Oscillations, Logic, and Dynamical Systems

van Benthem, J.

Publication date
2015

Document Version
Submitted manuscript

Published in
The facts matter: essays on logic and cognition in honour of Rineke Verbrugge

Oscillations, Logic, and Dynamical Systems

Johan van Benthem [1, 2]

University of Amsterdam & Stanford University
http://staff.fnwi.uva.nl/j.vanbenthem

Abstract

This is a short note with small observations about big questions. We discuss how fixed-point logics, modal and first-order, can describe natural and interesting kinds of dynamic limit behavior in social networks, not just convergence to one end state. We explore what new issues arise then, and how fixed-point logics interface with other mathematical views of dynamical systems. Finally, we discuss how to relate 'blind' network dynamics to behavior of conscious agents exercising their freedom.

1 Introduction: Social agency

Rineke Verbrugge has blazed a conspicuous trail from theory to reality (some recent samples of her road are [10] and[17]), taking dynamic-epistemic logics or logics of games out of their comfort zone to psychological and computational experiments, confronting logical fine-structure and precision with the actual facts of cognition in laboratory situations. But let's get even more real.

Society itself is one great experiment, where individual rationality is rocked by the storms of public opinion, and where long-term and large-group patterns keep emerging, far beyond our individual environment. The interface of individual rationality and statistical large-scale behavior raises difficult, and sometimes disturbing questions. [3]

Now, can the tools of logic play a role in understanding this situation we find ourselves in, say, by taking a look at comprehensible global reasoning about

[1] Rineke Verbrugge has already created an impressive intellectual trail, from the logical foundations of mathematics to computational and experimental studies of human agents. While her topics of research may be variable, her standards of quality are constant, winning the minds of many colleagues. However, what wins their hearts is Rineke's character and collegial behavior. Thus, having been won over twice, I am happy to congratulate Rineke, and write in this book in her honor.

[2] I thank the audience at the Workshop 'Trends in Logic' (Beijing, July 2014) for their responses, especially, Samson Abramsky, Paolo Galeazzi, and Phokion Kolaitis. I also thank Yu Junhua (Tsinghua University) for his careful reading of a draft, and Alexandru Baltag for several congenial responses. Two referees also gave helpful comments. Some further debts on specific points are acknowledged in the text.

[3] Just consider current debates about the basis of morality: is good versus bad a matter of deliberative principle, or merely a population equilibrium between predators and prey?

long-term behavior of agents in dynamical systems, and if so, how should we go about this endeavor? After all, this area has long been the preserve of dynamical system behavior, computational simulation, and evolutionary game theory. In this brief paper, I will make some observations about ways to go – and despite their extreme simplicity, try to convince the reader that there may be something of structure and value here to pursue.

2 Dynamics in networks

In recent work such as [11,2,5], long-term belief and behavior dynamics has been studied by logical methods in a setting of social networks, where agents' behavior is determined by that of their neighbors according to a given update rule stated in some logical language, a rule which is then applied iteratively. What will happen in the long run?

To develop more concrete intuitions, we look at a few simple cases where a finite network starts with an initial value for some predicate p of nodes, which is then updated according to a logical rule of the form

$$p := \varphi(p)$$

where $\varphi(p)$ is a formula (often taken from a simple modal language) whose universal modality quantifies over all neighbors of the current point in a network. Often p is interpreted as a belief of the agent, but it could stand for any property or short-term behavior.

Example 2.1 A network with a modal influence rule In any network, the modal formula $\Box p$ says that p is currently true at all neighboring nodes. We will see what happens with different initial predicates p in the following simple network, driven by the update rule $p := \Box p$ applied iteratively:

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & 2 \arrow[d] \\ 3 \arrow[ur] & 4 \arrow[l] \end{tikzcd}$$

In this dynamics, agents follow what all their neighbors do. Here are some runs that can easily be computed from the above picture with the given rule:

Case 1: initial $p = \{1\}$. The second stage has $p = \emptyset$, and this remains the outcome ever after.

Case 2: initial $p = \{2\}$. The next successive stages are $\{3\}$, $\{4\}$, $\{2\}$, and from this stage onward, we loop.

