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# Separation of Timescales in a Quantum Newton's Cradle

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## SUPPLEMENTAL MATERIAL

### A. Matrix elements and initial state on the ring

The initial state after the Bragg pulse is easily obtained from the matrix elements of the operator

$$\hat{U}_B(q, A) = e^{-iA \int dx \cos(qx) \hat{\Psi}^\dagger(x) \hat{\Psi}(x)}. \quad (1)$$

In the Tonks-Girardeau (TG) limit on a ring geometry we use the eigenstates

$$|\boldsymbol{\lambda}\rangle = \frac{1}{\sqrt{N!}} \int_0^L d^N x \psi_N(\mathbf{x}|\boldsymbol{\lambda}) \hat{\Psi}^\dagger(x_1) \dots \hat{\Psi}^\dagger(x_N) |0\rangle, \quad (2)$$

with wavefunctions given by

$$\psi_N(\mathbf{x}|\boldsymbol{\lambda}) = \frac{1}{\sqrt{N!}} \det [e^{ix_l \lambda_j}] \prod_{1 \leq l < j \leq N} \text{sgn}(x_j - x_l). \quad (3)$$

By commuting  $\hat{U}_B(q, A)$  through the creation operators, the matrix elements can be expressed as

$$\begin{aligned} \langle \boldsymbol{\lambda} | \hat{U}_B(q, A) | \boldsymbol{\mu} \rangle &= \frac{1}{N!} \int d^N x d^N x' \psi_N(\mathbf{x}|\boldsymbol{\lambda})^* \psi_N(\mathbf{x}'|\boldsymbol{\mu}) \\ &\times e^{-iA \sum_n \cos(qx_n)} \langle 0 | \prod_n \hat{\Psi}(x'_n) \prod_j \hat{\Psi}^\dagger(x_j) | 0 \rangle. \end{aligned} \quad (4)$$

The expectation value of the bosonic operators conspires with the signs in the Tonks-Girardeau wavefunctions leading to a determinant of  $\delta$ -functions. Treating the coordinates as dummy variables under the integral sign, it is easy to rewrite the integral in factorized form as a determinant of integrals of the form

$$\frac{1}{L} \int_0^L dx e^{ix(\lambda_j - \mu_k) - iA \cos(qx)} = I_{\frac{\lambda_j - \mu_k}{q}}(-iA) \delta_{\lambda_j, \mu_k}^{(q)}, \quad (5)$$

where we define  $\delta_{\lambda, \mu}^{(q)} = \delta_{(\lambda - \mu) \bmod q, 0}$  and where we used that  $\lambda_j - \mu_k$  and  $q$  lie on the momentum lattice ( $2\pi\alpha/L$ ) with  $\alpha \in \mathbb{Z}$ . This results in the matrix elements

$$\frac{\langle \boldsymbol{\mu} | \hat{U}_B(A) | \boldsymbol{\lambda} \rangle}{L^N} = \det_N \left[ \left( I_{\frac{\lambda_j - \mu_k}{q}}(-iA) \delta_{\lambda_j, \mu_k}^{(q)} \right)_{j,k} \right]. \quad (6)$$

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The initial state

$$|\psi_{q,A}\rangle = \hat{U}_B(q, A) |\psi_{GS}\rangle \quad (7)$$

is easily expressed in the Tonks-Girardeau eigenbasis using the matrix elements  $\langle \boldsymbol{\mu} | \hat{U}_B(q, A) | \psi_{GS} \rangle$ .

### B. The stationary state on a ring from a GGE and the Quench Action approach

In order to implement the GGE logic [1, 2], one starts with computing the conserved charges on the initial state. Let us focus on the case  $q > 2\lambda_F$ , for which the overlaps  $\langle \boldsymbol{\lambda} | \psi_{q,A} \rangle$  coming from Eq. (6) reduce to a simple product of  $N$  modified Bessel functions. While odd charges are trivially zero, for the even charges we find at finite system size

$$\begin{aligned} \langle \psi_{q,A} | \hat{Q}_{2\alpha} | \psi_{q,A} \rangle &= \sum_{j=1}^N \sum_{\beta \in \mathbb{Z}} |I_\beta(iA)|^2 (\lambda_j(\beta))^{2\alpha} \\ &= \sum_{j=1}^N \sum_{\beta \in \mathbb{Z}} |I_\beta(iA)|^2 \sum_{l=0}^{\alpha} \binom{2\alpha}{2l} (\lambda_j^{\text{GS}})^{2(\alpha-l)} (q\beta)^{2l} \\ &= \sum_{j=1}^N \sum_{l=0}^{\alpha} \binom{2\alpha}{2l} (\lambda_j^{\text{GS}})^{2(\alpha-l)} q^{2l} B_{2l,0}(A), \end{aligned} \quad (8)$$

where we defined  $\lambda_j(\beta) = \lambda_j^{\text{GS}} + q\beta$  and where the coefficients  $B_{2l,0}$  come from the sum over the order of the Bessel functions and are known recursively [3]. The sum over particles  $j$  can be performed, after which the thermodynamic limit can be taken,

$$\begin{aligned} \lim_{\text{th}} \langle \psi_{q,A} | \hat{Q}_{2\alpha} / N | \psi_{q,A} \rangle &= \sum_{l=0}^{\alpha} \binom{2\alpha}{2l} \frac{(n\pi)^{2(\alpha-l)} q^{2l}}{2(\alpha-l)+1} B_{2l,0}(A), \end{aligned} \quad (9)$$

where  $n$  is the average particle density. For example, the energy density pumped into the system by an instantaneous Bragg pulse is given by

$$\lim_{\text{th}} (\langle \psi_{q,A} | \hat{Q}_2 / N | \psi_{q,A} \rangle - \langle \psi_{GS} | \hat{Q}_2 / N | \psi_{GS} \rangle) = \frac{q^2 A^2}{2}. \quad (10)$$

