b(a/p)} \right)^T + \mu_X \left( \frac{\sum_{i=0}^{\infty} G_{n+2i}(a/p)c_i[\alpha,0]}{H_n.E.b(a/p)} \right)^T$$

$$\begin{align*}
\quad &+ \frac{1}{H_n.E.b(a/p)} \left[ \frac{\partial H_n.E.b(a/p)}{\partial t} \bigg|_{t=0} \right] \mu_X + \frac{1}{H_n.E.b(a/p)} \left[ \frac{\partial^2 H_n.E.b(a/p)}{\partial t^2} \bigg|_{t=0} \right] \mu_X
\quad &+ \frac{1}{H_n.E.b(a/p)} \left[ \frac{\partial H_n.E.b(a/p)}{\partial t} \bigg|_{t=0} \right] \mu_X + \frac{1}{H_n.E.b(a/p)} \left[ \frac{\partial^2 H_n.E.b(a/p)}{\partial t^2} \bigg|_{t=0} \right] \mu_X, \quad (13)
\end{align*}$$

where $c_{i,[\alpha,0]} \equiv \left[ \frac{\partial^2}{\partial t^2} \bigg|_{t=0} \right]$, and $c_{i,[\alpha,0]}$ is a matrix with $(j,k)$-th component $\left[ \frac{\partial^2}{\partial t^2} \bigg|_{t=0} \right]$.

We calculate $c_{0,[\alpha,0]}$: Let,

$$\left[ \frac{\partial^2 t^2}{\partial t_j \partial t_k} \right]_{t_j,t_k=0} = 2 \left( K_{j,\cdot}^{-1} P^{-1} \Sigma_{X(i,:)} \right) \left( K_{j,\cdot}^{-1} P^{-1} \Sigma_{X(i,:)} \right)^T$$

where $\Sigma_{X(i,:)}$ is the $k$-th column of the matrix $\Sigma_X$. Then,

$$c_{0,[\alpha,0]} = \exp \left( -\frac{1}{2} b_0^T b_0 \right) \prod_{j=1}^n (p/e_j)^{1/2} \left( \left( b_0^T K^{-1} P^{-1} \Sigma_X \right) \left( b_0^T K^{-1} P^{-1} \Sigma_X \right)^T - \left( K^{-1} P^{-1} \Sigma_X \right) \left( K^{-1} P^{-1} \Sigma_X \right)^T \right), \quad (14)$$

$$c_{i,[\alpha,0]} = (2i)^{-1} \sum_{k=0}^{\infty} \left( d_{i-k,[\alpha,0]} e_{k,0} + d_{i-k,[\alpha,0]} e_{k,[\alpha,0]}^T + d_{i-k,[\alpha,0]} e_{k,0} \right), \quad (15)$$

$$d_{i,[\alpha,0]} = 2ip \left( (\Lambda \odot K^{-1}) P^{-1} \Sigma_X \right)^T \left( K^{-1} P^{-1} \Sigma_X \right), \quad (16)$$

where $\Lambda = (\lambda, \ldots, \lambda)$ is a $n \times n$ matrix with $\lambda$ on each column. The terms in (14), (15), and (16) are matrices. Substituting the definition of $L$ in (13) yields the result.

3. Analytic expressions for the MGF of an elliptical truncated multivariate Student’s $t$ distribution

Proposition 3.1. Let $Z$ have the MVN distribution with pdf (7). Let $\eta$ have a gamma distribution with pdf,

$$f_\eta(x; \alpha, \beta) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} \exp(-x/\beta),$$

5
with shape parameter \( \alpha = \nu/2 \), and scale parameter \( \beta = 2/\nu \). Define \( X \) as the scale mixture,
\[
X^\top = (X_1, \ldots, X_n) = \eta^{-1/2}Z^\top.
\]
Then \( X \) has a multivariate standard Student’s \( \nu \)-distribution with \( \nu \) degrees of freedom.

**Proof.** Let us define the scale mixture of a gamma distribution and an MVN distribution as \( X_i = \eta^{-1/2}Z_i \).

Then the distribution of \( X \) conditional on \( \eta \) is
\[
f_{\eta^{-1/2}Z|\eta} = (2\pi)^{-n/2}|\Sigma_X|^{-1/2} \exp \left( -\frac{1}{2\eta}x^\top \Sigma_X^{-1}x \right) \eta^{n/2}.
\]
But (17) is the pdf of \( N(0, \eta^{-1}\Sigma_X) \). We have that \( f_{\eta^{-1/2}Z} = f_{\eta^{-1/2}Z|\eta}f_\eta \) with \( f_\eta \) equal to (17) with parameters \( \alpha = \nu/2, \beta = 2/\nu \). Hence,
\[
f_{\eta^{-1/2}Z} = (2\pi)^{-n/2}|\Sigma_X|^{-1/2} \int_0^\infty \exp \left( -\frac{1}{2\eta}x^\top \Sigma_X^{-1}x \right) \eta^{\nu/2} \frac{1}{(\nu/2)^{\nu/2}} \eta^{\nu/2-1} \exp \left( -\frac{\eta}{2/\nu} \right) d\eta,
\]
which is the density function of the MST distribution.

