Countable dense homogeneous rimcompact spaces and local connectivity

van Mill, J.

DOI
10.2298/FIL1501179M

Publication date
2015

Document Version
Final published version

Published in
Filomat

Citation for published version (APA):
Countable Dense Homogeneous Rimcompact Spaces and Local Connectivity

Jan van Mill

Abstract. We prove that every nonmeager connected Countable Dense Homogeneous space is locally connected under some additional mild connectivity assumption. As a corollary we obtain that every Countable Dense Homogeneous connected rimcompact space is locally connected.

1. Introduction

All spaces under discussion are separable metric.

A space $X$ is Countable Dense Homogeneous (abbreviated: CDH) provided that for all countable dense subsets $D$ and $E$ of $X$ there is a homeomorphism $f: X \to X$ such that $f(D) = E$. For more information on this concept, see Arhangel’skii and van Mill [2]. Bennett [3] proved that every connected CDH-space is homogeneous.

In 1972, Fitzpatrick [6] proved that every locally compact, connected CDH-space is locally connected.

Fitzpatrick and Zhou [7] asked in 1992 whether every Polish, connected CDH-space is locally connected. This problem is one of the few problems in [7] that is still open and was the motivation for the current investigations.

For a space $X$ and $x \in X$ we let $Q(x, X)$ denote the quasi-component of $x$ in $X$. That is, $Q(x, X)$ is the intersection of all clopen subsets of $X$ that contain $x$. Observe that if $x \in X$, and $X$ is a subspace of $Y$, then $Q(x, X) \subseteq Q(x, Y)$.

Theorem 1.1. Let $X$ be a nonmeager connected CDH-space and assume that for some point $x$ in $X$ we have that for every open neighborhood $W$ of $x$, $Q(x, W) \setminus \{x\}$ is nonempty. Then $X$ is locally connected.

Corollary 1.2. Every rimcompact connected CDH-space is locally connected.

This corollary generalizes the result of Fitzpatrick just quoted. Observe that we do not require our space to be nonmeager.
2. Preliminaries

As usual, for a subset $U$ of a space $X$, we put $\text{Fr } U = \overline{U} \setminus \text{Int } U$; it is called the boundary of $U$.

A space $X$ is meager if it can be expressed as a countable union of nowhere dense sets. Clearly, every Baire space (see below) is nonmeager.

A space $X$ is called rimcompact if there exists an open base $\mathcal{B}$ for $X$ such that $\text{Fr } B$ is compact for each $B \in \mathcal{B}$. For more information on this concept, see Aarts and Nishiura [1].

For a space $X$ we let $\mathcal{H}(X)$ denote its group of homeomorphisms. If $A \subseteq X$, then $\mathcal{H}(X; A)$ denotes \{ $f \in \mathcal{H}(X)$ : $h$ restricts to the identity on $A$ \}.

We will need the following result.

**Proposition 2.1 (van Mill [11, Proposition 3.1]).** Let $X$ be CDH. If $F \subseteq X$ is finite and $D, E \subseteq X \setminus F$ are countable and dense in $X$, then there is an element $f \in \mathcal{H}(X; F)$ such that $f(D) \subseteq E$.

A space $X$ is a $\lambda$-set if every countable subspace is $G_\delta$. It was shown by Fitzpatrick and Zhou [7, Theorem 3.4] that every meager CDH-space is a $\lambda$-set. There are such CDH-spaces, see [5] and [8].

A space is Polish if it has an admissible complete metric. A space is Baire if the intersection of any countable family of dense open sets in the space is dense. A space is analytic if it is a continuous image of the space of irrational numbers.

3. Proof of Theorem 1.1

Let $X$ be any nonmeager CDH-space which is connected and contains a point $x$ such that for every open neighborhood $W$ of $x$, $Q(x, W) \setminus \{ x \}$ is nonempty. By Bennett [3], $X$ is homogeneous. Hence this property of the point $x$ is shared by all points.

**Lemma 3.1.** For every open neighborhood $V$ of a point $x$ in $X$ we have that the interior of $Q(x, V)$ is nonempty.

**Proof.** Striving for a contradiction, assume that for some open $V$ in $X$ containing $x$ we have that $Q(x, V)$ has empty interior in $X$. Since $V$ is open in $X$, and $Q(x, V)$ is closed in $V$, this clearly implies that $Q(x, V)$ is nowhere dense in $X$.

For every $n$ pick an open neighborhood $U_n$ of $x$ such that $\text{diam } U_n < 2^{-n}$. The assumptions imply that for every $n$, there exists $y_n \in Q(x, U_n) \setminus \{ x \}$.

