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Clean Quantum and Classical Communication Protocols

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By how much must the communication complexity of a function increase if we demand that the parties not only correctly compute the function but also return all registers (other than the one containing the answer) to their initial states at the end of the communication protocol? Protocols that achieve this are referred to as clean and the associated cost as the clean communication complexity. Here we present clean protocols for calculating the inner product of two $n$-bit strings, showing that (in the absence of preshared entanglement) at most $n + 3$ qubits or $n + O(\sqrt{n})$ bits of communication are required. The quantum protocol provides inspiration for obtaining the optimal method to implement distributed CNOT gates in parallel while minimizing the amount of quantum communication. For more general functions, we show that nearly all Boolean functions require close to $2n$ bits of classical communication to compute and close to $n$ qubits if the parties have access to preshared entanglement. Both of these values are maximal for their respective paradigms.

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In a communication task, two players, Alice and Bob, receive inputs $x$ and $y$ and wish to calculate the value of some function $f$. To achieve this, messages will have to be exchanged between them, and, depending on the resources available to them, these may consist of classical or quantum communication in the form of bits and qubits, respectively. Typically, in such scenarios one is interested in minimizing the amount of communication that has to take place to evaluate the function, and the number of bits (qubits) that must be exchanged to do this is referred to as the classical (quantum) communication complexity [1,2].

A protocol for calculating a function will act on three distinct types of registers. Each player will receive an input register, containing $x$ or $y$, and an ancillary working space, initialized in some standard state such as a string of bits all set to 0, a number of qubits provided in the $|0\rangle$ state, or possibly containing entangled states shared between the parties. The final type of register is the answer register, which will contain the value of $f(x,y)$ at the end of the protocol. On the completion of a generic protocol for computing $f$, the input and ancillary registers will no longer be in their starting states and will depend upon both $x$ and $y$.

However, leaving these registers in such states can be problematic. First, if Alice and Bob wish to keep private the particular protocol that they ran, then discarding these unclean states may leak information regarding this to a third party. Second, in the quantum setting, if the players wish to run the protocol over a superposition of input states (perhaps as a subroutine of a larger computation), then allowing the ancillary registers to end up in some unclean, input-dependent state and then discarding them can lead to a loss of coherence in the superposition over answers. Finally, the players’ computational space may be in short supply, and without knowing the registers’ final states they cannot easily use them for future calculations.

To avoid such issues, we can demand that a protocol (in addition to computing $f$) returns the input and ancillary registers to their starting state. Following Ref. [3], we call such a protocol clean, and the minimum number of bits or qubits that a clean protocol needs to exchange to compute a given function is the clean communication complexity. We shall denote these quantities by $C_{\text{clean}}(f)$ and $Q_{\text{clean}}(f)$.

In the case where the players have access to preshared entanglement (which they must restore at the end of the protocol), the associated cost will be written $Q_{\text{clean}}$. We focus on the scenario where the players must compute the function exactly.

In all three scenarios, an unclean communication protocol can be converted into a clean one at the cost of doubling the communication. To do this, the players run the unclean protocol, copy the output to another location, and then run the unclean protocol backwards. At first glance, it may appear that clean, classical protocols are even easier to construct: The players keep a copy of their input and then simply erase all ancillary bits once the protocol is complete. However, Landauer’s principle [4–6] implies that such irreversible manipulations will generate heat or else cost work. As such, if one is interested in avoiding such costs, it makes sense to consider protocols where all operations must be reversible. In light of these constructions, it is natural to ask: Do more efficient clean protocols, without this doubling in communication, exist?

We first focus on the clean communication complexity of computing the inner product of two distributed bit strings of length $n$, showing that (without preshared entanglement) this can be done by exchanging $n + 3$ qubits. As a clean...
follows the initial state at the beginning of the protocol, this is again close to maximal. Whether similarly linear in privacy among honest players, to bound the amount of inequalities that carefully account for the way protocols can be combined. In quantum computation, and for constructing resource required to implement distributed quantum computation, we find that the method of generating clean protocols discussed above is near optimal. On the quantum side, we find that clean, quantum protocols for computing IPn exactly (Example 1.29 in Ref. [24]). For quantum protocols that are allowed to err with a fixed probability less than 1/2, the complexity is still \( \Omega(n) \) [24].

Here we examine the clean communication complexity of IPn without entanglement assistance. To this end, we first consider the quantum communication complexity of implementing the transformation:

\[
|x\rangle_A |y\rangle_B \mapsto (-1)^{xy}|x\rangle_A |y\rangle_B ,
\]

i.e., the distributed computation of the inner product of x and y in the phase. Such a transformation corresponds to performing controlled-Z gates across n pairs of qubits, and by a suitable local basis change this can be converted into an implementation of n-fold CNOTs.