One can show that the saddle-point density

$$\rho_{q,A}^{\text{sp}}(\lambda) = \frac{1}{2\pi} \sum_{\beta \in \mathbb{Z}} [\theta(\lambda - \beta q + \lambda_F) - \theta(\lambda - \beta q - \lambda_F)] |I_\beta(iA)|^2 \quad (11)$$

reproduces these values of the conserved charges, i.e.  $L \int_{-\infty}^{\infty} d\lambda \rho_{q,A}^{\text{SP}}(\lambda) \lambda^{2\alpha} = \lim_{\text{th}} \langle \psi_{q,A} | \hat{Q}_{2\alpha} | \psi_{q,A} \rangle$  for all  $\alpha \in \mathbb{N}$ , by performing the integral and recasting the infinite sum into the coefficients  $B_{2\alpha,0}$ . Since the local conserved charges (if well defined) uniquely determine the saddle point, we have thus found the saddle-point density after a Bragg pulse for  $q > 2\lambda_F$ . For smaller Bragg momenta  $q < 2\lambda_F$  the computation becomes considerably more difficult due to the determinant structure of the overlaps, but one can show that the saddle-point density given in Eq. (11) is still correct.

The Quench Action (QA) approach [4, 5] reproduces this saddle-point density for  $q > 2\lambda_F$ . As a consequence of working in the Tonks-Girardeau regime, there are many microstates with exactly the same overlap. We can rephrase the overlaps as

$$\langle \{\lambda_j(\beta_j)\}_{j=1}^N | \psi_{q,A} \rangle = L^N \prod_{\alpha=-\infty}^{\infty} [I_{\alpha}(-iA)]^{n_{\alpha}}, \quad (12)$$

where  $n_{\alpha}$  is the number of rapidities  $j$  with  $\beta_j = \alpha$  and  $\alpha \in \mathbb{Z}$ . In the thermodynamic limit these numbers are given by

$$n_{\alpha} = L \int_{\alpha q - \lambda_F}^{\alpha q + \lambda_F} d\lambda \rho(\lambda). \quad (13)$$

The normalized overlap coefficients  $S_{\{\beta_j\}} = -\ln(\langle \{\lambda_j(\beta_j)\}_{j=1}^N | \psi_{q,A} \rangle / L^N)$  have a well-defined thermodynamic limit,

$$S[\rho] = \lim_{\text{th}} \text{Re } S_{\{\beta_j\}} \quad (14)$$

$$= -L \sum_{\alpha=-\infty}^{\infty} \int_{\alpha q - \lambda_F}^{\alpha q + \lambda_F} d\lambda \rho(\lambda) \ln [|I_{\alpha}(-iA)|] \quad (15)$$

$$= L \int_{-\infty}^{\infty} d\lambda \rho(\lambda) \sum_{\alpha=-\infty}^{\infty} [\theta(\lambda - \alpha q - \lambda_F) - \theta(\lambda - \alpha q + \lambda_F)] \log [|I_{\alpha}(iA)|], \quad (16)$$

where  $\theta$  is the Heaviside step function and we used that  $|I_n(-z)| = |I_n(z)|$ . Furthermore, in the thermodynamic limit  $\lambda_F = \pi n$ , where  $n$  is the average particle density. The noncontinuous integrand will serve as the driving term of the GTBA equations. Note that in the second line we implicitly assume that  $\rho(\lambda) = 0$  when  $\lambda \notin [\lambda - \alpha q - \lambda_F, \lambda - \alpha q + \lambda_F]$  for any  $\alpha \in \mathbb{Z}$ . The reason is that for Bethe states that do not obey this condition, the overlap is exactly zero (rapidities will never end up in those regions) and therefore  $S[\rho] = \infty$ . These states are therefore infinitely suppressed in the Quench Action saddle-point equations. Another way of seeing this is that originally the functional integral in the quench action approach is a sum over states with non-zero overlaps and these states are not in that sum.

Even when you restrict the support of the density function to these intervals, in this ensemble of states there are still many microstates that have zero overlap with

the Bragg-pulsed ground state. The reason is that when a rapidity  $\lambda_j^{\text{GS}}$  has moved to an interval  $\alpha = \beta_j$ , it is not in the other intervals  $\alpha \neq \beta_j$  and therefore leaves a hole there. This alters the usual form of the Yang-Yang entropy significantly. Given the fillings  $\{n_{\alpha}\}_{\alpha=-\infty}^{\infty}$ , the finite-size entropy is

$$e^{S_{\text{YY}}(\{n_{\alpha}\})} = \frac{N!}{\prod_{\alpha=-\infty}^{\infty} (n_{\alpha}!)}, \quad (17a)$$

which leads to a modified Yang-Yang entropy for the Bragg pulse from the ground state,

$$S_{\text{YY}}[\rho] = -L \int_{-\infty}^{\infty} d\lambda \rho(\lambda) \log [2\pi\rho(\lambda)], \quad (18)$$

where we used the Tonks-Girardeau Bethe equation  $2\pi[\rho(\lambda) + \rho_h(\lambda)] = 1$ . The variation of the quench action should be restricted to densities for which  $\rho(\lambda) = 0$  when  $\lambda \notin [\lambda - \alpha q - \lambda_F, \lambda - \alpha q + \lambda_F]$  for any  $\alpha \in \mathbb{Z}$ . Also, a Lagrange multiplier  $h$  is added to fix the particle density to  $n = N/L$ . The resulting GTBA equation is not an integral equation because of the Tonks-Girardeau limit,

$$0 = 2 \sum_{\alpha=-\infty}^{\infty} [\theta(\lambda - \alpha q - \lambda_F) - \theta(\lambda - \alpha q + \lambda_F)] \log [|I_{\alpha}(iA)|] + \log \left( 2\pi\rho_{q,A}^{\text{SP}}(\lambda) \right) + 1 - h. \quad (19)$$