Case 3: initial $p = \{1, 2\}$. The next stage is $\{3\}$, and we get an oscillation as before in Case 2.

Case 4: initial $p = \{1, 2, 3\}$. We get $\{1, 3, 4\}$, $\{2, 4\}$, $\{2, 3\}$, $\{1, 3, 4\}$, and an oscillation starts here.

We see how network update dynamics can stabilize in one single state (witness Case 1), but also oscillate in loops of successive predicates. These oscillations come in different forms. Some-times, successive models in the loop are very similar, in fact isomorphic (Cases 2 and 3 have all irreflexive single points) – sometimes the loop runs through different non-isomorphic network configurations (this happened in Case 4, with predicates of different sizes).

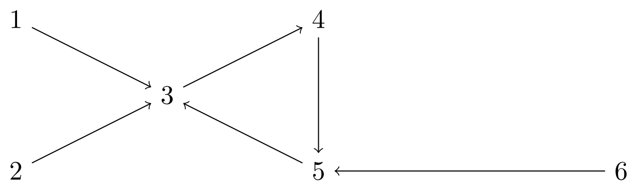
3 Oscillation and its laws

Let us now look directly at what happens in such update dynamics. For simplicity, we will consider *finite models* **M** and *ω-sequences* only. [4]

An update rule defines a function F on the power set of the domain of a model **M** like above. On finite sets, such functions all have the same pattern:

Fact 3.1 *For any function F on a finite set, there exists a finite family of disjoint loops, at each point of which there may be incoming disjoint F-sequences or F-trees arriving.*

Example 3.2 A function on a finite set Here is a simple example of a loop with incoming arrows:



We are especially interested in the structure of the loops, representing system behavior in the long run. Here 1-loops are fixed-points, a well-known form of system stability, but larger cycles, too, model natural phenomena that are stable in a more general sense, such as periodic swings in public opinion.

To describe this, we explore just one very simple notion:

Definition 3.3 Oscillation operator Given any subset (or viewed slightly differently, any unary predicate) q in a model **M**, we define

$$OSCp \bullet (\varphi(p), q)$$

as the subset that is the first $F^{\mathbf{M}}_{\varphi}$ oscillation point starting from q. [5]

The oscillation operator satisfies natural fixed-point principles.

Fact 3.4 $OSCp \bullet (\varphi(p), q) \leftrightarrow OSCp \bullet (\varphi(p), OSCp \bullet (\varphi(p), q))$ *is a valid law.*

Further appealing principles of reasoning emerge when we define the following notion that is independent from the starting point:

$OSCp \bullet \varphi(p)$ for 'occurring in some predicate of an oscillation loop of $\varphi(p)$'.

[4] The finiteness restriction is a very serious limitation to our approach in this paper, that should be overcome eventually. Some pointers as to how can be found in later passages below.

[5] Further stages of the loop are then definable from this via successive substitutions into φ.

For instance, we have the following valid 'pre-fixed-point law':

$$\varphi(OSCp \bullet \varphi(p)) \rightarrow OSCp \bullet \varphi(p)$$

The preceding observations suggest that there may be a systematic logic to oscillation, a theme that we will explore below. Moreover, studying oscillations is not at odds with studying fixed-points.

Discussion. Fixed-points of set functions The oscillation operator relates naturally to well-known notions from the literature on fixed-points. To see this, consider maps on our models. We call a set X a 'fixed-point' for a function F if $F(X) = X$. A widely used fact in logic is that, for all inclusion-monotonic maps F on a power set, there are smallest and greatest fixed-points, as stated by the well-known Tarski-Knaster Theorem. As with oscillation, we can think of such F as defined by special predicates $\varphi(p)$, this time with p occurring only *positively*. Then we get, e.g., the following observation:

Fact 3.5 *Smallest fixed-points $\mu p.\varphi(p)$ can be defined as follows for formulas $\varphi(p)$ with p occurring only positively: $\mu p.\varphi(p) := OSCp \bullet (\varphi(p), \bot)$*

