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Kennedy, J.; Shelah, S.; Väänänen, J.A.

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Regular Ultrapowers at Regular Cardinals

Juliette Kennedy∗
Department of Mathematics and Statistics
University of Helsinki, Finland

Saharon Shelah †
Institute of Mathematics
Hebrew University, Jerusalem, Israel
Rutgers University, New Jersey, USA

Jouko Väänänen‡
Department of Mathematics and Statistics
University of Helsinki, Finland
ILLC, University of Amsterdam
Amsterdam, Netherlands

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Abstract
In earlier work of the second and third author the equivalence of a finite square principle □_{\lambda,D}^{fin} with various model theoretic properties of structures of size \lambda and regular ultrafilters was established. In this paper we investigate the principle □_{\lambda,D}^{fin}, and thereby the above model theoretic properties, at a regular cardinal. By Chang’s Two-Cardinal Theorem, □_{\lambda,D}^{fin} holds at regular cardinals for all regular filters D if we assume GCH. In this paper we prove in ZFC that for certain regular

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filters that we call doubly regular, $\Box^{\kappa_n}_{\lambda,D}$ holds at regular cardinals, with no assumption about GCH. Thus we get new positive answers in ZFC to Open Problems 18 and 19 in the book Model Theory by Chang and Keisler.

1 Introduction

In [7] and [8] the equivalence of the following finite square principle $\Box^{\kappa_n}_{\lambda,D}$ with various model theoretic properties of regular reduced powers of models was established:

$\Box^{\kappa_n}_{\lambda,D} : D$ is a filter on a cardinal $\lambda$ and there exist finite sets $C^\xi_\alpha$ and integers $n_\xi$ for each $\alpha < \lambda^+$ and $\xi < \lambda$ such that for each $\xi, \alpha$

(i) $C^\xi_\alpha \subseteq \alpha + 1$

(ii) If $B \subseteq \lambda^+$ is a finite set of ordinals and $\alpha < \lambda^+$ is such that $B \subseteq \alpha + 1$, then $\{\xi : B \subseteq C^\xi_\alpha\} \in D$

(iii) $\beta \in C^\xi_\alpha$ implies $C^\xi_\beta = C^\xi_\alpha \cap (\beta + 1)$

(iv) $|C^\xi_\alpha| < n_\xi$

The model theoretic properties were the following: Firstly, if $D$ is an ultrafilter, then $\Box^{\kappa_n}_{\lambda,D}$ is equivalent to $\mathcal{M}^\lambda/D$ being $\lambda^{++}$-universal for each model $\mathcal{M}$ in a vocabulary of size $\leq \lambda$. To formulate the second model theoretic property, let us say that two models are $EF_\alpha$-equivalent if the second player (i.e. the “isomorphism” player) has a winning strategy in the Ehrenfeucht-Fraïssé game of length $\alpha$ on the two models\(^1\). Now $\Box^{\kappa_n}_{\lambda,D}$ is equivalent to $\mathcal{M}^\lambda/D$ and $\mathcal{N}^\lambda/D$ being $EF_{\lambda^+}$-equivalent for any elementarily equivalent models $\mathcal{M}$ and $\mathcal{N}$ (w.l.o.g. of cardinality $\leq \lambda^+$) in a vocabulary of size $\leq \lambda$. The existence of such ultrafilters and models is related to Open Problems 18 and 19 in the Chang-Keisler model theory book [1].

The consistency of the failure of $\Box^{\kappa_n}_{\lambda,D}$ for a regular ultrafilter $D$ at a singular strong limit cardinal $\lambda$ was proved in [8] relative to the consistency of a supercompact cardinal. In [9] this was improved to the failure of $\Box^{\kappa_n}_{\lambda,D}$ for a regular ultrafilter $D$ at a singular strong limit cardinal $\lambda$ relative to the

\(^1\)The usual elementary equivalence in a finite relational vocabulary is thus $EF_n$-equivalence for all $n < \omega$, and $L_{\infty,\omega}$-equivalence is the same as $EF_\omega$-equivalence. For models of cardinality $\leq \kappa$, $EF_\kappa$-equivalence is equivalent to isomorphism.
consistency of a strongly compact cardinal. The failure of \( \square^{\text{fin}}_{\lambda,D} \) for an ultrafilter implies the failure of \( \lambda^{++}\)-universality of \( \mathcal{M}^\lambda/D \) for some \( \mathcal{M} \), as well as the failure of isomorphism of some regular ultrapowers \( \mathcal{M}^\lambda/D \) and \( \mathcal{N}^\lambda/D \). Thus [9] answered negatively the following problems listed in [1] modulo large cardinal assumptions:

Problem 18 ([1]) Let \(|M|, |N|, |L| \leq \alpha\) and let \( D \) be a regular ultrafilter over \( \alpha \). If \( \mathcal{M} \equiv \mathcal{N} \), then \( \prod_D \mathcal{M} \cong \prod_D \mathcal{N} \).

Problem 19 ([1]) If \( D \) is a regular ultrafilter of \( \alpha \), then for all infinite \( \mathcal{M} \), \( \prod_D \mathcal{M} \) is \( \alpha^{++}\)-universal.

The use of large cardinals is justified by [7], [8] and [12] as the failure of \( \square^{\text{fin}}_{\lambda,D} \) for singular strong limit \( \lambda \) implies the failure of \( \square_{\lambda} \), which implies the consistency of large cardinals.

