Supersymmetric lattice models: Field theory correspondence, integrability, defects and degeneracies
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CHAPTER 3

Bethe ansatz solvability and supersymmetry of the $M_2$ model on a closed chain

In this chapter a detailed study of the $M_2$ model is presented. We identify a submanifold in the space of parameters of the model where it is Bethe ansatz solvable. The relation between this manifold and the existence of additional, so-called dynamic, supersymmetries is discussed. This chapter is based on the results published in [1].

3.1 Introduction

Integrability is arguably one of the most powerful analytical tools available for the study of $d = 2$ classical or $d = 1$ quantum mechanical systems [30]. Integrable models are very special and enjoy a high degree of analytical structure. The integrable supersymmetric models provide an interesting opportunity to study the interplay between the structure due to integrability and that due to supersymmetry (see also [31]). This is the topic of this chapter.

We study the staggered version of the $M_2$ model with interactions which are site-dependent. For a chain of length $N$ these interactions can be parametrised by $2N$ parameters: $\lambda_x$ and $\mu_x$, where $x = 1, \ldots, N$ labels the sites of the chain. For this model we identify a special two-parameter submanifold in the $(\lambda_x, \mu_x)$-parameter space where the model possesses two types of dynamic supersymmetry in addition to the explicit $\mathcal{N} = 2$ supersymmetry. Furthermore, from completely independent considerations we find that the model is solvable via a generalised coordinate Bethe ansatz [20] on precisely this same special submanifold in parameter space.

The special submanifold is given by

$$\lambda_{x+2} = \lambda_x, \quad \mu_{x+2} = \mu_x, \quad \mu_x^2 + \mu_{x+1}^2 = 1. \quad (3.1)$$

The homogeneous model, i.e. the original $M_2$ model with site-independent interactions, intersects the special submanifold at the point: $\lambda_x = 1$ and $\mu_x = \mu = 1/\sqrt{2}$ for all $x$. These are indeed precisely the parameters for which the original $M_2$ model was found to be integrable [7]. In the original model the constraint on $\mu$ follows from spin-reversal symmetry upon mapping the model to the spin–1 chain (as seen in sections 2.5 and 2.6). Here we will see that the constraint $\mu_x^2 + \mu_{x+1}^2 = 1$ is also related to a type of spin-reversal symmetry, even though for site-dependent interactions a mapping to a spin model is not known at present.
Figure 3.1: We plot the special two-parameter submanifold (3.1), where the model is integrable and enjoys additional supersymmetry, in a three-parameter subspace for which $\lambda_{x+2} = \lambda_x$, $\mu_{x+2} = \mu_x$ for all $x$. We define $\lambda = \lambda_x/\lambda_{x+1}$ and plot the range $0 < \lambda < 2$ and $\mu_x > 0$. The red line indicates the parameters for which the original $M_2$ model is recovered: $\lambda = 1$ and $\mu_x = \mu_{x+1} = \mu$. The red point lies at the intersection of the red line with the special submanifold: $\mu = 1/\sqrt{2}$. It is the point where the original model is integrable and maps to the spin-1 chain. The blue point is a generic point on the special submanifold, and the green point is a generic point that is neither on the homogeneous line nor on the special submanifold. In figure 3.7 we will compare numerical data for these three points. Note that the model possesses the explicit $\mathcal{N} = 2$ supersymmetry for general $\lambda_x$ and $\mu_x$ and thus, in particular, everywhere in the plotted parameter space.
We investigate how the supersymmetry translates into the structure of the Bethe ansatz. We find, in particular, a relation between the action of the supercharges, the generators of the supersymmetry and exact complete strings of Bethe roots.

In section 3.2 we extend the spin-reversal symmetry discussed in chapter 2 to the inhomogeneous $M_2$ model. In section 3.3 we discuss three dynamic supersymmetries which exist on the special submanifold. In section 3.4 we study the model by means of a generalised coordinate Bethe ansatz solution: as we shall see, the requirement that the model be Bethe ansatz solvable leads again to a restriction to the special submanifold. In section 3.5 we study the relation between the dynamical supersymmetries and the Bethe ansatz.

### 3.2 Extension of spin reversal symmetry to the inhomogeneous chain

We have seen above that we can have site dependent amplitudes $\lambda_{[a,b],j}$ in the model. It is an interesting question what the spin-reversal symmetry (see 2.4.1) will give in the staggered case. We compare the amplitudes for the single hop and partner swap processes in the $M_2$ model Hamiltonian see 2.8. As both amplitudes are position-dependent we need a rule to fix their positions on the lattice. We impose the following: the single particle which changes its position hops between the same sites in both processes. For the example at hand, we impose that the following amplitudes are equal:

\[
\lambda_x \lambda_{x+1} (1 - \mu_x^2) = \lambda_x \lambda_{x+1} \mu_{x-1} \mu_{x+1}
\]

We see that this implies $\mu_{x+1} \mu_{x-1} = 1 - \mu_x^2$. Applying this strategy to the left- and right-split-join processes, that is equating the amplitudes for the processes

\[
\lambda_x \lambda_{x+1} \mu_{x+1} = \lambda_x \lambda_{x+1} \mu_{x-1}
\]

leads to $\mu_{x-1} = \mu_{x+1}$ and hence we get

\[
\mu_x^2 + \mu_{x+1}^2 = 1
\]

which is the second equation of (3.1). It should be emphasized that while it leads to a mod–2 staggering for the $\mu_x$, the present argument does not impose a constraint on the $\lambda_x$. 
3.3 Dynamical supersymmetries of the M$_2$ model

3.3.1 Dynamic supersymmetry $Q_-$

We recall the definition of the supercharge $Q_+$ as discussed in chapter 2. The action of the supercharge $Q_{+,x}$ in the staggered case, is

\[
Q_{+,x}|\cdots \circ \cdots \circ \cdots \rangle = 0, \\
Q_{+,x}|\cdots \circ \bullet \circ \cdots \rangle = (\pm) \lambda_x |\cdots \circ \circ \circ \circ \cdots \rangle, \\
Q_{+,x}|\cdots \circ \bullet \bullet \circ \cdots \rangle = (\pm) \lambda_x \mu_x |\cdots \circ \circ \bullet \circ \cdots \rangle, \\
Q_{+,x}|\cdots \circ \bullet \bullet \circ \cdots \rangle = (\pm) \lambda_x \mu_{x-1} |\cdots \circ \circ \bullet \circ \cdots \rangle,
\]

where the sign $(\pm)$ is the usual fermionic string, i.e. minus one to the number of fermions located to the left of the site $x$. The supercharge of the model is defined as the sum of these operators over all sites

\[
Q_+ = \sum_{x=1}^{N} Q_{+,x}. \quad (3.4)
\]

The discussion of spin reversal symmetry in the last section leads naturally to the question if the “spin-reversal” transformation can be applied to the supercharge $Q_+$ in order to obtain a second copy of the $\mathcal{N} = 2$ supersymmetry algebra. Let us illustrate this idea with an example. We consider the local action of $Q_+$ on a filled site, $x$, with empty neighbouring sites. We map this configuration and its image to spin configurations, perform a spin reversal transformation and map the result back to fermion configurations:

```
0 0 0
fermions to spins spin reversal spins to fermions
```

The last column suggests that there might be a dynamic supercharge, which we called $Q_-$, that inserts 3 fermions and 4 sites. We show in this section that this supercharge indeed exists, provided that the staggering parameters are periodic with period two and the model is restricted to subsectors of the Hilbert space which are invariant under translations by four sites.

**Definition and properties.** As suggested by our example, the dynamic supercharge inserts four consecutive sites with three particles into the system: $Q_- : \mathcal{H}_{N,f} \rightarrow \mathcal{H}_{N+4,f+3}$. It is given as a linear superposition of locally acting...
operators $Q_{-,x}$ which perform the insertion process between sites $x$ and $x+1$:

$$Q_- = \sqrt{\frac{N}{N+4}} \sum_{j=-3}^{N} Q_-, j.$$ 

The terms with $j = -3, -2, \ldots, 0$ take care of the insertion process near the “boundary” (near sites $N$ and 1) and need to be present as the four new sites may have $N+4$ different locations on a chain with $N+4$ sites. Similarly $\bar{Q}_- = Q_+^\dagger$ is defined as

$$\bar{Q}_- = \sqrt{\frac{N}{N-4}} \sum_{j=1}^{N-4} \bar{Q}_-, j.$$ 

The application of the spin-reversal rules to the elementary processes induced by the supercharge $Q_+$ leads to the following actions on basis vectors:

$$Q_{-,x}| \cdots \circ \circ \cdots \rangle = (\pm) \left( a_x | \cdots \circ \circ \circ \cdots \rangle + b_x | \cdots \circ \circ \circ \cdots \rangle \right),$$

$$Q_{-,x}| \cdots \circ \circ \cdots \rangle = (\pm) c_x | \cdots \circ \circ \circ \cdots \rangle,$$

$$Q_{-,x}| \cdots \circ \circ \cdots \rangle = 0.$$ 

The action of $Q_{-,x}$ on configurations which contain a particle on site $x$ is always zero. The $\pm$ sign corresponds to the fermionic string, it is given by minus one to the number of particles located to the left of $x$.

