Supersymmetric lattice models: Field theory correspondence, integrability, defects and degeneracies
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CHAPTER 1

Introduction

One of the main challenges of theoretical condensed matter physics is to understand many particle systems in a regime where the interactions between the particles are strong. For weak interactions the method of approach is perturbation theory. This leads to accurate descriptions as long as the system can be approximated by a non-interacting system plus perturbations in a parameter that is small. In some special cases, when it is possible to resum an infinite number of terms in the perturbative series, this can also be used to describe a strongly interacting system. However, to be able to describe most materials in condensed matter physics where strongly correlated particles play a role, other methods have to be used.

Recently the anti-de Sitter/conformal field theory correspondence, which has been developed in the context of string theory, has been used to describe the strongly correlated phases in a condensed matter system by means of a weakly interacting gravitational theory. A much older approach to understand strongly correlated materials is to resort to a class of one-dimensional theoretical models which are exactly solvable. These are models in which all eigenstates and their energies can be found exactly, for example by the Bethe ansatz, and with some effort even their correlations functions can be calculated exactly, or in very good approximation using numerical methods.

Integrable models have an infinite number of conserved quantities which makes them very restrictive and simple enough to be solved exactly. Still there are many features of a general integrable model which are hard to calculate, especially for large system sizes, when finding a solution to the Bethe equations becomes cumbersome. In the continuum limit it is therefore a common practice to describe the model by a quantum field theory. But in general it is hard to make an exact correspondence between quantities in a finite size model and the corresponding field theory. In this thesis we study a class of models in which we require an extra symmetry to be present: supersymmetry. Defining the model to be supersymmetric gives the finite size model a lot of structure and makes a precise correspondence with a field theory possible.

Supersymmetry was introduced in particle physics and relates bosonic and fermionic particles to each other. In a field theory it extends the Poincaré algebra to the super-Poincaré algebra in which the generators of supersymmetry, the supercharges, anti-commute to the space-time momenta $P_\mu$, which are the generators of translations in time and space.

In this thesis we will study the so-called $M_k$ supersymmetric lattice models in one spatial dimension, which were originally introduced in [6, 7]. In the continuum limit they can be described by two-dimensional relativistic field theories. The $M_k$ model describes spin-less fermions on a lattice with a constraint that a maximum
of \( k \) particles can be next to each other. The supersymmetry relates states with \( f \) and \( f+1 \) particles to each other.

The \( M_k \) models are defined with an explicit supersymmetry on the lattice. The models can, for values of their parameters where they describe a critical phase, be mapped to a Heisenberg spin chain. It was known before that certain spin chains can correspond to a superconformal field theory but after the introduction of lattice models with explicit supersymmetry an additional (hidden) supersymmetry in the Heisenberg spin chain model at \( \Delta = -1/2 \) was discovered [8]. Unlike the Heisenberg spin chain, the \( M_k \) models are supersymmetric per definition and can be tuned off-critical by changing some parameters. Off criticality the integrability of the lattice model can be preserved and this makes it possible to study also the correspondence of the off-critical lattice model with an integrable massive field theory.

The main theme in this thesis is the relation between the \( M_k \) lattice models and the corresponding field theories. An overview of all relations studied in this thesis is given in figure 1.1. The focus will be always on the \( M_2 \) model but we will also mention the \( M_1, M_3 \) and higher \( M_k \) models.

In chapter 2 we introduce the \( M_k \) lattice models and describe their properties and some of the previous research done on these models. We discuss the free parameters in the \( M_k \) models and how these can be tuned to a critical point or a massive phase. In chapter 3 we establish an integrable deformation called staggering of the \( M_2 \) model away from the critical point. On the line where the \( M_2 \) model is integrable we find also two extra, dynamical supersymmetries.