In this paper we investigate the principle \( \square^{\text{fin}}_{\lambda,D} \), and thereby the above model theoretic problems, at a regular cardinal. The following result is proved in [4]: Assume \( \kappa \) is regular and \( \lambda^{<\kappa} = \lambda \). Suppose \( \mathcal{M} \) and \( \mathcal{N} \) are structures for a finite vocabulary such that \( \mathcal{M} \) and \( \mathcal{N} \) are \( EF_{\alpha}\)-equivalent for each \( \alpha < \kappa \). Suppose \( D \) is a filter on \( \xi \times \lambda, \xi \leq \lambda \), extending \( F' \times F \), where \( F' \) is a \( \kappa \)-descendingly incomplete filter on \( \xi \) and \( F \) is a \( \kappa \)-semigood filter on \( \lambda \) (the concept is defined in [4]). Then \( \mathcal{M}^\lambda/D \) and \( \mathcal{N}^\lambda/D \) are \( EF_{\lambda^+}\)-equivalent. For \( \kappa = \omega \) this, combined with the existence proof of semigood filters in [4], yields filters \( D \) with \( \square^{\text{fin}}_{\lambda,D} \).

The structure of the paper is the following: In Section 2 we prove weaker versions of \( \square^{\text{fin}}_{\lambda,D} \) in the case where the filter \( D \) extends the club filter on \( \lambda \). Naturally this case is in spirit quite far from the case of regular \( D \), which is our prime interest. However, this result is useful in the sequel. Note that there are many regular (ultra)filters extending the club filter. In Section 3 we define the concept of \textit{doubly regular} filter and show that such filters \( D \) on regular \( \lambda > \aleph_0 \) satisfy \( \square^{\text{fin}}_{\lambda,D} \). Thus we get new positive answers in \( ZFC \) to the above Problem 18 (with isomorphism replaced, in the absence of \( 2^\lambda = \lambda^+ \), by \( EF_{\lambda^+}\)-equivalence) and the above Problem 19. In Section 4 we prove results to the effect that not all regular filters are doubly regular. In Section 5 we compare our concept of double regularity to Keisler’s concept of goodness of a filter. In Section 6 we present some open questions.
2 Filters extending the club filter

We can get provable cases of a weaker form of $\Box_{\lambda,D}^{\kappa}$, when $D$ extends the club filter. This will prove useful in the next section, where we will use Theorem 1 in the proof of Theorem 5. The original $\Box_{\lambda,D}^{\kappa}$ is equivalent to reduced powers of elementarily equivalent models of cardinality $\lambda$ being $EF_{\lambda^+}$-equivalent. The weaker form which we shall prove below will give the $EF_{\lambda^+}$-equivalence of reduced powers of models of power $\lambda$ that are not just elementarily equivalent but even $EF_{\lambda}$-equivalent.

**Theorem 1** Suppose

(a) $\lambda$ is regular $> \aleph_0$,

(b) $D$ is a filter on $\lambda$.

(c) $D$ extends the club filter.

If $M$ and $N$ are $EF_{\lambda}$-equivalent, then $M^\lambda/D$ and $N^\lambda/D$ are $EF_{\lambda^+}$-equivalent.

**Proof.** If $\alpha < \lambda^+$, $\lambda$ regular, let $\{u^i_\alpha : i < \lambda\}$ be a continuously increasing sequence of subsets of $\alpha$ such that $|u^i_\alpha| < \lambda$ for all $i < \lambda$ and $\alpha = \bigcup_{i<\lambda} u^i_\alpha$. Let

$$D_\alpha = \{ i < \lambda : \forall \beta \in u^i_\alpha (u^i_\beta = u^i_\alpha \cap \beta) \}. \tag{1}$$

It is easy to see that $D_\alpha$ is a club of $\lambda$ (recall that $\lambda$ is regular).

Now we can proceed, as in [7] to prove that if $M$ and $N$ are $EF_{\lambda}$-equivalent, then $M^\lambda/D$ and $N^\lambda/D$ are $EF_{\lambda^+}$-equivalent:

Let $L$ be a finite vocabulary and for each $i < \lambda$ let $\mathcal{M}_i$ and $\mathcal{N}_i$ be $EF_{\lambda}$-equivalent $L$-structures. We show that $\Pi$ has a winning strategy in the game $EF_{\lambda^+}$ on the models $\mathcal{M} = \prod_i \mathcal{M}_i$ and $\mathcal{N} = \prod_i \mathcal{N}_i$.

The crucial idea of the proof is the following: When the Ehrenfeucht-Fraïssé game $EF_{\lambda^+}(\mathcal{M},\mathcal{N})$ is played, the players are actually playing $\lambda$ Ehrenfeucht-Fraïssé games simultaneously, namely the games $EF_{\lambda}(\mathcal{M}_i,\mathcal{N}_i)$, $i < \lambda$.

For each $i < \lambda$ let $\sigma_i$ be a winning strategy for $\Pi$ in the game $EF_{\lambda}$ on the models $\mathcal{M}_i$ and $\mathcal{N}_i$. A good position is a sequence $\langle (f_\beta, g_\beta) : \beta < \alpha \rangle$ for some $\alpha < \lambda^+$, together with a club $C \subseteq D_\alpha$, such that for all $\beta < \alpha$ we have $f_\beta \in \prod_i \mathcal{M}_i$, $g_\beta \in \prod_i \mathcal{N}_i$, and if $i \in C$, then

$$\langle (f_\eta(i), g_\eta(i)) : \eta \in u^i_\alpha \rangle$$
is a play according to $\sigma_i$ on the models $\mathcal{M}_i$ and $\mathcal{N}_i$. In a good position the equivalence classes of the functions $f_\beta$ and $g_\beta$ determine a partial isomorphism of the reduced products: Suppose $\alpha$ rounds have been played and we are in a good position. Let $\phi_\gamma([f_{\beta_1}], \ldots, [f_{\beta_k}])$ be an atomic formula holding in $\prod_i \mathcal{M}_i / D$, where $\beta_1 < \ldots < \beta_k < \alpha$, and let $A = \{i \in D_\alpha : \{\beta_1, \ldots, \beta_k\} \subseteq u^i_\alpha\}$. By assumption, $A \in D$. Since also $B = \{i < \lambda : \mathcal{M}_i \models \phi_\gamma(f_{\beta_1}(i), \ldots, f_{\beta_k}(i))\} \in D$, we have $A \cap B \in D$. For $i \in A \cap B$ we have $\beta_1, \ldots, \beta_k \in u^i_\alpha$, hence

