Classical simulation of entanglement swapping with bounded communication

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In this Supplementary material we prove that Protocols 1 and 2 generate the desired correlation (equation (1) of the main text), or in other terms, \( P(a = b|x, y) = \frac{1}{\pi} \).

Correlation from Protocol 1

We first note that the definitions of \( a = \text{sign}(\sin(\phi_A)) \) and of \( \phi'_A = (\phi_A - jA \frac{\pi}{4} \mod \pi) \in [0, \frac{\pi}{4}] \) on Alice’s side, and the definitions of \( \beta = \text{sign}(\sin(\phi_B - jA \frac{\pi}{4})) \) (and then of \( b = \pm \beta \)) and of \( \phi'_B = (\phi_B - jA \frac{\pi}{4} \mod \pi) \in [0, \pi] \) on Bob’s side, ensure that the following relations hold, as required:

\[
P(a = b|\phi_A, \phi_B) = P(a \neq b|\phi_A, \phi_B + \pi) = P(a \neq b|\phi_A + \pi, \phi_B) = P(a = b|\phi_A + \pi, \phi_B + \pi)
\]

\[
= P(a = b|\phi_A + j\frac{\pi}{4}, \phi_B + j\frac{\pi}{4}) \quad \text{for any } j \in \mathbb{Z}.
\]

It is therefore sufficient to check that the correct correlation is obtained for \( \phi_A \in [0, \frac{\pi}{4}] \) and \( \phi_B \in [0, \pi] \).

For such values of \( \phi_A, \phi_B \) (for which \( \phi'_A = \phi_A \) and \( \phi'_B = \phi_B \)), the probability \( P(a = b|\phi_A, \phi_B) \) obtained from Protocol 1 can be calculated as follows:

\[
P(a = b|\phi_A < \phi_B - jB \frac{\pi}{4}) = \frac{16}{\pi} \left( \int_{0}^{\phi_A} d\lambda_{AR} \int_{0}^{\lambda_{AR}} d\lambda_{RB} \psi_{010}^{\phi_{AR}}(\phi_B - \lambda_{RB}) \right.
\]

\[
\left. + \int_{0}^{\phi_A} d\lambda_{AR} \int_{\lambda_{AR}}^{\phi_{AR} - jB \frac{\pi}{4}} d\lambda_{RB} \psi_{010}^{\phi_{BR}}(\phi_B - \lambda_{RB}) + \int_{0}^{\phi_A} d\lambda_{AR} \int_{\phi_{BR} - jB \frac{\pi}{4}}^{\phi_{AR} - jB \frac{\pi}{4}} d\lambda_{RB} \psi_{010}^{\phi_{BR}}(\phi_B - \lambda_{RB}) \right)
\]

\[
+ \int_{\phi_{BR} - jB \frac{\pi}{4}}^{\phi_{AR}} d\lambda_{AR} \int_{0}^{\phi_{AR} - jB \frac{\pi}{4}} d\lambda_{RB} \psi_{010}^{\phi_{BR}}(\phi_B - \lambda_{RB}) + \int_{0}^{\phi_{AR}} d\lambda_{AR} \int_{\phi_{BR} - jB \frac{\pi}{4}}^{\phi_{AR} - jB \frac{\pi}{4}} d\lambda_{RB} \psi_{010}^{\phi_{BR}}(\phi_B - \lambda_{RB})
\]

\[
+ \int_{\phi_{AR}}^{\phi_{BR} - jB \frac{\pi}{4}} d\lambda_{AR} \int_{0}^{\phi_{BR} - jB \frac{\pi}{4}} d\lambda_{RB} \psi_{010}^{\phi_{BR}}(\phi_B - \lambda_{RB}) + \int_{0}^{\phi_{AR}} d\lambda_{AR} \int_{\phi_{BR} - jB \frac{\pi}{4}}^{\phi_{AR} - jB \frac{\pi}{4}} d\lambda_{RB} \psi_{010}^{\phi_{BR}}(\phi_B - \lambda_{RB})
\]