The requirement that $Q_-^2 = 0$ can be analysed locally by acting with the supercharge twice on the simple configurations shown in (3.5). A detailed analysis shows the only non-vanishing terms come from the first configuration. Imposing that this term vanishes results in the constraint

$$a_x c_{x+3} + b_x c_x = 0.$$ 

This is a clear analogue of equation (2.8), which constrains the parameters in the definition of the supercharge $Q_+$. After having imposed this first restriction, we would like to establish $Q_-$ as a symmetry of our model. Hence we look for values of the parameters $a_x, b_x, c_x$ such that locally it generates the same Hamiltonian as $Q_+$ (i.e. the hopping amplitudes and rules for the potential energy are the same):

$$H = \{Q_-, \bar{Q}_-\}$$

where $\bar{Q}_- = Q_+^\dagger$. The explicit comparison of the two sides is a cumbersome task. It leads to a system of quadratic difference equations for the parameters $a_x, b_x, c_x$ which allow to express them in terms of $\lambda_x, \mu_x$. Out of the many equations, let us just write

$$a_{x+2}^2 = a_x^2, \quad b_{x+2}^2 = b_x^2, \quad c_x^2 = a_x^2 + b_x^2 + 1.$$
which show that once more a staggering with period 2 is obtained. We skip the
details, and report only the solution to the complete set of difference equations
which is given by
\[ a_x = \mu_x \lambda_{x+1}, \quad b_x = -\mu_x \lambda_x, \quad c_x = \lambda_{x+1}. \] (3.8)

Its insertion into (3.7) implies that for real positive \( \lambda_x, \mu_x \) the parameters of the
model lie on the line (3.1).

We note that starting from the definition
\[ Q_\pi = SQ_\pi S, \] (3.9)
where \( S \) denotes the spin-reversal operation, does not give the amplitudes above
in the case that \( \mu_x \neq \mu_{x+1} \). One would instead get (by just a simple calculation)
the amplitudes
\[ \tilde{a}_x = \mu_{x+1} \lambda_{x+1}, \quad \tilde{b}_x = -\mu_{x+1} \lambda_x, \quad \tilde{c}_x = \lambda_{x+1} \] (3.10)
which we denote by a tilde to distinguish them from the ones above. The difference
with eq. (3.8) is that \( \mu_x \) is shifted by one site. The reason for this is exactly the
shift that we noted in section 3.2 and the amplitudes \( \tilde{a}, \tilde{b}, \tilde{c} \) do therefore not make
the anti-commutator \( \{ Q_\pi, \bar{Q}_\pi \} \) locally equal to the Hamiltonian. In the case
\( \mu_x = \mu_{x+1} \) the two prescriptions are equal, this is what we will use in chapter 7.

**Translation invariance.** So far we ignored the boundary terms \( Q_{-,x} \) with
\( x = -3, \ldots, 0 \), because all the constraints on the parameters in the definition of
\( Q_- \) could be derived from local considerations in the bulk. We will now consider
the boundary terms and find that it leads to a restriction on the Hilbert space.
That is, we find that the supersymmetry \( Q_- \) generates the Hamiltonian only
in a subsector of the Hilbert space. This feature is quite different from the supersymmetry generated by \( Q_+ \) which exists on the entire Hilbert space.

To see this we first define the boundary terms by taking advantage of the
periodicity of the staggering parameters. To this end, we use the translation
operator \( T \). Clearly, the staggering with period 2 implies that, for the local
operators in the bulk,
\[ T^2 Q_{-,x} T^{-2} = Q_{-,x+2}, \quad x = 1, \ldots, N - 2. \] (3.11)

Note that the translation operator on the left (right) of \( Q_{-,x} \) acts on states of a
chain of length \( N + 4 \) (\( N \)). We now use this equation to define the local operators,
\( Q_{-,x} \), for \( x = -3, \ldots, 0 \) and thus extend (3.11) to \( x = -3, \ldots, 0 \). This definition
now also implies that for \( x = -3, \ldots, 0 \) we have \( Q_{-,x+N} = T^N Q_{-,x} T^{-N} \). Using
\( T^{-N} = 1 \) when it acts on a chain of length \( N \) and similarly \( T^N = T^{-4} \) for a chain
of length \( N + 4 \), we can rewrite this as
\[ T^4 Q_{-,N+x} = Q_{-,x}, \quad x = -3, \ldots, 0. \]
3.3 Dynamical supersymmetries of the $M_2$ model

From this relation we conclude, in particular, that $Q_-$ is a well-defined mapping between eigenspaces of $T^4$ only when $T^4 \equiv 1$ (one easily checks this by computing $T^4 Q_- T^{-4}$, using the relations above and imposing it to be equal to $Q_-$). Since the Hamiltonian $H = \{Q_+, \bar{Q}_+\}$ of our model commutes with $T^4$, we conclude that $H = \{Q_-, \bar{Q}_-\}$ can only hold on subspaces where the translation operator by four sites acts like the identity. Finally, let us mention that as for the non-dynamic supercharge $Q_+$ one may show that the construction of $Q_-$ only works for non-zero twist angles.

**Extended supersymmetry algebra.** As we have two copies of the $\mathcal{N} = 2$ supersymmetry algebra in the case of periodic boundary conditions, it appears natural to find two distinct fermion numbers in order to characterise the system. We define the following two operators in terms of the system size $N$ and the number of fermions on the lattice $f$:

$$F_V = N - f, \quad \text{and} \quad F = f - N/2.$$  

Using the mapping to the spin chain, $F_V$ and $F$ correspond to the length of the spin chain and its magnetisation, respectively. With these two fermion numbers, we obtain a lattice representation of the $\mathcal{N} = (2,2)$ supersymmetry algebra [32] (up to a sign convention for $F_V$) with vanishing central charges

$$Q^2_\pm = \bar{Q}^2_\pm = 0, \quad \{Q_\pm, Q_\mp\} = \{Q_\pm, \bar{Q}_\mp\} = 0,$$

$$\{Q_\pm, \bar{Q}_\pm\} = H \pm P,$$

$$[F_V, Q_\pm] = Q_\pm, \quad [F_V, \bar{Q}_\pm] = -\bar{Q}_\pm,$$

$$[F, Q_\pm] = \mp Q_\pm, \quad [F, \bar{Q}_\pm] = \pm \bar{Q}_\pm,$$

and zero momentum

$$P = 0.$$  

For these commutators and anti-commutators, the action on chains of appropriate length is implied. Moreover, the algebra exists of course only in translation subsectors with $T^4 = 1$, where the presence of the dynamic supersymmetry is guaranteed. The fact that the momentum $P$ is zero is consistent with this restriction as we shall see in chapter 4 where the relation between the lattice model and its field-theory limit is discussed.

3.3.2 Dynamic supersymmetry $Q_0$

The dynamic supercharge defined in the last section increases the length of the chain by four sites. A priori, the periodicity of the staggering parameters does not exclude the existence of another length-changing operator which adds only two sites to the system. We will show here that such an operator $Q_0$ exists indeed in certain subsectors of the Hilbert space. The existence of such a symmetry is somewhat expected. In [33] it was shown that the Fateev-Zamolodchikov chain possesses a dynamic supersymmetry of the same structure as the one constructed below.
Supercharge and Hamiltonian. Let us show how the operator \( Q_0 : \mathcal{H}_{N,f} \to \mathcal{H}_{N+2,f+1} \), which inserts two sites and one particle, is constructed. As for the dynamic supercharge \( Q_- \), this operator can be written as a sum over local operators
\[
Q_0 = \sqrt{\frac{N}{N+2}} \sum_{x=-1}^{N} (-1)^x Q_{0,x}.
\]
(3.13)

Unlike for \( Q_- \) there is an additional site-dependent string \((-1)^x\): it implies that in the homogeneous limit \( Q_0 \) inserts a particle with momentum \( \pi \) into the system. We will see later that this picture harmonises well with the Bethe ansatz interpretation of this supersymmetry. The operator \((Q_0)^\dagger = \bar{Q}_0\) is only defined on \( N-2 \) sites
\[
\bar{Q}_0 = \sqrt{\frac{N}{N-2}} \sum_{x=1}^{N-2} (-1)^x \bar{Q}_{0,x}.
\]
(3.14)

The \( Q_{0,x} \) are fermionic operators which insert two sites and one particle. Like for \( Q_-\), their action is non-vanishing only if the site \( x \) is empty. In this case, the action depends on the occupation of the subsequent sites. Let us illustrate the three different possible scenarios by the action on basis vectors for \( x = 1, \ldots, N \):
\[
\begin{align*}
Q_{0,x}|\cdots \circ \cdots \rangle & = (\pm) \left( \alpha_x |\cdots \circ \circ \cdots \rangle + \beta_x |\cdots \circ \circ \circ \cdots \rangle \right) \\
Q_{0,x}|\cdots \circ \circ \cdots \rangle & = (\pm) \left( \gamma_x |\cdots \circ \circ \circ \cdots \rangle + \delta_x |\cdots \circ \circ \circ \cdots \rangle \right) \\
Q_{0,x}|\cdots \circ \circ \circ \cdots \rangle & = (\pm) \left( \epsilon_x |\cdots \circ \circ \circ \cdots \rangle + \eta_x |\cdots \circ \circ \circ \circ \cdots \rangle \right)
\end{align*}
\]
(3.15)

The sign \( \pm \) represents the fermionic string, it is given by \(-1\) to the number of fermions located to the left of \( x \) for \( x > 1 \), and if \( x = 1 \) it is given by \(+1\).

