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Regular Ultrapowers at Regular Cardinals

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Abstract
In earlier work of the second and third author the equivalence of a finite square principle $\square_{\lambda,D}^{\text{fin}}$ with various model theoretic properties of structures of size $\lambda$ and regular ultrafilters was established. In this paper we investigate the principle $\square_{\lambda,D}^{\text{fin}}$, and thereby the above model theoretic properties, at a regular cardinal. By Chang’s Two-Cardinal Theorem, $\square_{\lambda,D}^{\text{fin}}$ holds at regular cardinals for all regular filters $D$ if we assume GCH. In this paper we prove in $ZFC$ that for certain regular

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filters that we call \textit{doubly} \textsuperscript{+} \textit{regular}, $\square_{\lambda,D}^{\fin}$ holds at regular cardinals, with no assumption about GCH. Thus we get new positive answers in ZFC to Open Problems 18 and 19 in the book \textit{Model Theory} by Chang and Keisler.

1 Introduction

In [7] and [8] the equivalence of the following finite square principle $\square_{\lambda,D}^{\fin}$ with various model theoretic properties of regular reduced powers of models was established:

$\square_{\lambda,D}^{\fin}$: $D$ is a filter on a cardinal $\lambda$ and there exist finite sets $C_\alpha^\xi$ and integers $n_\xi$ for each $\alpha < \lambda^+$ and $\xi < \lambda$ such that for each $\xi, \alpha$

(i) $C_\alpha^\xi \subseteq \alpha + 1$

(ii) If $B \subseteq \lambda^+$ is a finite set of ordinals and $\alpha < \lambda^+$ is such that $B \subseteq \alpha + 1$, then $\{\xi : B \subseteq C_\alpha^\xi \} \in D$

(iii) $\beta \in C_\alpha^\xi$ implies $C_\beta^\xi = C_\alpha^\xi \cap (\beta + 1)$

(iv) $|C_\alpha^\xi| < n_\xi$

The model theoretic properties were the following: Firstly, if $D$ is an ultrafilter, then $\square_{\lambda,D}^{\fin}$ is equivalent to $M^\lambda / D$ being $\lambda^{++}$-universal for each model $M$ in a vocabulary of size $\leq \lambda$. To formulate the second model theoretic property, let us say that two models are \textit{EF}_\alpha-equivalent if the second player (i.e. the “isomorphism” player) has a winning strategy in the Ehrenfeucht-Fraïssé game of length $\alpha$ on the two models\textsuperscript{1}. Now $\square_{\lambda,D}^{\fin}$ is equivalent to $M^\lambda / D$ and $N^\lambda / D$ being $EF_{\lambda^+}$-equivalent for any elementarily equivalent models $M$ and $N$ (w.l.o.g. of cardinality $\leq \lambda^+$) in a vocabulary of size $\leq \lambda$. The existence of such ultrafilters and models is related to Open Problems 18 and 19 in the Chang-Keisler model theory book [1].

The consistency of the failure of $\square_{\lambda,D}^{\fin}$ for a regular ultrafilter $D$ at a singular strong limit cardinal $\lambda$ was proved in [8] relative to the consistency of a supercompact cardinal. In [9] this was improved to the failure of $\square_{\lambda,D}^{\fin}$ for a regular ultrafilter $D$ at a singular strong limit cardinal $\lambda$ relative to the

\textsuperscript{1}The usual elementary equivalence in a finite relational vocabulary is thus $EF_n$-equivalence for all $n < \omega$, and $L_{\infty\omega}$-equivalence is the same as $EF_\kappa$-equivalence. For models of cardinality $\leq \kappa$, $EF_\kappa$-equivalence is equivalent to isomorphism.
consistency of a strongly compact cardinal. The failure of $\Box^{\text{fin}}_{\lambda,D}$ for an ultrafilter implies the failure of $\lambda^{++}$-universality of $\mathcal{M}\lambda/D$ for some $\mathcal{M}$, as well as the failure of isomorphism of some regular ultrapowers $\mathcal{M}\lambda/D$ and $\mathcal{N}\lambda/D$. Thus [9] answered negatively the following problems listed in [1] modulo large cardinal assumptions:

**Problem 18 ([1])** Let $|\mathcal{M}|, |\mathcal{N}|, |L| \leq \alpha$ and let $D$ be a regular ultrafilter over $\alpha$. If $\mathcal{M} \equiv \mathcal{N}$, then $\prod_D \mathcal{M} \cong \prod_D \mathcal{N}$.

**Problem 19 ([1])** If $D$ is a regular ultrafilter of $\alpha$, then for all infinite $\mathcal{M}$, $\prod_D \mathcal{M}$ is $\alpha^{++}$-universal.

The use of large cardinals is justified by [7], [8] and [12] as the failure of $\Box^{\text{fin}}_{\lambda,D}$ for singular strong limit $\lambda$ implies the failure of $\Box_{\lambda}$, which implies the consistency of large cardinals.

In this paper we investigate the principle $\Box^{\text{fin}}_{\lambda,D}$, and thereby the above model theoretic problems, at a regular cardinal. The following result is proved in [4]: Assume $\kappa$ is regular and $\lambda^{<\kappa} = \lambda$. Suppose $\mathcal{M}$ and $\mathcal{N}$ are structures for a finite vocabulary such that $\mathcal{M}$ and $\mathcal{N}$ are $EF_\alpha$-equivalent for each $\alpha < \kappa$. Suppose $D$ is a filter on $\xi \times \lambda$, $\xi \leq \lambda$, extending $F' \times F$, where $F'$ is a $\kappa$-descendingly incomplete filter on $\xi$ and $F$ is a $\kappa$-semigood filter on $\lambda$ (the concept is defined in [4]). Then $\mathcal{M}\lambda/D$ and $\mathcal{N}\lambda/D$ are $EF_{\lambda^{+}}$-equivalent. For $\kappa = \omega$ this, combined with the existence proof of semigood filters in [4], yields filters $D$ with $\Box^{\text{fin}}_{\lambda,D}$.