Since $Q(x, V)$ is nowhere dense in $X$, we may pick a countable dense subset $E \subseteq X \setminus Q(x, V)$. Put $D = E \cup \{ y_n : n \in \mathbb{N} \}$. By Proposition 2.1, there exists $f \in \mathcal{H}(X)$ such that $f(x) = x$ and $f(D) \subseteq E$. Pick $n$ so large that $f(U_n) \subseteq V$. Since $y_n \in Q(x, U_n) \setminus \{ x \}$ we have that $f(y_n) = Q(x, f(U_n)) \setminus \{ f(x) \} = Q(x, f(U_n)) \setminus \{ x \} \subseteq Q(x, V) \setminus \{ x \}$. Since $f(y_n) \in E$ and $E \cap Q(x, V) = \emptyset$, this is a contradiction.

**Corollary 3.2.** For every open subset $V$ of $X$ and $x \in V$, we have that the interior of $Q(x, V)$ is dense in $Q(x, V)$.

**Proof.** Assume that the interior $W$ of $Q(x, V)$ is not dense in $Q(x, V)$. Then there are $y \in Q(x, V)$ and an open subset $U$ of $x$ such that $y \in U \subseteq V$ and $U \cap W = \emptyset$. By Lemma 3.1, the interior $P$ of $Q(y, U)$ is nonempty. However, $Q(y, U) \subseteq Q(y, V) = Q(x, V)$, hence $P \subseteq Q(x, V)$ and hence $P \subseteq W$. This is a contradiction since $\emptyset \neq P \subseteq U \cap W = \emptyset$.

**Lemma 3.3.** There is a point $x \in X$ with the following property: for every open neighborhood $V$ of $x$, the quasi-component $Q(x, V)$ is a neighborhood of $x$.

**Proof.** Let $\mathcal{U}_1$ be a maximal pairwise disjoint collection of nonempty open subsets of $X$ each of diameter less than $2^{-1}$. Clearly, $\bigcup \mathcal{U}_1$ is dense. Fix $U \in \mathcal{U}_1$. Each quasi-component of $U$ has dense interior by Corollary 3.2. Hence the interiors of all the quasi-components of elements of $\mathcal{U}_1$ form a pairwise disjoint open (and hence countable) collection with dense union. Let $\mathcal{U}_2$ be a maximal pairwise disjoint collection of nonempty open subsets of $X$ each of diameter less than $2^{-2}$ and having the property that every element $V \in \mathcal{U}_2$ is contained in some quasi-component of some member from $\mathcal{U}_1$. It is clear that $\mathcal{U}_2$ has dense
union. Hence we can continue the same construction with all the quasi-components of members from \( U_3 \), thus creating the family \( U_3 \). Etc. At the end of the construction, we have a sequence \( \{ U_n : n \in \mathbb{N} \} \) of subfamilies of pairwise disjoint nonempty open subsets of \( X \) such that for every \( n \),

1. \( \bigcup U_n \) is dense in \( X \),
2. if \( V \in U_{n+1} \), then there exist \( U \in U_n \) and \( p \in U \) such that \( V \subseteq Q(p, U) \),
3. \( \text{mesh } U_n < 2^{-n} \).

Since \( X \) is nonmeager, the collection \( [X \setminus \bigcup U_n : n \in \mathbb{N}] \) does not cover \( X \). Hence there is a point \( x \in X \) for which there exists for every \( n \in \mathbb{N} \) an element \( U_n \in U_n \) such that \( x \in U_n \). We claim that \( x \) is as required. To this end, let \( V \) be any open neighborhood of \( x \). By (3), there exists \( n \) such that \( x \in U_n \subseteq V \). Since by (2), \( x \in U_{n+1} \subseteq Q(p, U_n) \) for some \( p \in U_n \), we have \( x \in U_{n+1} \subseteq Q(x, U_n) \). But \( Q(x, U_n) \subseteq Q(x, V) \), and so \( Q(x, V) \) is a neighborhood of \( x \).

Again by homogeneity, the property of the point \( x \) in Lemma 3.3 is shared by all points.

**Corollary 3.4.** Every quasi-component of an arbitrary open subset of \( X \) is open.

Now let \( V \) be a nonempty open subset of \( X \), and let \( W \) be a quasi-component of \( V \). Observe that \( W \) is a clopen subset of \( V \) since the quasi-components of \( V \) form a pairwise disjoint family. If \( W \) is not connected, then we can write \( W \) as \( A \cup B \), where \( A \) and \( B \) are disjoint nonempty open subsets of \( W \). But then \( A \) and \( B \) are clearly clopen in \( V \), which implies that \( W \) is not a quasi-component. Hence quasi-components of open subsets of \( X \) are both open and connected. So we arrive at the conclusion that \( X \) is locally connected. This completes the proof of Theorem 1.1.

