Clean Quantum and Classical Communication Protocols

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By how much must the communication complexity of a function increase if we demand that the parties not only correctly compute the function but also return all registers (other than the one containing the answer) to their initial states at the end of the communication protocol? Protocols that achieve this are referred to as clean and the associated cost as the clean communication complexity. Here we present clean protocols for calculating the inner product of two $n$-bit strings, showing that (in the absence of preshared entanglement) at most $n + 3$ qubits or $n + O(\sqrt{n})$ bits of communication are required. The quantum protocol provides inspiration for obtaining the optimal method to implement distributed CNOT gates in parallel while minimizing the amount of quantum communication. For more general functions, we show that nearly all Boolean functions require close to $2n$ bits of classical communication to compute and close to $n$ qubits if the parties have access to preshared entanglement. Both of these values are maximal for their respective paradigms.

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In a communication task, two players, Alice and Bob, receive inputs $x$ and $y$ and wish to calculate the value of some function $f$. To achieve this, messages will have to be exchanged between them, and, depending on the resources available to them, these may consist of classical or quantum communication in the form of bits and qubits, respectively. Typically, in such scenarios one is interested in minimizing the amount of communication that has to take place to evaluate the function, and the number of bits (qubits) that must be exchanged to do this is referred to as the classical (quantum) communication complexity [1,2].

A protocol for calculating a function will act on three distinct types of registers. Each player will receive an input register, containing $x$ or $y$, and an ancillary working space, initialized in some standard state such as a string of bits all set to 0, a number of qubits provided in the $|0\rangle$ state, or possibly containing entangled states shared between the parties. The final type of register is the answer register, which will contain the value of $f(x, y)$ at the end of the protocol. On the completion of a generic protocol for computing $f$, the input and ancillary registers will no longer be in their starting states and will depend upon both $x$ and $y$.

However, leaving these registers in such states can be problematic. First, if Alice and Bob wish to keep private the particular protocol that they ran, then discarding these unclean states may leak information regarding this to a third party. Second, in the quantum setting, if the players wish to run the protocol over a superposition of input states (perhaps as a subroutine of a larger computation), then allowing the ancillary registers to end up in some unclean, input-dependent state and then discarding them can lead to a loss of coherence in the superposition over answers. Finally, the players’ computational space may be in short supply, and without knowing the registers’ final states they cannot easily use them for future calculations.

To avoid such issues, we can demand that a protocol (in addition to computing $f$) returns the input and ancillary registers to their starting state. Following Ref. [3], we call such a protocol clean, and the minimum number of bits or qubits that a clean protocol needs to exchange to compute a given function is the clean communication complexity. We shall denote these quantities by $C_{\text{clean}}(f)$ and $Q_{\text{clean}}(f)$. In the case where the players have access to preshared entanglement (which they must restore at the end of the protocol), the associated cost will be written $Q'_{\text{clean}}$. We focus on the scenario where the players must compute the function exactly.

In all three scenarios, an unclean communication protocol can be converted into a clean one at the cost of doubling the communication. To do this, the players run the unclean protocol, copy the output to another location, and then run the unclean protocol backwards. At first glance, it may appear that clean, classical protocols are even easier to construct: The players keep a copy of their input and then simply erase all ancillary bits once the protocol is complete. However, Landauer’s principle [4–6] implies that such irreversible manipulations will generate heat or else cost work. As such, if one is interested in avoiding such costs, it makes sense to consider protocols where all operations must be reversible. In light of these constructions, it is natural to ask: Do more efficient clean protocols, without this doubling in communication, exist?

We first focus on the clean communication complexity of computing the inner product of two distributed bit strings of length $n$, showing that (without preshared entanglement) this can be done by exchanging $n + 3$ qubits. As a clean
protocol for this function must exchange at least \( n + 1 \) qubits, this is very close to tight. We also provide a clean, classical protocol that computes the inner product while exchanging only \( n + O(\sqrt{n}) \) bits. This provides a saving over the most obvious protocol which, as we shall show, is close to optimal for the clean, classical computation of most functions.

A variation on our quantum protocol can be used to implement \( n \) copies of a CNOT gate in parallel by exchanging \( n + 1 \) qubits. In a quantum computing architecture consisting of distributed clusters of highly controllable qubits linked by quantum communication (such as that envisaged in Ref. [7]), it is prudent to minimize the number of qubits exchanged. Our implementation is optimal.

