Supersymmetric lattice models: Field theory correspondence, integrability, defects and degeneracies
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CHAPTER 2

The supersymmetric $M_k$ models

In this chapter we introduce the $M_k$ models and give an overview of the literature on $M_k$ models. This chapter is based on the introductory parts of [1, 2] and the references cited in the text.

2.1 Definition of the supersymmetric $M_k$ lattice models

The $M_k$ models are a family of models introduced in [7, 6]. They are lattice models of interacting particles with an explicit $N = 2$ supersymmetry. The particles on the lattice are fermions without spin. The models can be defined on general graphs but we will only consider the model defined on a one-dimensional open or closed chain of length $L$. In the $M_k$ model the spinless fermions are subject to an exclusion rule which allows a group of at most $k$ fermions on neighbouring sites:

The Hamiltonian of the model is defined in terms of fermion creation and annihilation operators via the supercharges. The supercharge $Q_+$ decreases the fermion number $f \to f - 1$ and its hermitian conjugate $Q_+^\dagger = \bar{Q}_+$ increases the fermion number $f \to f + 1$. The operator $\bar{Q}_+$ is written in terms of constrained fermionic creation operators $d_{[a,b],j}^\dagger$ which create a particle at lattice site $j$ in such a way that a string of $a$ particles is formed, with the newly created particle at position $b$. This process has an amplitude given by $\lambda_{[a,b],j}$.

$$\bar{Q}_+ = \sum_{j=1}^L \sum_{a,b} \lambda_{[a,b],j} d_{[a,b],j}^\dagger$$  \hspace{1cm} (2.1)

where the sum is over the sites $j$ on the lattice. The operators $d_{[a,b],j}^\dagger$ can be written in terms of the usual fermion creation and annihilation operators $c_j, c_j^\dagger$ which satisfy $\{c_i, c_j^\dagger\} = \delta_{i,j}, \{c_i, c_j\} = \{c_i^\dagger, c_j^\dagger\} = 0$. For this we use the projection operator $P_j = 1 - c_j^\dagger c_j$. For the $M_1$ model we only need the constrained fermion creation operator $d_{[1,1],j}^\dagger$ which is given by

$$d_{[1,1],j}^\dagger = P_j - 1 c_j^\dagger P_j + 1.$$  \hspace{1cm} (2.2)
For the $M_2$ model also $d_{[2,1],j}^\dagger$ and $d_{[2,2],j}^\dagger$ are needed and they are given by

\[

d_{[2,1],j}^\dagger = \mathcal{P}_j c_{j+1}^\dagger c_{j+1}^\dagger \mathcal{P}_{j+2}
\]
\[

d_{[2,2],j}^\dagger = \mathcal{P}_j c_{j-1}^\dagger c_{j-1}^\dagger \mathcal{P}_{j+1}.
\]

Similarly all $d_{[a,b],j}^\dagger$ are defined for the $M_k$ models. The Hamiltonian of the $M_k$ models are defined by the anti-commutator of the supercharge $\bar{Q}_+$ and its hermitian conjugate $Q_+$

\[
H = \{Q_+, \bar{Q}_+\},
\]

with the restriction $(Q_+)^2 = (\bar{Q}_+)^2 = 0$. This restriction is not true for general values of the parameters $\lambda_{[a,b],j}$, we will address the freedom we have in the choice of parameters in the next section. Although $Q_+$ and $\bar{Q}_+$ are nonlocal, taking the anti-commutator leads to a local Hamiltonian with an interaction range of a maximum of $k$ sites. In sections 2.8 we will consider the possible processes for the $M_2$ model in detail.

### 2.2 Free parameters

The $M_k$ models were first introduced with the parameters $\lambda_{[a,b],j} = \lambda_{[a,b]}$, thus independent of the lattice site $j$. In [10] staggering was introduced for the $M_1$ model (it was further studied in [11, 12] and [13]), in ref. [1] the staggered $M_2$ model has been considered (see also chapter 3). In the case the amplitudes do not depend on the site $j$, we call the model homogeneous. In the case where the amplitudes $\lambda_{[a,b],j}$ have an explicit site dependence we say that the amplitudes are staggered and we call the model inhomogeneous.