This is solved by the normalized saddle-point density of Eq. (11), with  $h = 1$ . For smaller Bragg momenta  $q < 2\lambda_F$  the derivation of the saddle-point distribution using the QA approach remains an open problem, since the determinant structure of the overlaps prevents obtaining a straightforward thermodynamic limit of the overlap coefficients that is expressible in terms of a root density  $\rho(\lambda)$ .

Moreover, one could question whether the time evolution of simple observables obtained from the QA approach using the saddle-point density of Eq. (11) is valid also for small Bragg momenta  $q < 2\lambda_F$ . However, the analysis of the FB mapping does not show any qualitative differences for Bragg momenta smaller or bigger than  $2\lambda_F$  and the agreement with the time evolution of the QA approach is excellent for all  $q > 0$ . It therefore seems safe to assume that the time evolution from the QA approach is valid for all Bragg momenta.

### C. Non-linear Luttinger liquid theory: time evolution of the density on a ring

The expectation value of the density operator after the Bragg pulse in the thermodynamic limit is defined as

$$\langle \hat{\rho}(x, t) \rangle = \lim_{\text{th}} \langle \psi_{q,A} | \Psi^{\dagger}(x, t) \Psi(x, t) | \psi_{q,A} \rangle. \quad (20)$$

We can compute this function in the TG limit in three equivalent ways: (i) using the saddle point state and the

Quench Action approach it is possible to perform the full calculation in the bosonic basis; (ii) since the Fermi-Bose mapping leaves the density  $\Psi^\dagger(x)\Psi(x)$  invariant, we can also work in the fermionic basis for which the combined effects of the Bragg pulse and time-evolution are easily accounted for; (iii) finally, we can treat the same problem in the Tomonaga-Luttinger model by including a quadratic band curvature term. The nonlinear Luttinger liquid (nLL) Hamiltonian is then given by

$$H_{\text{nLL}}^{(0)} = \sum_{\alpha} \Psi_{\alpha}^{\dagger} \left[ -i\alpha v_F \partial_x - \frac{1}{2m} \partial_x^2 \right] \Psi_{\alpha} \quad (21)$$

where  $\alpha = R, L = +, -$ ,  $v_F = \lambda_F/m = \pi n/m$  and  $\Psi_{R,L}$  denotes the annihilation operator of fermionic right- or left-mover fields.

The Hamiltonian in (21) can be obtained from the fermionic dual of the Lieb-Liniger model in the TG limit by expanding the dispersion relation around the Fermi points. The left and right movers simply correspond to positive and negative momenta, but with momenta shifted by  $\mp k_F$ . The ground state corresponds to all  $k < 0$  modes of  $\Psi_R$  occupied and all  $k > 0$  modes of  $\Psi_L$ .

The interesting aspect of using the model in terms of left and right movers is that there is a straightforward way to generalize to an interacting model. A number of recent advances in the equilibrium theory for one-dimensional systems show that the most relevant terms of a short range interaction can be accounted for in a fashion similar to Fermi liquid theory by going to a quasi-particle basis. The corresponding fermions  $\tilde{\Psi}_{R,L}$  remain weakly interacting, and can to first approximation be considered free with dynamics governed by the Hamiltonian [6]

$$H_{\text{nLL}}^{(0)} = \sum_{\alpha} \tilde{\Psi}_{\alpha}^{\dagger} \left[ -i\alpha v_s \partial_x - \frac{1}{2m^*} \partial_x^2 \right] \tilde{\Psi}_{\alpha}. \quad (22)$$

Here  $v_s$  and  $m^*$  correspond to the renormalized dispersion (dressed energy) of excitations at the Fermi point in the equilibrium Lieb-Liniger model. The relation between the fermionic quasi-particles and the bare fermions is given by

$$\Psi_{\alpha}(x) = F_{\alpha}(x) \tilde{\Psi}_{\alpha}, \quad (23)$$

$$F_{\alpha}(x) = e^{-2\pi i \int^x \left[ 1 - \frac{1}{2\sqrt{K}} - \frac{\sqrt{K}}{2} \right] \tilde{\Psi}_{\alpha} \tilde{\Psi}_{\alpha} + \left[ \frac{1}{2\sqrt{K}} - \frac{\sqrt{K}}{2} \right] \tilde{\Psi}_{-\alpha} \tilde{\Psi}_{-\alpha}}.$$

Together, Eq. (22) and (23) can be used to compute dynamical correlations at zero temperature in the vicinity of the Fermi points for general gapless one-dimensional systems which is one of the successes of the past years in the study of one-dimensional systems beyond the Luttinger liquid paradigm.