Next, we turn to the clean communication complexity of random functions on inputs of length \( n \). We show here that, in contrast to the inner product, nearly all functions are such that \( C_{\text{clean}}(f) \) is close to the maximal \( 2n \): The simple method of generating clean protocols discussed above is near optimal. On the quantum side, we find that \( Q_{\text{clean}}(f) \) is close to \( n \) for most functions. As superdense coding [8] allows all functions to be uncleanly computed while exchanging \( n/2 \) qubits when the players preshare entanglement, this is again close to maximal. Whether similarly \( Q_{\text{clean}}(f) \) is close to \( 2n \) remains an open question.

Appendices can be found in Supplemental Material [9].

**Clean protocols.**—Clean protocols have a long history in proving bounds in the model of quantum communication complexity with free entanglement assistance [16]. For example, considering clean, quantum protocols for the inner product function was used to imply that any entanglement-assisted quantum protocol for this function must use at least \( \lceil n/2 \rceil \) qubits [3]. Clean protocols have also been used to lower bound the entanglement-assisted, quantum communication complexity [17] and to show that, in this model of communication, most functions have complexity that scales linearly in \( n \) [18]. Cleanliness has also been used to analyze privacy among honest players [19], to bound the amount of quantum communication required to implement distributed quantum computation [20], and for constructing resource inequalities that carefully account for the way protocols can be combined [21,22].

More formally, a clean, quantum protocol for computing a function \( f : \{0,1\}^n \times \{0,1\}^n \rightarrow \{0,1\} \) is defined as follows [3]. The initial state at the beginning of the protocol is of the form

\[
|x\rangle_A |\tilde{0}\rangle_A |y\rangle_B |\tilde{0}\rangle_B |\Phi\rangle_{A_0 A_1 B_0} \ |z\rangle_{B_{\text{ans}}},
\]

where \( |x\rangle_A = \otimes_{i=1}^n |x_i\rangle_A \) and \( |y\rangle_B = \otimes_{i=1}^n |y_i\rangle_B \) are Alice and Bob’s respective inputs stored in \( n \) qubits, \( |\tilde{0}\rangle_A \) and \( |\tilde{0}\rangle_B \) their qubit ancillae, \( |\Phi\rangle_{A_0 A_1 B_0} \) their preshared entanglement (if supplied), and \( |z\rangle_{B_{\text{ans}}} \) is the initial state of the answer register with \( z \in \{0,1\} \). Throughout this letter, we will assume that at the beginning and end of a protocol the answer register is held by Bob.

Players then take turns to act on their share of the qubits. In each turn, a player will apply a unitary transformation to the qubits in their possession and then send some subset of them to the other player. The protocol computes \( f \) cleanly if the final state of the qubits is

\[
|x\rangle_A |\tilde{0}\rangle_A |y\rangle_B |\tilde{0}\rangle_B |\Phi\rangle_{A_0 A_1 B_0} \ |z\rangle_{B_{\text{ans}}},
\]

where the addition in the answer register is modulo 2. Clean classical protocols are defined similarly but with registers and communication given in terms of bits rather than qubits and no entanglement. All transformations must be reversible.

**Inner product.**—The specific function that we shall focus on in this paper is the inner product function \( IP_n \). This is defined by

\[
IP_n : \{0,1\}^n \times \{0,1\}^n \rightarrow \{0,1\},
\]

\[
IP_n(x,y) = \sum_{i=1}^n x_i \cdot y_i \mod 2. \tag{3}
\]

It is well known that, for both players to know the answer, at least \( n \) bits of classical communication are needed to uncleanly compute \( IP_n \) exactly (Example 1.29 in Ref. [23]). For quantum strategies in which the players preshare entanglement, \( \lceil n/2 \rceil \) qubits must be sent [3] to achieve the same goal. In Ref. [3], it is also shown that clean, quantum protocols for computing \( IP_n \) must exchange at least \( n \) qubits. The quantum communication required to uncleanly compute \( IP_n \) without prior entanglement is unknown (though it must lie between \( \lceil n/2 \rceil \) and \( n \)). For quantum protocols that are allowed to err with a fixed probability less than 1/2, the complexity is still \( \Omega(n) \) [24].

Here we examine the clean communication complexity of \( IP_n \) without entanglement assistance. To this end, we first consider the quantum communication complexity of implementing the transformation:

\[
|x\rangle_A |\tilde{0}\rangle_B \mapsto (-1)^{x \cdot y} |x\rangle_A |y\rangle_B, \tag{4}
\]

i.e., the distributed computation of the inner product of \( x \) and \( y \) in the phase. Such a transformation corresponds to performing controlled-Z gates across \( n \) pairs of qubits, and by a suitable local basis change this can be converted into an implementation of \( n \)-fold CNOTS.