Still, this is just a start, and there is more going on here in terms of valid laws than may be obvious at a first glance. For instance, with some slight abuse of notation, smallest fixed-points satisfy the equation

$$F(\mu p.\varphi(p)) = \mu p.\varphi(p) \qquad \text{where } F(X) = \{s \in \mathbf{M} \mid \mathbf{M}[p := X], s \models \varphi(p)\}$$

Now it is interesting to see that, despite initial appearances, the earlier law $OSCp \bullet (\varphi(p), q) \leftrightarrow OSCp \bullet (\varphi(p), OSCp \bullet (\varphi(p), q))$ that we noted for oscillation is not of this kind. Its underlying approximation procedure rather refers to a binary function $F(X, Y)$ where X is the current stage, and Y the initial stage, and its format is about replacing the initial Y by some other predicate, not the running X. The final version of this paper will contain further observations about this issue of unary versus binary functions, and matching different kinds of fixed-points – but for now, it is only meant as an appetizer.

More important still is the following issue concerning a natural generalization.

Discussion. From finite to infinite models In infinite models, approximation can go on beyond the first ω steps, and the question then arises how to define the limit stages. The usual stipulations in fixed-point logics such as taking unions or intersections seem to make little sense when we allow oscillation, and we need other ideas. There are interesting analogies here with similar liftings to the infinite in philosophy, logic, and game theory. [6]

[6] Some obvious analogies are with the limit steps required in Kripke-style and Gupta-Hertzberger revision theories of truth [21,13,16], that take lim-sups or lim-infs. Related issues of generalization arise in game theory with iterative solution concepts on infinite games (e.g., iterated removal of strictly dominated strategies): cf. [22,4], and for a general analysis [1]. Also related is work on common knowledge in iterations beyond the ordinal ω: cf. [15,29]. As for a more radical logical treatment, Alexandru Baltag (p.c.) has suggested making the definition of the limit jump itself an explicit parameter in the language.

Still, when we are interested in the behavior of dynamical systems, only the first ω evolution steps matter, since there are no further stages in the behavior of real systems over time. Though this seems at odds with standard logics of the sort to be discussed now, it generates interesting issues of its own – some of which are touched upon in Section 6 below. However, we do not pretend to solve the issue of the proper infinite perspective in this paper.

4 Stability and fixed-point logics

The idea of approximation to reach a stable state of some logically defined operator on models is not at all new. It underlies well-known logics of fixed-points in the literature, of which there are two main varieties. We will take these as role models for an 'oscillation logic'.

The system $\mathit{LFP}(\mathit{FO})$ enriches first-order logic with operators for smallest and greatest fixed-points of monotonic operations, which exist in any model by the Tarski-Knaster Theorem. These operations are defined syntactically by formulas $\varphi(P)$ of the formal language in which all occurrences of the predicate P in φ are syntactically positive: see [9], while [12] provides broader background in the theory of infinite computations. Whereas $\mathit{LFP}(\mathit{FO})$ is of high computational complexity (its satisfiability problem is Π^1_1-complete), a modal version of the same idea gives rise to the well-known decidable system of the modal *μ-calculus* (cf. [31]) whose syntax works as follows.

A smallest fixed-point formula $\mu p.\varphi(p)$ (with p occurring only positively in φ) denotes the smallest fixed-point of the following operation in the lattice of all subsets of a given model $\mathbf{M}$:

$$F^{\mathbf{M}}_{\varphi}(X) = \{s \in \mathbf{M} \mid \mathbf{M}[p := X], s \models \varphi\}$$

One can view smallest fixed points of such a function as the first stage in a possibly infinite cumulating approximation sequence where applying the function F no longer changes the current set:

$\emptyset, F(\emptyset), F^2(\emptyset), \ldots, F^{\alpha}(\emptyset)$
where at limit ordinals α, we take the union of all preceding stages.

The modal μ-calculus has been axiomatized completely, with proof principles:

$\varphi(\mu p.\varphi(p)) \leftrightarrow \mu p.\varphi(p)$	Fixed-point axiom
if $\vdash \varphi(\alpha) \to \alpha$, then $\vdash \mu p.\varphi(p) \to \alpha$	Smallest fixed-point rule

Similar laws govern reasoning with dual operators $\nu p.\varphi(p)$ for greatest fixed-points, definable as $\neg\varphi(\neg\mu p.\varphi(\neg p))$. In this case, the approximation sequence starts at the whole universe of the model.