**Proposition 3.2.** Let \( X \) have an MST distribution as in,
\[
f(x, \mu_X, \Sigma_X, \nu) = \frac{\Gamma((\nu+n)/2)}{\Gamma(1/2)\nu^{\nu/2}\Sigma_X^{1/2}} \left( 1 + \frac{1}{\nu}(x - \mu_X)^\top \Sigma_X^{-1}(x - \mu_X) \right)^{-(\nu+n)/2}.
\]
Define \( C(x, a) = \{ x \in \mathbb{R}^n : a \leq (x - \mu_A)^\top A(x - \mu_A) \} \), then \( X \) has an approximate elliptical truncated MGF over the region \( C(X, a) \) denoted by,
\[
m(t, C(x, a)) = \mathbb{E}_\eta \left[ L_\eta^{-1} \exp \left( \eta^{-1/2}t^\top \mu_Z + \eta^{-1/2}t^\top \Sigma_Z t \right) H_{\eta}^{n,E,b(\eta^{-1/2}t)}(\eta a/p) \right],
\]
where,
\[
H_{\eta,n,E,b(\eta^{-1/2}t)}(\eta a/p) = \sum_{i=0}^\infty G_{n+2i}(\eta a/p)c_i, \eta^{-1/2}t,
\]
\[
L_\eta = H_{\eta,n,E,b(\eta^{-1/2}t)}(\eta a/p) = \sum_{i=0}^\infty G_{n+2i}(\eta a/p)c_i, 0, \eta.
\]
with coefficients \( c_i, \eta^{-1/2}t \) equal to coefficients \( c_i \) as in (3) and (4) substituting \( b(t) \) by \( b(\eta^{-1/2}t) \) defined by,
\[
b(\eta^{-1/2}t) = \mathbf{K}^{-1}\mathbf{P}^{-1} \left( \eta^{1/2}\mu_A - \mu_X - \eta^{-1/2}\Sigma_X t \right),
\]
and coefficients \( c_i, 0, \eta \) equal to \( c_i \) as in (3) and (4) substituting \( b(t) \) by \( b(0) \) defined by,
\[
b(0) = (b_{1,0}, \ldots, b_{n,0}) = \mathbf{K}^{-1}\mathbf{P}^{-1} (\eta^{1/2}\mu_A - \mu_X).
\]
Assume without loss of generality that \( \mu_X = 0 \). The elliptical truncated zeroth-, first- and second-order moments of \( X \) at \( C(X, a) \) are,
\[
\Pr[X|a \leq (X - \mu_A)^\top A(X - \mu_A)] = L = \sum_{i=0}^\infty \sum_{j=0}^i \zeta_{ij}c_{ij, 0},
\]
\[
\mathbb{E}[X|a \leq (X - \mu_A)^\top A(X - \mu_A)] = L^{-1} \sum_{i=0}^\infty \sum_{j=0}^i \zeta_{ij}d_{ij, 0},
\]
\[
\mathbb{E}[XX^\top|a \leq (X - \mu_A)^\top A(X - \mu_A)] = L^{-1} \sum_{i=0}^\infty \sum_{j=0}^i \zeta_{ij}e_{ij, 0},
\]

where,

\[ \zeta_{j,i} = (1 + b_{nj} b_0/\nu)^{-\nu/2-j} \left( \frac{2}{\nu} \right)^{\nu/2} \frac{\Gamma \left( \frac{n+2j+2}{2} \right)}{\Gamma \left( \nu/2 \right)} \frac{\Gamma \left( \frac{n+2i}{2} \right)}{\Gamma \left( \nu/2 \right)} B \left( \frac{\nu}{2} + j \frac{n+2i}{2} \right) \times \]

\[ I_{\nu} \left( \frac{1+b_{nj} b_0/\nu}{\left( \nu + b_0 a_0 + a \right)^2} \right) \left( \frac{1}{2} + j \frac{n+2i}{2} \right), \]

and \( c_{m,j,0} \) are numerical coefficients calculated by solving the recurrence (4) for the MST distribution case, with

\[ B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt, \]
\[ B(x; a, b) = \int_0^x t^{a-1} (1-t)^{b-1} dt, \]
\[ I_\nu(a, b) = \frac{B(x; a, b)}{B(a, b)}, \]

the beta, incomplete beta, and the regularized incomplete beta functions.

Proof. The truncated MGF of \( X \) at the ellipsoid \( C(X, a) \) can be approximated as,

\[ E_\eta \left[ \exp(t^\top X) | C(X, a) \right] = m(t, C(X, a)) = E_\eta \left[ E_\eta \left[ \exp(t^\top \eta^{-1/2} Z) | \eta, C(X, a) \right] \right], \quad (20) \]

where the condition \( C(X, a) \) can be transformed as,

\[ C(X, a) = a \leq (X - \mu_A)^\top A (X - \mu_A) = a \leq \eta^{-1} (Z - \eta^{1/2} \mu_A)^\top A (Z - \eta^{1/2} \mu_A) = \eta a \leq (Z - \eta^{1/2} \mu_A)^\top A (Z - \eta^{1/2} \mu_A) = C(\eta^{-1/2} Z, a). \]

But using the results of Section 2 the internal expression of (20), \( E_\eta \left[ \exp(t^\top \eta^{-1/2} Z) | \eta, C(\eta^{-1/2} Z, a) \right] \), is the MGF of an MVN distribution, and it can be calculated as,

\[ E_\eta \left[ \exp(t^\top \eta^{-1/2} Z) | \eta, C(\eta^{-1/2} Z, a) \right] = L_\eta^{-1} \exp \left( \eta^{-1/2} t^\top \mu_Z + \eta^{-1/2} \frac{1}{2} t^\top \Sigma_Z t \right) H_{\mu \eta; \mathcal{E}_b(\eta^{-1/2} \Sigma)}, (21) \]

Then, the MGF expression (20) can be approximated by,

\[ m(t, C(X, a)) = E_\eta \left[ L_\eta^{-1} \exp \left( \eta^{-1/2} t^\top \mu_Z + \eta^{-1/2} \frac{1}{2} t^\top \Sigma_Z t \right) H_{\mu \eta; \mathcal{E}_b(\eta^{-1/2} \Sigma)} \right]. (22) \]

If we set \( t = 0 \) in (22), we have as a result that,

\[ 1 = E_\eta \left[ L_\eta^{-1} H_{\mu \eta; \mathcal{E}_b(\eta^{-1/2} \Sigma)} \right], (23) \]

and equality in (18) will hold if \( E_\eta \left[ \cdot \right] \) exists. In the case \( a - \mu_A \leq 0 \), the integral of the MGF approximation in (22) and the integral (23) are the integrals of a continuous density function over a compact set and we have a convergence of the integral. Otherwise, if the MGF of the non-truncated variable is not convergent the MGF over the truncated region can be not convergent.

The truncated zeroth-order moment (probability) of \( X \) at the ellipsoid \( C(X, a) \) is,

\[ L = E_\eta \left[ L_\eta \right]. (24) \]
To calculate (24), we need to develop the series (18),

\[ E_\eta \left[ \sum_{i=0}^{\infty} G_{n+2i}(\eta a/p)c_{i;0,0} \right] = E_\eta [G_n(\eta a/p)c_{0,0,0}] + E_\eta [G_{n+2}(\eta a/p)c_{1;0,0}] + E_\eta [G_{n+4}(\eta a/p)c_{2;0,0}] + \ldots. \]

\[ = E_\eta \left[ G_n(\eta a/p) \exp \left( -\frac{1}{2} b_{0;0}^\top b_{0;0} \right) \prod_{j=1}^{n} (p/e_j)^{1/2} \right] + \]

\[ E_\eta \left[ G_{n+2}(\eta a/p) 2^{-1} \exp \left( -\frac{1}{2} b_{0;0}^\top b_{0;0} \right) \prod_{j=1}^{n} (p/e_j)^{1/2} d_{1;0,0} \right] + \]

\[ E_\eta \left[ G_{n+4}(\eta a/p) 4^{-1} \exp \left( -\frac{1}{2} b_{0;0}^\top b_{0;0} \right) \prod_{j=1}^{n} (p/e_j)^{1/2} (d_{2;0,0} + 2^{-1} d_{1;0,0}^2) \right] + \ldots. \]