$$u^i_{\beta_j} = u^i_\alpha \cap \beta_j.$$ 

Since we are in a good position, $\langle(f_{\eta}(i), g_{\eta}(i)) : \eta \in u^i_\alpha\rangle$ is a play according to winning strategy $\sigma_i$. Hence $\langle(f_{\xi}(\xi), g_{\xi}(\xi)) : \epsilon \in u^i_\alpha\rangle$ determines a partial isomorphism of the structures $\mathcal{M}_i$ and $\mathcal{N}_i$. Since this was the case for all $i \in A \cap B \in D$, we get $\prod_i \mathcal{N}_i / D \models \phi_\gamma([g_{\beta_1}], \ldots, [g_{\beta_k}])$.

The strategy of II is to keep the position of the game good and thereby win the game. So suppose $\beta$ rounds have been played and II has been able to keep the position good. Then for all $\gamma < \beta$ there is a club $C_{\gamma} \subseteq D_{\gamma}$ such that for $i \in C_{\gamma}$, $\langle(f_{\eta}(i), g_{\eta}(i)) : \eta \in u^i_\gamma\rangle$ is a play according to $\sigma_i$.

**Case 1:** $\beta = \cup \beta$. Let $C = \bigcap_{\gamma < \beta} C_{\gamma}$. Since $\lambda$ is regular, this is still a club. We show that $\langle(f_{\gamma}, g_{\gamma}) : \gamma < \beta\rangle$ is good. Let $i \in C$. Let us look at $\langle(f_{\eta}(i), g_{\eta}(i)) : \eta \in u^i_\gamma\rangle$. Since $i \in D_{\beta}$, every initial segment of this play is a play according to $\sigma_i$. Hence so is the entire play $\langle(f_{\gamma}, g_{\gamma}) : \gamma < \beta\rangle$. We have shown that II can maintain a good position.

**Case 2:** $\beta = \delta + 1$. Let $C \subseteq \bigcap_{\gamma \leq \delta} C_{\gamma}$ such that $\delta \in u^i_\beta$ for $i \in C$. Now suppose I plays $f_\delta$. We show that II can play $g_\delta$ so that $\langle(f_{\gamma}, g_{\gamma}) : \gamma < \beta\rangle$ remains good. Let $i \in C$. Let us look at $\langle(f_{\eta}(i), g_{\eta}(i)) : \eta \in u^i_\delta\rangle$. This is a play according to the strategy $\sigma_i$. Since $i \in D_{\beta}$ and $\delta \in u^i_\beta$, $u^i_\delta = u^i_\gamma \cap \delta$, so after the moves $\langle(f_{\eta}(i), g_{\eta}(i)) : \eta \in u^i_\delta\rangle$ II can play one more move in $EF_\lambda$ on $\mathcal{M}_i$ and $\mathcal{N}_i$ with I playing the element $f_\delta(i)$. Let $g_\delta(i)$ be the answer of II in this game according to $\sigma_i$. The values $g_\delta(i)$, $i \in C$, constitute the function $g_\delta$ mod $D$. We have shown that II can maintain a good position. □

We do not know whether the conditions (a)-(c) of Theorem 1 are necessary for the conclusion.

**Remark 2** We point out some variants of Theorem 1:

1. We can define a version $\square^i_{\lambda,D}$ of $\square^0_{\lambda,D}$ which is equivalent to: “If $\mathcal{M}$ and $\mathcal{N}$ are $EF_\gamma$-equivalent, then $\mathcal{M}^\lambda / D$ and $\mathcal{N}^\lambda / D$ are $EF_{\lambda^+}$-equivalent”:
\( \square^\gamma_{\lambda,D} : D \) is a filter on a cardinal \( \lambda \) and there exist finite sets \( C^\xi_\alpha \) and ordinals \( \gamma_\xi < \gamma \) for each \( \alpha < \lambda^+ \) and \( \xi < \lambda \) such that for each \( \xi, \alpha \)

(i) \( C^\xi_\alpha \subseteq \alpha + 1 \)

(ii) If \( B \subseteq \lambda^+ \) is a set of ordinals with \( \text{otp}(B) < \gamma \) and \( \alpha < \lambda^+ \) is such that \( B \subseteq \alpha + 1 \), then \( \{ \xi : B \subseteq C^\xi_\alpha \} \in D \).

(iii) \( \beta \in C^\xi_\alpha \) implies \( C^\xi_\beta = C^\xi_\alpha \cap (\beta + 1) \).

(iv) \( \text{otp}(C^\xi_\alpha) < \gamma_\xi \).

If clauses (a), (b) and (c) of Theorem 1 are assumed, then \( \square^\lambda_{\lambda,D} \).

