Classical simulation of entanglement swapping with bounded communication
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Published in:
Physical Review Letters

DOI:
10.1103/PhysRevLett.109.100401

Citation for published version (APA):
In this Supplementary material we prove that Protocols 1 and 2 generate the desired correlation (equation (1) of the main text), or in other terms, $P(a = b|\mathbf{x}, \mathbf{y}) = \frac{1-x-y}{2}$.

Correlation from Protocol 1

We first note that the definitions of $a = \text{sign} (\sin \phi_A)$ and of $\phi_A' = (\phi_A - j A \pi \mod \pi)$ (and then of $b = \pm \beta$) and of $\phi_B' = (\phi_B - j B \pi \mod \pi) \in [0, \pi]$ on Bob’s side, ensure that the following relations hold, as required:

$$P(a = b|\phi_A, \phi_B) = P(a \neq b|\phi_A, \phi_B + \pi) = P(a \neq b|\phi_A + \pi, \phi_B) = P(a = b|\phi_A + \pi, \phi_B + \pi)$$

$$= P(a = b|\phi_A + j \pi 4, \phi_B + j \pi 4) \quad \text{for any } j \in \mathbb{Z}.$$ 

It is therefore sufficient to check that the correct correlation is obtained for $\phi_A \in [0, \pi]$ and $\phi_B \in [0, \pi]$.

For such values of $\phi_A, \phi_B$ (for which $\phi_A' = \phi_A$ and $\phi_B' = \phi_B$), the probability $P(a = b|\phi_A, \phi_B)$ obtained from Protocol 1 can be calculated as follows:

$$P(a = b|\phi_A < \phi_B - j B \pi 4) = \frac{16}{\pi} \left( \int_0^{\phi_A} d\lambda_A \int_0^{\lambda_A} d\lambda_{RB} \left( \phi_B - \lambda_{RB} \right) \right.$$

$$+ \int_0^{\phi_A} d\lambda_A \int_{\lambda_A}^{\phi_B - j B \pi 4} d\lambda_{RB} \left( \phi_B - \lambda_{RB} \right) + \int_0^{\phi_A} d\lambda_A \int_{\phi_B - j B \pi 4}^{\phi_B} d\lambda_{RB} \left( \phi_B - \lambda_{RB} \right)$$

$$+ \int_{\phi_A}^{\phi_B - j B \pi 4} d\lambda_A \int_0^{\lambda_A} d\lambda_{RB} \left( \phi_B - \lambda_{RB} \right) + \int_{\phi_A}^{\phi_B - j B \pi 4} d\lambda_A \int_{\phi_B - j B \pi 4}^{\phi_B} d\lambda_{RB} \left( \phi_B - \lambda_{RB} \right)$$

$$+ \int_{\phi_B - j B \pi 4}^{\phi_B} d\lambda_A \int_0^{\lambda_A} d\lambda_{RB} \left( \phi_B - \lambda_{RB} \right) + \int_{\phi_B - j B \pi 4}^{\phi_B} d\lambda_A \int_{\phi_B - j B \pi 4}^{\phi_B} d\lambda_{RB} \left( \phi_B - \lambda_{RB} \right)$$

$$\left. + \int_{\phi_B - j B \pi 4}^{\phi_B} d\lambda_A \int_{\phi_B - j B \pi 4}^{\phi_B} d\lambda_{RB} \left( \phi_B - \lambda_{RB} \right) \right).$$
and

\[ P(a = b|\phi_A \geq \phi_B - j_B \frac{\pi}{4}) = \frac{16}{\pi^2} \left( \int_0^{\phi_B - j_B \frac{\pi}{4}} d\lambda_{AR} \int_0^{\lambda_{AR}} d\lambda_{RB} \psi_{001}^{j_B}(\phi_B - \lambda_{RB}) \right. \]

\[ + \left. \int_0^{\phi_B - j_B \frac{\pi}{4}} d\lambda_{AR} \int_0^{\phi_B - j_B \frac{\pi}{4}} d\lambda_{RB} \psi_{001}^{j_B}(\phi_B - \lambda_{RB}) \right) \]

One can then check that with the choice of functions \( \psi_{\phi_B - \phi_{RB}}^{j_B} \in [0, 1] \) indicated in Table I (see main text), this leads (for all values of \( j_B \)) to

\[ P(a = b|\phi_A, \phi_B) = \frac{1 - \cos(\phi_A - \phi_B)}{2}, \]

as desired.

**Correlation from Protocol 2**

After running Protocol 2 for inputs \( x = (\sin \theta_A \cos \phi_A, \sin \theta_A \sin \phi_A, \cos \theta_A) \) and \( y = (\sin \theta_B \cos \phi_B, \sin \theta_B \sin \phi_B, \cos \theta_B) \), the probability that Alice and Bob’s outputs are the same is

\[ P(a = b|x, y) = \sum_{a_0, b_0 = \pm 1} P(a_0, b_0|\phi_A, \phi_B) \sum_{a_0, b_0 = \pm 1} P(a_0, b_0|\phi_A, \phi_B) \frac{1 - \cos(a_0 \theta_A + b_0 \theta_B)}{2} \]

\[ = P(a_0 = b_0, \phi_A, \phi_B) \frac{1 - \cos(\theta_A + \theta_B)}{2} + P(a_0 \neq b_0, \phi_A, \phi_B) \frac{1 - \cos(\theta_A - \theta_B)}{2} \]

\[ = \frac{1 - \cos(\phi_A - \phi_B)}{2} \frac{1 - \cos(\theta_A + \theta_B)}{2} + \frac{1 + \cos(\phi_A - \phi_B)}{2} \frac{1 - \cos(\theta_A - \theta_B)}{2} \]

\[ = \frac{1 - \cos(\theta_A \cos \theta_B - \sin \theta_A \sin \theta_B \cos(\phi_A - \phi_B))}{2} = \frac{1 - x \cdot y}{2}. \]

Protocol 2 thus reproduces the desired entanglement swapping correlation (equation (1) of the main text).