The site-dependent parameters, \( \alpha_x, \beta_x, \ldots, \eta_x \), are constrained by the requirement \( Q_0^2 = 0 \). This can be done locally, by acting twice on the three configurations which we use in order to define the action of \( Q_{0,x} \), and imposing that the result vanishes locally up to boundary terms. An explicit calculation shows then that this is possible if and only if all parameters are periodic in \( x \) with period 2. Furthermore, it leads to
\[
\beta_x = -\alpha_{x+1}, \quad \eta_x = \alpha_{x+1}, \quad \epsilon_x = -\alpha_x
\]
(3.16)

and to the relation
\[
\alpha_x \delta_x + \alpha_{x+1} \gamma_{x+1} = 0,
\]
(3.17)

which can be thought of as an analogue of (2.8). The periodicity of the parameters leads to the following relation
\[
T^2 Q_{0,x} T^{-2} = Q_{0,x+2}, \quad x = 1, \ldots, N-2.
\]
The remaining operators $Q_{0,-1}, Q_{0,0}$ which take into account the insertion of a pair of sites between $x = N$ and $x = 1$ are defined by extending this relation to $x = -1, 0$. Using a similar reasoning as for the dynamic supercharge $Q_-$ presented above, we find that they can be equivalently expressed as

$$T^2 Q_{0,x+N} = Q_{0,x}, \quad x = -1, 0.$$  

In complete analogy with the case of $Q_-$ we conclude from this equation that the supercharge $Q_0$ is a well-defined mapping between translation sectors only if $T^2 \equiv 1$. In these subsectors it generates a supersymmetric Hamiltonian

$$H = \{Q_0, \bar{Q}_0\}. \quad (3.18)$$

Let us count how many remaining parameters there are at this stage: we are left with $\alpha_x, \gamma_x, \delta_x$ which are periodic under $x \to x + 2$, and subject to (3.17). Taking into account that we are free to rescale them, we find thus three free parameters. Their number is furthermore reduced if we impose that the Hamiltonian $H$ coincide with the one for the $M_2$ model. Quite interestingly, it implies also a restriction to the line of couplings (3.1), and the 2-periodicity for $\lambda_x, \mu_x$. Indeed, the analysis of the potential energies on both sides of (3.18) shows that equality can hold only if

$$\gamma_x^2 + \delta_x^2 = 4\alpha_x^2 + 1. \quad (3.19)$$

Adjusting the hopping terms one finds that the parameters are related to the original staggering parameters according to

$$\alpha_x = \lambda_x, \quad \beta_x = -\frac{\lambda_{x+1}}{2}, \quad \gamma_x = \lambda_{x+1}\mu_{x+1},$$

$$\epsilon_x = -\frac{\lambda_x}{2}, \quad \eta_x = \frac{\lambda_{x+1}}{2}, \quad \delta_x = -\lambda_{x+1}\mu_x. \quad (3.20)$$

This implies periodicity, and via reinsertion into (3.19) the special line (3.1). Furthermore, we checked explicitly that in the subsectors where $T^2 \equiv 1$, the dynamic supercharge $Q_0$ anti-commutes with the other ones:

$$\{Q_0, Q_\pm\} = 0, \quad \{\bar{Q}_0, Q_\pm\} = 0.$$  

### 3.4 The coordinate Bethe ansatz for the staggered $M_2$ model

In this section, we show that the Hamiltonian of the $M_2$ model can be diagonalised by means of the coordinate Bethe ansatz along the two-parameter submanifold in the space of staggering parameters $\mu_x^2 + \mu_{x+1}^2 = 1$ which coincides precisely with the submanifold with enhanced supersymmetry identified in the previous section.

The coordinate Bethe ansatz was introduced by Hans Bethe [34]. He found the exact eigenstates and energies of the one-dimensional spin-$\frac{1}{2}$ Heisenberg
model. The technique employed here is a combination of the Bethe ansatz for
the staggered $M_1$ model found by Nienhuis and Blom [20], the coordinate Bethe
ansatz for higher spin XXX chains [35], and an asymptotic analysis which allows
an easy determination of the integrable manifold in the space of parameters.
We start by specifying the basis, and Bethe ansatz form for the wave function
in section 3.4.1. In section 3.4.2 we analyse the one-particle problem: it is not
directly solvable for general staggering, but will allow to determine some useful
asymptotic expansions of the single-particle wave function. The two-particle
problem is solved in section 3.4.3. We show that it fixes the choice of admissible
staggering parameters as well as the period of the staggering to the special
submanifold. Furthermore, we introduce a useful elliptic parametrisation for it.
The many-particle case is considered afterwards, and leads to the Bethe ansatz
equations.

3.4.1 Basis vectors and Bethe ansatz form of the wave function

Basis vectors. Our principal goal is to diagonalise the Hamiltonian $H$, i.e. to
solve the Schrödinger equation

$$H|\psi\rangle = E|\psi\rangle. \quad (3.21)$$

To this end we need to choose a suitable basis in the fermion Hilbert space which
simplifies our problem as much as possible. The most natural choice appears
to be the canonical occupation number basis: a basis vector is labelled by the
positions of the particles in a given configurations:

$$|x_1, \ldots, x_f\rangle = \text{\(\cdots\)} \circ \cdots \bullet \circ \cdots \circ \circ \cdots \circ, \quad (3.22)$$

This basis is orthonormal which is convenient for many applications. However,
the Bethe ansatz is more conveniently formulated when using basis vectors $|\{x_1, \ldots, x_f\}\rangle$ which differ from the canonical ones by configuration-dependent
factors. Let us denote by $x'_1, x'_2, \ldots$ the positions of the first members of pairs in
the configuration $x_1, x_2, \ldots, x_f$. We introduce the non-orthonormal basis

$$|\{x_1, x_2, \ldots, x_f\}\rangle = \left(\prod_j C_{x'_j}\right) |x_1, x_2, \ldots, x_f\rangle \quad (3.22)$$

with normalisation factors $C_x$ to be determined. This modified basis is similar to
the basis used in the coordinate Bethe ansatz solution for the higher-spin XXX
chains studied in [35]. It allows to absorb a trivial part of the wave function into
the basis itself.

Bethe ansatz. The Hamiltonian of our model, twisted or not, commutes
obviously with the fermion number operator $F$, and can therefore be diagonalised
3.4 The coordinate Bethe ansatz for the staggered $M_2$ model

separately in each subsector $\mathcal{H}_{N,f}$. We expand its eigenstates in $\mathcal{H}_{N,f}$ in the modified basis

$$|\psi\rangle = \sum_{\{x\}} \psi(x_1, x_2, \ldots, x_f)|x_1, x_2, \ldots, x_f\rangle. \quad (3.23)$$

Here, the sum is taken over all positions $1 \leq x_1 < x_2 < \cdots < x_f \leq N$ of the particles which respect the exclusion constraint of the $M_2$ model, and the boundary conditions. For the wave function $\psi(x_1, \ldots, x_f)$ we make the Bethe ansatz by writing it as a linear combination of products of single-particle wave functions $\varphi(x; z)$:

$$\psi(x_1, \ldots, x_f) = \sum_{\sigma \in S_f} B_\sigma \varphi(x_1; z_{\sigma(1)}) \cdots \varphi(x_f; z_{\sigma(f)}). \quad (3.24)$$

The sum is over all permutations $\sigma$ of $f$ objects, weighted by certain amplitudes $B_\sigma$. The variables $z_1, \ldots, z_f$ are the rapidities of the particles. In order to give meaning to these rapidity variables we need to specify the structure of the single-particle wave functions. In fact, we shall assume that the weights of the model are periodic: $\lambda_{x+p} = \lambda_x$, $\mu_{x+p} = \mu_x$. Hence, single particles are described by Bloch wave functions

$$\varphi(x; z) = A_x(z) z^x, \quad A_{x+p}(z) = A_x(z). \quad (3.25)$$

Our main objective here is to show that the model is Bethe ansatz solvable if and only if $p = 2$, and the staggering parameters are chosen from the submanifold of equation (3.1).

Given the basis and Bethe ansatz form of the wave function we proceed now through a series of standard steps for the solution of the Schrödinger equation (3.21). We project this equation on simple basis vectors, and resolve the resulting system of difference equations for the wave functions $\psi(x_1, \ldots, x_f)$. In the following sections, we address first the case of $f = 1$ and 2 particles, mention briefly the case of $f = 3$ and 4 particles, and deduce the result for general $f$ through a standard argument.