The restriction $(Q_+)^2 = 0$ gives relations on the coefficients $\lambda_{[a,b],j}$ reducing the number of free parameters. This restriction is equivalent to setting the amplitudes of a process in which from a string of length $a$ the particle at position $b$ and then the particle at position $c$ ($b < c$) is removed equal to the process in which these particles are annihilated in the opposite order. The particle at position $c$ is of course actually a particle at position $c$ of a string of length $a - b$ after a particle at position $b$ has been removed. This leads to the recursion relation:

\[
\lambda_{[a,b],j} \lambda_{[a-b,c-b],j+c-b} = \lambda_{[a,c],j+c-b} \lambda_{[c-1,b],j} \quad 1 \leq b < c \leq a
\]

This can be solved by [14, 7]

\[
\lambda_{[a,b],j} = \left( \prod_{k=1}^{b-1} \frac{\lambda_{[a-k+1,1],j-b+k}}{\lambda_{[b-k,1],j-b+k}} \right) \lambda_{[a-b+1,1],j}.
\]

In the homogeneous case, $\lambda_{[a,b],j} = \lambda_{[a,b]}$, this gives

\[
\lambda_{[a,b]} = \frac{\lambda_{[a,1]} \lambda_{[a-1,1]} \cdots \lambda_{[a-b+1,1]}}{\lambda_{[b-1,1]} \lambda_{[b-2,1]} \cdots \lambda_{[1,1]}}
\]
so only $\lambda_{[1,1]}$, $\lambda_{[2,1]}$, ..., $\lambda_{[k,1]}$ are left as free parameters. Since we can choose a normalisation of the Hamiltonian one of these parameters can be set to one, which gives a total of $k - 1$ free parameters for the homogeneous $M_k$ model.

**M$_2$ model**

For the $M_2$ model equation (2.6) gives

$$\lambda_{[1,1],j-1} \lambda_{[2,2],j} = \lambda_{[1,1],j} \lambda_{[2,1],j-1}$$

(2.8)

which can be conveniently parametrized by

$$\lambda_{[1,1],j} = \lambda_j$$

$$\lambda_{[2,1],j} = \lambda_j \mu_j$$

(2.9)

$$\lambda_{[2,2],j} = \lambda_j \mu_{j-1}.$$

We will use this parametrization when discussing the integrability in chapter 3. It follows that in the homogeneous case $\lambda_{[2,1]} = \lambda_{[2,2]}$, so in this case there is a symmetry between annihilating the first and the second particle of a pair of two particles. If we want this property also in the staggered case we have to set $\mu_j = \mu$ for all $j$. Another special point in the parameter space of the $M_2$ model is the point where $\lambda_{[1,1]} = 0$. At this point the amplitude for the creation and annihilation of single particles is zero. Hence, the only processes that are possible involve pairs of particles. The model at $\lambda_{[1,1]} = 0$ can be mapped to the supersymmetric $t-J$ model [7, 8].

**M$_3$ model**

For the $M_3$ model equation (2.5) gives

$$\lambda_{[3,2],j} \lambda_{[1,1],j-1} = \lambda_{[2,1],j} \lambda_{[3,1],j-1}$$

$$\lambda_{[3,3],j} \lambda_{[1,1],j-1} \lambda_{[2,1],j-2} = \lambda_{[1,1],j} \lambda_{[2,1],j-1} \lambda_{[3,1],j-2}.$$  

(2.10)

We can add to the parametrization of equation (2.9) the following relations to satisfy $(Q_+)^2 = 0$ for the $M_3$ model:

$$\lambda_{[3,1],j} = \lambda_j \mu_j \nu_j$$

$$\lambda_{[3,2],j} = \lambda_j \mu_j \mu_{j-1} \nu_{j-1}$$

(2.11)

$$\lambda_{[3,3],j} = \lambda_j \mu_{j-1} \nu_{j-2}.$$

### 2.3 Translation symmetry

For periodic boundary conditions there exists a translation operator $T$ on the closed, homogeneous fermion chain, thus choosing the parameters $\lambda_{[a,b],j} = \lambda_{[a,b]}$. If the total number of particles on the chain is even and the last site is occupied, the translation operator includes the fermionic minus sign per definition, for example

$T(\text{●●●○}) = \text{○●○●}, \quad T(\text{○●●○}) = \text{●○●●}.$

(2.12)
The eigenvalues $t$ of the translation operator satisfy $t^L = 1$ for a chain of length $L$. The translation operator commutes with supercharges $Q_+, \bar{Q}_+$ and thus with the Hamiltonian of the $M_k$ models. The eigenstates of the Hamiltonian can for periodic boundary conditions be chosen as eigenstates of the translation operator.