In Ref. [25], it was shown that two qubits of communication together with sharing four ebits is exactly equivalent as a resource to the ability to implement two CNOT gates and sharing four ebits. As such, this provides a protocol for implementing \( IP_n \) in the phase using \( n + 8 \) qubits of communication and eight ancilla qubits (for even \( n \)). This can be adapted to give a protocol requiring \( n + 2 \) qubits of communication for even \( n \) and \( n + 3 \) qubits when \( n \) is odd. In the following lemma, we give an improved, optimal protocol.
implementing cannot achieve this. □

information must flow in each direction. Without preshared the concept of information complexity[26] to show that in
each round, a player cleans up the message they sent
previously, applies the relevant global phase, and communicates the
next bit of their input string.

\[ Q_{\text{clean}}(IP_n^{\text{phase}}) = n + 1. \] (5)

One ancilla qubit is required. (Without using ancilla qubits,
n + 1 qubits for odd \( n \) and \( n + 2 \) for even \( n \) suffice.)

Proof.—The \( n + 1 \) qubit protocol for even \( n \) is as
follows. Alice initially prepares an ancilla qubit in the
state \( |x_1\rangle \) and sends it to Bob, who applies a phase of
\((-1)^{x_1\cdot y_1}\). He then adds \( y_2 \) to the communication qubit and
sends it back to Alice in the state \( |x_1 \oplus y_2\rangle\). Now, Alice
cleans up her previous communication by subtracting \( x_1 \)
from the communication and then uses the value of \( y_2 \) to
apply the phase \((-1)^{x_2 \cdot y_2}\). She then adds \( x_3 \) to
the communication qubit to leave it in the state \( |y_2 \oplus x_3\rangle\)
and sends it back to Bob. A schematic of these first rounds
is given in Fig. 1.

The players then proceed similarly, with each round of
communication being used to convey a new bit to the other
party and send a received bit back in order to clean the
ancilla qubit. After \( n \) rounds, the global phase will be
\((-1)^{x_1 \cdot y_1}\), and Alice will hold the communication qubit in
the state \( |y_n\rangle\). She sends this back to Bob, who cleans it,
completing the protocol using \( n + 1 \) qubits of communication
and the change in ownership of one ancilla qubit. For
odd \( n \), Alice will perform the final cleaning step. The
protocol to implement the transformation without an ancilla
qubit is given in Appendix B1[9].

The lower bound is proved in Appendix C3[9]. It uses
the concept of information complexity [26] to show that in
a clean protocol for implementing Eq. (4) \( n \) bits of
information must flow in each direction. Without preshared
entanglement, we show that \( n \) qubits of communication
cannot achieve this.

The above lemma provides the optimal method for
implementing \( n \) \( CZ \) gates in parallel while exchanging \( n + 1 \) qubits. Such a protocol would prove useful for quantum
computing architectures where quantum communication is
used to interface and implement gates between clusters of
highly controllable qubits. As an example, in quantum error
correction one could imagine using the Steane code [27] to
protect two logical qubits using two spatially separated
clusters of seven physical qubits. To implement a \( CZ \) gate
between the logical qubits requires seven \( CZ \)s to be
performed in parallel between the physical qubits.

Our protocol achieves this while exchanging only
eight qubits, whereas the naive protocol would send 14.
Protocols based solely on shared entanglement and
classical communication [28–30] use seven pairs of ebits,
14 bits of communication, and the implementation of 14
measurements, while their coherent counterpart [25]
requires one shared ebit and eight qubits of communication.

In Appendix B2[9], we give a clean quantum protocol
for computing \( IP_n \).

Theorem 2.—The clean, quantum communication complexity of exactly computing \( IP_n \) satisfies

\[ n + 1 \leq Q_{\text{clean}}(IP_n) \leq \begin{cases} n + 3 & \text{for } n \text{ odd,} \\ n + 2 & \text{for } n \text{ even.} \end{cases} \] (6)

No ancilla qubits are required.

By adapting the protocol from Lemma 1, \( IP_n \) can be
computed cleanly using two qubits and \( n + 1 \) bits. We give
this protocol in Appendix B3[9].

Our novel quantum communication protocols inspire a
classical protocol for the inner product (given in
Appendix B4[9]) which is near optimal and for which
only the naive \( 2n \) protocol was known before.