3.4.2 The one-particle problem

In the subsector of the Hilbert space where no particles are present the diagonalisation of the Hamiltonian is trivial: the empty state $|\cdots\rangle$ is an eigenvector of $H$ with eigenvalue $E = \sum_{x=1}^{N} \lambda_x^2$. It serves as a reference state for the Bethe ansatz and will allow to build eigenstates with non-zero fermion numbers. Let us start with a single particle $f = 1$. We write thus the eigenvalue as

$$E = \sum_{x=1}^{N} \lambda_x^2 + \epsilon(z),$$
where \( \epsilon(z) \) denotes the excitation energy for a (pseudo-)particle with rapidity \( z \) above the reference-state level. The corresponding Bethe ansatz wave function is in fact simply given by \( \varphi(x; z) \). Using its Bloch-wave structure (3.25) we find the difference equation

\[
(\epsilon(z) + \lambda_{x-1}^2(1 - \mu_{x-1}^2) + \lambda_{x+1}^2(1 - \mu_{x+1}^2))A_x(z) = \lambda_x\lambda_{x+1}(1 - \mu_x^2)A_{x+1}(z)z + \lambda_x\lambda_{x-1}(1 - \mu_{x-1}^2)A_{x-1}(z)z^{-1}
\]

Given the periodicity in \( x \), we conclude that this leads to a system of \( p \) homogeneous linear equations for the quantities \( A_1(z), \ldots, A_p(z) \). In order to have a non-trivial solution its coefficient matrix needs to have zero determinant. This leads to a polynomial equation of order \( p \) for the excitation energy, and determines the dispersion relation, \( \epsilon = \epsilon(z) \). Without knowing \( p \) it is not very useful to write down this system and its solution explicitly. Instead, we will analyse it in the formal limit where \( z \rightarrow 0 \), and \( z \rightarrow \infty \), in order to obtain an asymptotic expansion for \( \epsilon(z) \) and the amplitude ratio

\[
f_x(z) = \frac{\lambda_x A_{x+1}(z)z}{\lambda_{x+1}A_x(z)}.
\]

Let us start with large rapidity. It is clear from (3.26) that both the excitation energy, and the amplitude ratio diverge linearly for large \( z \). Writing

\[
\epsilon(z) = \gamma_1z + \gamma_0 + O(z^{-1}), \quad f_x(z) = \alpha_xz(1 + \beta_xz^{-1} + O(z^{-2}))
\]

we find that the coefficients \( \alpha_x, \beta_x \) are given by

\[
\alpha_x = \frac{\gamma_1 - \lambda_{x+1}^2(1 - \mu_{x+1}^2)}{\lambda_{x+1}^2(1 - \mu_{x+1}^2)} , \quad \beta_x = \frac{\gamma_0 + \lambda_{x+1}^2(1 - \mu_{x+1}^2) + \lambda_{x-1}^2(1 - \mu_{x-1}^2)}{\gamma_1}.
\]

Here the constants \( \gamma_1, \gamma_0 \) may in principle be determined from the periodicity \( A_{x+p}(z) = A_x(z) \) which leads to \( f_1(z)f_2(z) \cdots f_p(z) = 1 \), but we will not need their explicit form. The limit of small rapidity leads to similar results. We find that the excitation energy and amplitude ratio have the expansions

\[
\epsilon(z) = \delta_1z^{-1} + \delta_0 + O(z), \quad f_x(z) = \rho_xz(1 + \eta_xz + O(z^2))
\]

where the coefficients \( \rho_x, \beta_x \) are given by

\[
\rho_x = \frac{\lambda_x^2(1 - \mu_x^2)}{\delta_1}, \quad \eta_x = -\frac{\delta_0 + \lambda_x^2(1 - \mu_x^2) + \lambda_{x+2}^2(1 - \mu_{x+2}^2)}{\gamma_1}.
\]

It would be well justified to question the use of these expansions at this point. The idea is the following. For the two-particle problem we will derive the \( S \)-matrix of the model as a complicated combination of the amplitude ratios \( f_j(z_j), j = 1, 2 \). The expression carries an \( x \)-dependence which should however be spurious. The formal limit where one of the rapidities tends to zero or infinity allows to understand the (pole) structure of the \( S \)-matrix, and determine conditions on the staggering parameters which yield a position-independent \( S \)-matrix. The expressions derived above will be instrumental in this procedure.
3.4.3 The two-particle problem and the S-matrix

Next, we consider the case of two particles \( f = 2 \). As long as we project the Schrödinger equation on configurations where the two particles are far apart (and far from the boundaries), the Bethe ansatz wave function (3.24) solves the resulting difference equation with the eigenvalue

\[
E = \sum_{x=1}^{N} \lambda_x^2 + \epsilon(z_1) + \epsilon(z_2). \tag{3.26}
\]

If, however, the particles are next-to-nearest or nearest neighbours, we have to take into account that the action of the Hamiltonian induces pair formation, pair splitting, and pair hopping.

Next-to-nearest neighbours

We start with the projection of the Schrödinger equation on a configuration where two particles are next-to-nearest neighbours \( \cdots \circ \circ \circ \circ \circ \cdots \), i.e. on some basis vector \(|x, x+2\rangle\). For this case we find the rather long equation

\[
E\psi(x, x+2) = \lambda_x \lambda_{x-1} (1 - \mu_x^2) \psi(x-1, x+2) \\
+ \lambda_{x+2} \lambda_{x+3} (1 - \mu_{x+2}^2) \psi(x, x+3) \\
+ \lambda_x \lambda_{x+1} \mu_{x+1} C_{x+1} \psi(x+1, x+2) \\
+ \lambda_{x+1} \lambda_{x+2} \mu_x C_x \psi(x, x+1) \\
+ \left( \sum_{y=1}^{N} \lambda_y^2 - \lambda_{x-1}^2 (1 - \mu_{x-1}^2) - \lambda_{x+1}^2 - \lambda_{x+3}^2 (1 - \mu_{x+2}^2) \right) \psi(x, x+2).
\]

If the Bethe ansatz holds together with the form of the eigenvalue as written in (3.26) then each of the arguments of the wave function can formally be treated as for isolated particles. This leads to a second eigenvalue equation. Equating the two expressions we find the difference equation

\[
\lambda_x \lambda_{x+1} (\mu_{x+1} C_{x+1} + \mu_x^2 - 1) \psi(x+1, x+2) \\
+ \lambda_{x+1} \lambda_{x+2} (\mu_x C_x + \mu_{x+1}^2 - 1) \psi(x+1, x+2) \\
+ \lambda_{x+1}^2 (1 - \mu_x^2 - \mu_{x+1}^2) \psi(x, x+2) = 0.
\]

Given this equation there are two ways to proceed. We could impose the equation as a constraint on the wave function, leaving the normalisation factors \( C_x \) undetermined, but fixing the structure of the \( S \)-matrix. However, one can show that this leads to a contradiction for other particle arrangements. Therefore, we will instead require that the coefficients multiplying the wave functions in this equation be identically zero, so that the equation does not lead to any constraints for the \( S \)-matrix [35]. In the present case, the only non-trivial solution is

\[
C_x = \mu_x, \quad \text{and} \quad \mu_x^2 + \mu_{x+1}^2 = 1, \tag{3.27}
\]
for arbitrary $x$. We conclude that our requirement implies the 2-periodicity of the staggering parameters $\mu_x$ while it does not fix the $\lambda_x$. As we shall see, the latter will be constrained by the nearest-neighbour problem.

**Nearest neighbours**

Next, we project the Schrödinger equation for two particles onto a configuration with a single pair $\cdots \circ \bullet \circ \circ \cdots$, i.e. on the basis vector $|x, x+1\rangle$. We follow the same procedure as in the last section and find the following difference equation for the wave function

$$
\lambda_{x-1} \psi(x - 1, x) - \lambda_x \psi(x - 1, x + 1) + \lambda_{x-1} \psi(x, x + 1) + \lambda_x \lambda_{x+1} \psi(x, x) + \lambda_{x+2} (\lambda_{x+2} \psi(x, x + 1) - \lambda_{x+1} \psi(x, x + 2) + \lambda_x \psi(x + 1, x + 2)) + \lambda_x \lambda_{x+1} \psi(x + 1, x + 1) = 0.
$$

(3.28)

Unlike in the case of next-to-nearest neighbours this equation cannot vanish identically for non-trivial choices of the staggering parameters. Hence it will lead to constraints on the parameters in the Bethe ansatz wave function. Indeed, using (3.24) we find that the two amplitudes $B_{12}$ and $B_{21}$ are related by

$$
B_{12} P_x(z, w) + B_{21} P_x(w, z) = 0,
$$

(3.29)

where $P_x(z, w)$ denotes the complicated expression

$$
P_x(z, w) = \lambda^2_{x-1} \left( \frac{1 - f_x(w)}{f_{x-1}(z)} + f_x(w) \right) + \lambda^2_x + \lambda^2_{x+1} f_x(z) f_x(w) + \lambda^2_{x+2} (f_x(w)f_{x+1}(w)(f_x(z) - 1) + f_x(w)).
$$

Let us suppose that $z$ and $w$ are such that both $P_x(z, w)$ and $P_x(w, z)$ are non-vanishing for all $x$. In this case we find the $S$-matrix of the model

$$
S(z, w) = \frac{B_{12}}{B_{21}} = - \frac{P_x(w, z)}{P_x(z, w)}.
$$

The fact that this expression is rather implicit, since we have not yet found a general expression for $f_x(z)$, is perhaps less dramatic than its $x$-dependence. Indeed, the Bethe ansatz assumes that the amplitudes $B_\sigma$ are position-independent. For generic choices of the staggering parameters the formula for $S(z, w)$ leads, however, to an $x$-dependent expression. It follows that the only possible choice for the model to be Bethe ansatz solvable is to impose the site-independence for arbitrary $z, w$. One may try to do this by working directly with the given expression, but the resulting equations are quite involved. Hence, we choose to take advantage of our asymptotic expansions for $f_x(z)$ derived in section 3.4.2, and analyse the limit $z \to \infty$. We find

$$
S(z, w) = z \sigma(w) + O(1), \quad \text{with} \quad \sigma(w) = \frac{\gamma^{-1}(1 - f_x(w))}{\mu^2 f_x(w)(\lambda^2_{x+1} + \lambda^2_{x+2} f_{x+1}(w))}.
$$
3.4 The coordinate Bethe ansatz for the staggered $M_2$ model

Like $S(z, w)$ the function $\sigma(w)$ has to be position-independent. Finding which staggering parameters lead to this independence is still a delicate task, and hence we analyse again only the leading terms of the expansion of $\sigma(w)$ as $w \to 0$:

$$\sigma(w) = \frac{\delta_{-1}}{\mu_x^2 \mu_{x+1}^2 \lambda_x^2 \lambda_{x+1}^2} \left( \delta_{-1} w^{-1} - \delta_0 + O(w) \right).$$

Every term in this expansion needs to be independent of $x$. Using the 2-periodicity for $\mu_x$, we see that this can at leading order only be true if the product $\lambda_x^2 \lambda_{x+1}^2$ is independent of $x$. This condition implies trivially that for real positive staggering parameters we have

$$\lambda_{x+2} = \lambda_x.$$

Hence, we find that the expression for the $S$-matrix found above is independent of the position $x$ only if the staggering parameters are 2-periodic, and satisfy (3.1), which confirms that the latter is a necessary condition for Bethe ansatz solvability.