The structure of the paper is the following: In Section 2 we prove weaker versions of $\Box^{\text{fin}}_{\lambda,D}$ in the case where the filter $D$ extends the club filter on $\lambda$. Naturally this case is in spirit quite far from the case of regular $D$, which is our prime interest. However, this result is useful in the sequel. Note that there are many regular (ultra)filters extending the club filter. In Section 3 we define the concept of doubly regular filter and show that such filters $D$ on regular $\lambda > \aleph_0$ satisfy $\Box^{\text{fin}}_{\lambda,D}$. Thus we get new positive answers in $ZFC$ to the above Problem 18 (with isomorphism replaced, in the absence of $2^\lambda = \lambda^+$, by $EF_{\lambda^{+}}$-equivalence) and the above Problem 19. In Section 4 we prove results to the effect that not all regular filters are doubly regular. In Section 5 we compare our concept of double regularity to Keisler’s concept of goodness of a filter. In Section 6 we present some open questions.
2 Filters extending the club filter

We can get provable cases of a weaker form of $\square^{\kappa}_{\lambda, D}$, when $D$ extends the club filter. This will prove useful in the next section, where we will use Theorem 1 in the proof of Theorem 5. The original $\square^{\kappa}_{\lambda, D}$ is equivalent to reduced powers of elementarily equivalent models of cardinality $\lambda$ being $EF_{\lambda^+}$-equivalent. The weaker form which we shall prove below will give the $EF_{\lambda^+}$-equivalence of reduced powers of models of power $\lambda$ that are not just elementarily equivalent but even $EF_{\lambda}$-equivalent.

**Theorem 1** Suppose

(a) $\lambda$ is regular $> \aleph_0$,

(b) $D$ is a filter on $\lambda$.

(c) $D$ extends the club filter.

If $\mathcal{M}$ and $\mathcal{N}$ are $EF_{\lambda}$-equivalent, then $\mathcal{M}^\lambda/D$ and $\mathcal{N}^\lambda/D$ are $EF_{\lambda^+}$-equivalent.

**Proof.** If $\alpha < \lambda^+$, $\lambda$ regular, let $\{u^i_\alpha : i < \lambda\}$ be a continuously increasing sequence of subsets of $\alpha$ such that $|u^i_\alpha| < \lambda$ for all $i < \lambda$ and $\alpha = \bigcup_{i<\lambda} u^i_\alpha$.

Let

$$D_\alpha = \{i < \lambda : \forall \beta \in u^i_\alpha (u^i_\beta = u^i_\alpha \cap \beta)\}.$$  \hspace{1cm} (1)

It is easy to see that $D_\alpha$ is a club of $\lambda$ (recall that $\lambda$ is regular).

Now we can proceed, as in [7] to prove that if $M$ and $N$ are $EF_{\lambda}$-equivalent, then $M^\lambda/D$ and $N^\lambda/D$ are $EF_{\lambda^+}$-equivalent:

Let $L$ be a finite vocabulary and for each $i < \lambda$ let $\mathcal{M}_i$ and $\mathcal{N}_i$ be $EF_{\lambda}$-equivalent $L$-structures. We show that II has a winning strategy in the game $EF_{\lambda^+}$ on the models $\mathcal{M} = \prod_D \mathcal{M}_i$ and $\mathcal{N} = \prod_D \mathcal{N}_i$.

The crucial idea of the proof is the following: When the Ehrenfeucht-Fraïssé game $EF_{\lambda^+}(\mathcal{M}, \mathcal{N})$ is played, the players are actually playing $\lambda$ Ehrenfeucht-Fraïssé games simultaneously, namely the games $EF_{\lambda}(\mathcal{M}_i, \mathcal{N}_i)$, $i < \lambda$.

For each $i < \lambda$ let $\sigma_i$ be a winning strategy for II in the game $EF_{\lambda}$ on the models $\mathcal{M}_i$ and $\mathcal{N}_i$. A good position is a sequence $\langle (f_\beta, g_\beta) : \beta < \alpha \rangle$ for some $\alpha < \lambda^+$, together with a club $C \subseteq D_\alpha$, such that for all $\beta < \alpha$ we have $f_\beta \in \prod_i \mathcal{M}_i$, $g_\beta \in \prod_i \mathcal{N}_i$, and if $i \in C$, then

$$\langle (f_\eta(i), g_\eta(i)) : \eta \in u^i_\alpha \rangle$$
is a play according to $\sigma_i$ on the models $M_i$ and $N_i$. In a good position the equivalence classes of the functions $f_\beta$ and $g_\beta$ determine a partial isomorphism of the reduced products: Suppose $\alpha$ rounds have been played and we are in a good position. Let $\phi_\gamma([f_{\beta_1}], \ldots, [f_{\beta_k}])$ be an atomic formula holding in $\prod_i M_i/D$, where $\beta_1 < \ldots < \beta_k < \alpha$, and let $A = \{i \in D_\alpha : \{\beta_1, \ldots, \beta_k\} \subseteq u^i_\alpha\}$. By assumption, $A \in D$. Since also $B = \{i < \lambda : M_i \models \phi_\gamma(f_{\beta_1}(i), \ldots, f_{\beta_k}(i))\} \in D$, we have $A \cap B \in D$. For $i \in A \cap B$ we have $\beta_1, \ldots, \beta_k \in u^i_\alpha$, hence

$$u^i_{\beta_j} = u^i_\alpha \cap \beta_j.$$  

Since we are in a good position, $\{(f_\eta(i), g_\eta(i)) : \eta \in u^i_\alpha\}$ is a play according to winning strategy $\sigma_i$. Hence $\{(f_\epsilon(\xi), g_\epsilon(\xi)) : \epsilon \in u^i_\alpha\}$ determines a partial isomorphism of the structures $M_i$ and $N_i$. Since this was the case for all $i \in A \cap B \in D$, we get $\prod_i N_i/D \models \phi_\gamma([g_{\beta_1}], \ldots, [g_{\beta_k}]).$