When we stagger the parameters $\lambda_{[a,b],j}$ they become site dependent. We will always consider this modulation to be periodic with a certain period $p$, such that $\lambda_{[a,b],j+p} = \lambda_{[a,b],j}$ (for the $M_2$ model we always take $p = 2$). In this case the translation symmetry over one site is broken, but the supercharges still commute with $T^p$. And thus $[H, T^p] = 0$, so we can still characterise the eigenvalues of the Hamiltonian by their value of $T^p$.

Another type of boundary condition is the so-called twisted boundary condition. In this case a fermion that hops over the boundary acquires a phase $e^{i\phi}$ and the translation operator becomes $T' = T e^{i\phi n_L}$ where $n_L$ is the number operator on site $L$. The twisted translation operator can also be chosen in a way that it contains a phase factor for every translation, in this way ‘smearing out’ the phase factor over the whole chain. Then the definition is (which we use in chapter 3)

$$T' = T e^{i\phi(n_L-f/L)}$$

(2.13)

The supercharges do not commutes with $T'$ and therefore the supersymmetry is broken on the lattice. In the field theory (which we will consider in the next chapters) inserting a boundary twist angle $\phi$ corresponds to a procedure called spectral flow (see for example [11]). For periodic boundary conditions we associate the Ramond sector of a superconformal field theory to the lattice model. In this case the basic $\mathcal{N} = 2$ supersymmetry algebra is generated by the zero modes of the supercurrents $G_0^+, G_0^-$. Spectral flow change the modes of the supercurrents $G^\pm$ as $G_\ell^+ \rightarrow G_{\ell-\phi}^+$, $G_\ell^- \rightarrow G_{\ell+\phi}^-$. Since the modes change independently and connect states with different energy the $\mathcal{N} = 2$ supersymmetry algebra is broken. For $\phi = \pi$, anti-periodic boundary conditions, the supercharges, which had zero modes in the Ramond sector become $G_{-1/2}^+$ and $G_{1/2}^-$. They are thus in the NS sector of the CFT and the spectrum of the lattice model with periodic boundary conditions corresponds to the NS sector of the field theory.

### 2.4 Mapping to a Heisenberg spin chain

The homogeneous $M_k$ model can be mapped to a Heisenberg spin chain [7]. This mapping maps the edge between empty sites to a spin with $S_z = k/2$, while one particle between empty sites is mapped to a spin with $S_z = k/2 - 1$, etc.,

\[
\begin{align*}
\begin{array}{c}
\vline \\
\end{array} & \leftrightarrow & S_z = k/2 \\
\begin{array}{c}
\vline \\
\end{array} & \leftrightarrow & S_z = k/2 - 1 \\
\begin{array}{c}
\vline \\
\end{array} & \leftrightarrow & S_z = k/2 - 2 \\
\vdots & \vdots & \vdots
\end{align*}
\]
2.4 Mapping to a Heisenberg spin chain

Under this mapping a fermion chain with \( L \) sites and \( f \) particles is mapped to a spin chain with \( L - f \) sites.

**M\(_1\) model** From the above map it can be shown that up to an additive constant the Hamiltonian of the M\(_1\) model is equal to the Hamiltonian of the spin \( \frac{1}{2} \) XXZ Heisenberg model at \( \Delta = -1/2 \).

\[
H_{\text{XXZ}} = \frac{1}{2} \sum_{j=1}^{L} \left( \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y - \Delta \sigma_j^z \sigma_{j+1}^z \right). \tag{2.14}
\]

The spectra of the two models correspond for appropriately twisted periodic boundary conditions.