**Reduced equation.** If all staggering parameters have period two, then the equation for the wave function can be simplified. Indeed, notice that for $\lambda_{x+2} = \lambda_x$ the first four terms equal the last four terms of the left-hand side of (3.28) up to a shift $x \to x + 1$. We may formalise this by introducing the shift operator $T$ defined through $T f(x) = f(x + 1)$. Then we find

$$(1 + T) \left[ \lambda_{x-1} (\lambda_{x+1} \psi(x - 1, x) - \lambda_x \psi(x - 1, x + 1) + \lambda_{x-1} \psi(x, x + 1)) + \lambda_x \lambda_{x+1} \psi(x, x) \right] = 0$$

Hence the expression within brackets lies in the kernel of $(1 + T)$, i.e. it is of the form $(-1)^x \times \text{const}$. In fact, one can show that this constant needs to be zero (we omit this tedious and not very illuminating discussion here). Using this result we find the reduced equation

$$\lambda_{x+1} (\psi(x - 1, x) + \psi(x, x + 1)) + \lambda_x (\psi(x, x) - \psi(x - 1, x + 1)) = 0. \quad (3.30)$$

This relation has two advantages. First of all, it allows to write a somewhat simpler version of (3.29). We find that

$$B_{12} R_x(z, w) + B_{21} R_x(w, z) = 0 \quad (3.31a)$$

where $R_x(z, w)$ is given by the expression

$$R_x(z, w) = 1 + \frac{\lambda_{x+1}^2}{\lambda_x^2} \left( \frac{1 - f_x(w)}{f_{x-1}(z)} + f_x(w) \right). \quad (3.31b)$$

This leads to a simplified expression for the $S$-matrix, and will be used below to derive a closed expression for it in terms of Jacobi theta functions. Second, (3.30) proves to be quite useful in order to show that processes involving three and four particles are indeed coherent.
Elliptic parametrisation

In the previous section, we saw that it is necessary to restrict the parameters of the model to the special submanifold. The aim of this and the following section is to show that this restriction is also sufficient for the model to be Bethe ansatz solvable. To this end, it is convenient to re-examine the one-particle problem, and determine an explicit parametrisation for the rapidities and the excitation energy, which will lead to an explicit and simple form for the $S$-matrix.

Staggering parameters. We need a suitable parametrisation of the parameters $\lambda_x$ and $\mu_x$. It turns out that a convenient choice is to write them in terms of Jacobi theta functions

$$\vartheta_j(u) = \vartheta_j(u, q), j = 1, \ldots, 4$$

where $q$ is the so-called elliptic nome. We follow the conventions of Whittaker and Watson [36]. In fact, it is sufficient to define

$$\vartheta_1(u, q) = -i \sum_{j=-\infty}^{\infty} (-1)^j q^{(j+1/2)^2} e^{(2j+1)iu}.$$

The other theta functions are obtained by shifting the argument by $\pi/2$ and $\pi \tau /2$ and $\pi / 2 + \pi \tau /2$ where $\tau$ is related to the elliptic nome by $q = e^{i \pi \tau}$. For instance, $\vartheta_4(u) = iq^{1/4} e^{-iu} \vartheta_1(u - \pi \tau /2)$. These functions can be thought of as generalisations of the trigonometric or the exponential functions. They satisfy a host of identities, in particular various addition theorems which are at the heart of the simplifications in the following.

The staggering parameters are given as functions of two real parameters: $t$ and the elliptic nome $0 \leq q < 1$. In terms of theta functions they read

$$\mu^2_x = \left( \frac{\vartheta_1(\theta)}{\vartheta_1(2\theta)} \right)^2 \frac{\vartheta_4(t + 2x\theta)^2}{\vartheta_4(t + (2x - 1)\theta)\vartheta_4(t + (2x + 1)\theta)}, \quad \theta = \frac{\pi}{4},$$

and

$$\lambda^2_x = 2 \left( \frac{\vartheta_1(\theta)}{\vartheta_1(2\theta)} \right)^2 \frac{\vartheta_4(t + (2x - 1)\theta)^2}{\vartheta_4(t + 2(x - 1)\theta)\vartheta_4(t + 2x\theta)}, \quad \theta = \frac{\pi}{4}.$$  

One may check that this choice is compatible with both periodicity, and the equation $\mu^2_x + \mu^2_{x+1} = 1$. The second equation implies that $\lambda^2_x + \lambda^2_{x+1} = 2$, a normalisation which we are free to choose. The latter is designed to recover $\lambda_x = 1$ and $\mu_x = 1/\sqrt{2}$ in the trigonometric limit where the elliptic module $q$ tends to zero, and the Hamiltonian becomes translation invariant. For non-zero $q$, we note that the transformation $t \to t + 2\theta$ is equivalent to a shift $x \to x + 1$, and thus a translation of the system by one site. The elliptic parametrisation given here can be justified and derived in a systematic analysis of the $M_k$ models for all $k = 1, 2, 3, \ldots$ [14]. In figure 3.2 we plot the parameters as a function of $t$ with $0 \leq t \leq 2\theta$ for $q = 0, 0.05, \ldots, 0.5$. It is clear that this parametrisation maps out the special submanifold.
3.4 The coordinate Bethe ansatz for the staggered $M_2$ model

**Figure 3.2:** We plot the parameters $\mu_x, \mu_{x+1}$ and $\lambda = \lambda_x / \lambda_{x+1}$ as a function of $t$ with $0 \leq t \leq 2\theta$ for $q = 0, 0.05, \ldots, 0.5$. The black dot corresponds to $q = 0$, the outer red line corresponds to $q = 0.5$. Finally, we also show the special submanifold given by (3.1), where the model is integrable and enjoys additional supersymmetry. The constant $q$ lines lie on the special submanifold.

**Excitation energy, rapidity and one-particle wave function.** Now let us use this parametrisation in order to find convenient expressions for the excitation energy $\epsilon(z)$. As we know, it is determined by the homogeneous linear $p \times p$ system (3.26). The period $p = 2$ of the staggering parameters implies that the periodic part of the single-particle Bloch wave function satisfies $A_{x+2}(z) = A_x(z)$. We may use this in the recursion relation (3.26) which becomes of first order (as opposed to second order for arbitrary $p > 2$):

$$(\epsilon(z) + \lambda_{x+1}^2)A_x(z) = \lambda_x \lambda_{x+1} \left( \mu_{x+1}^2 z + \mu_x^2 z^{-1} \right) A_{x+1}(z)$$

Applying the periodicity property after shifting $x \to x + 1$, we obtain that $\epsilon(z)$ solves the second-order polynomial equation

$$\epsilon(z)(\epsilon(z) + 2) = \Lambda^2 (z - z^{-1})^2, \quad \Lambda = 2 \left( \frac{\vartheta_1(\theta)}{\vartheta_1(2\theta)} \right)^4 .$$

Notice that this equation is independent of the parameter $t$. As we shall see the spectrum of the Hamiltonian is $t$-independent as a consequence. Solving this
equation for \( \epsilon(z) \) leads to roots of quartic polynomials in \( z \) which are neither elegant nor useful. Instead, we will uniformise this equation through the introduction of theta-function parametrisations of the rapidities. One checks that the choice

\[
z(u) = -\frac{\vartheta_1(u + \theta)}{\vartheta_1(u - \theta)}
\]

leads to

\[
\epsilon(u) = \epsilon(z(u)) = -2 \left( \frac{\vartheta_1(\theta)}{\vartheta_1(2\theta)} \right)^2 \frac{\vartheta_1(u)^2}{\vartheta_1(u - \theta)\vartheta_1(u + \theta)}.
\]

The main tools in all these calculations are the addition theorems for the Jacobi theta functions mentioned above. Using these two relations, one may determine the functions \( A_x(z) \) as a function of the parameter \( u \) from the recursion relation given above up to an overall factor. We fix the latter by the requirement \( A_x(z = 1) = \lambda_x \). This gives

\[
A_x(u) = A_x(z(u)) = \frac{\lambda_x \vartheta_4(t - u + (2x - 1)\theta)}{\vartheta_4(t + (2x - 1)\theta)}.
\]