2. We can also define a version \( \square^{<\delta}_{\lambda,D} \) of \( \square^\text{fin}_{\lambda,D} \) which is equivalent to “If \( \mathcal{M} \) and \( \mathcal{N} \) are \( EF_\gamma \)-equivalent for all \( \gamma < \delta \), then \( \mathcal{M}^\lambda/D \) and \( \mathcal{N}^\lambda/D \) are \( EF_{\lambda^+} \)-equivalent”. If clauses (a), (b) and (c) of Theorem 1 are assumed, then \( \square^{<\lambda}_{\lambda,D} \) holds, where (c)\(^+\) says that (c) holds and there are functions \( f_\alpha, \alpha \leq \lambda^+ \), such that \( \alpha < \beta \leq \lambda^+ \) implies \( \{ i < \lambda : f_\alpha(i) < f_\beta(i) \} \in D \) (For \( D = \) the club filter this is the so called assumption of the existence of the \( \lambda^+ \)'th canonical function, see e.g. [5, p. 445].)

3. Note that

\[
\square^\text{fin}_{\lambda,D} \Rightarrow \square^\gamma_{\lambda,D} \Rightarrow \square^{<\lambda}_{\lambda,D} \Rightarrow \square^\lambda_{\lambda,D}
\]

for \( \gamma < \lambda \).

4. We get a variant of Theorem 1 also by showing, assuming (a), (b) and (c), that \( \prod_D \mathcal{M}_i \) and \( \prod_D \mathcal{N}_i \) are \( EF_{\lambda^+} \)-equivalent, if for all \( \beta < \lambda \):

\( \{ i < \lambda : \mathcal{M}_i \text{ and } \mathcal{N}_i \text{ are } EF_\beta \text{-equivalent} \} \in D \).

5. We can weaken clause (c) of the theorem to the assumption that \( D \) is unreasonable ([14]) in the following sense: There is a partition \( \{ w_i : i < \lambda \} \) of \( \lambda \) such that \( \bigcup_{i \in E} w_i \in D \) for every club \( E \) of \( \lambda \).

3 Doubly regular filters

We define the concept of a doubly regular filter, give examples of such on regular cardinals, and prove that \( \square^\text{fin}_{\lambda,D} \) holds for such filters. Recall that
A family of sets is a regular family if finite intersections of members of the family are non-empty, but all infinite intersections are empty, a filter is called $\mu$-regular if it contains a regular family of size $\mu$, and a filter on $\lambda$ is called regular if it is $\lambda$-regular.

**Definition 3** Suppose $D$ is a filter on a regular cardinal $\lambda$.

1. $D$ is called doubly regular, if there are pairwise disjoint sets $u_i \subseteq \lambda$, $i < \lambda$, each of cardinality $\lambda$, and regular filters $D_i$ on $u_i$ such that for all $A \subseteq \lambda$:

   $\forall^\infty i < \lambda (A \cap u_i \in D_i) \Rightarrow A \in D$.

   (“$\forall^\infty i < \lambda$” means “for all but boundedly many $i$”.)

2. The filter $D$ is called doubly $^+$ regular if the above holds with “$\forall^\infty i < \lambda$” replaced by “for a club of $i$”.

Let us make some easy observations about doubly regular filters:

**Observation 4**

1. A doubly regular filter is necessarily regular: Let $\{A^\alpha_i : \alpha < \lambda\}$ be a regular family in $D_i$. Let

   $$B^\alpha = \bigcup_{i<\lambda} A^\alpha_i.$$  

   Then $\{B^\alpha : \alpha < \lambda\}$ is a regular family in $D$. We will show below (Theorem 7) that the converse need not be true.

2. A doubly $^+$ regular filter is always doubly regular.

3. It is easy to construct doubly($^+$) regular filters. Indeed, if the sets $u_i \subseteq \lambda$, $i < \lambda$, are disjoint, each of cardinality $\lambda$, $\lambda = \bigcup_i u_i$, and we have regular filters $D_i$ on $u_i$, then the set $\{A \subseteq \lambda : \forall^\infty i < \lambda (A \cap u_i \in D_i)\}$ is a doubly regular filter on $\lambda$, and the larger set $\{A \subseteq \lambda : \text{For a club of } i < \lambda (A \cap u_i \in D_i)\}$ is a doubly $^+$ regular filter on $\lambda$. Both double regularity and double $^+$ regularity are closed under extensions of the filter, so we get also ultrafilter examples of both.

Here is the main point of doubly $^+$ regular filters, at least from the point of view of this paper:
Theorem 5 If $D$ is a doubly regular filter on a regular cardinal $\lambda > \aleph_0$, then $\square^{\alpha}_{\lambda,D}$ holds.

Proof. Let the sets $u_i$ and the filters $D_i$ be as in Definition 3. Let $D^*$ be the club filter of $\lambda$, and

$$D' = \{ A \subseteq \lambda : \{ i < \lambda : A \cap u_i \in D_i \} \in D^* \}.$$ 

We prove $\square^{\alpha}_{\lambda,D'}$. From this $\square^{\alpha}_{\lambda,D}$ follows, as $D' \subseteq D$. It suffices to prove that if $M_\alpha$ and $N_\alpha$, $\alpha < \lambda$, are elementarily equivalent, with a vocabulary of size $\leq \lambda$, then $M = \prod_{D'} M_\alpha$ and $N = \prod_{D'} N_\alpha$ are $EF_{\lambda^+}$-equivalent. Note that

(a) $\mathcal{M} \cong \prod_{i<\lambda} \mathcal{M}^i/D^*$, where $\mathcal{M}^i = \prod_{\alpha \in u_i} \mathcal{M}_\alpha/D_i$.

(b) $\mathcal{N} \cong \prod_{i<\lambda} \mathcal{N}^i/D^*$, where $\mathcal{N}^i = \prod_{\alpha \in u_i} \mathcal{N}_\alpha/D_i$.