(25)

and,

\[ d_{i;0,n} = \sum_{j=1}^{n} (1 - p/e_j)^i + ip \sum_{j=1}^{n} (b_{j,0,0}/e_j) (1 - p/e_j)^{i-1}. \]

Terms \( b_{0,0;0} \), \( d_i \) and \( G_{n+2i}(\eta a/p) \) in (25) contain \( \eta \), then for calculating the \( E_\eta \) we use a series expansion approach. First, considering \( \mu_X = 0 \) the following factors can be applied to terms dependent on \( \eta \),

\[ b_{0,0;0}^\top b_{0,0} = \eta b_0^\top b_0, \]

\[ d_{i;0,0} = \sum_{j=1}^{n} (1 - p/e_j)^i + ip \sum_{j=1}^{n} (b_{j,0,0}/e_j) (1 - p/e_j)^{i-1}, \]

with \( dA_i = \sum_{j=1}^{n} (1 - p/e_j)^i, dB_i = ip \sum_{j=1}^{n} (b_{j,0,0}/e_j) (1 - p/e_j)^{i-1} \). After applying (26) to the recurrence of \( c_{i;0,0} \) as in (3), (4), and (5) with the corresponding change (19), lead us to denote the coefficients \( c_{i;0,0} \) around two terms, \( -\frac{1}{2} \eta b_0^\top b_0 \) and \( \eta \) as,

\[ c_{i;0,0} = \exp \left( -\frac{1}{2} \eta b_0^\top b_0 \right) \left( c_{i,0} + c_{i+1,0} \eta + c_{i+2,0} \eta^2 + \cdots + c_{i,0} \eta^i \right), \]

(27)

with \( c_{i,j,0}, i, j \geq 0 \) that are coefficients not dependent on \( \eta \). The value of the coefficients \( c_{i,0} \) are found by equating (4) with (27) and substituting \( b(t) \) by \( b_{0,0;0} \), considering the relation derived in (26). Hence, the terms in the series (25) can be denoted by,

\[ E_\eta [G_{n+2i}(\eta a/p)c_{i;0,0}] = E_\eta \left[ G_{n+2i}(\eta a/p) \exp \left( -\frac{1}{2} \eta b_0^\top b_0 \right) \right] c_{i,0} + \]

\[ E_\eta \left[ G_{n+2i}(\eta a/p) \eta \exp \left( -\frac{1}{2} \eta b_0^\top b_0 \right) \right] c_{i+1,0} + \cdots + \]

\[ E_\eta \left[ G_{n+2i}(\eta a/p) \eta^i \exp \left( -\frac{1}{2} \eta b_0^\top b_0 \right) \right] c_{i,0}. \]

(28)

We calculate \( E_\eta [G_{n+2i}(\eta a/p) \eta^j \exp \left( -\frac{1}{2} \eta b_0^\top b_0 \right)] \) for \( j \in \{1, \ldots, i\} \) applying the definitions of the chi-
squared distribution and \( \mathbb{E}_\eta \),

\[
\mathbb{E}_\eta \left[ G_{n+2i}(\eta a/p) \eta^j \exp \left( -\frac{1}{2} \eta b_0^\top b_0 \right) \right] = \int_0^\infty \int_{\eta a/p}^\infty \frac{x^{(n+2i)/2 - 1} \exp \left( -\frac{x}{2} \right)}{2^{(n+2i)/2} \Gamma \left( \frac{n+2i}{2} \right)} \times \eta^{\nu/2 - 1} \exp \left( -\frac{\eta y}{2} \right) \times \eta^j \exp \left( -\frac{\eta y}{2} \eta b_0^\top b_0 \right) \, dx \, dy.
\]

(29)

where \( \nu/2 \) and \( 2/\nu \) are the shape and scale parameters of the \( \eta \) variable. Apply the change of variable \( \eta y = x \) and (29) becomes,

\[
= \int_0^\infty \int_{\eta a/p}^\infty \frac{(\eta y)^{(n+2i)/2 - 1} \exp \left( -\frac{1}{2} \eta y \right)}{2^{(n+2i)/2} \Gamma \left( \frac{n+2i}{2} \right)} \times \eta^{\nu/2 + j} \exp \left( -\frac{\eta y}{2} \right) \times \frac{\eta^j \exp \left( -\frac{\eta y}{2} \eta b_0^\top b_0 \right)}{(2/\nu)^{\nu/2} \Gamma \left( \nu/2 \right)} \, d\eta \, dy.
\]

(30)

Now apply the change of variables \( u^2 = y \) and later \( w = \frac{1}{2} \eta (u^2 + \nu + b_0^\top b_0) \), hence,

\[
= \int_0^\infty \int_{(\eta a/p)^{1/2}}^\infty \frac{(2w/(u^2 + \nu + b_0^\top b_0))^{(n+2i+2\nu)/(2-1)}u^{n+2i-1} \exp \left( -w \right)}{2^{(n+2i)/2} \Gamma \left( \frac{n+2i}{2} \right) (2/\nu)^{\nu/2} \Gamma \left( \nu/2 \right)} \times \frac{2(u^2 + \nu + b_0^\top b_0)^{-1}(2w) \, du \, dw}{2(u^2 + \nu + b_0^\top b_0)^{-1}(2w) \, du \, dw}.
\]

Applying Fubini and by definition of the function \( \Gamma(\cdot) \), we have,

\[
= \int_0^\infty \frac{2^{j+1} \nu^{\nu/2} (u^2 + \nu + b_0^\top b_0)^{-(n+2i+2\nu)/2} \, du}{\Gamma \left( \frac{n+2i}{2} \right) \Gamma \left( \nu/2 \right)} \times \int_0^\infty \eta^{\nu/2} (u^2 + \nu + b_0^\top b_0)^{-1} \, d\eta.
\]

(30)

To solve (30), we apply the change of variable \( s = \nu (u^2 + \nu + b_0^\top b_0)^{-1} (1 + b_0^\top b_0) / \nu \), then we have,

\[
= 2^{j+1} \frac{\Gamma \left( \frac{n+2i+2\nu}{2} \right)}{\Gamma \left( \frac{n+2i}{2} \right) \Gamma \left( \nu/2 \right)} \frac{\nu^{\nu/2} (u^2 + \nu + b_0^\top b_0)^{-(n+2i+2\nu)/2} \, du}{\Gamma \left( \frac{n+2i}{2} \right) \Gamma \left( \nu/2 \right)} \times \left( s (1 + b_0^\top b_0) / \nu \right) \exp \left( -s (1 + b_0^\top b_0) / \nu \right) \, ds.
\]

Therefore,

\[
\mathbb{E}_\eta \left[ G_{n+2i}(\eta a/p) \eta^j \exp \left( -\frac{1}{2} \eta b_0^\top b_0 \right) \right] = (1 + b_0^\top b_0) / \nu \times \frac{\nu^{\nu/2 - j} \left( \frac{2}{\nu} \right)^j \Gamma \left( \frac{n+2i+2\nu}{2} \right)}{\Gamma \left( \frac{n+2i}{2} \right) \Gamma \left( \nu/2 \right)} \times \frac{\nu^{j + \frac{n+2i}{2}} \Gamma \left( \nu + j \right)}{\Gamma \left( \frac{n+2i}{2} \right) \Gamma \left( \nu/2 \right)} \times B \left( \frac{\nu}{2}, j, \frac{n+2i}{2} \right).
\]

(31)
where $B(y, z)$ is the beta function and $I_x(y, z)$ is the lower incomplete beta function. Having (25), (27), (28), and (31) the solution for (24) is derived.