The S-matrix. We are now in the position to derive an explicit expression for the S-matrix in terms of Jacobi theta functions. To this end, we use (3.31), and express all the amplitude ratios \( f_x(z) \) in terms of the expressions given in the last paragraph. We find that

\[
R_x(z(u), z(v)) = -\frac{\vartheta_1(2\theta)^2 \vartheta_4(t + 2x\theta) \vartheta_4(t + 2(x - 1)\theta) \vartheta_4(t - (u + v) + (2x - 1)\theta) \vartheta_4(t - (u + v) + (2x - 1)\theta)}{\vartheta_1(\theta) \vartheta_4(t + (2x - 1)\theta) \vartheta_4(t - u + (2x - 1)\theta) \vartheta_4(t - v + (2x - 1)\theta)} r(u, v)
\]

where \( r(u, v) \) has the simple form

\[
r(u, v) = \frac{\vartheta_1(u - v + \theta)}{\vartheta_1(u + \theta) \vartheta_1(v - \theta)}.
\]

Notice that the \( x \)-dependent part is symmetric in \( u, v \), and that \( r(u, v) \) does not depend on the parameter \( t \). This implies that we are left with the rather simple equation \( B_{12} r(u, v) + B_{21} r(v, u) = 0 \), and hence find the S-matrix

\[
S(u, v) = S(z(u), z(v)) = \frac{z(u) \vartheta_1(u - v - \theta)}{z(v) \vartheta_1(u - v + \theta)}, \quad \theta = \frac{\pi}{4}.
\]

This expression does not have the difference property, i.e. it does not depend on \( u, v \) only through the difference \( u - v \). The reason for this is the exclusion constraint of the model, which leads to the prefactor \( z(u)/z(v) \). Otherwise, the expression is similar to the one for the eight-vertex model. The result of Blom and Nienhuis for the M_1 model is similar [20], one needs to choose \( \theta = \pi/3 \) instead of \( \theta = \pi/4 \).
3.4 The coordinate Bethe ansatz for the staggered $M_2$ model

3.4.4 Many particles: boundary conditions and the Bethe equations

**Consistency.** Considering the one- and two-particle problems is not sufficient to conclude that the model is Bethe ansatz solvable. The next level of difficulty comes from testing the Bethe ansatz for local three- and four-particle interactions, i.e. for projections on configurations $\cdots \circ \bullet \circ \circ \cdots$ and $\cdots \circ \circ \circ \circ \cdots$ in the sector with $f = 3$ particles, and $\cdots \circ \bullet \circ \bullet \circ \cdots$ in the sector with $f = 4$ particles.

The key to show consistency for these situations is the reduced difference equation (3.30), which we derived for the two-particle wave function. In fact, the nature of the Bethe ansatz implies that it holds in fact for any $f$ in the following way:

\[
\lambda_{x+1}(\psi(\ldots, x - 1, x, \ldots) + \psi(\ldots, x, x + 1, \ldots)) \\
+ \lambda_x(\psi(\ldots, x, x, \ldots) - \psi(\ldots, x - 1, x + 1, \ldots)) = 0. \tag{3.35}
\]

Let us start with $f = 3$ particles. If we consider (3.35) with an additional particle on the site $x + 1$ or $x + 2$, and re-apply it to the existing two particles, we obtain quite straightforwardly the following equations for the three-particle wave function:

\[
\psi(x - 1, x + 1, x + 1) + \psi(x, x + 1, x + 2) = 0 \\
\psi(x - 1, x, x + 1) + \psi(x, x, x + 2) = 0. \tag{3.36}
\]

These equations are sufficient to prove that the three-particle problem is consistent. Indeed, when projecting the Schrödinger equation onto the basis vector $|x, x + 2, x + 3\rangle$, and comparing it as for the one- and two-particle problems, to the case when all particles are treated as if they were free, one obtains the following constraint on the wave function:

\[
\lambda_{x+1}(\psi(x, x + 2, x + 4) - \psi(x, x + 2, x + 2)) \\
- \lambda_x(\psi(x, x + 3, x + 4) + \psi(x, x + 2, x + 3)) \\
= \lambda_x(\psi(x + 1, x + 1, x + 3) + \psi(x + 1, x + 2, x + 3)).
\]

The left-hand side vanishes as a consequence of (3.35) whereas the vanishing of the right-hand side is due to (3.36). The projection on the basis vector $|x, x + 1, x + 3\rangle$ leads to the same type of relation which holds identically. This exhausts all possible three-particle interactions, and shows that they can be reduced to two-particle processes via (3.35).

The case to be checked for $f = 4$ particles is the projection of the Schrödinger equation on the state $|x, x + 1, x + 3, x + 4\rangle$. The comparison to the case of free particles leads to the consistency constraint

\[
\mu_{x+1}^2(\lambda_x(\psi(x - 1, x + 1, x + 3, x + 4) - \psi(x, x, x + 3, x + 4))
\]
The left-hand side of this lengthy equation vanishes by applying (3.35) to the second two variables, whereas the right-hand side gives zero by application of (3.36) to the first and last three variables. Hence, also the four-particle problem is consistent. This exhausts all cases which need to be checked: the consistency for configurations with higher particle numbers can be reduced to linear superpositions the one-, two-, three- and four-particle situations. We conclude that the Bethe ansatz works consistently for the staggered $M_2$ model along the special line in parameter space (3.1).

**Translation symmetry.** We consider from now on an arbitrary number of fermions $f > 1$ on the chain. In order to discuss the Bethe ansatz equations we need to specify the boundary conditions of our model. We will consider the model with a twist. Because of the 2-periodicity of the staggering, the Hamiltonian commutes with the square of the twisted translation operator $T', \ [H, (T')^2] = 0$, as explained in section 2.3. Hence we impose the solutions of the Schrödinger equation to be eigenvectors of $(T')^2$: 

$$(T')^2 |\psi\rangle = t^2 |\psi\rangle.$$ 

The equation is written in a suggestive form: for the homogeneous model $t$ is simply the eigenvalue of $T'$, which becomes a proper symmetry of the Hamiltonian in this case. In the general, staggered case, we use (3.23) and project the resulting equation on a configuration $|x_1, \ldots, x_f\rangle$. Assuming that the first $m = 0, 1, 2$ particles are located on the first 2 lattice sites, the projection leads to 

$$(-1)^{m(f-1)} e^{i\phi(m-2f/N)} \psi(x_{m+1} - 2, \ldots, x_f - 2, x_1 + N - 2, \ldots, x_m + N - 2) = t^2 \psi(x_1, \ldots, x_f).$$

In the case $m = 0$, we use the Bethe ansatz (3.24) and conclude that the eigenvalue $t^2$ is given by

$$t^2 = e^{-2i\phi f/N} \prod_{j=1}^{f} z_j^{-2}. \quad (3.37)$$

If we require this result to be compatible with the other choices $m = 1, 2$ then we obtain a common constraint on the transformation behaviour of the amplitudes...
3.5 The coordinate Bethe ansatz for the staggered $M_2$ model

$B_\sigma$ under a cyclic shift of the permutation $\sigma$. Let $\pi$ be the cyclic shift, i.e. $\pi(1) = 2, \pi(2) = 3, \ldots \pi(f - 1) = f, \pi(f) = 1$, then we have

$$B_\sigma = (-1)^{f-1} e^{i\phi} z^{N}_{(1)} B_{\sigma \cdot \pi}. \quad (3.38)$$

**Bethe equations.** The cyclic shift property allows to derive the Bethe equations for the model. For $f$ particles the amplitudes $B_\sigma$ are required to solve the system of equations

$$B_{\sigma(i), \sigma(i+1), \ldots r(u_{\sigma(i)}, u_{\sigma(i+1)})} + B_{\sigma(i+1), \sigma(i), \ldots r(u_{\sigma(i+1)}, u_{\sigma(i)})} = 0, \quad \sigma \in S_f. \quad (3.39)$$

where $r(u, v)$ is the function defined in (3.34). The system can be solved by

$$B_\sigma = C^{-1} \frac{\tilde{\sigma}^{-1}}{\prod_{1 \leq m < n \leq f} b_{\sigma(m)\sigma(n)}},$$

Here $C$ is an arbitrary normalisation factor, and the numbers $b_{mn}$ are solutions to the equations

$$b_{mn}r(u_m, u_n) = b_{nm}r(u_n, u_m).$$

If all the $r(u_m, u_n)$ are finite and non-zero for all $m, n$ then this system of equations has the simple solution $b_{mn} = r(u_n, u_m)$. Using the cyclic shift property, we obtain in this case the Bethe equations of the model:

$$z(u_k)^{N-f} = e^{-i\phi} \prod_{j=1}^{f} \frac{\vartheta_1(u_j - \theta)}{\vartheta_1(u_j + \theta)} \frac{\vartheta_1(u_k - u_j - \theta)}{\vartheta_1(u_k - u_j + \theta)} \quad (3.40)$$

A solution of this equation leads to an eigenstate of the Hamiltonian whose energy is given by

$$E = N + \sum_{j=1}^{f} \epsilon(u_j) \quad (3.41)$$

where $\epsilon(u)$ is the elliptic form of the excitation energy found in (3.33). In particular, as the Bethe equations do not depend on the parameter $t$ which parametrises the constants $\mu_x, \lambda_x$, nor does the excitation energy itself, we conclude that the energy is $t$-independent. It follows that the spectrum does not change as one moves along the constant $q$ lines on the special submanifold in parameter space plotted in figure 3.2. The eigenstate itself can be reconstructed from the amplitudes $B_\sigma$. With an appropriate choice of normalisation, we find that they are given by

$$B_\sigma = \frac{\tilde{\sigma}^{-1}}{\prod_{n=1}^{f} z(u_{\sigma(n)})^{-n} \prod_{1 \leq m < n \leq f} \vartheta_1(u_{\sigma(n)} - u_{\sigma(m)} + \theta).}$$

There are however cases, where $r(u_m, u_n)$ vanishes or becomes infinite for certain pairs $m, n$. This happens for so-called exact strings or bound states,
which need to be treated separately. We will show in the next section, that these somewhat exceptional cases are actually quite relevant in order to understand the supersymmetry from the point of view of the Bethe ansatz.