**M\(_2\) model** In [7] it was shown that the M\(_2\) Hamiltonian is similar to the Hamiltonian of the Fateev-Zamolodchikov XXZ chain whose Hamiltonian is of the form [15, 16]

\[
H_{\text{FZ}} = \sum_{x=1}^{N} \left( \sum_{a=1}^{3} J_a (S_x^a S_{x+1}^a + 2(S_x^a)^2) - \sum_{a,b=1}^{3} A_{ab} S_x^a S_{x+1}^b S_x^a S_{x+1}^b \right). \tag{2.15}
\]

Here \( J_a, A_{ab} \) are constants with \( A_{aa} = J_a \) and \( A_{ab} = A_{ba} \). The case of the spin\( -1 \) XXZ chain corresponds to the choice \( J_1 = J_2 = 1, J_3 = \cos 2\theta \) and \( A_{12} = 1, A_{13} = A_{23} = 2 \cos \theta - 1 \) where \( \theta \) parametrizes the anisotropy of the model. The \( S^1, S^2, S^3 \) are the standard \( \mathfrak{su}(2) \) generators in the spin\( -1 \) representation. At \( \theta = \pi / 4 \) the Hamiltonians of the M\(_2\) model and the XXZ spin-1 chain are the same up to an overall constant potential energy and a twist in the boundary conditions.

### 2.4.1 Spin-reversal symmetry

In the spin chain there is a natural notion of spin-reversal symmetry. Via the above mapping, a spin-reversal operation can also be defined for the fermion chain. For the open chain we define the spin-reversal operation as first adding half sites \( \chi, \kappa \) at the left and right of the chain, then mapping the model to a spin model, then acting with the spin reversal operator and then mapping it back to a model of fermions on a chain and finally removing the half sites at the end. For the M\(_2\) model this procedure results in adding half sites at the end of the chain, then doing the mapping

\[
\begin{align*}
\begin{array}{c}
\chi \leftrightarrow \chi, \\
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\kappa \leftrightarrow \kappa, \\
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\kappa \leftrightarrow \kappa, \\
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\chi \leftrightarrow \chi, \\
\end{array}
\end{align*}
\]

(2.16)
and finally removing the half sites at the end of the chain. If we add an extra empty site to the first line of the equation above we get the state $\sigma\sigma\sigma$. Acting with spin reversal gives $\sigma\sigma\sigma \rightarrow \sigma\sigma\sigma\sigma\sigma\sigma\sigma$.

The spin reversal of an extra empty site is thus three extra sites and two extra particles, it can be seen that this is always the case. Similarly, by comparing eq. (2.17) with the second line of eq. (2.16) it can be seen that the spin reversal of adding an extra particle is minus four sites and minus three particles. It thus follows that under spin reversal a chain of length $L$ and $f$ transform as

$$(L, f) \rightarrow (2 + 3L - 4f, 2 + 2L - 3f).$$

Note that the dimension of the Hilbert space of states for $(L, f)$ and $(2 + 3L - 4f, 2 + 2L - 3f)$ is exactly the same for the $M_2$ model. In the section below we will at the critical point choose the parameters in the Hamiltonian such that spin reversal symmetry is obeyed locally in the lattice model. To find an explicit spin-reversal symmetry at the critical point of the open chain, we need to add a boundary term in the Hamiltonian such that also at the boundaries the symmetry is preserved. For the open chain, modifying the Hamiltonian to

$$H = \{Q_+, \bar{Q}_+\} + 1 - n_L - n_R,$$

makes the spin-reversal symmetry explicit. The spectra of $(L, f)$ are then identical to those of $(2 + 3L - 4f, 2 + 2L - 3f)$. This modified Hamiltonian breaks supersymmetry.

For the closed chain there are no two values of $(L, f)$ that can be related under spin-reversal and have the same dimension of the Hilbert space. An exact notion of spin-reversal for the closed chain would therefore require defects that change the dimension of the Hilbert space.

### 2.5 Criticality

In ref. [7] it was conjectured that the parameters $\lambda_{[a,b],j}$ can for all of the $M_k$ models be tuned to a critical point. At the critical point the $M_k$ model is in its continuum limit described by the $k$-th $\mathcal{N} = (2,2)$ superconformal minimal model with central charge $3k/(k + 2)$. These theories are the fermionic ‘cousins’ of the $SU(2)_k$ Wess-Zumino-Witten conformal field theories (see appendix A): they share the same central charge and a decomposition into a free boson factor ($c = 1$) and a $Z_k$ parafermion CFT with $c = 2(k - 1)/(k + 2)$. For the $M_k$ model at the critical point the parameters $\lambda_{[a,b],j}$ do not depend on the site $j$ so the model is homogeneous.