The strategy of II is to keep the position of the game good and thereby win the game. So suppose $\beta$ rounds have been played and II has been able to keep the position good. Then for all $\gamma < \beta$ there is a club $C_\gamma \subseteq D_\gamma$ such that for $i \in C_\gamma$, $\{(f_\eta(i), g_\eta(i)) : \eta \in u^i_\gamma\}$ is a play according to $\sigma_i$.

**Case 1:** $\beta = \cup \beta$. Let $C = \bigcap_{\gamma < \beta} C_\gamma$. Since $\lambda$ is regular, this is still a club. We show that $\{(f_\gamma, g_\gamma) : \gamma < \beta\}$ is good. Let $i \in C$. Let us look at $\{(f_\eta(i), g_\eta(i)) : \eta \in u^i_\beta\}$. Since $i \in D_\beta$, every initial segment of this play is a play according to $\sigma_i$. Hence so is the entire play $\{(f_\gamma, g_\gamma) : \gamma < \beta\}$. We have shown that II can maintain a good position.

**Case 2:** $\beta = \delta + 1$. Let $C \subseteq \bigcap_{\gamma \leq \beta} C_\gamma$ such that $\delta \in u^i_\beta$ for $i \in C$. Now suppose I plays $f_\delta$. We show that II can play $g_\delta$ so that $\{(f_\gamma, g_\gamma) : \gamma < \beta\}$ remains good. Let $i \in C$. Let us look at $\{(f_\eta(i), g_\eta(i)) : \eta \in u^i_\delta\}$. This is a play according to the strategy $\sigma_i$. Since $i \in D_{\delta}$ and $\delta \in u^i_\beta$, $u^i_\delta = u^i_\beta \cap \delta$, so after the moves $\{(f_\gamma(i), g_\gamma(i)) : \eta \in u^i_\delta\}$ II can play one more move in $EF_\delta$ on $M_i$ and $N_i$ with I playing the element $f_\delta(i)$. Let $g_\delta(i)$ be the answer of II in this game according to $\sigma_i$. The values $g_\delta(i), i \in C$, constitute the function $g_\delta \mod D$. We have shown that II can maintain a good position. □

We do not know whether the conditions (a)-(c) of Theorem 1 are necessary for the conclusion.

**Remark 2** We point out some variants of Theorem 1:

1. *We can define a version $\square_{\lambda,D}^\kappa$ of $\square_{\lambda,D}^\kappa$ which is equivalent to: “If $M$ and $N$ are $EF_\gamma$-equivalent, then $M^\lambda/D$ and $N^\lambda/D$ are $EF_{\lambda^+}$-equivalent”:*
□_{\lambda,D}^\gamma : D is a filter on a cardinal \( \lambda \) and there exist finite sets \( C_\alpha^\xi \) and ordinals \( \gamma_\xi < \gamma \) for each \( \alpha < \lambda^+ \) and \( \xi < \lambda \) such that for each \( \xi, \alpha 
abla \)

(i) \( C_\alpha^\xi \subseteq \alpha + 1 \)
(ii) If \( B \subseteq \lambda^+ \) is a set of ordinals with \( \text{otp}(B) < \gamma \) and \( \alpha < \lambda^+ \) is such that \( B \subseteq C_\alpha^\xi \) then \( \{ \xi : B \subseteq C_\alpha^\xi \} \in D \).

(iii) \( \beta \in C_\alpha^\xi \) implies \( C_{\beta}^\xi = C_\alpha^\xi \cap (\beta + 1) \).
(iv) \( \text{otp}(C_\alpha^\xi) < \gamma_\xi \).

If clauses (a), (b) and (c) of Theorem 1 are assumed, then \( \square_{\lambda,D}^\lambda \).

2. We can also define a version \( \square_{\lambda,D}^{< \delta} \) of \( \square_{\lambda,D}^{\text{fin}} \) which is equivalent to “If \( \mathcal{M} \) and \( \mathcal{N} \) are \( EF_\gamma \)-equivalent for all \( \gamma < \delta \), then \( \mathcal{M}^\lambda/D \) and \( \mathcal{N}^\lambda/D \) are \( EF_{\lambda^+} \)-equivalent”. If clauses (a), (b) and (c) of Theorem 1 are assumed, then \( \square_{\lambda,D}^{< \lambda} \) holds, where (c)\( ^+ \) says that (c) holds and there are functions \( f_\alpha \), \( \alpha \leq \lambda^+ \), such that \( \alpha < \beta \leq \lambda^+ \) implies \( \{ i < \lambda : f_\alpha(i) < f_\beta(i) \} \in D \) (For \( D \) = the club filter this is the so called assumption of the existence of the \( \lambda^+ \)th canonical function, see e.g. [5, p. 445]).