3.5 Supersymmetry and the Bethe ansatz

In this section we analyse the relation between the different symmetries of the model and the Bethe ansatz equations. We show that the action of the operators $Q_+$ and $Q_0$ can be derived rather straightforwardly from the Bethe equations. The dynamic supersymmetry generated by $Q_-$ is however more subtle. As we shall see it is related to the existence of so-called exact strings of Bethe roots, which are present in the model essentially because the parameter $\theta$ is a rational multiple of $\pi$.

3.5.1 Non-dynamic supersymmetry $Q_+$

Let us start with the supersymmetry that was originally used to define the model. We claim that the action of $\bar{Q}_+$ is equivalent to adding to a set of Bethe roots $u_1, \ldots, u_f$ that solve (3.40) an additional root $u_{f+1} = 0$, i.e. rapidity $z_{f+1} = 1$, without changing the number of sites. This is readily verified by comparing the Bethe equations at $f$ and $f + 1$ particles, which confirms our statement provided that the twist angle is $\phi = 0$. The relation between the wave functions with $f$ and $f + 1$ particles can be evaluated explicitly for $z_{f+1} = 1$:

$$\psi(x_1, \ldots, x_{f+1}) = \text{const.} \times \sum_{k=1}^{f+1} (-1)^{k-1} \lambda_{x_k} \psi(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{f+1})$$

The string $(-1)^{k-1}$ is a clear sign of a fermionic operator. Promoting this relation between wave functions to a relation between the corresponding states by multiplying each side with $||x_1, \ldots, x_{f+1}||$, followed by a summation over all allowed particle arrangements, leads straightforwardly to the definition of $\bar{Q}_+$ (up to a constant).

3.5.2 Dynamic supersymmetry $Q_0$

The neutral dynamic supersymmetry can also be understood through a simple addition of a Bethe root to a given solution $u_1, \ldots, u_f$ of (3.40). The new member has $u_{f+1} = \pi/2$, and corresponds therefore to a particle with rapidity $z_{f+1} = -1$. In addition to this particle insertion, one needs to increase the length of the chain by two.

From the Bethe ansatz form of the wave function we obtain a relation between the wave functions for $f + 1$ particles with $z_{f+1} = -1$, and $f$ particles with arbitrary rapidities:

$$\psi(x_1, \ldots, x_{f+1}) = \text{const.} \times \sum_{k=1}^{f+1} (-1)^{x_k+k-1} \lambda_{x_{k+1}} \psi(x_1, \ldots, x_{k-1}, x_{k+1}-2, \ldots, x_{f+1}-2)$$
It is not difficult to translate this equation into a relation between the corresponding eigenvectors of the Hamiltonian at $N$ and $N + 2$ sites. The corresponding operator is fermionic and carries “momentum” $\pi$ as can be seen from the string $(-1)^{x_k + k - 1}$. Working out its amplitudes leads precisely to the dynamic supercharge $Q_0$ discussed in section 3.3.2.

### 3.5.3 Dynamic supersymmetry $Q_-$

The Bethe equations for the staggered $M_2$ model resemble those of the eight-vertex model at so-called root-of-unity points, i.e. points where $\theta$ is a rational multiple of $\pi$. It is known that at such points so-called exact strings appear in finite-size systems, i.e. configurations of Bethe roots which are arranged in the pattern

$$u_j = u + (j - 1)\theta, \quad \theta = \pi/4,$$

with $j = 1, 2, 3, 4$. The aim of this section is to discuss a relation between these exact strings and the dynamic supercharge $Q_-$. To this end, we derive the wave function for a single exact string and then relate it to the action of $Q_-$ and $Q_+$ in the limit where the so-called string centre $u$ tends to zero.

Let us first discuss a few properties of an exact string of Bethe roots. From the elliptic parameterisation of the Bethe roots given in (3.32), we infer that its total rapidity is given by

$$\prod_{j=1}^{4} z(u_j) = 1,$$

and hence does not carry any net momentum. Moreover, the sum of the single particle excitation energies for its members yields

$$\sum_{j=1}^{4} \epsilon(u_j) = -4,$$

irrespectively of the value $u$ for the string centre. Comparing this with the expression of the total energy in (3.41), we conclude that adding an exact string to a configuration of Bethe roots decreases the energy by four. Therefore, if we add simultaneously four sites to the system, the total energy remains unchanged. This observation hints at a dynamic symmetry relating the Hamiltonians for chains of length $N$ and $N + 4$.

Here, we investigate the simplest case of a single exact string in order to establish a relation with dynamic supersymmetry. To this end, we compute the wave function by following Baxter’s calculation for the six-vertex model [37]. Concretely, for four particles we have to solve the equations, (3.39),

$$B_{\sigma(i),\sigma(i+1),\ldots} r(u_{\sigma(i)}, u_{\sigma(i+1)}) + B_{\sigma(i+1),\sigma(i),\ldots} r(u_{\sigma(i+1)}, u_{\sigma(i)}) = 0.$$

These give the relations between the different amplitudes of the wave function, $B_\sigma$, where $\sigma \in S_4$ in the present case. For the configuration of Bethe roots that form
the exact string (3.42) it can easily be seen that the function \( r(u_i, u_j) \) defined in (3.34) vanishes whenever \( j = i + 1 \) for \( i = 1, 2, 3 \), or \( i = 4, j = 1 \). Conversely, \( r(u_j, u_i) \) is nonzero for these cases. If we normalise the amplitudes such that \( B_{1234} \neq 0 \) then our equations imply that all other amplitudes are finite, and that \( B_{2134} = 0 \) because \( r(u_2, u_1) = 0 \), \( B_{1324} = 0 \) because \( r(u_3, u_2) = 0 \) etc. We find that all \( B_\sigma \) are zero except those for which the permutation \( \sigma \) is an integer power of the cyclic shift \( \pi = (1234) \). The remaining amplitudes are related because of the translation symmetry (3.38). For zero twist angle, we find the simple relation

\[
B_{j+1,...,4,1,...,j} = (-1)^j z_{j+1}^N \cdots z_4^N B_{1234}.
\]

We obtain the wave function for a single exact string by plugging into the Bethe wave function, (3.24), the expression for the non-vanishing \( B_\sigma \). Writing out the rapidities in their elliptic parametrisation, we find

\[
\psi(x_1, x_2, x_3, x_4) = C \sum_{j=0}^3 (-1)^{x_1+x_2} \prod_{k=1}^4 A_{x_k} \vartheta_1(u + (k + j - 2)\theta)^{x_{k+2}-x_k-3},
\]

where

\[
A_{x_k} = \lambda_{x_k} \frac{\vartheta_4(t - u_1 + (2x_k - (k + j))\theta)}{\vartheta_4(t + (2x_k - 1)\theta)}
\]

and \( C \) is some normalisation constant.

We now take the limit where the centre of the exact string \( u \) tends to zero and establish the relation with the supercharge \( Q_- \). To this end, observe that for \( u = 0 \) the wave function is only nonzero for a special set of configurations. In this case, the products in the exact-string wave function contain a factor \( \vartheta_1((k + j - 2)\theta) \) which is zero when \( k = 2 - j, 6 - j \), so for every \( j \) there is a case for which this becomes zero. The whole wave function vanishes therefore unless also \( x_{k+2} - x_k - 3 = 0 \) simultaneously. This means that the fermion configuration contains a pair of particles located at \( x_k \) and \( x_{k+2} \), such that \( x_{k+2} - x_k = 3 \). Now, recall that because the particles are ordered there needs to be one particle on the two sites in between them. This fixes the relative positions of three particles. The remaining one is itinerant: it can be anywhere as long as the exclusion constraints of the model hold. For simplicity, we consider only the configurations for which the first site is occupied. The non-vanishing values of the Bethe wave functions for a chain of \( N \) sites are given by:

\[
\psi(1, 3, 4, x) = C(\vartheta_1(\theta)\vartheta_1(2\theta))^{N-6} \lambda_1^2 \lambda_x^2 \lambda_0 \frac{\vartheta_4(t)^2}{\vartheta_4(t + \theta)^2}
\]

\[
\psi(1, 2, 4, x) = -C(\vartheta_1(\theta)\vartheta_1(2\theta))^{N-6} \lambda_1^2 \lambda_x^2 \frac{\vartheta_4(t)^2}{\vartheta_4(t + 3\theta)^2}
\]
3.5 Supersymmetry and the Bethe ansatz

\[ \psi(1, 3, 4, 6) = C (\partial_1(\theta) \partial_1(2\theta))^{N-6} \partial_4(t)^2 \lambda_0^2 \lambda_1^2 \left( \frac{1}{\partial_4(t+3\theta)^2} + \frac{1}{\partial_4(t+\theta)^2} \right) \]

\[ \psi(1, 2, 4, 5) = -C (\partial_1(\theta) \partial_1(2\theta))^{N-6} \lambda_1^2 \lambda_2^2 \left( \frac{\partial_4(t)^2}{\partial_4(t+3\theta)^2} + \frac{\partial_4(t+2\theta)^2}{\partial_4(t+3\theta)^2} \right). \]