At the critical point we can relate the \( M_k \) model to the \( k \)-th \( \mathcal{N} = 2 \) superconformal minimal model. We denote these conformal field theories (CFTs) in the diagram 1.1 as \( SU(2)_{k,1} \) CFT. These CFTs are part of a more general series of conformal field theories corresponding to specific quantum Hall states (see appendix A.7). In chapter 4 we study the relation of the open and closed \( M_1 \) and \( M_2 \) models with the first and second \( \mathcal{N} = 2 \) superconformal minimal models. At the end of chapter 4 we also look into the off-critical continuum limit of the \( M_2 \) model. We claim that the \( M_2 \) model, perturbed by an integrable staggering, corresponds in the continuum limit to the supersymmetric sine-Gordon model.

In chapter 5, the relation with the conformal field theory is made stronger by the introduction of defects in the lattice model which correspond to (para)fermion spin fields in the CFT. Using these defects all the fusion rules of the CFT can be found back in the lattice model. As mentioned above, the CFTs corresponding to the \( M_k \) models and specific quantum Hall states are related. In chapter 5 we find that indeed the \( M_2 \) model can be related to the Moore-Read quantum Hall state.

The staggering perturbation, which takes the model away from the critical point, varies the amplitudes of the model periodically along the sites of the chain. A limit of extreme staggering can be taken in which the eigenstates of the \( M_k \) models become simple. We study this limit in chapter 5. The defects, which correspond to (para)fermion spin-fields in the CFT, lead to ground state degeneracies in the massive phase that are protected by supersymmetry. At
Figure 1.1: Overview of the relations studied in this thesis. The chapters in which these relations are studied are indicated. For all relations we study the case $k = 2$ in detail. The $SU(2)_{k,1}$ CFT is the $k$-th $\mathcal{N} = 2$ superconformal minimal model. RR stands for Read-Rezayi quantum Hall states, NASS stands for the non-Abelian spin-singlet quantum Hall states.
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extreme staggering the excited states can be described by kinks in between the ground states. We can follow the eigenstates at extreme staggering numerically to the critical point where we know the relation of these states with the CFT. In this way a more direct link between the extreme staggering limit and the CFT can be established. We find that the formulas which count the number of kinks at extreme staggering can be $q$-deformed to find the corresponding CFT characters.

The arrow in the diagram that goes from the CFT to the lattice model, corresponds to chapter 6, an important chapter in this thesis. This chapter is inspired by the Haldane-Shastry model. Historically, the $M_k$ lattice models were discovered in ref. [6] along the lines described in detail in chapter 6. The Haldane-Shastry model is a very special integrable model with a lot of structure which makes it possible to relate every eigenstate of the lattice model exactly to a state in the conformal field theory. To establish this relation a so-called spinon basis of the CFT can be used. In chapter 6 we study a similar spinon basis for the CFTs corresponding to the $M_k$ models. We find that the $M_k$ models do not have as much structure as the Haldane-Shastry model but it is still possible to formulate a spinon basis of the corresponding CFT. By a method called finitisation we can relate it to the lattice model. We explore the algebraic structure of the CFT operators and their action on the spinon basis. The number of ground states in the lattice model, as well as their fermion numbers, can be found directly from the field theory. We also find the $k$ clustering constraint from the CFT as a restriction on the Hilbert space of the lattice model. In chapter 6 we furthermore consider the $SU(3)_{k=1, M=1}$ CFT which would be able to describe a version of the $M_1$ model with spin-full fermions. We formulate a spinon basis for this CFT and find explicit rules that describe the spin content of the states in the spinon basis. We also make the first steps which could lead to a new supersymmetric lattice model with spin-full fermions.

An integrable deformation analogous to the one for the $M_2$ model exists for all $M_k$ models [9]. For general $k$ we conjecture that the $M_k$ models, tuned away from the critical point by an integrable perturbation, can be related to the massive integrable field theories with Chebyshev superpotential $W_{k+2}$. We work this out in chapter 7, the last chapter of this thesis. For the $M_2$ model we establish a relation between the operators in the extreme staggering limit and the operators in the massive field theory.

Besides these chapters, this thesis contains two appendices in which the background material is provided, one on conformal field theory, appendix A, and one on the relevant features of integrable field theories, appendix B. Some technical details concerning massive integrable quantum field theories of Chebyshev type, which we need in chapter 7, are also presented in appendix B.