All three computations of  $\langle \hat{\rho}(x, t) \rangle$  are instructive, but we will focus on the nLL result as it is by far the most surprising in the present out-of-equilibrium context of the Bragg pulse. On the technical level, the calculations using the Fermi-Bose mapping or the Quench Action approach are similar and can be used to verify the result in the TG limit.

In the nLL theory, the slowly fluctuating components of the density operator can be expressed as[7]

$$\begin{aligned} \Psi^{\dagger} \Psi &\sim n + [\Psi_R^{\dagger} \Psi_R + \Psi_L^{\dagger} \Psi_L] \\ &\sim n + \sqrt{K} [\tilde{\Psi}_R^{\dagger} \tilde{\Psi}_R + \tilde{\Psi}_L^{\dagger} \tilde{\Psi}_L]. \end{aligned} \quad (24)$$

We will use this as an identity both in the unitary operator implementing the Bragg pulse as in the density as observable for which we want to compute the expectation value. The computation splits into separate contributions from the right and the left movers. Let us focus on the right movers.

One has the following behavior for the modes

$$U_B^{\dagger} \tilde{\Psi}_{R,k} U_B = \sum_{\beta} I_{\beta}(-i\sqrt{K}A) \tilde{\Psi}_{R,k-\beta q} \quad (25)$$

This leads in the thermodynamic limit to

$$\begin{aligned} \langle 0 | U_B^{\dagger} \tilde{\Psi}_R^{\dagger}(x, t) \tilde{\Psi}_R(x, t) U_B | 0 \rangle \\ = \sum_{\beta \neq 0} J_{\beta} \left( -2\sqrt{K}A \sin \frac{\beta q^2 t}{2m^*} \right) \frac{e^{-i\beta q[x-v_s t]}}{2\pi i \beta q t / m^*} \end{aligned} \quad (26)$$

where we used the quasi-particle dispersion for the right movers

$$\tilde{\xi}_{R,k} = vk + \frac{k^2}{2m^*} \quad (27)$$

and Graf's summation formula [8]. Here  $|0\rangle$  denotes the vacuum state in the appropriate model. Combining Eq. (26) with a similar expression for left movers in the thermodynamic limit we arrive at

$$\begin{aligned} \langle \hat{\rho}(x, t) \rangle = n + \sqrt{K} \sum_{\beta \neq 0} J_{\beta} \left( -2\sqrt{K}A \sin \frac{\beta q^2 t}{2m^*} \right) \times \\ \times \cos(\beta q x) \frac{\sin(\beta q v_s t)}{\pi \beta q t / m^*}. \end{aligned} \quad (28)$$

In the equilibrium case and the computation of dynamical correlations, the use of Eq. (22) is justified by Renormalization Group arguments. Here, there is no such justification. It is all the more surprising that the exact result is recovered in the TG limit. Because of this and since Eq. (22) does include the most relevant terms of a general interacting model, we expect that there will be a range of  $q$  and  $A$  for which Eq. (28) is a good approximation. The importance here is that this would hold beyond the Lieb-Liniger model. A more detailed discussion will feature in future work. Notice that the result in a linear Luttinger liquid calculation  $\langle \hat{\rho}(x, t) \rangle_{\text{LuttLiq}} = n - KAq \cos(qx) \sin(qv_s t) / \pi$ , which can be obtained by taking the limit  $1/m^* = 0$  in Eq. (28), only captures the behavior for very short times and gives unphysical results for intermediate to late times.

### D. Time evolution of the momentum distribution function on a ring from the Quench Action approach

In Ref. [9] the thermodynamic limit of the matrix elements of the one-body density matrix between states with a countable number  $n_e$  of particle-hole excitations  $\{h_j \rightarrow p_j\}_{j=1}^{n_e}$  on a thermodynamic state  $|\rho\rangle$  was computed. The result is obtained by decomposing the elements into a Fredholm determinant and a finite-size determinant accounting for the excitations. The result is

$$\begin{aligned} & \left\langle \rho \left| \hat{\Psi}^\dagger(x) \hat{\Psi}(0) \right| \rho, \{h_j \rightarrow p_j\}_{j=1}^{n_e} \right\rangle \\ &= L^{-n_e} e^{i\frac{x}{2} \sum_{j=1}^{n_e} (p_j - h_j)} \left\{ \text{Det}(1 + K' \rho) \det_{i,j=1}^{n_e} [W'(h_i, p_j)] \right. \\ & \left. - \text{Det}(1 + K \rho) \det_{i,j=1}^{n_e} [W(h_i, p_j)] \right\}, \end{aligned} \quad (29)$$

where  $\rho(\lambda)$  is the density of rapidities of the thermodynamic state. The Fredholm determinants are denoted by  $\text{Det}$ , where  $(K\rho)(\lambda, \mu) = K(\lambda, \mu)\rho(\mu)$ . The kernels are given by

$$K'(\lambda, \mu) = K(\lambda, \mu) + ne^{-i\frac{x}{2}(\lambda+\mu)}, \quad (30)$$

$$K(\lambda, \mu) = -4n \frac{\sin\left(\frac{x}{2}(\lambda - \mu)\right)}{\lambda - \mu}, \quad (31)$$

where  $n = N/L$  is the density. The function  $W$  is defined as

$$W(\lambda, \mu) = ((1 + K\rho)^{-1}K)(\lambda, \mu), \quad (32)$$

$$W'(\lambda, \mu) = ((1 + K'\rho)^{-1}K')(\lambda, \mu), \quad (33)$$

or equivalently via the integral equations

$$W(\lambda, \mu) + \int_{-\infty}^{\infty} d\nu K(\lambda, \nu)\rho(\nu)W(\nu, \mu) = K(\lambda, \mu), \quad (34)$$

$$W'(\lambda, \mu) + \int_{-\infty}^{\infty} d\nu K'(\lambda, \nu)\rho(\nu)W'(\nu, \mu) = K'(\lambda, \mu). \quad (35)$$