The elliptical truncated first-order moments of $X$ are calculated,

$$E[X|a \leq (X - \mu_A)^\top A(X - \mu_A)] = E_\eta[E[\eta^{-1/2}Z|\eta, a \leq (X - \mu_A)^\top A(X - \mu_A)]] = E_\eta[E[\eta^{-1/2}Z|\eta, a \leq \eta^{-1}(Z - \eta^{-1/2}\mu_A)^\top A(Z - \eta^{-1/2}\mu_A)]] = E_\eta[\eta^{-1/2}E[Z|\eta, \eta a \leq (Z - \eta^{-1/2}\mu_A)^\top A(Z - \eta^{-1/2}\mu_A)]] \quad (32)$$

The internal expression,

$$E[Z|\eta, \eta a \leq (Z - \eta^{-1/2}\mu_A)^\top A(Z - \eta^{-1/2}\mu_A)],$$

of (32) is the first-order expected value of a normal distribution truncated with the ellipsoid $\eta a \leq (Z - \eta^{-1/2}\mu_A)^\top A(Z - \eta^{-1/2}\mu_A)$ similar to (21), then the results of Section 2 can be used,

$$E[Z|\eta, \eta a \leq (Z - \eta^{-1/2}\mu_A)^\top A(Z - \eta^{-1/2}\mu_A)] = \mu_Z + L_\eta^{-1} \sum_{i=0}^\infty G_{n+2i}(\eta a/p)c_{i(0:0);\eta},$$

where $c_{i(0:0);\eta}$ are equal to $c_{i(0:0)}$ as in (12) substituting $b_0$ by $b_{0,0}$. Then, the elliptical truncated first-order moment is,

$$E[X|a \leq (X - \mu_A)^\top A(X - \mu_A)] = E_\eta\left[\eta^{-1/2}\mu_Z + \eta^{-1/2}L_\eta^{-1} \sum_{i=0}^\infty G_{n+2i}(\eta a/p)c_{i(0:0);\eta}\right] = \mu_X + \sum_{i=0}^\infty E_\eta\left[\eta^{-1/2}L_\eta^{-1}G_{n+2i}(\eta a/p)c_{i(0:0);\eta}\right]. \quad (33)$$

Using the definition and properties of the conditional expectation we have that,

$$E[X|a \leq (X - \mu_A)^\top A(X - \mu_A)] = \mu_X + \sum_{i=0}^\infty E_\eta\left[G_{n+2i}(\eta a/p)\eta^{-1/2}c_{i(0:0);\eta}\right]. \quad (34)$$

The expected value in (33), is solved introducing $\eta^{-1/2}$ inside the coefficients $c_{i(0:0);\eta}$ and applying the decomposition in (27), (28) for $c_{i(0:0);\eta}$,

$$c_{i(0:0);\eta}\eta^{-1/2} = \exp\left(-\frac{1}{2}\eta b_0^\top b_0\right)\left(c_{i_10(0:0)}\eta^{-1/2} + c_{i_11(0:0)}\eta^{1/2} + c_{i_2(0:0)}\eta^{3/2} + \cdots + c_{i_a(0:0)}\eta^{i-1/2}\right),$$

and,

$$E_\eta\left[G_{n+2i}(\eta a/p)\eta^{-1/2}c_{i(0:0);\eta}\right] = E_\eta\left[G_{n+2i}(\eta a/p)\eta^{-1/2} \exp\left(-\frac{1}{2}\eta b_0^\top b_0\right)\right]c_{i0(0:0)} + \quad (35)$$

$$E_\eta\left[G_{n+2i}(\eta a/p)\eta^{1/2} \exp\left(-\frac{1}{2}\eta b_0^\top b_0\right)\right]c_{i1(0:0)} + \cdots + \quad (35)$$

$$E_\eta\left[G_{n+2i}(\eta a/p)\eta^{i-1/2} \exp\left(-\frac{1}{2}\eta b_0^\top b_0\right)\right]c_{i_a(0:0)}.$$

The solutions to the internal integrals in (35) are solved as in (31),

$$E_\eta\left[G_{n+2i}(\eta a/p)\eta^{j-1/2} \exp\left(-\frac{1}{2}\eta b_0^\top b_0\right)\right] = (1 + b_0^\top b_0/n)^{-(\nu+1)/2-j} \left\{\frac{\nu + 1 + \nu^2 + 1}{\nu}\right\} \Gamma\left(\frac{n+2j+1}{2}\right) \Gamma\left(\frac{\nu}{2}\right) \times$$

$$\quad B\left(\frac{\nu + 1}{2} + j; \frac{n + 2j}{2}\right) I_{\nu+1+b_0^\top b_0/\nu} \left(\nu+1+b_0^\top b_0+1/\nu\right) B\left(\frac{\nu + 1}{2} + j; \frac{n + 2j}{2}\right) \left(\nu+1+b_0^\top b_0+1/\nu\right) \Gamma\left(\frac{n+2j+1}{2}\right) \Gamma\left(\frac{\nu}{2}\right), \quad (36)$$
therefore, having (33), (34), (35), and (36), yields the result on truncated first-order moments.