As we have noted, the emphasis in these logics is on reaching fixed-points, stable stages in the approximation process where the same set returns. However, this stability can be fragile, even with our special positive syntax. If we start the approximation sequence in an arbitrary initial predicate, there is no guarantee that even monotone transformations reach a fixed point.

Fact 4.1 *Monotone set transformations can oscillate forever when the initial input is non-trivial.*

A counterexample occurred in Section 2. Just notice that the positive modal formula $\Box p$ kept oscillating when started at non-trivial input predicates. [7]

Remark 4.2 Extended μ-calculus By the preceding fact, our oscillation perspective suggests a fresh look at existing logical systems. Alexandru Baltag (p.c.) has suggested an extended μ-calculus with operators $OSCp\bullet(\varphi(p), q)$ in which the formula $\varphi(p)$ has only positive occurrences of p. Modulo some definitional subtleties to be mentioned below, this extension makes sense, and there is some interesting structure here. For instance, the set of predicates in a loop forms an anti-chain, as is easy to see. [8]

Going still further, there have been generalizations of fixed-point logics which can deal with arbitrary formulas that need not induce monotone set transformation, just as in our network dynamics. However, such systems, such as *inflationary fixed-point logic IFP*, still enforce cumulative growth of successive approximations by means of the following stipulation:

$$F^{\mathbf{M}}_{IFP}(X) = F^{\mathbf{M}}(X) \cup X$$ [9]

Basic results about generalized fixed-point logics include the theorem that $IFP(FO)$ is equal in expressive power to $LFP(FO)$ (cf. [20]) – though there is still a procedural difference: recursion in the defining formulas runs over auxiliary predicates with higher arities. [10]

From the viewpoint of fixed-point logics, oscillations seem mostly like 'junk' or failure in an approximation process. What happens when we add systematic syntax for them, to get richer logical systems? In the following section, we explore this line of thought a little bit.

5 Oscillation in logical systems

The oscillation operator OSC seems a natural addition to the syntax of logical systems, and we will do so now. But caution is needed, as we have not given a general definition of OSC on arbitrary infinite models – due to problems at limit ordinals. [11] In what follows, we will stick with our earlier restriction to *finite models*. Still, many of the systems to be considered can also define loop structure in infinite models, in particular, *infinitary modal logic*. We leave it to the reader to see which of our observations generalize straightforwardly.

[7] Monotonicity only starts producing cumulation thanks to the starting inclusion $\emptyset \subseteq F(\emptyset)$.

[8] There may be connections here with 'partial fixed-point logics' in computer science, [19].

[9] We suppress the reference to the defining formula $\varphi(p)$ here for perspicuity of notation.

[10] However, adding inflationary fixed-points to the less expressive system of the modal μ-calculus does increase the latter system's expressive power, cf. [7].

[11] Failing a good transfer convention across limit ordinals, we couldn define a uniform 'finite-oscillation operator' $OSCp\bullet(\varphi(p), q)$ in all models, saying that the φ-approximation sequence starting from q reaches a finite loop at some finite stage. On infinite models, the latter need not always happen, as the first ω-sequence for φ might not loop.

5.1 Modal logic

Many natural cases of network dynamics work with modal update rules. Starting from this simple setting, then, we add an operator $OSCp\bullet(\varphi(p),\psi)$ to the syntax of basic modal logic, with a semantic meaning as given above. The oscillation operator fits well in a modal setting.

Fact 5.1 *Modal logic with an added oscillation operator is invariant for total bisimulations whose domain and range are the whole models.*

Proof. This can be proved by a direct argument, or by noting that the above truth definition of the oscillation operator can also be written explicitly in an infinitary modal logic with an added *universal modality*, a language which is invariant for total bisimulations. □

This modal character is reinforced by further features. In particular, using the oscillation operator as shown in Section 4, our oscillation logic extends the modal μ-calculus.

Fact 5.2 *Smallest fixed-points $\mu p.\varphi(p)$ can be defined as $OSCp\bullet(\varphi(p),\bot)$.*

Thus, the logically valid laws of oscillation immediately include the laws for fixed-points. We suspect that a converse definition is not possible, though we only have a loosely related observation.