The QA approach yields the following expression for the time evolution of the one-body density matrix in the thermodynamic limit

$$\begin{aligned} & \left\langle \psi_{q,A}(t) \left| \hat{\Psi}^\dagger(x) \hat{\Psi}(0) \right| \psi_{q,A}(t) \right\rangle \\ &= \Re \sum_{n_e=0}^{\infty} \frac{1}{(n_e!)^2} \left( \prod_{j=1}^{n_e} L^2 \int_{-\infty}^{\infty} dh_j dp_j \varphi_-^{(t)}(h_j) \varphi_+^{(t)}(p_j) \right) \\ & \times \left\langle \rho_{q,A} \left| \hat{\Psi}^\dagger(x) \hat{\Psi}(0) \right| \rho_{q,A}, \{h_j \rightarrow p_j\}_{j=1}^{n_e} \right\rangle, \end{aligned} \quad (36)$$

where the effective densities of holes and particles are given by

$$\varphi_-^{(t)}(h_j) = e^{\delta s(h_j) + i\delta\omega(h_j)t} \rho_{q,A}(h_j), \quad (37)$$

$$\varphi_+^{(t)}(p_j) = e^{-\delta s(h_j) - i\delta\omega(h_j)t} \rho_{q,A}^h(p_j), \quad (38)$$

with the density of holes of the saddle point given by  $\rho_{q,A}^h(p) = \frac{1}{2\pi} - \rho_{q,A}(p)$ . The differential overlap for a single particle-hole  $e^{-\delta s(p) + \delta s(h)}$  is obtained by taking a finite-size realization of the saddle point state  $|\lambda_{q,A}\rangle \rightarrow |\rho_{q,A}\rangle$  and the modified state obtained by performing a single particle-hole on  $|\lambda_{q,A}\rangle$ . The ratio of the two overlaps in the thermodynamic limit gives the differential overlap

$$e^{-\delta s(p) + \delta s(h)} = \lim_{N \rightarrow \infty} \frac{\langle \psi_{q,A} | \lambda_{q,A}, h \rightarrow p \rangle}{\langle \psi_{q,A} | \lambda_{q,A} \rangle}. \quad (39)$$

The same argument gives the energy of the single particle-hole excitations

$$e^{-i\delta\omega(p)t + i\delta\omega(h)t} = e^{-ip^2t + ih^2t}. \quad (40)$$

Since the overlaps only couple states such that the difference of their rapidities are multiples of  $q$ , the sum over the excitations on the saddle point state reduces to

$$\begin{aligned} & \frac{1}{(n_e!)^2} \prod_{j=1}^{n_e} L^2 \int_{-\infty}^{\infty} dh_j dp_j \varphi_-^{(t)}(h_j) \varphi_+^{(t)}(p_j) \\ & \rightarrow \frac{1}{n_e!} \prod_{j=1}^{n_e} L \int_{-\infty}^{\infty} dh_j \sum_{\beta_j \in \mathbb{Z}} \varphi_-^{(t)}(h_j) \frac{\varphi_+^{(t)}(h_j + \beta_j q)}{\rho_{q,A}^h(h_j + \beta_j q)}. \end{aligned} \quad (41)$$

After this substitution the sum in Eq. (36) becomes the definition of a Fredholm determinant, and using the definition of the saddle-point distribution in Eq. (11), the expression for the time evolution of the one-body density matrix can be rewritten as the difference of two Fredholm determinants of two infinite block matrices where each block  $S_{\alpha,\beta}$  and  $S'_{\alpha,\beta}$  for any  $\alpha, \beta \in \mathbb{Z}$  is an operator acting on  $\Lambda = [-\lambda_F, \lambda_F]$ ,

$$\begin{aligned} & \langle \Psi_{q,A}(t) | \Psi^\dagger(x) \Psi(0) | \Psi_{q,A}(t) \rangle \\ &= \Re \left[ \text{Det}_\Lambda (\mathbf{1}\delta_{\alpha,\beta} + S'_{\alpha,\beta})_{\alpha,\beta \in \mathbb{Z}} \right. \\ & \quad \left. - \text{Det}_\Lambda (\mathbf{1}\delta_{\alpha,\beta} + S_{\alpha,\beta})_{\alpha,\beta \in \mathbb{Z}} \right], \end{aligned} \quad (42)$$

with the operators given by

$$S'_{\alpha,\beta}(u, v) = \sum_{\gamma \in \mathbb{Z}} \zeta_\gamma^{(t)}(u + \alpha q) K'(u + \alpha q, v + (\beta + \gamma)q) \Phi_{\beta,\gamma}^{(t)},$$

$$S_{\alpha,\beta}(u, v) = \sum_{\gamma \in \mathbb{Z}} \zeta_\gamma^{(t)}(u + \alpha q) K(u + \alpha q, v + (\beta + \gamma)q) \Phi_{\beta,\gamma}^{(t)}. \quad (43)$$