\[
+ \left. \int_{\phi_{AR}}^{\phi_{BR} - jB \frac{\pi}{4}} d\lambda_{AR} \int_{\phi_{AR} - jB \frac{\pi}{4}}^{\phi_{AR}} d\lambda_{RB} \psi_{010}^{\phi_{BR}}(\phi_B - \lambda_{RB}) + \int_{0}^{\phi_{AR}} d\lambda_{AR} \int_{\phi_{BR} - jB \frac{\pi}{4}}^{\phi_{AR} - jB \frac{\pi}{4}} d\lambda_{RB} \psi_{010}^{\phi_{BR}}(\phi_B - \lambda_{RB}) \right).
\]
and

\[ P(a = b|\phi_A \geq \phi_B - j_B \frac{\pi}{4}) = \frac{16}{\pi^2} \left( \int_0^{\phi_B - j_B \frac{\pi}{4}} d\lambda_AR \int_0^{\lambda_{AN}} d\lambda_RB \, \psi_{001}^j(\phi_B - \lambda_{RB}) \right. \\
+ \left. \int_0^{\phi_B - j_B \frac{\pi}{4}} d\lambda_AR \int_{\lambda_{AN}}^{\phi_B - j_B \frac{\pi}{4}} d\lambda_RB \, \psi_{001}^j(\phi_B - \lambda_{RB}) \right) \\
+ \left. \int_0^{\phi_B - j_B \frac{\pi}{4}} d\lambda_AR \int_{\phi_B - j_B \frac{\pi}{4}}^{\lambda_{AN}} d\lambda_RB \, \psi_{000}^j(\phi_B - \lambda_{RB}) \right) \\
+ \left. \int_0^{\phi_B - j_B \frac{\pi}{4}} d\lambda_AR \int_{\phi_B - j_B \frac{\pi}{4}}^{\lambda_{AN}} d\lambda_RB \, \psi_{100}^j(\phi_B - \lambda_{RB}) \right) \\
+ \left. \int_0^{\phi_B - j_B \frac{\pi}{4}} d\lambda_AR \int_{\phi_B - j_B \frac{\pi}{4}}^{\lambda_{AN}} d\lambda_RB \, \psi_{101}^j(\phi_B - \lambda_{RB}) \right). \]

One can then check that with the choice of functions \( \psi_{CA\text{R}RB}^j \in [0, 1] \) indicated in Table I (see main text), this leads (for all values of \( j_B \)) to

\[ P(a = b|\phi_A, \phi_B) = \frac{1 - \cos(\phi_A - \phi_B)}{2}, \]

as desired.

**Correlation from Protocol 2**

After running Protocol 2 for inputs \( x = (\sin \theta_A \cos \phi_A, \sin \theta_A \sin \phi_A, \cos \theta_A) \) and \( y = (\sin \theta_B \cos \phi_B, \sin \theta_B \sin \phi_B, \cos \theta_B) \), the probability that Alice and Bob’s outputs are the same is

\[ P(a = b|x, y) = \sum_{a_0, b_0 = \pm 1} P(a_0, b_0|\phi_A, \phi_B) \, P(a = b) \left( \frac{1 - \cos(\theta_A + \theta_B)}{2} \right. \\
+ \left. \frac{1 - \cos(\theta_A - \theta_B)}{2} \right) \\
= \frac{1 - \cos(\phi_A - \phi_B)}{2} \left( \frac{1 - \cos(\theta_A + \theta_B)}{2} \right. \\
+ \left. \frac{1 + \cos(\phi_A - \phi_B)}{2} \right) \\
= \frac{1 - \cos \theta_A \cos \theta_B - \sin \theta_A \sin \theta_B \cos(\phi_A - \phi_B)}{2} = \frac{1 - x \cdot y}{2}. \]

Protocol 2 thus reproduces the desired entanglement swapping correlation (equation (1) of the main text).