Since each $D_i$ is $\lambda$-regular, the models $\mathcal{M}^i$ and $\mathcal{N}^i$ are $EF_{\lambda}$-equivalent by [13, Theorem VI.1.8]. By Theorem 1 the models $\mathcal{M}$ and $\mathcal{N}$ are now $EF_{\lambda^+}$-equivalent. □

4 On regular but non-doubly regular filters

Non-regular uniform filters do not necessarily exist. If there is a non-regular uniform ultrafilter on $\omega_1$, then $V \neq L$ by [11], $0^*$ exists by [10], and in fact $\omega_2$ is a limit of measurable cardinals in the Jensen-Dodd Core Model, by [2]. We show that we can always construct a regular but non-doubly regular filter. In this sense double regularity is easier to avoid than regularity.

If $E$ is an equivalence relation on $\lambda$ we denote the set of all $E$-classes by $\lambda/E$, and the $E$-class of $i$ by $i/E$.

First an equivalent condition for double regularity, one that fits better our present purpose:

Lemma 6 A filter $D$ is doubly regular if and only if there is an equivalence relation $E$ of $\lambda$ and $\bar{u} = \langle u_\alpha : \alpha \in \lambda \rangle$ such that:

(DR-a) $\{ u_\epsilon : \epsilon \sim_E i \}$ is a regular family of subsets of $i/E$ for each $i < \lambda$.

(DR-b) If $S \subseteq \lambda$ and $|S| < \lambda$, then $\bigcup \{ i/E : i \in S \} = \emptyset \mod D$,
(DR-c) \( |i/E| = \lambda \) for all \( i < \lambda \),

(DR-d) If \( f \) is a function such that \( \text{dom}(f) = \lambda/E \) and \( f(i/E) \sim_E i \) for all \( i \in \lambda/E \), then \( \bigcup_{i \in \lambda/E} u_f(i) \notin D \).

The proof is easy.

**Theorem 7** If \( 2^\lambda = \lambda^+ \), then there is a regular ultrafilter on \( \lambda \), which is not doubly regular.

**Proof.** Let \( \{ B_\alpha : \alpha \in \lambda^+ \} \) list \( \mathcal{P}(\lambda) \). Let \( \{(E^\alpha, \bar{u}_\alpha) : \alpha < \lambda^+ \} \) list potential candidates for double regularity i.e. \( E \) and \( \bar{u} = \langle u_\zeta : \zeta < \lambda \rangle \) such that \( \{u_\zeta : \zeta < i/E\} \) is a regular family on \( i/E \) for each \( i < \lambda \). This is only place where we use \( 2^\lambda = \lambda^+ \).

We construct by induction sets \( D_\alpha, \alpha < \lambda^+ \), such that the following conditions will hold:

(C-a) \( D_\alpha \subseteq \mathcal{P}(\lambda) \) is \( \subseteq \)-continuously increasing.

(C-b) \( |D_\alpha| = \lambda \).

(C-c) \( D_\alpha \) is closed under finite intersections. We use \( \text{Fil}(D_\alpha) \) to denote the filter \( D_\alpha \) generates.

(C-d) \( D_0 \) contains a regular family. (So necessarily, \( u \in [\lambda]^{<\lambda} \) implies \( u = \emptyset \) mod \( D \).)

(C-e) If \( \alpha = 2\beta + 1 \), then \( B_\beta \in D_\alpha \) or \( (\lambda \setminus B_\beta) \in D_\alpha \).

(C-f) If \( \alpha = 2\beta + 2 \), then either there is \( S \in [\lambda]^{<\lambda} \) such that \( \bigcup_{\epsilon \in S} \epsilon/E_\beta \neq \emptyset \) mod \( \text{Fil}(D_\alpha) \), or, letting \( \bar{u}_\beta = \langle u_{\beta,\epsilon} : \epsilon < \lambda \rangle \), there is \( f \) such that \( \text{dom}(f) = \lambda/E_\beta \) and \( f(i/E_\beta) \sim_{E_\beta} i \) for all \( i \in \lambda/E_\beta \), then \( \bigcup_{i \in \lambda/E_\beta} u_{\beta, f(i)} \in D_\alpha \).

Here is the construction:

**Case 1:** \( \alpha = 0 \). Let \( E \) be a regular family on \( \lambda \). (We can construct a regular family on \( \lambda \) in the standard way: Let \( J \) be the set of finite subsets of \( \lambda \). The family \( \{\{X \in J : \beta \in X\} : \beta < \lambda\} \) is a regular family on \( J \), and hence gives rise to one on \( \lambda \).) We extend \( E \) to \( D_0 \) by closing under finite intersections.

**Case 2:** \( \alpha = 2\beta + 1 \). We make a choice between \( B_\beta \in D_\alpha \) and \( (\lambda \setminus B_\beta) \in D_\alpha \) so that \( \emptyset \notin \text{Fil}(D_\alpha) \).
Case 3: $\alpha = 2\beta + 2$. Let $\{C^\alpha_l : l < \lambda\}$ list $D_{2\beta+1}$. If there is $S \in [\lambda]^{<\lambda}$ such that $\bigcup_{\epsilon \in S} \epsilon / E_\beta \neq \emptyset$ mod $\text{Fil}(D_{2\beta+1})$, we let $D_{2\beta+2} = D_{2\beta+1}$. So let us assume

(*) For all $S \in [\lambda]^{<\lambda}$ we have $\bigcup_{\epsilon \in S} \epsilon / E_\beta = \emptyset$ mod $\text{Fil}(D_{2\beta+1})$.

We prove the following auxiliary:

Subclaim: There are $(\epsilon_i, \gamma_i), i < \lambda$ such that

(a) $\epsilon_i \in \lambda \setminus \{\epsilon_j : j < i\}$.
(b) $\gamma_i \sim_{E_\beta} \epsilon_i$.
(c) $u_{\beta,\gamma_i} \not\supseteq C^\alpha_i \cap \epsilon_i / E_\beta$.