In Ref. [25], it was shown that two qubits of communication together with sharing four ebits is exactly equivalent as a resource to the ability to implement two CNOT gates and sharing four ebits. As such, this provides a protocol for implementing IPn in the phase using n + 8 qubits of communication and eight ancilla qubits (for even n). This can be adapted to give a protocol requiring n + 2 qubits of communication for even n and n + 3 qubits when n is odd. In the following lemma, we give an improved, optimal protocol.

\[
|x\rangle_A |y\rangle_B \mapsto (-1)^{xy}|x\rangle_A |y\rangle_B ,
\]
FIG. 1. Clean, quantum protocol for calculating \( IP_n \) in the phase. Here we illustrate the first four rounds of communication. In each round, a player cleans up the message they sent previously, applies the relevant global phase, and communicates the next bit of their input string.

**Lemma 1.**—The clean, quantum communication complexity of exactly implementing \( IP_n \) in the phase satisfies

\[
Q_{\text{clean}}(IP_n^{\text{phase}}) = n + 1.
\]

One ancilla qubit is required. (Without using ancilla qubits, \( n + 1 \) qubits for odd \( n \) and \( n + 2 \) for even \( n \) suffice.)

**Proof.**—The \( n + 1 \) qubit protocol for even \( n \) is as follows. Alice initially prepares an ancilla qubit in the state \( |x_1\rangle \) and sends it to Bob, who applies a phase of \((-1)^{x_1 y_1}\). He then adds \( y_2 \) to the communication qubit and sends it back to Alice in the state \(|x_1 \oplus y_2\rangle\). Now, Alice cleans up her previous communication by subtracting \( x_1 \) from the communication and then uses the value of \( y_2 \) to apply the phase \((-1)^{y_2 y_2}\). She then adds \( x_3 \) to the communication qubit to leave it in the state \(|y_2 \oplus x_3\rangle\) and sends it back to Bob. A schematic of these first rounds is given in Fig. 1.

The players then proceed similarly, with each round of communication being used to convey a new bit to the other party and send a received bit back in order to clean the ancilla qubit. After \( n \) rounds, the global phase will be \((-1)^{y_2 y_2}\), and Alice will hold the communication qubit in the state \(|y_n\rangle\). She sends this back to Bob, who cleans it, completing the protocol using \( n + 1 \) qubits of communication and the change in ownership of one ancilla qubit. For odd \( n \), Alice will perform the final cleaning step. The protocol to implement the transformation without an ancilla qubit is given in Appendix B1 [9].

The lower bound is proved in Appendix C3 [9]. It uses the concept of information complexity [26] to show that in a clean protocol for implementing Eq. (4) \( n \) bits of information must flow in each direction. Without preshared entanglement, we show that \( n \) qubits of communication cannot achieve this.

The above lemma provides the optimal method for implementing \( n \) \( CZ \) gates in parallel while exchanging \( n + 1 \) qubits. Such a protocol would prove useful for quantum computing architectures where quantum communication is used to interface and implement gates between clusters of highly controllable qubits. As an example, in quantum error correction one could imagine using the Steane code [27] to protect two logical qubits using two spatially separated clusters of seven physical qubits. To implement a \( CZ \) gate between the logical qubits requires seven \( CZ \)s to be performed in parallel between the physical qubits.

Our protocol achieves this while exchanging only eight qubits, whereas the naive protocol would send 14. Protocols based solely on shared entanglement and classical communication [28–30] use seven pairs of ebits, 14 bits of communication, and the implementation of 14 measurements, while their coherent counterpart [25] requires one shared ebit and eight qubits of communication.

In Appendix B2 [9], we give a clean quantum protocol for computing \( IP_n \).

**Theorem 2.**—The clean, quantum communication complexity of exactly computing \( IP_n \) satisfies

\[
n + 1 \leq Q_{\text{clean}}(IP_n) \leq \begin{cases} n + 3 & \text{for } n \text{ odd}, \\ n + 2 & \text{for } n \text{ even}. \end{cases}
\]

No ancilla qubits are required.

By adapting the protocol from Lemma 1, \( IP_n \) can be computed cleanly using two qubits and \( n + 1 \) bits. We give this protocol in Appendix B3 [9].

Our novel quantum communication protocols inspire a classical protocol for the inner product (given in Appendix B4 [9]) which is near optimal and for which only the naive \( 2n \) protocol was known before.

**Theorem 3.**—The clean, classical communication complexity of exactly computing \( IP_n \) satisfies

\[
n + 1 \leq C_{\text{clean}}(IP_n) \leq n + 4\sqrt{n} + \frac{1}{\sqrt{n} - 1} + 2.
\]

No ancilla bits are required.

**Generic functions.**—In contrast to Theorem 3, we will show that nearly all Boolean functions on \( n \)-bit inputs require \( 2n - O(\log n) \) bits of classical communication to compute cleanly. The proof follows from the following two lemmas. In what follows, \( X \) and \( Y \) are the random variables for Alice and Bob’s inputs, and \( A \) and \( B \) are the random variables received by Alice and Bob, respectively, over the course of the protocol. By \(|a|\) and \(|b|\), we denote the number of bits received by Alice and Bob.