Theorem 3.—The clean, classical communication complexity of exactly computing \( IP_n \) satisfies

\[ n + 1 \leq C_{\text{clean}}(IP_n) \leq n + 4 \sqrt{n} + \frac{1}{\sqrt{n} - 1} + 2. \] (7)

No ancilla bits are required.

Generic functions.—In contrast to Theorem 3, we will
show that nearly all Boolean functions on \( n \)-bit inputs
require \( 2n - O(\log n) \) bits of classical communication to
compute cleanly. The proof follows from the following two
lemmas. In what follows, \( X \) and \( Y \) are the random variables
for Alice and Bob’s inputs, and \( A \) and \( B \) are the random
variables received by Alice and Bob, respectively, over the
course of the protocol. By \(|a|\) and \(|b|\), we denote the
number of bits received by Alice and Bob.

Lemma 4.—Consider picking uniformly at random a
Boolean function \( f_n \) on \( n \)-bit inputs. Then with probability
\( 1 - o(1) \), all protocols that compute \( f_n \) exactly are such that
either 1. Alice must receive

\[ |a| \geq n - \log (n + 1) - 2 \] (8)

bits and there exists a uniform distribution over at least half
the pairs of inputs such that

\[ I(Y : AX) \geq n - \log (n + 1) - 3, \] (9)

or 2. Bob must receive

\[ |b| \geq n - \log (n + 1) - 2 \] (10)

bits and there exists a uniform distribution over at least half
the pairs of inputs such that
To prove the first two bounds, begin by noting that the protocol for computing $f$ that partitions $M_f$ into large rectangles, the Kolmogorov complexity of $M_f$ is low. (b) For $M_f$ to have high Kolmogorov complexity, all protocols for computing $f$ must partition $M_f$ into either very narrow or very thin rectangles. To produce the bound in Eq. (9), we take a distribution over the rectangles induced by any function of their inputs, local memory (denoted by $A_i$ and $B_i'$), and the last message received. To obtain (for example) Eq. (13), the chain rule for the conditional mutual information can then be used to write

$$I(X:Y) = I(X:Y) + I(X:B_i'|Y)$$

$$+ \sum_{i=1}^{r} I(X:A_i|YB'B_{i+1}B_{i+2}...B_r)$$

$$\leq I(X:Y) + 1 + |a|,$$

where in the last line we have used the fact that the protocol is clean and that the conditional mutual information can be upper bounded by the number of bits contained in $A_i$.

Combining these two lemmas, together with the fact that $I(X:Y) \leq 1$ for uniform distributions over at least half the possible inputs, we obtain the following theorem.

**Theorem 6.**—Consider exactly computing a Boolean function $f_n$ on $n$-bit inputs that has been picked uniformly at random. Then with probability $1 - o(1)$
\[ C_{\text{clean}}(f_n) \geq 2n - 2 \log (n + 1) - 7. \] (14)

In the case of quantum protocols, a similar result holds in the entanglement-assisted case. Proving this result (Appendix C2 [9]) makes use of the fully quantum notion of information complexity introduced in Ref. [26]. The proof follows a similar structure to the classical result: arguing that for most functions close to \( n \) bits of information have to flow from Alice to Bob and for the protocol to be clean an equivalent amount of information has to be returned.

**Theorem 7.**—Consider exactly computing a Boolean function \( f_n \) on \( n \)-bit inputs that has been picked uniformly at random. Then with probability \( 1 - o(1) \)

\[ Q^*_{\text{clean}}(f_n) \geq n - \log n. \] (15)

**Conclusion.**—In this Letter, we have initiated the study of how big an overhead in communication cleanliness requires. For the inner product function (and the task of implementing \( n \) CZ gates in parallel), we have exhibited quantum and classical protocols for which the overhead is low. For most Boolean functions, however, we have shown that the additional cost incurred by demanding cleanliness is close to maximal for the classical and entanglement-assisted complexities. Many questions remain. For example, what are the clean complexities of other notable functions such as equality and disjointness?

As Theorems 6 and 7 show that the clean, classical, and entanglement-assisted communication complexity for most functions is close to maximal, one can ask: Does something similar hold for \( Q_{\text{clean}}(f) \)? We leave this as an open question but conjecture it to be close to \( 2n \), as the inner product appears somewhat special in its ability to reuse a single bit efficiently. However, the concept of information cost is blind to sending bits, so the technique used for the entanglement-assisted case does not immediately generalize to proving a bound potentially larger than \( n \).

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