Fact 5.3 *The finite-oscillation operator is not definable in the μ-calculus.*

Proof. The reason is that, when added, the enlarged system loses the finite model property which the modal μ-calculus possesses. Here is a concrete counter-example in the enlarged language. The formula

$$\mu p.\Box p \wedge \neg OSCp\bullet(\Box p,\bot)$$

has infinite models, where in fact it forces the 'well-founded core' is infinite, but this formula lacks finite models. □

These are just simple observations, and open problems abound. In particular,

Question. *Is the modal oscillation calculus decidable, or is it at least axiomatizable, on the class of finite models?*

Remark 5.4 Inflationary μ-calculus Next, we can also embed the inflationary μ-calculus. We can mimic inflationary approximation for arbitrary formulas $\varphi(p)$ in our network dynamics by means of disjunctive formulas

$$p := \varphi(p) \vee p$$

Formulas $OSCp\bullet(\varphi(p)\vee p, q)$ then define smallest inflationary fixed-points, reached from an initial predicate q. We suspect that a converse still fails, and that the oscillation operator is undefinable even with inflationary fixed-points.

Discussion. Fine-structure: bisimulation loops One can also pursue new kinds of issue. As we saw in Section 2, larger loops can be of different kinds. Sometimes, they are close to fixed-points as all models in the loop are isomorphic, like in all our initial examples. More relevant to the modal setting:

The successive stages in a loop can be *bisimilar* models. [12]

Here, we are not saying that identity is a bisimulation in the loop: individual points may still behave differently from one stage to another. Nevertheless, at a certain description level, the models in the loop are indeed the same, having reached a stable theory modulo bisimulation. [13]

On finite models, the models in a bisimulation loop have the same collection of 'modal types', though they may differ in which object exemplifies which type. Bisimulation loops consist of models with the same theory in the following syntax. Take the basic modal language with an added universal modality, and consider only 'global formulas', true or false throughout a model. While world-dependent formulas can still change truth values at the same world in different models of a bisimulation loop, there is no detectable change in global syntax, since the set of available modal types does not change in the loop.

Here is one interesting question out of many that arise in this perspective:

Problem 5.5 *Is there special syntax for oscillation operators that guarantees generalized fixed-points in the form of bisimulation loops?* [14]

The more general point, however, is this:

Oscillation suggests the use of several logical languages, at different levels of detail, providing different invariants for the network dynamics. [15]

But one can also focus on the influence of the graph structure, and ask, for instance, for which graphs all modal formulas stabilize their oscillation loops when started anywhere.

Conjecture 5.6 *The graphs with guaranteed stabilization for all modal update rules are precisely the finite trees.*

Similar questions of oscillation logic arise for update formalisms for richer network update rules, such as 'graded modal logic' that counts numbers of neighbors, or modal logics of 'most' (cf. [24]). One special extension deserves separate attention here, as with fixed-point logics.

5.2 First-order logic

This time, we do not restrict attention to finite models, but take the other route mentioned in Footnote 11 above. First-order logic plus a finite-oscillation operator is of high complexity. We merely note two facts.

Fact 5.7 *The finite-oscillation operator on arbitrary models is definable in the infinitary first-order logic $L_{\omega 1 \omega}$.*

---

[12] Actually, bisimulation also makes sense in non-looping iteration sequences in infinite models, as a sort of generalized fixed-point. This, too, seems a natural concept.

[13] For this stability in higher languages, compare dynamical systems in biology where we consider a system stable when the percentages of different types of animal no longer change.

[14] We can also vary such issues, and ask which syntactic types of formulas guarantee the existence of 1-loops (i.e., fixed-points) when started at any predicate in any model.

[15] It may even be true that, at some appropriate higher level of description, in a lattice with other approximation operators, loops become ordinary monotone fixed-points again.