**Lemma 4.**—Consider picking uniformly at random a Boolean function \( f_n \) on \( n \)-bit inputs. Then with probability \( 1 - o(1) \), all protocols that compute \( f_n \) exactly are such that either 1. Alice must receive

\[
|a| \geq n - \log (n + 1) - 2
\]

bits and there exists a uniform distribution over at least half the pairs of inputs such that

\[
I(Y; AX) \geq n - \log (n + 1) - 3,
\]

or 2. Bob must receive

\[
|b| \geq n - \log (n + 1) - 2
\]

bits and there exists a uniform distribution over at least half the pairs of inputs such that
To prove the first two bounds, begin by noting that the communication matrix $M^f$ is low. For $M^f$ to have high Kolmogorov complexity, all protocols for computing $f$ must partition $M^f$ into either very narrow or very thin rectangles. To produce the bound in Eq. (9), we take a distribution over the input pairs, each of which has low Kolmogorov complexity. If one of these rectangles is large enough (which happens when the amount of communication that takes place in one direction is small), then the Kolmogorov complexity of $M^f$ will also be low. Such an $M^f$ is shown in Fig. 2(a). Comparing these two statements leads to the bounds on $|a|$ and $|b|$.

These bounds imply that the rectangles induced by any protocol for computing most input pairs must be either very short or very thick as shown in Fig. 2(b). In fact, they cannot be larger than $4(n+1) \times 2^n$ or $2^n \times 4(n+1)$. Either at least half the inputs will belong to very short rectangles or at least half the inputs will belong to very thin ones. By taking a distribution over the larger set, we induce a direction into the communication that occurs in the protocol to ensure that one of Eqs. (8) and (10) holds and bound the related mutual information. For example, consider the case where more than half the input pairs lie in rectangles of a size less than $2^n \times 4(n+1)$ (as shown in the figure) and the distribution over $x$ and $y$ is formed by picking Alice and Bob’s inputs uniformly at random from such rectangles. Then, at the end of the protocol, Alice will know that Bob received one of at most $4(n+1)$ inputs and Eq. (8) will hold. Hence,

$$I(Y:AX) = H(Y) - H(Y|AX) \geq n - \log (n + 1) - 3,$$

as required.

**Proof.**—The full proof is given in Appendix C1 [9]. It revolves around considering a protocol as a random Boolean function has large Kolmogorov complexity with high probability. However, a classical protocol for computing $f$ partitions the matrix into rectangles (see Appendix A2 [9]), each of which has low Kolmogorov complexity. If one of these rectangles is large enough (which happens when the amount of communication that takes place in one direction is small), then the Kolmogorov complexity of $M^f$ will also be low. Such an $M^f$ is shown in Fig. 2(a). Comparing these two statements leads to the bounds on $|a|$ and $|b|$.

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$$I(Y:AX) = H(Y) - H(Y|AX) \geq n - \log (n + 1) - 3,$$

as required.

**Theorem 6.**—Consider exactly computing a Boolean function $f_n$ on $n$-bit inputs that has been picked uniformly at random. Then with probability $1 - o(1)$
In the case of quantum protocols, a similar result holds in the entanglement-assisted case. Proving this result (Appendix C2 [9]) makes use of the fully quantum notion of information complexity introduced in Ref. [26]. The proof follows a similar structure to the classical result: arguing that for most functions close to $n$ bits of information have to flow from Alice to Bob and for the protocol to be clean an equivalent amount of information has to be returned.

**Theorem 7.**—Consider exactly computing a Boolean function $f_n$ on $n$-bit inputs that has been picked uniformly at random. Then with probability $1 - o(1)$

$$Q_{\text{clean}}^*(f_n) \geq n - \log n.$$  \hspace{1cm} (15)

**Conclusion.**—In this Letter, we have initiated the study of how big an overhead in communication cleanliness requires. For the inner product function (and the task of implementing $n$ CNOT gates in parallel), we have exhibited quantum and classical protocols for which the overhead is low. For most Boolean functions, however, we have shown that the additional cost incurred by demanding cleanliness is close to maximal for the classical and entanglement-assisted complexities. Many questions remain. For example, what are the clean complexities of other notable functions such as equality and disjointness?

As Theorems 6 and 7 show that the clean, classical, and entanglement-assisted communication complexity for most functions is close to maximal, one can ask: Does something similar hold for $Q_{\text{clean}}(f)$? We leave this as an open question but conjecture it to be close to $2n$, as the inner product appears somewhat special in its ability to reuse a single bit efficiently. However, the concept of information cost is blind to sending bits, so the technique used for the entanglement-assisted case does not immediately generalize to proving a bound potentially larger than $n$.

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