Let us first suppose the subclaim is true and we have such a sequence $(\epsilon_i, \gamma_i), i < \lambda$. Choose $f$ by letting $f(\epsilon_i) = \gamma_i$. So $\bigcup_{i \in \lambda / E_\beta} u_{\beta,f(i)}$ is a subset of $\lambda$, which includes no element of $D_{2\beta+1}$. So we let

$$D_\alpha = D_{2\beta+1} \cup \{A \setminus \bigcup_{i \in \lambda / E_\beta} u_{\beta,f(i)} : A \in D_{2\beta+1}\}.$$ 

This is clearly closed under finite intersections and does not contain $\emptyset$ and every set in $D_\alpha$ has cardinality $\lambda$.

Let us then prove the subclaim. Let $i < \lambda$ and

$$W_1 = \bigcup_{j < i} \epsilon_j / E_\beta.$$ 

By our assumption (*), $W_1 = \emptyset$ mod $\text{Fil}(D_{2\beta+1})$. Choose $\xi_i$ from the non-empty set $(\lambda \setminus W_1) \cap C_{\alpha,i}$. Then pick $\epsilon_i$ so that $\xi_i \sim_{E_\beta} \epsilon_i$. Finally, let

$$W_2 = \{\gamma < \lambda : \gamma \sim_{E_\beta} \epsilon_i \text{ and } \xi_i \in u_{\beta,\gamma}\}.$$ 

Since $A^\beta$ is a regular family, the set $W_2$ is finite. So there is $\gamma_i \in u_{\beta,\epsilon_i} \setminus W_2$. This ends the construction of the sequence $(\epsilon_i, \gamma_i), i < \lambda$, and thereby finishes the proof of the subclaim.

Finishing the proof: Now that we have constructed the sequence $D_\alpha, \alpha < \lambda^+$, we can let

$$D = \bigcup_{\alpha < \lambda^+} D_\alpha.$$
This is an ultrafilter on $\lambda$. It is regular by (C-d). Now we can easily see that $D$ is not doubly regular: Suppose $E_\beta$ and $\bar{u}_\beta$ witnesses that $D$ is doubly regular. Let us look at the construction of $D_{2\beta+2}$. In the first case we assumed that there is $S \in [\lambda]<\lambda$ with $\bigcup_{\varepsilon \in S} \varepsilon/E_\beta \neq \emptyset \mod \text{Fil}(D_{2\beta+1})$. So $\bigcup_{\varepsilon \in S} \varepsilon/E_\beta \neq \emptyset$ mod $D$, and (DR-b) is violated. In the second case we found $f$ such that $\bigcup_{i \in \lambda/E_\beta} u_{\beta,f}(i) = \emptyset$ mod $\text{Fil}(D_\alpha)$. Hence $\bigcup_{i \in \lambda/E_\beta} u_{\beta,f}(i) = \emptyset$ mod $D$, and (DR-d) is violated. $\square$

Note that double$^+$ regularity of $D$ implies $\square_\lambda^{\aleph_0}$ on a regular cardinal $\lambda > \aleph_0$ (Theorem 5), but in the light of the above Theorem, not conversely, as GCH implies $\square_\lambda^{\aleph_0}$ for regular $D$ and regular $\lambda$ ([7, Lemma 4]).

Theorem 7 has the assumption $2^\lambda = \lambda^+$, which may fail for all $\lambda$. We shall present next a slightly different construction under a different assumption, one that is always satisfied by a multitude of cardinals $\lambda$.

**Theorem 8** Assume the following two conditions:

(A1) $\text{cof}(\lambda) > \aleph_0$ or $\lambda > 2^{\aleph_0}$.

(A2) There is $A \subseteq P(\lambda)$ of cardinality $2^\lambda$ such that $|\{A \cap i : A \in A\}| \leq \lambda$ for all $i < \lambda$.

Then there is a regular but not doubly regular filter on $\lambda$.

Note a family $A$, as in (A2), always exists if $\lambda = 2^{<\lambda}$. Hence condition (A2) can be replaced by $\lambda = \beth_\alpha$, $\alpha$ limit.

**Proof.** Let $\{(E_\beta, \bar{u}_\beta) : \beta < 2^\lambda\}$ list all pairs where $E_\beta$ is an equivalence relation on $\lambda$ and $\bar{u}_i^\beta = \langle u_{\beta,\epsilon} : \epsilon \sim E_\beta i \rangle$ is a regular family of subsets of $i/E_\beta$ for each $i < \lambda$. Let $\{B_\alpha : \alpha < 2^\lambda\}$ list $P(\lambda)$.

We construct a sequence $(I_\alpha, D_\alpha), \alpha < 2^\lambda$ such that:

1. $|I_\alpha| \leq |\alpha|$, $I_\alpha \subseteq P(\lambda)$, $(I_\alpha)$ is continuously increasing,

2. $D_\alpha$ is the filter $D[I_\alpha] = \{A \subseteq \lambda : \exists J \in [I_\alpha]^{<\aleph_0}, \exists S \in [\lambda]^{<\lambda} (\bigcap J \subseteq A \cup S)\}$,

3. $D_{2\beta+1} = D_{2\beta} \cup \{B_\beta\}$ or $D_{2\beta+1} = D_{2\beta} \cup \{\lambda \setminus B_\beta\}$

4. $D_{2\beta+2}$ satisfies

   (a) There is some $W \in [\lambda]^{<\lambda}$ such that $\bigcup_{i \in W} i/E_\beta \neq \emptyset \mod D_{2\beta+1}$, or
(b) There is an \( f \) such that \( f(i/E_\beta) \in i/E_\beta \) for all \( i \) and \( \lambda \setminus \bigcup \{ u_{\beta,f(x)} : x \in \lambda/E_\beta \} \in I_\beta \) or
\[
\{ X \in \lambda/E_\beta : |X \cap B| = \lambda \} < \lambda \text{ for some } B \in D_{2\beta+1}.
\]

(c) \(|\{ X \in \lambda/E_\beta : |X \cap B| = \lambda \}| < \lambda \) for some \( B \in D_{2\beta+1} \).