The truncated second-order moment of $X$ over $C(X, a)$ is calculated,

$$
E \left[ X X^\top | a \leq (X - \mu_A)^\top A (X - \mu_A) \right] = m_2(C(x, a))
$$

$$
= E_\eta \left[ E \left[ \eta^{-1} Z Z^\top | \eta, a \leq (X - \mu_A)^\top A (X - \mu_A) \right] \right]
$$

$$
= E_\eta \left[ E \left[ \eta^{-1} Z Z^\top | \eta, a \leq \eta^{-1} (Z - \eta^{-1/2} \mu_A)^\top A (Z - \eta^{-1/2} \mu_A) \right] \right]
$$

$$
= E_\eta \left[ E \left[ \eta^{-1} E \left[ Z Z^\top | \eta, \eta a/p \leq (Z - \eta^{-1/2} \mu_A)^\top A (Z - \eta^{-1/2} \mu_A) \right] \right] \right]
$$

$$
= L^{-1} E_\eta \left[ \eta^{-1} \sum_{i=0}^\infty G_{n+2i}(\eta a/p) c_i(\mu; 0, \theta) \right]
$$

$$
= L^{-1} \sum_{i=0}^\infty E_\eta \left[ G_{n+2i}(\eta a/p) \eta^{-1} c_i(\mu; 0, \theta) \right],
$$

where coefficients $c_i(\mu; 0, \theta)$ are equal to $c_i(\mu; 0)$ as in (15) substituting $b_0$ by $b_{0, \theta}$. The expected value in (37), is solved by introducing $\eta^{-1}$ inside the coefficients $c_i(\mu; 0, \theta)$ and applying the decomposition in (27), (28) for $c_{i, 1}(\mu; 0, \theta)$,

$$
c_{i, 1}(\mu; 0, \theta) \eta^{-1} = \exp \left( -\frac{1}{2} \eta b_0^\top b_0 \right) \left( c_{i, 0}(\mu; 0) \eta^{-1} + c_{i, 1}(\mu; 0) + c_{i, 2}(\mu; 0) \eta^{-1/2} \right)
$$

and,

$$
E_\eta \left[ G_{n+2i}(\eta a/p) \eta^{-1} c_{i, 1}(\mu; 0) \right] = E_\eta \left[ G_{n+2i}(\eta a/p) \eta^{-1} \exp \left( -\frac{1}{2} \eta b_0^\top b_0 \right) \right] c_{i, 0}(\mu; 0)
$$

$$
+ E_\eta \left[ G_{n+2i}(\eta a/p) \eta^{-1} \exp \left( -\frac{1}{2} \eta b_0^\top b_0 \right) \right] c_{i, 1}(\mu; 0) + \cdots + E_\eta \left[ G_{n+2i}(\eta a/p) \eta^{-1} \exp \left( -\frac{1}{2} \eta b_0^\top b_0 \right) \right] c_{i, 2}(\mu; 0).
$$

The solutions to the internal integrals in (39) are solved as in (31), then having (37), (38), and (39), the result on the truncated second-order moments is derived.

4. Analytic expressions for the MGF of the elliptical truncated multivariate generalized hyperbolic distribution

**Proposition 4.1.** Let $X$ be a random vector with MGH distribution,

$$
f_X = \frac{\hat{\alpha}^{n/2} (1 - \beta^\top \beta)^{\lambda/2} K_{n-\lambda/2} \left( \hat{\alpha} \sqrt{1 + (x-\mu_X)^\top \Sigma_X^{-1} (x-\mu_X)} \right)}{(2\pi)^{n/2} K_{\lambda} \left( \hat{\alpha} \sqrt{1 - \beta^\top \beta} \right) \left( 1 + (x-\mu_X)^\top \Sigma_X^{-1} (x-\mu_X) \right)^{n/4 - \lambda/2}} \exp \left( \hat{\alpha} \beta^\top \Sigma_X^{-1/2} (x-\mu_X) \right),
$$

where $K_\lambda(\cdot)$ is the modified Bessel function of third-kind, $\mu \in \mathbb{R}^n$ is a location parameter, $\Sigma_X \in \mathbb{R}^{n \times n}$ is a positive definite dispersion parameter, $\beta \in \mathbb{R}^n$ is an asymmetry parameter, $\hat{\alpha} \in \mathbb{R}^+$ is a scale parameter, and $\lambda$ a parameter used to produce close distributions in the marginals under affine transformations. Let $Z$ be distributed as a generalized inverse Gaussian as in,

$$
Z \sim GIG(\lambda, \delta, \sqrt{\delta^2 (\hat{\alpha}^2 - \beta^\top \Sigma_X \beta)}),
$$
and $W$ be the conditional distribution of $W \equiv (X|Z = z)$. Define $C(x, a) = \{x \in \mathbb{R}^n : a \leq (x - \mu_A)^\top A(x - \mu_A)\}$, then $X$ has an approximate elliptical truncated MGF over the region $C(X, a)$ denoted by,

$$m(t, C(X, a)) = E_z \left[ L_z^{-1} \exp \left( t^\top \left( z^{1/2} \Delta_X \beta \right) + \frac{1}{2} t^\top \Sigma_X t \right) H_{n,E,b_i(t)}(z^{-1}a/p) \right],$$

where,

$$H_{n,E,b_i(t)}(z^{-1}a/p) = \sum_{i=0}^\infty G_{n+2i}(z^{-1}a/p)c_{i,t;z},$$

$$L_z = H_{n,E,b_{i0}}(z^{-1}a/p) = \sum_{i=0}^\infty G_{n+2i}(z^{-1}a/p)c_{i0;z},$$

(41)

with coefficients $c_{i,t;z}$ equal to coefficients $c_i$ as in (3) and (4) substituting $b(t)$ by $b_z(t)$ defined by,

$$b_z(t) = K^{-1}P^{-1}\left( z^{-1/2} \mu_A - 1/2 \Delta_X \beta \right),$$

and coefficients $c_{i0;z}$ equal to $c_i$ as in (3) and (4) substituting $b(t)$ by $b_{0z}$ defined by,

$$b_{0z} = (b_{0z1}, \ldots, b_{0zn}) = K^{-1}P^{-1}(z^{-1/2} \mu_A - 1/2 \Delta_X \beta).$$

(42)

Assume without loss of generality that $\mu_X = 0$. The elliptical truncated zeroth-, first-, and second-order moments are,

$$\Pr [X|a \leq (X - \mu_A)^\top A(X - \mu_A)] = L = \sum_{i=0}^\infty \sum_{j=-i}^i \zeta_{j,i}c_{i,j;0},$$

$$E [X|a \leq (X - \mu_A)^\top A(X - \mu_A)] = L^{-1}\left( \mu_W \sum_{i=0}^\infty \sum_{j=-i}^i \zeta_{j,i}c_{i,j;0} + \sum_{i=0}^\infty \sum_{j=-i}^{i+1} \zeta_{j,i}c_{i,j;\partial t;0} \right),$$

$$E [XX^\top|a \leq (X - \mu_A)^\top A(X - \mu_A)] = L^{-1}\left( \sum_{i=0}^\infty \sum_{j=-i}^i \zeta_{j+1,i}c_{i,j+1;0} \right) \mu_W \mu_W^\top + \sum_{i=0}^\infty \sum_{j=-i}^{i+1} \zeta_{j+1,i}c_{i,j+1;\partial t;0},$$

where,

$$\zeta_{j,i} = \frac{1}{2\Gamma(\frac{n+2i}{2})K_\lambda(\sqrt{\lambda b})} \Gamma\left( \frac{n+2i}{2} \right) \left( \frac{\lambda b}{\psi_B} \right)^{(\lambda+j+1)/2} K_{\lambda+j+1}\left( \sqrt{\lambda b}\psi_B \right) - \sum_{k=0}^{\infty} \frac{1}{\prod_{s=0}^{k}(\frac{n+2i+2s}{2}+s)} \left( \frac{\lambda b\psi_B}{\psi_B} \right)^{(\lambda+j-(n+2i)/2-1+k)/2} K_{\lambda+j-(n+2i)/2-1+k} \left( \sqrt{\lambda b}\psi_B \right) ;$$

and $c_{i,j;0}$ are numerical coefficients calculated by solving the recurrence (4) for the MGH distribution case.