Here  $u, v \in [-\lambda_F, \lambda_F]$  and  $\mathbf{1}$  is the identity operator. The coefficients  $\Phi_{\beta,\gamma}^{(t)}$  and the function  $\zeta_\gamma^{(t)}(u)$  are given by

$$\Phi_{\beta,\gamma}^{(t)} = \frac{I_\beta(iA) I_{\beta+\gamma}(-iA)}{2\pi} e^{-it(q\gamma)^2 + ixq\gamma/2}, \quad (44)$$

$$\zeta_\gamma^{(t)}(u) = e^{-2itq\gamma u}. \quad (45)$$

In order to obtain the time evolution of the momentum distribution  $\hat{n}(k, t)$  one needs to restrict the sum in Eq. (36) to excitations with zero total momentum, namely

$$\begin{aligned} \hat{n}(k, t) = & \\ \text{FT} \left\{ \sum_{n_e=0}^{\infty} \frac{1}{n_e!} \left( \prod_{j=1}^{n_e} \int_{-\infty}^{\infty} dh_j \sum_{\beta_j \in \mathbb{Z}} \varphi_{-}^{(t)}(h_j) \frac{\varphi_{+}^{(t)}(h_j + \beta_j q)}{\rho_{q,A}^h(h_j + \beta_j q)} \right) \right. & \\ \times L^{n_e} \langle \rho_{q,A} | \Psi^\dagger(x) \Psi(0) | \rho_{q,A}, \{h_j \rightarrow h_j + q\beta_j\}_{j=1}^{n_e} \rangle & \\ \left. \times \delta_{\sum_{j=1}^{n_e} \beta_j, 0} \right\} & \quad (46) \end{aligned}$$

where we denoted the Fourier transform as  $\text{FT}\{f(x)\} = \int_{-\infty}^{\infty} dx f(x) e^{-ikx}$ . Using the identity

$$\int_{-\pi}^{\pi} \frac{dv}{2\pi} e^{-i\beta v} = \delta_{\beta,0}, \quad (47)$$

we obtain

$$\begin{aligned} \hat{n}(k, t) = \text{FT} \left\{ \Re \int_{-\pi}^{\pi} \frac{d\kappa}{2\pi} \left[ \text{Det}_{\Lambda} \left( \mathbf{1}_{\alpha, \beta} + S^{(\kappa)'}_{\alpha, \beta} \right)_{\alpha, \beta \in \mathbb{Z}} - \text{Det}_{\Lambda} \left( \mathbf{1}_{\alpha, \beta} + S^{(\kappa)}_{\alpha, \beta} \right)_{\alpha, \beta \in \mathbb{Z}} \right] \right\}, & \\ S^{(\kappa)'}_{\alpha, \beta}(u, v) = \sum_{\gamma \in \mathbb{Z}} \zeta_{\gamma}^{(t)}(u + \alpha q) K'(u + \alpha q, v + (\beta + \gamma)q) \Phi_{\beta, \gamma}^{(t)} e^{-i\kappa \gamma}, & \\ S^{(\kappa)}_{\alpha, \beta}(u, v) = \sum_{\gamma \in \mathbb{Z}} \zeta_{\gamma}^{(t)}(u + \alpha q) K(u + \alpha q, v + (\beta + \gamma)q) \Phi_{\beta, \gamma}^{(t)} e^{-i\kappa \gamma}. & \quad (48) \end{aligned}$$

### E. Exact momentum distribution at $t = 0$ for arbitrary interactions

The one-body density matrix at  $t = 0$  (after the Bragg pulse) is given by

$$\begin{aligned} \langle \hat{U}_B^\dagger(q, A) \hat{\Psi}^\dagger(x) \hat{\Psi}(y) \hat{U}_B(q, A) \rangle & \\ = \langle \hat{\Psi}^\dagger(x) \hat{\Psi}(y) \rangle e^{-i2A \sin(q \frac{x-y}{2}) \sin(q \frac{x+y}{2})}. & \quad (49) \end{aligned}$$

The latter equality follows strictly from the commutation relations of the Bose fields with the density and thus holds irrespective of interaction or geometry. For the case of the ring geometry, the associated MDF is

$$\langle \hat{n}(k, t = 0) \rangle = \frac{1}{L} \int_0^L d\xi e^{ik\xi} I_0(i2A \sin(q\xi/2)) \langle \hat{\Psi}^\dagger(\xi) \hat{\Psi}(0) \rangle, \quad (50)$$

where we defined  $\xi = x - y$  and used the integral

$$\begin{aligned} \frac{1}{L} \int_0^L dy e^{-i2A \sin(q\xi/2) \sin(qy + q\xi/2)} & \\ = I_0(i2A \sin(q\xi/2)), & \quad (51) \end{aligned}$$

under the assumption that  $qL/2\pi$  is integer. Using the convolution theorem we obtain

$$\langle \hat{n}(k, t = 0) \rangle = \sum_{k'} f(k') \langle \hat{n}(k - k') \rangle_{\text{GS}} \quad (52)$$

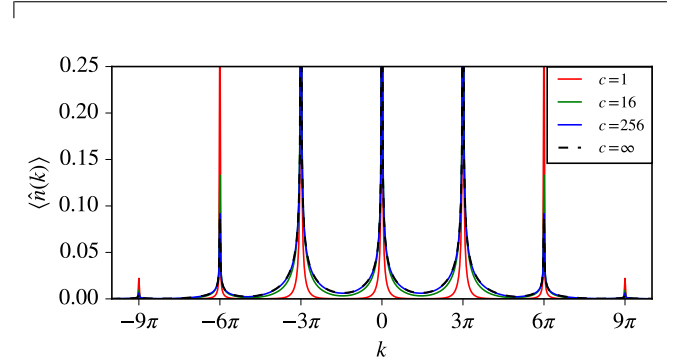


FIG. 1. The initial MDF ( $t = 0$ ) for  $A = 1.5$ ,  $q = 3\pi$  and different values of the interaction strength  $c$ . The finite- $c$  interactions cause a decrease of the width of the satellites but do not influence their relative heights.

where

$$f(k) = \frac{1}{L} \int_0^L dx e^{ikx} I_0(i2A \sin(qx/2)) \quad (53)$$

and  $\langle \hat{n}(k) \rangle_{\text{GS}}$  is the MDF of the ground state.