The construction now follows: Let us look at the case \( \alpha = 2\beta + 2 \). If we cannot form \( D_\alpha \) as required, then:

(N1) If \( W \in [\lambda]^{<\lambda} \), then \( \bigcup_{i \in W} i/E_\beta = \emptyset \mod D_{2\beta+1} \).

(N2) If \( f \) is a function such that \( \text{dom}(f) = \lambda/E_\beta \) and \( f(i/E_\beta) \sim_{E_\beta} i \) for all \( i < \lambda \), and
\[
A_{\beta,f} = \bigcup \{ u_{\beta,f(x)} : x \in \lambda/E_\beta \},
\]
then \( \emptyset \in D(I_{2\beta+1} \cup \{ \lambda \setminus A_{\beta,f} \}) \).

(N3) For \( B \in D_{2\beta+1} \), \(|\{ X \in \lambda/E_\beta : |X \cap B| = \lambda \}| = \lambda \).

We derive a contradiction. This will ensure that \( D_\alpha \) can be found. Let \( \langle x_\beta,i : i < \lambda \rangle \) list \( \lambda/E_\beta \). By our choice of \( A \), there are one-one functions \( b_i : \{ A \cap i : A \in \mathcal{A} \} \to x_\beta,i \) for each \( i < \lambda \). If \( s \subseteq \lambda \), let \( g_s \) be a function such that \( \text{dom}(g_s) = \lambda/E_\beta \) and
\[
g_s(x_\beta,i) = b_i(s \cap i)
\]
so that \( g_s(x_\beta,i) \in x_\beta,i \). By (N2) there are
\[
J_{\beta,s} \in [I_{2\beta+1}]^{<\aleph_0}, W_{\beta,s} \in [\lambda]^{<\lambda}
\]
such that
\[
\bigcap_{B \in J_{\beta,s}} B \subseteq A_{\beta,g_s} \cup W_{\beta,s}.
\]
Since \( |\mathcal{A}| = 2^\lambda \), there are \( J_* \in [I_{2\beta+1}]^{<\aleph_0} \) and \( \mu < \lambda \) such that if
\[
\mathcal{A}_1 = \{ s \in \mathcal{A} : J_{\beta,s} = J_*, |W_{\beta,s}| = \mu \},
\]
then \( |\mathcal{A}_1| = 2^\lambda \). Let \( B_* = \bigcap J_* \in D_{2\beta+1} \). By (N3),
\[
|\{ j < \lambda : |x_\beta,j \cap B_*| = \lambda \}| = \lambda.
\]
Claim: There are $s_n \in A_1$, $n < \omega$, and $i < \omega$ such that $s_n \cap i \neq s_m \cap i$ for all $n < m < \omega$.

Case 1: $\text{cof}(\lambda) > \aleph_0$. Pick distinct $s_n \in A_1$, $n < \omega$. Since $\text{cof}(\lambda) > \aleph_0$, there is $i < \lambda$ such that $s_n \cap i \neq s_m \cap i$ for all $n < m < \omega$.

Case 2: $\text{cof}(\lambda) = \aleph_0$, $\lambda > 2^{\aleph_0}$. Pick distinct $s_\xi \in A_1$, $\xi < (2^{\aleph_0})^+$. Let $C \subseteq \lambda$ be cofinal, $|C| = \aleph_0$. Let $\chi : [(2^{\aleph_0})^+]^2 \to C$ be defined by $\chi(\{\xi, \zeta\}) = \min\{c \in C : s_\xi \cap c \neq s_\zeta \cap c\}$. By the Erdős-Rado Theorem $(2^{\aleph_0})^+ \to (\aleph_1)^2$ there is $i \in C$ and an uncountable $H \subseteq (2^{\aleph_0})^+$ such that $\chi \upharpoonright [H]^2$ has constant value $i$.

The Claim is proved. By (2), there is $j > i$ such that $|B_\ast \cap x_{\beta,j}| = \lambda$. With the notation of (N2)

$$A_{\beta, g_{s_n}} \cap x_{\beta,j} = u_{\beta, b_j(s_n \cap j)}$$

and the sets $u_{\beta, b_j(s_n \cap j)}$ are distinct because $b_j$ is one-one. By regularity,

$$\bigcap_n u_{\beta, b_j(s_n \cap j)} = \emptyset. \quad (3)$$

Let $W = \bigcup\{W_{\beta, s_n} : n < \omega\}$. Clearly, $|W| = \mu$. Now

$$B_\ast \cap x_{\beta,j} \subseteq u_{\beta, b_j(s_n \cap j)} \cup W.$$  

This contradicts $|B_\ast \cap x_{\beta,j}| = \lambda$, since $|W| = \mu$ and (3) gives

$$B_\ast \cap x_{\beta,j} \subseteq \bigcap_n (u_{\beta, b_j(s_n \cap j)} \cup W) = W.$$

If we start with a model of $GCH$, we can use Easton forcing [3] to obtain a model in which $2^\lambda$ is—for all regular $\lambda$—anything not ruled out by the conditions $\kappa \leq \lambda \Rightarrow 2^\kappa \leq 2^\lambda$ and $\text{cof}(2^\lambda) > \lambda$. In the arising forcing extension $V[G]$ the tree $(\lessdot^\lambda 2)^V$, $\lambda$ regular, has cardinality $\lambda$ and $2^\lambda$ branches. Hence we have in $V[G]$ a set $A_\lambda$ of cardinality $2^\lambda$—for all regular $\lambda$—such that $\forall i < \lambda(|A \cap i : A \in A_\lambda|) \leq \lambda)$, which is exactly the assumption $(A2)$ of Theorem 8.