**Proof.** Let $Z$ be a random vector with a GIG distribution, with pdf,

$$f_Z = \frac{(\bar{p}/\delta)^{\lambda/2}}{2K_\lambda(\sqrt{\bar{p}})} z^{\lambda-1} \exp\left(-\frac{1}{2} (\delta z^{-1} + \bar{p} z)\right),$$

$$\psi_B = \bar{p} \frac{(\lambda b\psi_B)}{\psi_B}$$

$$\delta = \psi_B - \bar{p}$$

$$\lambda = \frac{\psi_B}{\bar{p}}$$
where \( \tilde{p} = \sqrt{\delta^2(\alpha^2 - \beta^\top \Sigma X \beta)} \) and \( \delta = \|\Sigma X\|^{1/n} \). Define \( W = (X|Z = z) \), then \( W \) is multivariate normal distributed,

\[
W \sim N(\mu_X + z\Delta X \beta, z\Delta X),
\]

and the unconditional distribution of \( X \) is MGH as (40) by Barndorff-Nielsen [1].

Before calculating the MGF and the moments, we introduce a change of variable for the convenience of future calculations. Let \( V \) be a random vector distributed as the multivariate standard normal (MVSN) distribution. By the properties of the MGH distribution, \( X \) will have the same distribution law of,

\[
X \overset{d}{=} z^{1/2} \left( z^{-1/2} \mu_X + z^{-1/2} \Delta X \beta + PV \right),
\]

where \( PP^\top = \Delta X \). Noting that \( \mu_X = 0 \), and defining \( Y = z^{1/2} \Delta X \beta + PV \), the variable \( X \) can be denoted as,

\[
X \overset{d}{=} z^{1/2} Y,
\]

where \( Y \) is MGH distributed. Considering (43) the elliptical truncated region \( C(X, a) \) can be transformed as,

\[
a \leq (X - \mu_A)^\top A (X - \mu_A) = a \leq z^{1/2}(Y - z^{-1/2} \mu_A)^\top A (Y - z^{-1/2} \mu_A)z^{1/2}
\]

\[
\leq z^{-1} a \leq (Y - z^{-1/2} \mu_A)^\top A (Y - z^{-1/2} \mu_A) = z^{-1} a \leq (Y - \mu_{A,Y})^\top A (Y - \mu_{A,Y}),
\]

then,

\[
C(X, a) \overset{d}{=} C(z^{1/2}Y, a),
\]

where \( \mu_{A,Y} = z^{-1/2} \mu_A \). Considering (43) and (44), define \( W^\top_Y = (W_1, \ldots, W_{nY}) \equiv z^{1/2} Y | Z = z \equiv X | Z = z \), then \( W_Y \) is a random vector with MVN distribution,

\[
z^{1/2} Y | Z = z \equiv W_Y \sim N(z^{1/2} \Delta X \beta, \Delta X),
\]

and

\[
C(z^{1/2} Y, a) | Z = z \equiv C(W_Y, a).
\]

The truncated MGF of \( X \) at the ellipsoid \( C(X, a) \) applying (45) and (46) can be approximated as,

\[
E[\exp(t^\top X)|C(X, a)] = m(t, C(X, a)) = E_z[E[\exp(t^\top W_Y)|C(W_Y, a)]].
\]

Using the results of Section 2 the internal expression of (47),

\[
E[\exp(t^\top W_Y)|C(W_Y, a)],
\]

is the MGF of the MVN distribution, and can be calculated as,

\[
E[\exp(t^\top W_Y)|C(W_Y, a)] = L_z^{-1} \exp \left( t^\top \left( z^{1/2} \Delta X \beta \right) + \frac{1}{2} t^\top \Sigma X t \right) H_{n, E, b_z(t)}(z^{-1} a/p).
\]

Then, the MGF expression (47) can be approximated by,

\[
m(t, C(X, a)) = E_z \left[ L_z^{-1} \exp \left( t^\top \left( z^{1/2} \Delta X \beta \right) + \frac{1}{2} t^\top \Sigma X t \right) H_{n, E, b_z(t)}(z^{-1} a/p) \right].
\]
If we set $t = 0$ in (48), we have as a result that,

$$1 = E_z \left[ L^{-1} H_{n,E} b_{0,1} (z^{-1}a/p) \right] ,$$  

(49)

and equality in (41) will hold if $E_z$ [•] exists. The truncated zeroth-order moment (probability) of $X$ at the ellipsoid $C(X,a)$ is,

$$L = E_z[L_z].$$  

(50)

To calculate (50), we need to develop the series (41),

$$E_z \left[ \sum_{i=0}^{\infty} G_n(z^{-1}a/p) c_{i,0:z} \right] =$$

$$= E_z \left[ G_n(z^{-1}a/p) c_{0,0:z} \right] + E_z \left[ G_n(z^{-1}a/p) c_{1,0:z} \right] + E_z \left[ G_n(z^{-1}a/p) c_{2,0:z} \right] + \ldots ,$$

$$= E_z \left[ G_n(z^{-1}a/p) \exp \left( -\frac{1}{2} b_{0:z}^T b_{0:z} \right) \prod_{j=1}^{n} (p/e_j)^{1/2} \right] +$$

$$+ E_z \left[ G_n(z^{-1}a/p) 2^{-1} \exp \left( -\frac{1}{2} b_{0:z}^T b_{0:z} \right) \prod_{j=1}^{n} (p/e_j)^{1/2} d_{1:0:z} \right] +$$

$$+ E_z \left[ G_n(z^{-1}a/p) 4^{-1} \exp \left( -\frac{1}{2} b_{0:z}^T b_{0:z} \right) \prod_{j=1}^{n} (p/e_j)^{1/2} (d_{2:0:z} + 2^{-1} d_{1:0:z}^2) \right] + \ldots .$$  