It turns out that the $M_k$ models are critical at the point where they can be mapped to a Heisenberg spin $S = k/2$ chain at its supersymmetric point. The critical point of the $M_k$ model can be found by demanding that the Hamiltonian is (locally) spin-reversal symmetric. Thus we want for example that the amplitude
2.5 Criticality

for the spin exchange process of spins \( s_1 \) and \( s_2 \) is equal to the amplitude for the spin exchange process of spin \(-s_1\) and \(-s_2\). Starting with \( s_1 = k/2, s_2 = k/2 - 1 \) corresponds to the following process for the fermions:

\[
\frac{k}{2}(k/2 - 1) \leftrightarrow \frac{k}{2} - 1(k/2)
\]

(2.20)

The spin reversed process is given by

\[
\frac{k}{2} + 1(-k/2 + 1) \leftrightarrow \frac{k}{2} + 1(-k/2)
\]

(2.21)

Equating the amplitudes for the processes (2.20) and (2.21) gives

\[
\lambda_{[1,1]}^2 - \lambda_{[2,1]}^2 = \lambda_{[k,1]}^2.
\]

(2.22)

Continuing with the process for spin \( s_1 = k/2 - 1 \) and \( s_2 = k/2 - 2 \) gives

\[
(k/2 - 1)(k/2 - 2) \leftrightarrow \frac{k}{2} - 1(k/2 - 2)
\]

(2.23)

which has as spin reversal

\[
\frac{k}{2} + 1(-k/2 + 2) \leftrightarrow \frac{k}{2} + 2(-k/2 + 1)
\]

(2.24)

equation these amplitudes gives

\[
\lambda_{[2,1]}^2 - \lambda_{[4,2]}^2 = \lambda_{[k-1,1]}^2 - \lambda_{[2k-2,k]}^2 \lambda_{[2k-2,k-1]}.
\]

(2.25)

In general we get for \( a \in \{1, \ldots, k\} \)

\[
\lambda_{[a,1]}^2 - \lambda_{[2a,a]}^2 \lambda_{[2a,a+1]}^2 = \lambda_{[k-a,1]}^2 - \lambda_{[2k-2a+2,k-1]}^2 \lambda_{[2k-2a+2,k-2]}^2.
\]

(2.26)

where of course the amplitudes \( \lambda_{[i,j]} \) with \( i \) or \( j \) larger than \( k \) are zero.

Equation (2.26) together with the equation that solves \((Q_+)^2 = 0\) in the homogeneous case (equation (2.7)) gives the values of the parameters at the critical point. For the first three models these values are:

**M_1**

\[
\lambda_{[1,1],j} = 1
\]

(2.27)

**M_2**

\[
\lambda_{[1,1],j} = \sqrt{2}, \quad \lambda_{[2,1],j} = \lambda_{[2,2],j} = 1
\]

(2.28)
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$M_3$

\[
\begin{align*}
\lambda_{[1,1],j} & = y, & \lambda_{[2,1],j} & = \lambda_{[2,2],j} = 1, \\
\lambda_{[3,1],j} & = \lambda_{[3,3],j} = 1/y, & \lambda_{[3,2],j} & = 1/y^2
\end{align*}
\]

(2.29)

with $y = \sqrt{1 + \sqrt{5}}/2$.

At the critical point the $M_2$ model actually has $\mathcal{N} = (3, 3)$ supersymmetry. The operators and states in the CFT can be directly related to the lattice model. For the $M_1$ model this has been considered in [17, 18]. For the $M_2$ and $M_3$ model we look into this in chapters 4 and 5 which are based on [1, 2]. In chapter 6 we will also show the relation with the CFT by discussing a scheme that relates a finite part of the Hilbert spaces of the CFT to the Hilbert space of the lattice model.