Let us return to the question whether every connected Polish \( \text{CDH} \)-space is locally connected. Theorem 1.1 implies that a counterexample is very tricky. It is connected, yet its properties resemble those of complete Erdős space in [4].

**4. Proof of Theorem 1.2**

To begin with, let us prove the following simple but curious fact.

**Proposition 4.1.** Every meager \( \text{CDH} \)-space which has an open base \( U \) such that \( \text{Fr } U \) is analytic for every \( U \in U \), is zero-dimensional.

**Proof.** By the result of Fitzpatrick and Zhou quoted in §2, it follows that \( X \) is a \( \lambda \)-set. Observe that by the Baire Category Theorem, a countable dense subspace of a Cantor set \( K \) is not a \( G_\delta \)-subset of \( K \). This implies that \( X \) does not contain a copy of the Cantor set. Let \( U = \{ U_n : n \in \mathbb{N} \} \) be an open basis for \( X \) such that \( \text{Fr } U_n \) is analytic for every \( n \). Clearly, every \( \text{Fr } U_n \) is countable since every uncountable analytic space contains a copy of the Cantor set, [10, Corollary 1.5.13]. Let \( D = \bigcup_n \text{Fr } U_n \). Then \( D \) is countable and hence \( G_\delta \) and so \( X \setminus D \) can be written as \( \bigcup_n F_n \), where every \( F_n \) is closed in \( X \). Since \( F_n \cap \overline{U_m} = F_n \cap U_m \) for all \( n \) and \( m \), it follows that each \( F_n \) is zero-dimensional. So the cover

\[
\{ \{ d \} : d \in D \} \cup \{ F_n : n \in \mathbb{N} \}
\]

of \( X \) consists of countably many closed and zero-dimensional subsets. Hence \( X \) is zero-dimensional by the Countable Closed Sum Theorem [10, Theorem 3.2.8].

Let \( X \) be any \( \text{CDH} \)-space which is connected and rimcompact. Then \( X \) is nonmeager by the previous result.

Pick an arbitrary \( x \in X \).

**Lemma 4.2.** For every open neighborhood \( V \) of \( x \) we have that \( Q(x, V) \setminus \{ x \} \neq \emptyset \).
Proof. Pick an open set $A$ such that $x \in A \subseteq A \subseteq V$ while moreover $\text{Fr} A$ is compact. We claim that $Q(x, V)$ meets $\text{Fr} A$. Indeed, pick an arbitrary (relatively) clopen $E \subseteq V$ that contains $x$. Then $E \cap \overline{A}$ is clopen in $\overline{A}$, hence closed in $X$, and contains $x$. Suppose that $(E \cap \overline{A}) \cap \text{Fr} A = \emptyset$. Then $E \cap \overline{A} = E \cap A$ is nonempty and clopen in $X$ which contradicts connectivity. Hence the collection 

$$\{E \cap \text{Fr} A : E \text{ is a (relatively) clopen subset of } V \text{ that contains } x\}$$

is a family of closed subsets of $\text{Fr} A$ with the finite intersection property. By compactness of $\text{Fr} A$, the set $Q(x, V)$ consequently meets $\text{Fr} A$. \hfill \Box

So $X$ is as in Theorem 1.1, and we are done.

It was noted by Lyubomyr Zdomskyy that if a connected, CDH, rim-$\sigma$-compact space $X$ has dimension greater than 1, then it is locally connected. Striving for a contradiction, assume that $X$ is not locally connected. From Theorem 1.1 it follows that there is a base $\mathcal{U}$ at a point $x$ in $X$ such that $Q(x, U) = \{x\}$ for all $U \in \mathcal{U}$. Hence for every $U \in \mathcal{U}$, $\{x\}$ is a countable intersection of clopen subsets of $U$. This together with the homogeneity of $X$ easily implies that every compact subspace of $U$ is zero-dimensional. As a result, every compact subspace of $X$ must be zero-dimensional. Then the rim-$\sigma$-compactness yields that there is a base with zero-dimensional boundaries, and hence the space $X$ must have dimension 1.

In the light of Proposition 4.1, the question whether every rimcompact connected CDH-space is Polish, is natural. It was shown by Hrušák and Zamora Avilés [9] that every Borel CDH-space is Polish. As a consequence, a counterexample to this question is not Borel. The answer is in the negative, at least consistently. Let $X$ be an $\aleph_1$-dense subset of the 2-sphere $S^2$. The proof of the main theorem in Steprāns and Watson [12] shows that $Y = S^2 \setminus X$ is CDH under $\text{MA}_{\aleph_1}$ for $\sigma$-centered posets. It is clear that $Y$ is connected and locally connected. It is also clear that $Y$ not Polish since $\aleph_1 < c$. Moreover, every $y \in Y$ has a neighborhood base the boundary of every element of which misses $X$ so that $Y$ is rimcompact.

References