(51)

By definition we can apply the following factors over the terms dependent on $z$ in (51),

$$b_{0:z}^T b_{0:z} = \left( K^{-1} P^{-1} (z^{-1/2} \mu_A - z^{1/2} \Delta_X \beta) \right)^T \left( K^{-1} P^{-1} (z^{-1/2} \mu_A - z^{1/2} \Delta_X \beta) \right)$$

$$= z^{-1} (\mu_A^T \Delta_X K^{-2} \mu_A) - 2 (\Delta_X \beta)^T (\Delta_X K^{-2} \mu_A) + z (\Delta_X \beta)^T (\Delta_X K^{-2}) (\Delta_X \beta),$$

$$= z^{-1} (\mu_A^T \Delta_X^{-1} \mu_A) - 2 (\Delta_X \beta)^T (\Delta_X^{-1}) (\Delta_X \beta),$$

$$= z^{-1} B_{0,0} + B_{0,0} = z B_{0,0},$$

and,

$$d_{i,0:z} = dA_i + z^{-1} dB_i + z dC_i,$$  

(52)

where,

$$B_{0,0} = \mu_A^T \Delta_X^{-1} \mu_A,$$

$$B_{0,0} = -2 \beta^T \mu_A,$$

$$B_{0,1} = \beta^T \Delta_X \beta,$$

with,

$$dA_i = \sum_{j=1}^{n} \left( 1 - p/e_i \right)^i + ipB_{0,0} \sum_{j=1}^{n} \left( 1/e_i \right) \left( 1 - p/e_j \right)^{i-1},$$

$$dB_i = ipB_{0,0} \sum_{j=1}^{n} \left( 1/e_i \right) \left( 1 - p/e_j \right)^{i-1},$$

$$dC_i = ipB_{0,1} \sum_{j=1}^{n} \left( 1/e_i \right) \left( 1 - p/e_j \right)^{i-1}.$$
and in consequence developing the recurrence \(c_{i:0:z}\) as in (3), (4), and (5) with the corresponding change (42), the coefficients \(c_{i:0:z}\) can be denoted by a polynomial over two terms, exp \((-\frac{1}{2} (z\lambda - B_{0:0} + B_{0:1})\) and \(z\) by,

\[
c_{i:0:z} = \exp \left( -\frac{1}{2} \left( z^{-1} B_{0:0} + B_{0:0} + z B_{0:1} \right) \right) \times 
\left( c_{i:0:0} \sum_{j=0}^{i} c_{i-j:0} z^{-j} + \cdots + c_{i:0} z^0 + \cdots + c_{i:0} z^i \right),
\]

(53)

with \(c_{a:0,i,j} \geq 0\) that are coefficients not dependent on \(z\). The value of the coefficients \(c_{i:0,0}\) is found by equating (4) with (53) and substituting \(b(t)\) by \(b_{0:z}\), considering the relation derived in (52). Hence, the terms in the series (51) can be denoted by,

\[
E_z \left[ G_{n+2i}(z^{-1} a/p) c_{i:0:z} \right] = \int_0^\infty \int_0^\infty \frac{x^{(n+2i)/2} \exp \left( -\frac{1}{2} (\chi z^{-1} + \psi z) \right) \lambda/\sqrt{\chi \psi}}{2 K_n(\sqrt{\chi \psi})} \left( \frac{a}{p} \right)^{\frac{n+2i}{2}} \exp \left( -\frac{1}{2} (z\lambda - B_{0:0} + B_{0:1}) \right) dx dz,
\]

(55)

where \(\chi = \hat{\delta}, \psi = \hat{p}\), and \(\lambda = \hat{\lambda}\) is a different parametrisation of GIG variables commonly used in the literature. Applying the change of variable \(z^{-1} y = x\) to (55) we have,

\[
= \int_0^\infty \int_0^\infty \frac{(z\lambda - B_{0:0} + B_{0:1})^{(n+2i)/2-1} \exp \left( -\frac{1}{2} (\chi z^{-1} + \psi z) \right) \lambda/\sqrt{\chi \psi}}{2 K_n(\sqrt{\chi \psi})} \left( \frac{a}{p} \right)^{\frac{n+2i}{2}} \exp \left( -\frac{1}{2} (\chi z^{-1} - B_{0:0} + B_{0:1}) \right) dy dz
\]

\[
= \frac{(\psi/\chi)^{\lambda/2} \exp \left( -\frac{1}{2} B_{0:0} \right)}{2^{(n+2i)/2+1} K_n(\sqrt{\chi \psi})} \int_0^\infty \left( \int_0^\infty \frac{y^{(n+2i)/2-1} \exp \left( -\frac{1}{2} yz^{-1} \right) \lambda/\sqrt{\chi \psi}}{2 K_n(\sqrt{\chi \psi})} \exp \left( -\frac{1}{2} (\chi z^{-1} + \psi) \right) \right) dy dz
\]

(56)

Introduce the change of variable \( t = \frac{1}{2} yz^{-1}\),

\[
= \frac{(\psi/\chi)^{\lambda/2} \exp \left( -\frac{1}{2} B_{0:0} \right)}{2^{(n+2i)/2+1} K_n(\sqrt{\chi \psi})} \int_0^\infty \left( \int_0^\infty \frac{(2z)^{(n+2i)/2-1} (n+2i)/2-1 \exp (-t) /2 dt}{2 K_n(\sqrt{\chi \psi})} \right) \exp \left( -\frac{1}{2} (\chi z^{-1} + \psi) \right) \right) dz.
\]

(56)
and the internal integral in (56) over \( t \) is an upper-incomplete gamma function\(^3\) that is denoted using its properties as,

\[
\Gamma(x, y) = \int_y^\infty t^{x-1} \exp(-t),
\]

and it can be denoted as,

\[
\Gamma(x, y) = \Gamma(x) - \gamma(x, y) = \Gamma(x) - \sum_{k=0}^{\infty} \frac{y^{x+k} \exp(-y)}{x(x+1)\cdots(x+k)}.
\]