\[\square\]
5 Good ultrafilters

Keisler [6] introduced the concept of \( \kappa \)-goodness of ultrafilters and proved that if \( 2^\lambda = \lambda^+ \) and \( D \) is a \( \lambda^+ \)-good (i.e. good) countably incomplete ultrafilter on \( \lambda \), then \( \prod_D \mathcal{M}_i \cong \prod_D \mathcal{N}_i \) for any models \( \mathcal{M}_i \equiv \mathcal{N}_i \) of cardinality \( \leq \lambda^+ \) in a vocabulary of cardinality \( \leq \lambda \). This raises the question whether there is a connection between goodness and double regularity. It turns out that these concepts are independent of each other.

**Proposition 9** Suppose \( \lambda > \aleph_0 \). There is a doubly regular ultrafilter on \( \lambda \) which is not good. If \( 2^\lambda = \lambda^+ \), then there is a good countably incomplete ultrafilter on \( \lambda \) which is not doubly regular.

**Proof.** For the first claim, let \( D_1 \) be a doubly regular ultrafilter on \( \lambda \) (exists by Observation 4) and \( D_2 \) a countably incomplete ultrafilter of \( \omega \) which is not \( \aleph_2 \)-good. (exists by [6, 5.1]). Let \( D = D_1 \times D_2 \). This is an ultrafilter on the set \( \lambda \times \omega \) of size \( \lambda \). Since \( D_2 \) is not \( \lambda^+ \)-good, neither is \( D \) ([13, VI.3.7]). Double regularity is inherited from \( D_1 \) as follows: Suppose we have pairwise disjoint sets \( u_i, i < \lambda \), on \( \lambda \), each of cardinality \( \lambda \), and regular filters \( F_i \) on \( u_i \) such that for all \( A \subseteq \lambda \):

\[
[\forall \infty i < \lambda (A \cap u_i \in F_i)] \rightarrow A \in D_1.
\]

Let \( G_i \subseteq F_i \) be a regular family on \( u_i \). Let \( u_i^* = u_i \times \omega \) and \( G_i^* = \{ A \times \omega : A \in G_i \} \). Let \( F_i^* \) be the filter on \( u_i^* \) generated by \( \{ A \times \omega : A \in F_i \} \). Now \( G_i^* \) is a regular family \( \subseteq F_i^* \) and if \( A \subseteq \lambda \times \omega \), then

\[
[\forall \infty i < \lambda (A \cap u_i^* \in F_i^*)] \rightarrow A \in D_1 \times D_2.
\]

This ends the proof that \( D \) is doubly regular.

For the second claim we use a combination of the construction of the proof of Theorem 7 and Keisler’s construction of a good ultrafilter in [6, 4.4]. The construction of Keisler, as presented in [1, Chapter 6, p. 387] proceeds in stages, generating a continuously increasing sequence \( F_\alpha, \alpha < 2^\lambda \), of filters such that the following condition holds (for unexplained terminology we refer to [1, Chapter 6, p. 387]): For the first (in a fixed well-ordering) monotone \( f : [\lambda]^{<\aleph_0} \rightarrow F_\alpha \) for which there is no additive extension \( [\lambda]^{<\aleph_0} \rightarrow F_\alpha \), there is an additive extension \( g : [\lambda]^{<\aleph_0} \rightarrow F_{\alpha+1} \). To make sure that such \( g \) and \( F_{\alpha+1} \) always exist an auxiliary sequence is simultaneously defined, namely a
descending sequence \( \Pi_\alpha, \alpha < 2^\lambda \), of partitions of \( \lambda \), starting from a carefully chose initial set \( \Pi_0 \) with \( |\Pi_0| = 2^\lambda \). There is no problem in interleaving the inductive construction of the filters \( F_\alpha \) into the construction in the proof of Theorem 7. The resulting ultrafilter is good but not doubly regular. \( \square \)

6 Concluding remarks

We proved that \( \square^{\text{fin}}_{\lambda,D} \) holds if \( \lambda \) is a regular cardinal and \( D \) is a doubly regular filter. This naturally raises the question whether \( \square^{\text{fin}}_{\lambda,D} \) can fail at a regular cardinal for some regular, but not doubly regular, filter. We know it can fail at a singular cardinal [8].

**Conjecture 1:** Consistently, \( \square^{\text{fin}}_{\lambda,D} \) fails for some regular \( \lambda > \omega \) and some regular filter \( \lambda \) generated by \( \lambda \) sets.

**Conjecture 2:** If \( D \) is a regular ultrafilter on \( \mathbb{N}_1 \) such that \( \neg \square^{\text{fin}}_{\mathbb{N}_1,D} \), then for any increasing continuous \( \langle \alpha_i : i < \omega_1 \rangle \) with \( \alpha_i < \omega_1 \), there is \( A \in D \) such that \( A \cap [\alpha_i, \alpha_{i+1}) \) is finite for all \( i < \omega_1 \).

Note that if

\[
D = \{ A \subseteq \omega_1 : \forall^\infty i < \lambda (A \cap [\alpha_i, \alpha_{i+1}) \in D_i) \},
\]

\( D_i \) ultrafilter on \( [\alpha_i, \alpha_{i+1}) \), then the answer to Conjecture 2 is positive. This may indicate that looking for counterexamples for \( \square^{\text{fin}}_{\mathbb{N}_1,D} \) can be hard.

References