3. Note that
\[ \boxtimes_{\lambda,D}^\gamma \Rightarrow \square_{\lambda,D}^\gamma \Rightarrow \square_{\lambda,D}^{< \lambda} \Rightarrow \square_{\lambda,D}^\lambda \]
for \( \gamma < \lambda \).

4. We get a variant of Theorem 1 also by showing, assuming (a), (b) and (c), that \( \prod_D \mathcal{M}_i \) and \( \prod_D \mathcal{N}_i \) are \( EF_{\lambda^+} \)-equivalent, if for all \( \beta < \lambda \):
\[ \{ i < \lambda : \mathcal{M}_i \text{ and } \mathcal{N}_i \text{ are } EF_\beta \text{-equivalent} \} \in D. \]

5. We can weaken clause (c) of the theorem to the assumption that \( D \) is unreasonable ([14]) in the following sense: There is a partition \( \{ w_i : i < \lambda \} \) of \( \lambda \) such that \( \bigcup_{i \in E} w_i \in D \) for every club \( E \) of \( \lambda \).

3 Doubly regular filters

We define the concept of a doubly regular filter, give examples of such on regular cardinals, and prove that \( \square_{\lambda,D}^{\text{fin}} \) holds for such filters. Recall that
a family of sets is a regular family if finite intersections of members of the family are non-empty, but all infinite intersections are empty, a filter is called $\mu$-regular if it contains a regular family of size $\mu$, and a filter on $\lambda$ is called regular if it $\lambda$-regular.

**Definition 3** Suppose $D$ is a filter on a regular cardinal $\lambda$.

1. $D$ is called doubly regular, if there are pairwise disjoint sets $u_i \subseteq \lambda$, $i < \lambda$, each of cardinality $\lambda$, and regular filters $D_i$ on $u_i$ such that for all $A \subseteq \lambda$:
   \[ \forall i < \lambda (A \cap u_i \in D_i) \Rightarrow A \in D. \]
   ("$\forall \infty i < \lambda$" means "for all but boundedly many $i$".)

2. The filter $D$ is called doubly$^+$ regular if the above holds with "$\forall \infty i < \lambda$" replaced by "for a club of $i$".

Let us make some easy observations about doubly regular filters:

**Observation 4**

1. A doubly regular filter is necessarily regular: Let $\{A^\alpha_i : \alpha < \lambda\}$ be a regular family in $D_i$. Let
   \[ B^\alpha = \bigcup_{i<\lambda} A^\alpha_i. \]
   Then $\{B^\alpha : \alpha < \lambda\}$ is a regular family in $D$. We will show below (Theorem 7) that the converse need not be true.

2. A doubly$^+$ regular filter is always doubly regular.

3. It is easy to construct doubly($^+$) regular filters. Indeed, if the sets $u_i \subseteq \lambda, i < \lambda$, are disjoint, each of cardinality $\lambda$, $\lambda = \bigcup_i u_i$, and we have regular filters $D_i$ on $u_i$, then the set $\{A \subseteq \lambda : \forall \infty i < \lambda (A \cap u_i \in D_i)\}$ is a doubly regular filter on $\lambda$, and the larger set $\{A \subseteq \lambda : \text{For a club of } i < \lambda (A \cap u_i \in D_i)\}$ is a doubly$^+$ regular filter on $\lambda$. Both double regularity and double$^+$ regularity are closed under extensions of the filter, so we get also ultrafilter examples of both.

Here is the main point of doubly$^+$ regular filters, at least from the point of view of this paper:
**Theorem 5** If $D$ is a doubly regular filter on a regular cardinal $\lambda > \aleph_0$, then $\square_{\lambda,D}^{fin}$ holds.

**Proof.** Let the sets $u_i$ and the filters $D_i$ be as in Definition 3. Let $D^*$ be the club filter of $\lambda$, and $D' = \{A \subseteq \lambda : \{i < \lambda : A \cap u_i \in D_i\} \in D^*\}$.

We prove $\square_{\lambda,D'}^{fin}$. From this $\square_{\lambda,D}^{fin}$ follows, as $D' \subseteq D$. It suffices to prove that if $\mathcal{M}_\alpha$ and $\mathcal{N}_\alpha$, $\alpha < \lambda$, are elementarily equivalent, with a vocabulary of size $\leq \lambda$, then $\mathcal{M} = \prod_{D'} \mathcal{M}_\alpha$ and $\mathcal{N} = \prod_{D'} \mathcal{N}_\alpha$ are $EF_{\lambda^+}$-equivalent. Note that

(a) $\mathcal{M} \cong \prod_{i<\lambda} \mathcal{M}^i/D^*$, where $\mathcal{M}^i = \prod_{\alpha \in u_i} \mathcal{M}_{\alpha}/D_i$.

(b) $\mathcal{N} \cong \prod_{i<\lambda} \mathcal{N}^i/D^*$, where $\mathcal{N}^i = \prod_{\alpha \in u_i} \mathcal{N}_{\alpha}/D_i$.

Since each $D_i$ is $\lambda$-regular, the models $\mathcal{M}^i$ and $\mathcal{N}^i$ are $EF_{\lambda}$-equivalent by [13, Theorem VI.1.8]. By Theorem 1 the models $\mathcal{M}$ and $\mathcal{N}$ are now $EF_{\lambda^+}$-equivalent. $\square$

### 4 On regular but non-doubly regular filters

Non-regular uniform filters do not necessarily exist. If there is a non-regular uniform ultrafilter on $\omega_1$, then $V \neq L$ by [11], $0^\#$ exists by [10], and in fact $\omega_2$ is a limit of measurable cardinals in the Jensen-Dodd Core Model, by [2]. We show that we can always construct a regular but non-doubly regular filter. In this sense double regularity is easier to avoid than regularity.