\(^3\)The definition of the upper-incomplete gamma function is,

\[
\Gamma(x, y) = \int_y^\infty t^{x-1} \exp(-t),
\]
where coefficients \( c_{i,[\alpha t,0]:z} \) are equal to \( c_{i,[\alpha t,0]} \) as in (12) substituting \( b_0 \) by \( b_{0,z} \). The expected value in (60), is solved introducing \( z^{1/2} \) inside the coefficients \( c_{i,[\alpha t,0]:z} \) and applying the decomposition in (53), (54) for \( c_{i,[\alpha t,0]:z} \),

\[
c_{i,[\alpha t,0]:z} z^{1/2} = \exp \left( -\frac{1}{2} (z^{-1} B_{0,-1} + B_{0,0} + z B_{0,1}) \right) \left( c_{i_{n-1},[\alpha t,0]:z} z^{-i+1} + c_{i_{n-1-i-1},[\alpha t,0]:z} z^{-(i-2)} + \ldots \right.
\]

\[
+ c_{i_{n-1},[\alpha t,0]:z} z^{1/2} + \cdots + c_{i_{n-1},[\alpha t,0]:z} z^{1-1/2} + c_{i_{n-1},[\alpha t,0]:z} z^{1+1/2},
\]

and,

\[
E_z \left[ G_{n+2i}(z^{-1}a/p) z^{1/2} c_{i,[\alpha t,0]:z} \right] =
\]

\[
E_z \left[ G_{n+2i}(z^{-1}a/p) z^{-i+1/2} \exp \left( -\frac{1}{2} (z^{-1} B_{0,-1} + B_{0,0} + z B_{0,1}) \right) \right] c_{i_{n-1},[\alpha t,0]:z} + \cdots +
\]

\[
E_z \left[ G_{n+2i}(z^{-1}a/p) z^{1/2} \exp \left( -\frac{1}{2} (z^{-1} B_{0,-1} + B_{0,0} + z B_{0,1}) \right) \right] c_{i_{n-1},[\alpha t,0]:z} + \cdots +
\]

\[
E_z \left[ G_{n+2i}(z^{-1}a/p) z^{+1/2} \exp \left( -\frac{1}{2} (z^{-1} B_{0,-1} + B_{0,0} + z B_{0,1}) \right) \right] c_{i_{n-1},[\alpha t,0]:z}. \]

(62)

The solutions to the internal integrals in (62) are solved as in (59),

\[
E_z \left[ G_{n+2i}(z^{-1}a/p) z^{+1/2} \exp \left( -\frac{1}{2} (z^{-1} B_{0,-1} + B_{0,0} + z B_{0,1}) \right) \right] =
\]

\[
\frac{(\psi/\chi)^{\lambda/2}}{2 \Gamma(n+2i/2)} \frac{\chi_B}{\psi_B} K_{\lambda+j+1/2} \left( \sqrt{\chi_B \psi_B} \right) -
\]

\[
\sum_{k=0}^{\infty} \frac{(1/a/p)^{(n+2k)/2+k}}{\prod_{s=0}^{k} \left( \frac{n+2i}{2} + s \right)} \frac{\chi_B}{\psi_B} K_{\lambda+(n+2k)/2-k+1/2} \left( \sqrt{\chi_B \psi_B} \right), \]

(63)

therefore, having (60), (61), (62), and (63), the result on truncated first-order moments is obtained.

Finally, the elliptically truncated second-order moment is calculated,

\[
E [XX' | a \leq (X - \mu_A)^T A (X - \mu_A)] = E_z [zE [YY' | z, z^{-1} a \leq (Y - \mu_{A,Y})^T A (Y - \mu_{A,Y})]].
\]

Applying the results of Proposition (2.1) we have,

\[
E [XX' | a \leq (X - \mu)^T \Sigma^{-1}_X (X - \mu)] = E_z \left[ zL_z^{-1} \sum_{i=0}^{\infty} G_{n+2i}(z^{-1}a/p) c_{i,[\alpha t,0]:z} \right],
\]

(64)

where coefficients \( c_{i,[\alpha t,0]:z} \) are equal to \( c_{i,[\alpha t,0]} \) as in (15) substituting \( b_0 \) by \( b_{0,z} \). The expected value in (64) is solved introducing \( z \) inside the coefficients \( c_{i,[\alpha t,0]:z} \) and applying the decomposition in (53) and (54) for \( c_{i,[\alpha t,0]:z} \),

\[
c_{i,[\alpha t,0]:z} z = \exp \left( -\frac{1}{2} (z^{-1} B_{0,-1} + B_{0,0} + z B_{0,1}) \right) \left( c_{i_{n-1},[\alpha t,0]:z} z^{-i+1} + c_{i_{n-1-(i-1)},[\alpha t,0]:z} z^{-(i-2)} + \ldots \right.
\]

\[
+ c_{i_{n-1},[\alpha t,0]:z} z^{1/2} + \cdots + c_{i_{n-1},[\alpha t,0]:z} z^{1-1/2} + c_{i_{n-1},[\alpha t,0]:z} z^{1+1/2},
\]

(65)
and,

\[
E_z \left[ G_{n+2i}(z^{-1}a/p)zc_i;[\alpha;\theta]_0]:z \right] =
E_z \left[ G_{n+2i}(z^{-1}a/p)z^{-i+1} \exp \left( -\frac{1}{2} \left( z^{-1}B_{0,-1} + B_{0,0} + zB_{0,1} \right) \right) \right] c_{i_n};[\alpha;\theta]_0 + \cdots +
E_z \left[ G_{n+2i}(z^{-1}a/p)z^i \exp \left( -\frac{1}{2} \left( z^{-1}B_{0,-1} + B_{0,0} + zB_{0,1} \right) \right) \right] c_{i_0};[\alpha;\theta]_0 + \cdots +
E_z \left[ G_{n+2i}(z^{-1}a/p)z^{i+1} \exp \left( -\frac{1}{2} \left( z^{-1}B_{0,-1} + B_{0,0} + zB_{0,1} \right) \right) \right] c_{i_{i+1}};[\alpha;\theta]_0.
\] (66)

The solutions to the internal integrals in (66) are solved as in (59),

\[
E_z \left[ G_{n+2i}(z^{-1}a/p)z^{j+1} \exp \left( -\frac{1}{2} \left( z^{-1}B_{0,-1} + B_{0,0} + zB_{0,1} \right) \right) \right] =
\frac{\left( \psi/\chi \right)^{\lambda/2}}{2\Gamma \left( \frac{n+2i}{2} \right)} K_\lambda \left( \sqrt{\chi \psi} \right) \frac{\left( \lambda+j+1 \right) / 2}{K_{\lambda+j+1} \left( \sqrt{\chi \psi} \right)}
\sum_{k=0}^{\infty} \frac{\left( \frac{1}{2}(a/p) \right)^{n+2i}/2+k}{\prod_{s=0}^{k} \left( \frac{n+2i}{2} + s \right)} \frac{\chi B_{ap \psi} B}{\psi B} \frac{\left( \lambda+j-(n+2)/2-k+1 \right)}{K_{\lambda+j-(n+2)/2-k+1} \left( \sqrt{\chi \psi} \right)},
\] (67)

then having (64), (65), (66), and (67) yields the result on truncated second-order moments.

References