2.6 Integrability

The $M_k$ models are integrable at their critical points [7] and there also exists a whole manifold in the $\lambda_{[a,b],j}$ parameter space that is integrable [1, 14]. Based on the exact results obtained for the ground state of the period three staggered $M_1$ model, the integrability of the staggered model was conjectured [10, 19]. For the $M_1$ model it was shown explicitly in [20] that it is integrable for all values of $\lambda_{[1,1],j}$ as long as these values are periodic with three sites. The continuum limit of the staggered $M_1$ lattice model is described by the sine-Gordon field theory at the value of its coupling $\beta$ where the theory has $\mathcal{N} = 3$ supersymmetry (see appendix B).

For the $M_2$ model we will see in chapter 3 that a staggering with period two satisfying $\mu_j^2 + \mu_{j+1}^2 = 1$ is integrable. As we explain in chapter 3 this restriction on the parameters can be found by demanding the presence of a spin-reversal symmetry for the staggered lattice model. For $\mu_j^2 + \mu_{j+1}^2 = 1$, the lattice model also exhibits two extra dynamical supersymmetries. For the case where $\lambda_{[2,1],j} = \lambda_{[2,2],j}$ the integrable staggering reduces to $(\mu_j = \mu_{j-1} = 1/\sqrt{2})$. The continuum limit of the staggered $M_2$ model corresponds to the supersymmetric sine-Gordon model as we show in chapter 4.

For the general $M_k$ models an integrable staggering was found by Hagendorf and Huijse [14]. For the $M_3$ model we will not use the general integrable staggering but only consider a limit in which the parameters $\lambda_{[a,b],j}$ on some of the sites tend to zero, we call this the extreme staggering limit. For the $M_3$ model in the limit of extreme staggering the following pattern modulo five for $\lambda_j, \mu_j, \nu_j$ is integrable

\[
\begin{align*}
\lambda_j : & \ldots & 1/\sqrt{2} & 1 & \lambda & 1/\sqrt{2} & \ldots \\
\mu_j : & \ldots & 1 & 1/\sqrt{2} & 1/\sqrt{2} & 1 & \lambda & \ldots \\
\nu_j : & \ldots & \sqrt{2} & 1/\sqrt{2} & \sqrt{2} & 1 & 1 & \ldots,
\end{align*}
\]

(2.30)

where $\lambda \to 0$. 

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2.7 Spectrum

Because of the presence of supersymmetry in the $M_k$ models the spectrum has a lot of structure, particularly the number of ground states can be calculated for various boundary conditions. In ref. [7] the number of ground states were conjectured using the Witten index, in refs. [21, 9] the number of ground states for various boundary conditions have been calculated exactly using cohomology techniques.

The supercharges commute with the Hamiltonian by construction, $[H, Q_+] = [H, \bar{Q}_+] = 0$. The fermion number operator $\mathcal{F}$ has as eigenvalue the number of fermions $f$ on the lattice. The Hamiltonian commutes with the fermion number $[H, \mathcal{F}] = 0$ and can thus be diagonalized in sectors of the Hilbert space with a fixed fermion number. The commutation relation of the fermion operator with the supercharges give

$$[\mathcal{F}, \bar{Q}_+] = \bar{Q}_+, \quad [\mathcal{F}, Q_+] = -Q_+. \quad (2.31)$$

Therefore all states in the spectrum are doublets with $(f, f+1)$ particles, with the exception of the supersymmetric ground states at $E = 0$, which are annihilated by both $Q_+$ and $\bar{Q}_+$, see figure 2.1. The supersymmetry allows by use of the Witten index to easily identify the number of ground states. The Witten index [22] is defined as

$$W = \text{tr}(-1)^f, \quad (2.32)$$

where $f$ is the number of fermions on the chain. The contribution to $W$ of all states with $E > 0$ cancel. A nonzero Witten index therefore means that there are at least $|W|$ ground states at energy $E = 0$. In the cases studied in this thesis the number of ground states is precisely $|W|$ but in general this is not always the case [9].

Clearly, a Hilbert space formed by all possible ways of putting spin-less fermions on a lattice leads to a vanishing Witten index. However, the definitions of the $M_k$ models include constraints ruling out certain configurations where particles occupy nearest neighbour sites on the lattice. This then does lead to non-vanishing values for $W$. The general pattern is that, for a given model, $W$ is largest for a choice of periodic boundary conditions that is compatible with the periods of the bulk supersymmetric ground states. For the 1D $M_k$ models this maximal number turns out to be $W_k = k+1$.