If $E$ is an equivalence relation on $\lambda$ we denote the set of all $E$-classes by $\lambda/E$, and the $E$-class of $i$ by $i/E$.

First an equivalent condition for double regularity, one that fits better our present purpose:

**Lemma 6** A filter $D$ is doubly regular if and only if there is an equivalence relation $E$ of $\lambda$ and $\bar{u} = \langle u_\alpha : \alpha \in \lambda \rangle$ such that:

**DR-a** $\{u_\epsilon : \epsilon \sim_E i\}$ is a regular family of subsets of $i/E$ for each $i < \lambda$.

**DR-b** If $S \subseteq \lambda$ and $|S| < \lambda$, then $\bigcup\{i/E : i \in S\} = \emptyset \mod D$,
\( |i/E| = \lambda \) for all \( i < \lambda \),

\textbf{(DR-d)} If \( f \) is a function such that \( \text{dom}(f) = \lambda/E \) and \( f(i/E) \sim_E i \) for all \( i \in \lambda/E \), then \( \bigcup_{i \in \lambda/E} u_{f(i)} \notin D \).

The proof is easy.

\textbf{Theorem 7} If \( 2^\lambda = \lambda^+ \), then there is a regular ultrafilter on \( \lambda \), which is not doubly regular.

\textbf{Proof.} Let \( \{B_\alpha : \alpha \in \lambda^+\} \) list \( \mathcal{P}(\lambda) \). Let \( \{(E_\alpha, \bar{u}_\alpha) : \alpha < \lambda^+\} \) list potential candidates for double regularity i.e. \( E \) and \( \bar{u} = \langle u_\zeta : \zeta < \lambda \rangle \) such that \( \{u_\zeta : \zeta < i/E\} \) is a regular family on \( i/E \) for each \( i < \lambda \). This is only place where we use \( 2^\lambda = \lambda^+ \).

We construct by induction sets \( D_\alpha, \alpha < \lambda^+ \), such that the following conditions will hold:

\textbf{(C-a)} \( D_\alpha \subseteq \mathcal{P}(\lambda) \) is \( \subseteq \)-continuously increasing.

\textbf{(C-b)} \( |D_\alpha| = \lambda \).

\textbf{(C-c)} \( D_\alpha \) is closed under finite intersections. We use \( \text{Fil}(D_\alpha) \) to denote the filter \( D_\alpha \) generates.

\textbf{(C-d)} \( D_0 \) contains a regular family. (So necessarily, \( u \in [\lambda]^{<\lambda} \) implies \( u = \emptyset \) mod \( D \).)

\textbf{(C-e)} If \( \alpha = 2\beta + 1 \), then \( B_\beta \in D_\alpha \) or \( (\lambda \setminus B_\beta) \in D_\alpha \).

\textbf{(C-f)} If \( \alpha = 2\beta + 2 \), then either there is \( S \in [\lambda]^{<\lambda} \) such that \( \bigcup_{\epsilon \in S} \epsilon/E_\beta \neq \emptyset \) mod \( \text{Fil}(D_\alpha) \), or, letting \( \bar{u}_\beta = \langle u_{\beta,\epsilon} : \epsilon < \lambda \rangle \), there is \( f \) such that \( \text{dom}(f) = \lambda/E_\beta \) and \( f(i/E_\beta) \sim_{E_\beta} i \) for all \( i \in \lambda/E_\beta \), then \( \bigcup_{i \in \lambda/E_\beta} u_{\beta,f(i)} \notin D_\alpha \).

Here is the construction:

\textbf{Case 1:} \( \alpha = 0 \). Let \( E \) be a regular family on \( \lambda \). (We can construct a regular family on \( \lambda \) in the standard way: Let \( J \) be the set of finite subsets of \( \lambda \). The family \( \{\{X \in J : \beta \in X\} : \beta < \lambda\} \) is a regular family on \( J \), and hence gives rise to one on \( \lambda \).) We extend \( E \) to \( D_0 \) by closing under finite intersections.

\textbf{Case 2:} \( \alpha = 2\beta + 1 \). We make a choice between \( B_\beta \in D_\alpha \) and \( (\lambda \setminus B_\beta) \in D_\alpha \) so that \( \emptyset \notin \text{Fil}(D_\alpha) \).
Case 3: $\alpha = 2\beta + 2$. Let $\{C^\alpha_l : l < \lambda\}$ list $\mathcal{D}_{2\beta+1}$. If there is $S \in [\lambda]^{<\lambda}$ such that $\bigcup_{\varepsilon \in S} \varepsilon/E_\beta \neq \emptyset \mod \text{Fil}(\mathcal{D}_{2\beta+1})$, we let $\mathcal{D}_{2\beta+2} = \mathcal{D}_{2\beta+1}$. So let us assume

(\star) For all $S \in [\lambda]^{<\lambda}$ we have $\bigcup_{\varepsilon \in S} \varepsilon/E_\beta = \emptyset \mod \text{Fil}(\mathcal{D}_{2\beta+1})$.

We prove the following auxiliary:

Subclaim: There are $(\varepsilon_i, \gamma_i), i < \lambda$ such that

(a) $\varepsilon_i \in \lambda \setminus \{\varepsilon_j : j < i\}$.
(b) $\gamma_i \sim_{E_\beta} \varepsilon_i$.
(c) $u_{\beta, \gamma_i} \nsubseteq C^\alpha_i \cap \varepsilon_i/E_\beta$.