Using the expansion  $I_0(z) = \sum_{n=0}^{\infty} (\frac{1}{4}z^2)^n / (n!)^2$  one

finds

$$I_0(i2A \sin(qx/2)) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} A^{2n} \sin^{2n}(qx/2) \\ = \sum_{n=0}^{\infty} \sum_{l=-n}^n \frac{(-1)^{n+l} (2n)!}{(n!)^2 (n-l)! (n+l)!} \left(\frac{A}{2}\right)^{2n} e^{ilqx}, \quad (54)$$

where we used the binomial to expand in plane waves.

The order of the sums can now be interchanged. Defining the coefficients

$$c_l(A) = \sum_{n=|l|}^{\infty} \frac{(-1)^{n+l} (2n)!}{(n!)^2 (n-l)! (n+l)!} \left(\frac{A}{2}\right)^{2n} \quad (55)$$

we obtain  $f(k) = \sum_l c_l \delta_{k,lq}$ . The coefficients  $c_l(A)$  can in fact be resummed and expressed in terms of a Bessel function

$$c_l(A) = J_l(A)^2. \quad (56)$$

The  $t = 0$  post-pulse MDF can therefore be exactly expressed in terms of the MDF before the pulse  $\langle \hat{n}(k) \rangle_{\text{GS}}$  as

$$\langle \hat{n}(k, t = 0) \rangle = \sum_{l=-\infty}^{\infty} J_l(A)^2 \langle \hat{n}(k + lq) \rangle_{\text{GS}}. \quad (57)$$

Note that this result holds for arbitrary interaction strength  $c$  with  $\langle \hat{n}(k) \rangle_{\text{GS}}$  the appropriate ground state MDF. The result is plotted in Fig. 1 for different values of  $c$ . The influence of the finite interactions resides solely in the groundstate MDF  $\langle \hat{n}(k + lq) \rangle_{\text{GS}}$ , leading to a decreasing width of the peaks as one goes from the hard-core limit ( $c \rightarrow \infty$ ) to the BEC limit ( $c \rightarrow 0$ ). In contrast, Eq. (57) shows that their relative heights are completely determined by the value of  $A$ .

## F. Equilibrated momentum distribution on a ring

In Fig. 2 the equilibrated momentum distribution function (MDF) is shown for different values of  $q$  and  $A$ , computed using the Fermi-Bose (FB) mapping for  $N = 50$ . The width of the satellites is independent of the bragg momentum  $q$ , and is only determined by the value of the area of the pulse  $A$ .

## G. Time-evolved single-particle states in the trap

The propagator for the quantum harmonic oscillator (Mehler kernel) is given by

$$K(x, y; t) = \sqrt{\frac{m\omega}{2\pi i \sin(\omega t)}} \times \\ \times \exp\left(\frac{-m\omega(x^2 + y^2) \cos(\omega t) + 2m\omega xy}{2i \sin(\omega t)}\right). \quad (58)$$

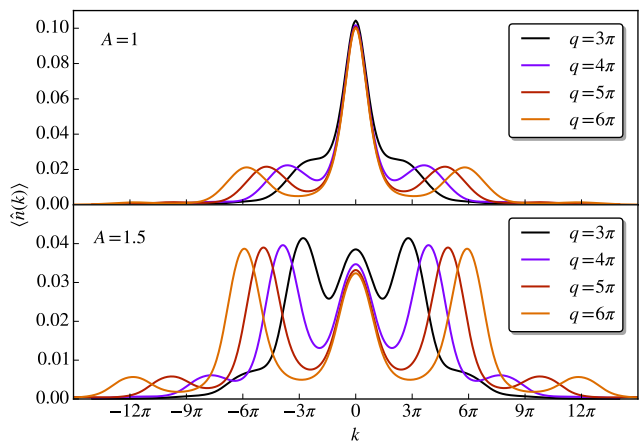


FIG. 2. The relaxed MDF for  $N = 50$  particles ( $t = 0.4$ ) as a function of the Bragg momentum  $q$ . In the upper panel  $A = 1$ , and in the lower panel  $A$  is fixed to 1.5.