For periodic boundary conditions also the eigenvalue $t = e^{i\pi}$ of the translation operator $T$, with $T^L = 1$ can be calculated using the Witten index. There are $k+1$ ground states at length $L = 0 \mod (k+2)$ which have translation eigenvalues

$$t = (-1)^{L+1} \exp\left(\frac{2\pi i}{k+2} j\right), \quad (2.33)$$

for $j = 1, \ldots, k+1$. At all other lengths there is one ground state at translation eigenvalue

$$t = (-1)^{L+1}. \quad (2.34)$$
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Figure 2.1: Spectrum of the $M_2$ model, open chain, $L = 16$, $f = 7, 8, 9, 10$. All nonzero energy states form doublets.

For open boundary conditions the situation is rather different. There is a single supersymmetric ground state for $L \equiv 0, -1 \mod (k + 2)$ and there are none otherwise.

2.8 Explicit $M_2$ model Hamiltonian

The Hamiltonian of the $M_2$ model is written down easiest by considering the various hopping terms separately. Using the parametrization above the hopping terms and their amplitudes for the $M_2$ model are:

- Single hop
  \[ \bullet \quad \text{with amplitude} \quad \lambda_j \lambda_{j+1}(1 - \mu_j^2), \quad (2.35a) \]

- Pair hop
  \[ \bullet \quad \text{with amplitude} \quad -\lambda_j \lambda_{j+2} \mu_j \mu_{j+1}, \quad (2.35b) \]

- Split-join on the right and left
  \[ \bullet \quad \text{with amplitude} \quad \lambda_{j+1} \lambda_{j+2} \mu_j, \quad (2.35c) \]
  \[ \bullet \quad \text{with amplitude} \quad \lambda_j \lambda_{j+1} \mu_{j+1}, \quad (2.35d) \]

- Partner swap
  \[ \bullet \quad \text{with amplitude} \quad \lambda_{j+1} \lambda_{j+2} \mu_j \mu_{j+2}. \quad (2.35e) \]
The potential energy per site is given by:

- $\lambda_j^1$ if it is possible to create or annihilate an isolated particle at site $j$,
- $\lambda_j^2 \mu_{j-1}$ if it is possible to create or annihilate a particle at site $j$ which is part of a pair on sites $j - 1, j$,
- $\lambda_j^2 \mu_j^2$ if it is possible to create or annihilate a particle at site $j$ which is part of a pair on sites $j, j + 1$.

### 2.9 Two dimensions

In [23] the $M_1$ model on a 2-dimensional lattice was first studied. It turned out to have many special properties. Different types of lattices have been studied. On the majority of possible lattices the number of ground state of the $M_1$ model either possess a number of zero-energy supersymmetric ground states that is exponential in the circumference or in the area (number of sites), these are called sub-extensive and extensive ground state entropies, respectively [24, 25, 26, 21, 17, 27]. The phenomenon of a system having an extensive ground state entropy is called superfrustration. The 2-dimensional square lattice with periodic boundary conditions provides an example of a sub-extensive ground state entropy, the 2D triangular, hexagonal and kagome lattices exhibit extensive ground state entropy [28, 29]. The precise ground state counting is known in some of the sub-extensive cases (2D square and octagon-square), and for some particular cases with extensive ground state entropy. For more general cases the ground state entropy is not known in analytic form (see [28] for partial results for the 2D triangular lattice).

For example, an $L \times L$ patch of the 2D square lattice with periodic boundary conditions in two directions has a Witten index growing as [24]

$$W_{L \times L} = 1, -1, 4, 7, -9, 1, 7, 40, 9, 1, 166, \ldots \quad L = 1, 2, \ldots 12, \ldots$$ (2.36)

where asymptotically the Witten index $W_{L \times L}$ grows exponentially with the length $L$ of the system. We have seen above that $|W|$ is a lower bound for the number of ground states. For the one-dimensional chain it turned out that the number of ground states is equal to $|W|$ but in this two-dimensional case the Witten index only gives a lower bound. Still the number of ground states depends exponentially on the length of the system [21]. On the contrary, an $L \times L$ patch of the 2D square lattice with open boundary conditions has a single supersymmetric ground state.