Let us first suppose the subclaim is true and we have such a sequence $(\varepsilon_i, \gamma_i), i < \lambda$. Choose $f$ by letting $f(\varepsilon_i) = \gamma_i$. So $\bigcup_{i \in \lambda/E_\beta} u_{\beta, f(i)}$ is a subset of $\lambda$, which includes no element of $\mathcal{D}_{2\beta+1}$. So we let

$$\mathcal{D}_\alpha = \mathcal{D}_{2\beta+1} \cup \{A : A \in \bigcup_{i \in \lambda/E_\beta} u_{\beta, f(i)} : A \in \mathcal{D}_{2\beta+1}\}.$$ 

This is clearly closed under finite intersections and does not contain $\emptyset$ and every set in $\mathcal{D}_\alpha$ has cardinality $\lambda$.

Let us then prove the subclaim. Let $i < \lambda$ and

$$W_1 = \bigcup_{j < i} \varepsilon_j/E_\beta.$$ 

By our assumption (\star), $W_1 = \emptyset \mod \text{Fil}(\mathcal{D}_{2\beta+1})$. Choose $\xi_i$ from the non-empty set $(\lambda \setminus W_1) \cap C_{\alpha, i}$. Then pick $\varepsilon_i$ so that $\xi_i \sim_{E_\beta} \varepsilon_i$. Finally, let

$$W_2 = \{\gamma < \lambda : \gamma \sim_{E_\beta} \varepsilon_i \text{ and } \xi_i \in u_{\beta, \gamma}\}.$$ 

Since $\mathcal{A}^\beta$ is a regular family, the set $W_2$ is finite. So there is $\gamma_i \in u_{\beta, \varepsilon_i} \setminus W_2$. This ends the construction of the sequence $(\varepsilon_i, \gamma_i), i < \lambda$, and thereby finishes the proof of the subclaim.

Finishing the proof: Now that we have constructed the sequence $\mathcal{D}_\alpha, \alpha < \lambda^+$, we can let

$$D = \bigcup_{\alpha < \lambda^+} \mathcal{D}_\alpha.$$
This is an ultrafilter on $\lambda$. It is regular by (C-d). Now we can easily see that $D$ is not doubly regular: Suppose $E_\beta$ and $\bar{u}_\beta$ witnesses that $D$ is doubly regular. Let us look at the construction of $D_{2\beta+2}$. In the first case we assumed that there is $S \in [\lambda]^{< \lambda}$ with $\bigcup_{\epsilon \in S} \epsilon/E_\beta \neq \emptyset \mod \text{Fil}(D_{2\beta+1})$. So $\bigcup_{\epsilon \in S} \epsilon/E_\beta \neq \emptyset \mod D$, and (DR-b) is violated. In the second case we found $f$ such that $\bigcup_{i \in \lambda/E_\beta} u_{\beta,f}(i) = \emptyset \mod \text{Fil}(D_\alpha)$. Hence $\bigcup_{i \in \lambda/E_\beta} u_{\beta,f}(i) = \emptyset \mod D$, and (DR-d) is violated. □

Note that double$^+$ regularity of $D$ implies $\square_\lambda^{\aleph_0}$ on a regular cardinal $\lambda > \aleph_0$ (Theorem 5), but in the light of the above Theorem, not conversely, as GCH implies $\square_\lambda^{\aleph_0}$ for regular $D$ and regular $\lambda$ ([7, Lemma 4]).

Theorem 7 has the assumption $2^\lambda = \lambda^+$, which may fail for all $\lambda$. We shall present next a slightly different construction under a different assumption, one that is always satisfied by a multitude of cardinals $\lambda$.

**Theorem 8** Assume the following two conditions:

(A1) $\text{cof}(\lambda) > \aleph_0$ or $\lambda > 2^{\aleph_0}$.

(A2) There is $\mathcal{A} \subseteq \mathcal{P}(\lambda)$ of cardinality $2^\lambda$ such that $|\{A \cap i : A \in \mathcal{A}\}| \leq \lambda$ for all $i < \lambda$.

Then there is a regular but not doubly regular filter on $\lambda$.

Note a family $\mathcal{A}$, as in (A2), always exists if $\lambda = 2^{< \lambda}$. Hence condition (A2) can be replaced by $\lambda = \beth_\alpha$, $\alpha$ limit.

**Proof.** Let $\langle (E_\beta, \bar{u}_\beta) : \beta < 2^\lambda \rangle$ list all pairs where $E_\beta$ is an equivalence relation on $\lambda$ and $\bar{u}_\beta = \langle u_{\beta,\epsilon} : \epsilon \sim E_\beta i \rangle$ is a regular family of subsets of $i/E_\beta$ for each $i < \lambda$. Let $\{B_\alpha : \alpha < 2^\lambda\}$ list $\mathcal{P}(\lambda)$.

We construct a sequence $(I_\alpha, D_\alpha), \alpha < 2^\lambda$ such that:

1. $|I_\alpha| \leq |\alpha|$, $I_\alpha \subseteq \mathcal{P}(\lambda)$, $(I_\alpha)$ is continuously increasing,

2. $D_\alpha$ is the filter $\mathcal{D}[I_\alpha] = \{A \subseteq \lambda : \exists J \in [I_\alpha]^{< \aleph_0} \exists S \in [\lambda]^{< \lambda}(\exists J \subseteq A \cup S)\},$

3. $D_{2\beta+1} = D_{2\beta} \cup \{B_\beta\}$ or $D_{2\beta+1} = D_{2\beta} \cup \{\lambda \setminus B_\beta\}$

4. $D_{2\beta+2}$ satisfies

   (a) There is some $W \in [\lambda]^{< \lambda}$ such that $\bigcup_{i \in W} i/E_\beta \neq \emptyset \mod D_{2\beta+1}$, or
(b) There is an $f$ such that $f(i/E_\beta) \in i/E_\beta$ for all $i$ and $\lambda \setminus \bigcup \{ u_{\beta,f(x)} : x \in \lambda/E_\beta \} \in I_\beta$ or

(c) $| \{ X \in \lambda/E_\beta : |X \cap B| = \lambda \} | < \lambda$ for some $B \in D_{2\beta+1}$.