The single particle (SP) wavefunctions after the Bragg pulse can then be time evolved by integrating the initial wavefunctions including the cosine phase with the propagator:

$$\psi_j(x; t) = \int_{-\infty}^{\infty} dy K(x, y; t) e^{-iA \cos(qx)} \psi_j(y) \\ = \sum_{\beta=-\infty}^{\infty} I_{\beta}(-iA) e^{-i\beta q \cos(\omega t) (x + \frac{\beta q}{2m\omega} \sin(\omega t))} \\ \psi_j(x + \frac{\beta q}{m\omega} \sin(\omega t)) e^{-i\omega(j + \frac{1}{2})t}, \quad (59)$$

where  $\psi_j(x)$  are the groundstate harmonic eigenfunctions

$$\psi_j(x) = \frac{1}{\sqrt{2^j j!}} \left(\frac{m\omega}{\pi}\right)^{1/4} e^{-\frac{m\omega x^2}{2}} H_j(\sqrt{m\omega} x). \quad (60)$$

The result in Eq. (59) has been obtained by using the following two identities

$$e^{-iz \cos(\phi)} = \sum_{n=-\infty}^{\infty} I_n(-iz) e^{-in\phi}, \quad (61)$$

$$\int_{-\infty}^{\infty} dx e^{-(x-y)^2} H_j(\alpha x) = \sqrt{\pi} (1 - \alpha^2)^{j/2} H_j\left(\frac{\alpha y}{\sqrt{1 - \alpha^2}}\right). \quad (62)$$

## H. Local density approximation

The local density approximation (LDA) for the gas in a parabolic trap amounts to replacing the value for the mean density in the Quench Action result for the short distance fluctuations with a space-dependent density profile corresponding to the ground state in the trap. This result is considerably improved when one introduces the classical harmonic motion of the density profile in accordance with the exact  $t = 0$  MDF Eq. (57).

In the thermodynamic limit the ground state density profile in a harmonic trap is given by

$$\rho_{\text{GS}}(x) = \langle \hat{\rho}(x) \rangle_{\text{GS}} = \frac{1}{\pi} \sqrt{mN\omega - m^2\omega^2 x^2}. \quad (63)$$

Writing  $\rho(x, t; n)$  for the function in Eq. (28) for a gas with mean density  $n$  in the TG limit  $K = 1$ ,  $v_s = \pi n/m$ ,  $m^* = m$ , our result for the improved LDA in the trap reads

$$\rho_{\text{LDA}}(x, t) = \sum_l J_l(A)^2 \times \left( x - \frac{lq}{\omega m} \sin(\omega t), t; \rho_{\text{GS}}\left(x - \frac{lq}{\omega m} \sin(\omega t)\right) \right), \quad (64)$$

### I. Time-dependent Hartree-Fock approximation

The Lieb-Liniger model for  $\delta$ -function interacting bosons has a dual fermionic model for which the Hamiltonian acts on the fermionic many-body wavefunction as [10, 11]

$$\begin{aligned} (H_F \psi_F)(x_1, \dots, x_N) = & - \sum_{i=1}^N \frac{1}{2m} \frac{\partial^2}{\partial x_i^2} \psi_F(x_1, \dots, x_N) \\ & - \frac{2}{m^2 c} \sum_{1 \leq i < j \leq N} \int dx'_i dx'_j \delta'(x_i - x_j) \delta'(x'_i - x'_j) \times \\ & \times \delta\left(\frac{x_i + x_j - x'_i - x'_j}{2}\right) \psi_F(x_1, \dots, x'_i, \dots, x'_j, \dots, x_N). \end{aligned} \quad (65)$$

For Hartree-Fock approximations to this fermionic model, an equivalent interaction potential can be used given by [12]  $V(x_i - x_j) = -1/(m^2 c) \delta''(x_i - x_j)$ . The equilibrium Hartree-Fock approximation amounts to finding the eigenmodes of the Fock operator given by [10, 11]

$$\mathcal{F}(x) = -\frac{1}{2m} \frac{\partial^2}{\partial x^2} + \frac{1}{m^2 c} \left[ n(x) \frac{\partial^2}{\partial x^2} + 2\mathcal{P}(x) i \frac{\partial}{\partial x} - \mathcal{T}(x) \right], \quad (66)$$

with

$$\begin{aligned} n(x) &= \sum_j n_j \phi_j^*(x) \phi_j(x) \\ \mathcal{P}(x) &= -i \sum_j n_j \phi_j^{*'}(x) \phi_j(x) \\ \mathcal{T}(x) &= \sum_j n_j [\phi_j^*(x) \phi_j''(x) + 2\phi_j^{*'}(x) \phi_j'(x)], \end{aligned} \quad (67)$$

The fermionic occupation numbers  $n_j$  are determined by the eigenenergies of the Hartree-Fock orbitals which obey  $\mathcal{F}(x) \phi_j(x) = \epsilon_j \phi_j(x)$ . The occupation numbers are then given by the Fermi-Dirac distribution  $n_j = 1/\{\exp[(\epsilon_j - \mu)/(k_B T)] + 1\}$ . The HF orbitals for the ring geometry are given by planewaves, with  $j$  associated to the momentum of the particle. At  $T = 0$  their dispersion relation is given by

$$\epsilon_p = \frac{p^2}{2m^*} - \frac{\pi^2 n^3}{3m^2 c} \quad (68)$$

with the effective mass  $m^* = m/(1 - 2n/mc)$ .

The time-dependent Hartree-Fock approximation then uses the Hartree-Fock orbitals as initial conditions to the time-dependent differential equation  $i d\phi_j(x, t)/dt = \mathcal{F}(x, t) \phi_j(x, t)$ . For a finite duration Bragg pulse an external time-dependent cosine potential is added to eq. (66), bringing the system out of equilibrium. When specializing to the instantaneous Kapitza-Dirac pulse, this reduces to simply multiplying the initial condition with a cosine phase as is done in eq. 3 of the main text.

The differential equation for the single particle wavefunctions  $\phi_j(x, t)$  is then numerically evaluated using the Crank-Nicolson method, and observables such as the density and the momentum distribution (both the fermionic and bosonic one) can be constructed in the same way as in the  $c \rightarrow \infty$  case.

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