All other non-vanishing amplitudes can be recovered from these either from invariance under translation by two sites, or by shifting \( t \to t + 2\theta \), which amounts to a translation by one site. We now normalise the amplitudes by setting

\[ C = -\frac{1}{(\partial_1(\theta) \partial_1(2\theta))^{N-6}} \frac{\partial_4(t+\theta)^2}{\partial_4(t)^2} \frac{1}{\lambda_1^2} \frac{\mu_0}{\mu_1}. \]

The Bethe wave function gives the amplitudes of the non-orthonormal basis we defined in (3.22). To find the amplitudes of the configurations in the basis in which we defined \( Q_-(3.5) \) we have to include an extra normalisation factor \( \mu_x \) for any pair starting a site \( x \). When comparing with the definitions of the non-dynamic and dynamic supercharges in section 3.3 and 3.3 we observe that we obtain after some algebra precisely the amplitudes of \( \bar{Q} \bar{Q} + (= -\bar{Q} + Q) \) acting on the empty chain:

\[ \mu_3 \psi(1, 3, 4, x) = -\mu_0 \lambda_0 \lambda_x, \quad \mu_1 \psi(1, 2, 4, x) = \mu_0 \lambda_1 \lambda_x, \quad \text{for} \quad 5 < x < N, \]

\[ \mu_3 \psi(1, 3, 4, 6) = -2\mu_0, \quad \mu_1 \mu_4 \psi(1, 2, 4, 5) = \lambda_1^2. \]

Remember that we showed that \( \bar{Q}_+ \) acts on Bethe states by adding a Bethe root \( u = 0 \) to a configuration. We conclude that \( Q_- \) adds a set of three Bethe roots which is centred around the Bethe root \( u = 0 \), in a pattern of an exact string with exactly the central root missing. To be more precise, the above argument only shows this for \( Q_- \) acting on an empty chain, but we expect the action to be the same when starting from a general configuration of Bethe roots. Another possible way to prove this statement might be to study the insertion of the three Bethe roots \( u = \pi/4, \pi/2, 3\pi/4 \) to a given solution of the Bethe equations, and increase the number of sites by four. However, it appears that this has to be done by employing a suitable limiting procedure which involves breaking translation symmetry, and is therefore technically very challenging.

3.5.4 Multiplet structure

The eigenstates of the Hamiltonian organise in representations of its symmetry algebra. From our discussion in section 3.3, we conclude that the amount of supersymmetry of the model on the special submanifold actually depends on the translation subsector of the Hilbert space we consider. Here we provide a brief analysis of the resulting multiplet structure in the case of periodic boundary conditions.
First of all, by definition the model always contains a copy of the $\mathcal{N} = 2$ supersymmetry algebra with non-dynamic supercharges $Q_+, \bar{Q}_+$. As the Hamiltonian is given by their anticommutator, and thus a positive operator, its eigenvalues are either zero or positive. It is well-known that the zero-energy eigenstates correspond to supersymmetry singlets, i.e. eigenstates which are annihilated by both supercharges. Conversely, all positive eigenvalues are doubly-degenerate, and the corresponding eigenvalues come as doublets $(|\psi\rangle, Q_+|\psi\rangle)$ where $\bar{Q}_+|\psi\rangle = 0$.

Second, in translation sectors where $T^4 \equiv 1$, the multiplets have a richer structure because of the presence of a second $\mathcal{N} = 2$ algebra with dynamic supercharges $Q_-, \bar{Q}_-$. Since $[T^2, H] = 0$, we need to distinguish two cases here. (i) If $T^2 \equiv -1$, which can be the case only if the number of sites is a multiple of four, we find that states with positive energy organise in quartets $(|\psi\rangle, Q_+|\psi\rangle, Q_-|\psi\rangle, Q_+Q_-|\psi\rangle)$ where $|\psi\rangle$ is annihilated by both $Q_+$ and $\bar{Q}_-$. So-called short (or BPS) multiplets are not present, because the central charges of the algebra generated by the supercharges are all zero. (ii) If on the other hand $T^2 \equiv 1$, then a third dynamic copy of the supersymmetry algebra generated by $Q_0, \bar{Q}_0$ is present. In this case, the non-zero energy states organise in octets, which are generated from a cyclic state, that is, a state annihilated by all the adjoint supercharges $Q_+, \bar{Q}_-, Q_0$.
Hamiltonian are taken at the homogeneous point, i.e. $\lambda_x = 1, \mu_x = 1/\sqrt{2} \ \forall \ x$. To reveal the multiplet structure we split up the spectrum of a single chain length by fermion number. The symmetry generated by $Q_+^\dagger$ is very obvious in this plot, but also the full multiplets, octets in this case, can be detected upon closer examination. For one of the multiplets, with the cyclic state at $N = 2, f = 1, E = 2$, we indicate the action of the supercharges. This can be compared directly with figure 3.3. The multiplets that start at length $N = 8$ or larger are incomplete, because we truncate the spectrum at length $N = 12$.

To show that the multiplet structure survives when we move away from the homogeneous point, while remaining on the manifold with dynamical supersymmetry, we show the same plot for $\lambda_x = 4/5\lambda_{x+1} = 1, \mu_x = 1/2$ and $\mu_{x+1} = \sqrt{3}/2$ for all even $x$ for the translation sectors $t^2 = 1$ and $t^2 = -1$ (see figures 3.5 and 3.6). The supermultiplets in the translation sector $t^2 = -1$ consist of at most 4 elements, since $Q_0$ is absent in this sector. Finally, in figure 3.7 we plot the spectra for a choice of parameters away from the manifold with dynamical supersymmetry. The parameters are staggered with period 2, since this is required in order for $T^2$ to commute with the Hamiltonian. It is clear that the doublet structure generated by $Q_+^\dagger$ is preserved, while the multiplets corresponding to the dynamical supersymmetry are absent.

Finally, we point out that the supersymmetries still do not account for all the degeneracies in the spectrum (see for instance the degeneracies of the levels with $E = 4$ in figure 3.4). This suggests the possible presence of further symmetries.

### 3.6 Conclusion

In this chapter we provided a detailed analysis of the one-dimensional $M_2$ model of strongly-interacting fermions and pairs. In particular, we determined a submanifold in the space of parameters for which the model presents two hidden dynamic supersymmetries. We showed that this symmetry enhancement is present precisely for the choice of parameters for which the model is also diagonalisable by the Bethe ansatz. This allowed to understand its various symmetries in terms of the Bethe equations: in particular, we pointed out a relation between the existence of a dynamic supersymmetry and the existence of exact strings.

Furthermore, there are a few aspects beyond the $M_2$ model. One of the key features to detect the Bethe ansatz solvability was an asymptotic analysis for small and large rapidities within the one- and two-particle problem of the Bethe ansatz. This strategy has potential to be applied to other systems, and help find integrable points when only a Hamiltonian (without additional structures such as transfer matrices etc.) is given. In fact, the reasoning presented here can be extended to all $M_k$ models with $k = 1, 2, 3, \ldots$ [14].
Figure 3.4: The plot shows the energy levels of chains of even lengths $N = 2, 4, \ldots, 12$ for the translation sector $t^2 = 1$. The parameters in the Hamiltonian are taken at the homogeneous point, i.e. $\lambda_x = 1, \mu_x = 1/\sqrt{2} \forall x$. To reveal the multiplet structure we split up the spectrum of a single chain length by fermion number. At the bottom of the plot we indicate a range containing the levels of a chain of a certain length. At the top of the plot we indicate the fermion number, the grey dashed and drawn lines serve as a guide to the eye, where the drawn lines correspond to fermion number $f = 0$. The vertical axis labels the energy. We plot each level as a box, where the numerical value of the energy corresponds to the centre of the box. The numbers inside the boxes give the degeneracy of the level and the colours of the boxes are one-to-one with the fermion numbers.
Figure 3.5: The plot shows the energy levels of chains of even lengths $N = 2, 4, \ldots, 12$ for the translation sector $t^2 = 1$. The parameters in the Hamiltonian are taken at some point on the manifold with dynamical supersymmetry, specifically $\lambda_x = 4/5\lambda_{x+1} = 1$, $\mu_x = 1/2$ and $\mu_{x+1} = \sqrt{3}/2$ for all even $x$. See figure 3.4 for details on how to read the plot.
Figure 3.6: The plot shows the energy levels of chains of even lengths $N = 4, 8, \ldots, 16$ for the translation sector $t^2 = -1$. The parameters in the Hamiltonian are taken at some point on the manifold with dynamical supersymmetry, specifically $\lambda_x = 4/5\lambda_{x+1} = 1$, $\mu_x = 1/2$ and $\mu_{x+1} = \sqrt{3}/2$ for all even $x$. The spectrum is cut-off at $E = 4.1$. See figure 3.4 for details on how to read the plot.
Figure 3.7: The plot shows the energy levels of chains of even lengths $N = 2, 4, \ldots, 12$ for the translation sector $t^2 = 1$. The parameters in the Hamiltonian are staggered with period 2, but do not lie on the manifold with dynamical supersymmetry, specifically $\lambda_x = 3/4\lambda_{x+1} = 1$, $\mu_x = 1/4$ and $\mu_{x+1} = 1/2$ for all even $x$. See figure 3.4 for details on how to read the plot.