The construction now follows: Let us look at the case $\alpha = 2\beta + 2$. If we cannot form $D_\alpha$ as required, then:

(N1) If $W \in [\lambda]^{<\lambda}$, then $\bigcup_{i \in W} i/E_\beta = \emptyset \bmod D_{2\beta+1}$.

(N2) If $f$ is a function such that $\text{dom}(f) = \lambda/E_\beta$ and $f(i/E_\beta) \sim_{E_\beta} i$ for all $i < \lambda$, and

$$A_{\beta,f} = \bigcup \{ u_{\beta,f(x)} : x \in \lambda/E_\beta \},$$

then $\emptyset \in D(I_{2\beta+1} \cup \{ \lambda \setminus A_{\beta,f} \})$.

(N3) For $B \in D_{2\beta+1}$, $| \{ X \in \lambda/E_\beta : |X \cap B| = \lambda \} | = \lambda$.

We derive a contradiction. This will ensure that $D_\alpha$ can be found. Let $\langle x_{\beta,i} : i < \lambda \rangle$ list $\lambda/E_\beta$. By our choice of $A$, there are one-one functions $b_i : \{ A \cap i : A \in A \} \to x_{\beta,i}$ for each $i < \lambda$. If $s \subseteq \lambda$, let $g_s$ be a function such that $\text{dom}(g_s) = \lambda/E_\beta$ and

$$g_s(x_{\beta,i}) = b_i(s \cap i)$$

so that $g_s(x_{\beta,i}) \in x_{\beta,i}$. By (N2) there are

$$J_{\beta,s} \in [I_{2\beta+1}]^{<\aleph_0}, W_{\beta,s} \in [\lambda]^{<\lambda}$$

such that

$$\bigcap_{B \in J_{\beta,s}} B \subseteq A_{\beta,f} \cup W_{\beta,s}.$$

Since $|A| = 2^\lambda$, there are $J_\ast \in [I_{2\beta+1}]^{<\aleph_0}$ and $\mu < \lambda$ such that if

$$A_1 = \{ s \in A : J_{\beta,s} = J_\ast, |W_{\beta,s}| = \mu \},$$

then $|A_1| = 2^\lambda$. Let $B_\ast = \bigcap J_\ast \in D_{2\beta+1}$. By (N3),

$$| \{ j < \lambda : |x_{\beta,j} \cap B_\ast| = \lambda \} | = \lambda.$$
Claim: There are \( s_n \in A_1, n < \omega \), and \( i < \omega \) such that \( s_n \cap i \neq s_m \cap i \) for all \( n < m < \omega \).

Case 1: \( \text{cof}(\lambda) > \aleph_0 \). Pick distinct \( s_n \in A_1, n < \omega \). Since \( \text{cof}(\lambda) > \aleph_0 \), there is \( i < \lambda \) such that \( s_n \cap i \neq s_m \cap i \) for all \( n < m < \omega \).

Case 2: \( \text{cof}(\lambda) = \aleph_0 \), \( \lambda > 2^{\aleph_0} \). Pick distinct \( s_\xi \in A_1, \xi < (2^{\aleph_0})^+ \). Let \( C \subseteq \lambda \) be cofinal, \(|C| = \aleph_0 \). Let \( \chi : [(2^{\aleph_0})^+]^2 \to C \) be defined by \( \chi(\{\xi, \zeta\}) = \min\{c \in C : s_\xi \cap c \neq s_\zeta \cap c\} \). By the Erdős-Rado Theorem \((2^{\aleph_0})^+ \to (\aleph_1)^2\) there is \( i \in C \) and an uncountable \( H \subseteq (2^{\aleph_0})^+ \) such that \( \chi \upharpoonright [H]^2 \) has constant value \( i \).

The Claim is proved. By (2), there is \( j > i \) such that \(|B_\ast \cap x_{\beta,j}| = \lambda \).

With the notation of (N2)

\[ A_{\beta,g_{s_n}} \cap x_{\beta,j} = u_{\beta,b_j(s_n \cap j)} \]

and the sets \( u_{\beta,b_j(s_n \cap j)} \) are distinct because \( b_j \) is one-one. By regularity,

\[ \bigcap_n u_{\beta,b_j(s_n \cap j)} = \emptyset. \tag{3} \]

Let \( W = \bigcup \{W_{\beta,s_n} : n < \omega\} \). Clearly, \(|W| = \mu \). Now

\[ B_\ast \cap x_{\beta,j} \subseteq u_{\beta,b_j(s_n \cap j)} \cup W. \]

This contradicts \(|B_\ast \cap x_{\beta,j}| = \lambda \), since \(|W| = \mu \) and (3) gives

\[ B_\ast \cap x_{\beta,j} \subseteq \bigcap_n (u_{\beta,b_j(s_n \cap j)} \cup W) = W. \]

\[ \square \]

If we start with a model of \( GCH \), we can use Easton forcing [3] to obtain a model in which \( 2^\lambda \) is—for all regular \( \lambda \)—anything not ruled out by the conditions \( \kappa \leq \lambda \Rightarrow 2^\kappa \leq 2^\lambda \) and \( \text{cof}(2^\lambda) > \lambda \). In the arising forcing extension \( V[G] \) the tree \((^{<\lambda}2)^V\), \( \lambda \) regular, has cardinality \( \lambda \) and \( 2^\lambda \) branches. Hence we have in \( V[G] \) a set \( A_\lambda \) of cardinality \( 2^\lambda \)—for all regular \( \lambda \)—such that \( \forall i < \lambda(\{|A \cap i : A \in A_\lambda\}| \leq \lambda) \), which is exactly the assumption (A2) of Theorem 8.
5 Good ultrafilters

Keisler [6] introduced the concept of \( \kappa \)-goodness of ultrafilters and proved that if \( 2^\lambda = \lambda^+ \) and \( D \) is a \( \lambda^+ \)-good (i.e. good) countably incomplete ultrafilter on \( \lambda \), then \( \prod_D \mathcal{M}_i \cong \prod_D \mathcal{N}_i \) for any models \( \mathcal{M}_i \equiv \mathcal{N}_i \) of cardinality \( \leq \lambda^+ \) in a vocabulary of cardinality \( \leq \lambda \). This raises the question whether there is a connection between goodness and double regularity. It turns out that these concepts are independent of each other.

**Proposition 9** Suppose \( \lambda > \aleph_0 \). There is a doubly regular ultrafilter on \( \lambda \) which is not good. If \( 2^\lambda = \lambda^+ \), then there is a good countably incomplete ultrafilter on \( \lambda \) which is not doubly regular.

**Proof.** For the first claim, let \( D_1 \) be a doubly regular ultrafilter on \( \lambda \) (exists by Observation 4) and \( D_2 \) a countably incomplete ultrafilter of \( \omega \) which is not \( \aleph_2 \)-good. (exists by [6, 5.1]). Let \( D = D_1 \times D_2 \). This is an ultrafilter on the set \( \lambda \times \omega \) of size \( \lambda \). Since \( D_2 \) is not \( \lambda^+ \)-good, neither is \( D \) ([13, VI.3.7]).

Double regularity is inherited from \( D_1 \) as follows: Suppose we have pairwise disjoint sets \( u_i, i < \lambda \), on \( \lambda \), each of cardinality \( \lambda \), and regular filters \( F_i \) on \( u_i \) such that for all \( A \subseteq \lambda \):

\[
[\forall^\infty i < \lambda (A \cap u_i \in F_i)] \rightarrow A \in D_1.
\]

Let \( G_i \subseteq F_i \) be a regular family on \( u_i \). Let \( u_i^* = u_i \times \omega \) and \( G_i^* = \{ A \times \omega : A \in G_i \} \). Let \( F_i^* \) be the filter on \( u_i^* \) generated by \( \{ A \times \omega : A \in F_i \} \). Now \( G_i^* \) is a regular family \( \subseteq F_i^* \) and if \( A \subseteq \lambda \times \omega \), then

\[
[\forall^\infty i < \lambda (A \cap u_i^* \in F_i^*)] \rightarrow A \in D_1 \times D_2.
\]

This ends the proof that \( D \) is doubly regular.

For the second claim we use a combination of the construction of the proof of Theorem 7 and Keisler’s construction of a good ultrafilter in [6, 4.4]. The construction of Keisler, as presented in [1, Chapter 6, p. 387] proceeds in stages, generating a continuously increasing sequence \( F_\alpha, \alpha < 2^\lambda \), of filters such that the following condition holds (for unexplained terminology we refer to [1, Chapter 6, p. 387]): For the first (in a fixed well-ordering) monotone \( f : [\lambda]^{<\kappa_0} \rightarrow F_\alpha \) for which there is no additive extension \( [\lambda]^{<\kappa_0} \rightarrow F_\alpha \), there is an additive extension \( g : [\lambda]^{<\kappa_0} \rightarrow F_{\alpha+1} \). To make sure that such \( g \) and \( F_{\alpha+1} \) always exist an auxiliary sequence is simultaneously defined, namely a
descending sequence $\Pi_\alpha$, $\alpha < 2^\lambda$, of partitions of $\lambda$, starting from a carefully chose initial set $\Pi_0$ with $|\Pi_0| = 2^\lambda$. There is no problem in interleaving the inductive construction of the filters $F_\alpha$ into the construction in the proof of Theorem 7. The resulting ultrafilter is good but not doubly regular. $\square$

6 Concluding remarks

We proved that $\Box_{\lambda,D}^{\text{fin}}$ holds if $\lambda$ is a regular cardinal and $D$ is a doubly regular filter. This naturally raises the question whether $\Box_{\lambda,D}^{\text{fin}}$ can fail at a regular cardinal for some regular, but not doubly regular, filter. We know it can fail at a singular cardinal [8].

**Conjecture 1:** Consistently, $\Box_{\lambda,D}^{\text{fin}}$ fails for some regular $\lambda > \omega$ and some regular filter $\lambda$ generated by $\lambda$ sets.

**Conjecture 2:** If $D$ is a regular ultrafilter on $\aleph_1$ such that $\neg\Box_{\aleph_1,D}^{\text{fin}}$, then for any increasing continuous $\langle \alpha_i : i < \omega_1 \rangle$ with $\alpha_i < \omega_1$, there is $A \in D$ such that $A \cap [\alpha_i, \alpha_{i+1})$ is finite for all $i < \omega_1$.

Note that if

$$D = \{A \subseteq \omega_1 : \forall^{\omega} \alpha_i, \alpha_{i+1} \in D_i \},$$

$D_i$ ultrafilter on $[\alpha_i, \alpha_{i+1})$, then the answer to Conjecture 2 is positive. This may indicate that looking for counterexamples for $\Box_{\aleph_1,D}^{\text{fin}}$ can be hard.

References