Proof. By direct description. For every formula $\varphi(p)$ and predicate q, one can define the n-th iteration φ^n for any finite n by successive substitution. Now one says there is some n and k where for all objects in the model, φ^n holds iff φ^{n+k} holds, and then that n is the smallest number with this property. All this can be formulated in $L_{\omega 1 \omega}$. $\Box$

Fact 5.8 *First-order logic with the oscillation operator is non-axiomatizable.*

Proof. Consider the modal μ-calculus formula $\mu p.\Box p$ that defines the upward well-founded part of the accessibility relation on any model for the modality $\Box$. Under the standard translation for modal logic, its modal part $\Box p$ is first-order. Without loss of generality, we can think of this modality as looking downward in the ordering. By an earlier observation, the formula $\mu p.\Box p$ is definable using the oscillation operator. Now consider the statement that

"all objects satisfying $\mu p.\Box p$ satisfy $OSCp \bullet (\Box p, \bot)$"

This says that every object in the well-founded part is admitted after finitely many iteration steps. But this can only happen when the upward well-founded chains are finite. And this property enforces, on models satisfying the first-order theory of 'greater than' on the natural numbers, that the model actually is a copy of the natural numbers. But then, the validities of the logic encode arithmetical truth, which is non-axiomatizable – and in fact Π^1_1-complete. $\Box$

6 Further logical perspectives

We pursued one straightforward way of adding oscillation operators to standard languages. However, there are also other natural technical perspectives on what is going on. We pursue this a little bit to show the broader circle of ideas that we have entered in this paper.

6.1 Dynamic logic of substitutions

An alternative approach would focus on the basic dynamic act itself that drives the above network dynamics, which is a *predicate substitution*

$$p_{new} := \varphi(p_{old})$$

One can study dynamic substitutions like this in a system of dynamic-epistemic logic (cf. [30]) with dynamic modal operators

$$\langle p := \varphi(p) \rangle$$

The valid laws for the basic predicate substitution modality form a simple decidable calculus *DEL(subst)* whose axioms mirror the usual recursive clauses for syntactic substitution. [16] In more complex versions, substitution actions can also be sequentially composed and even finitely iterated. The resulting system can define the notion of oscillation as defined above.

[16] We claim no originality for this system. Various dynamic-epistemic logics that deal with substitutions occur in the literature.

Fact 6.1 *The oscillation operator* $OSCp\bullet(\varphi(p), q)$ *is definable in DEL(subst).*

Adding step by step sequential composition still leaves this calculus simple. But adding arbitrary finite iteration of substitutions introduces complexity.

Fact 6.2 *DEL(subst*) is non-axiomatizable, and in fact* Π^1_1*-complete.*

Proof. The reason is that, using a simple translation, the logic *DEL(subst*)* faithfully embeds the better-known system of public announcement logic with iterations PAL^*, whose complexity is of this sort (cf. [23]). $\Box$

Even so, fragments of dynamic substitution logics with iteration might well be good tools for analyzing network limit dynamics driven by special formulas. Also relevant is the following observation made by Alexandru Baltag (p.c.): substitution logic meshes well with oscillation logic.

Fact 6.3
The equivalence $\langle q := \psi\rangle OSCp\bullet(\varphi(p), q) \leftrightarrow OSCp\bullet(\langle q := \psi\rangle\varphi(p), \psi)$ *is valid.*

6.2 Modal logic of dynamical systems

Fixed point logics are natural candidates for describing dynamical systems – since their laws are often simple, and yet pack quite a lot of explanatory power. [17] But there are alternatives. An earlier approach to dynamical systems is the system DTL [18], with a simple modal language that capture basic results on dynamical systems such as the Poincaré fixed-point theorem. The base language is more global than ours, with operators

$O\varphi, \Box\varphi$

These are a temporal operator $O\varphi$ for the next state of some continuous operator on the state space, plus a modality $\Box\varphi$ for topological interior. The handbook chapter [18] surveys the resulting logics on special spaces, as well as language extensions such as finite iterations of the system dynamic operator O. This modal zooming out on basic structures in dynamical systems lies at an abstraction level above our fixed-point or substitution logics in the above.

There is a challenge of how to interface perspectives, since DTL adds important structure that we have left out. In particular, our networks with neighborhoods also support DTL's topological structure, and this seems important since limit behavior is definitely influenced by two factors: (a) the logical form of the update rule, and (b) the network structure that these work on.

For more about interfacing logic and dynamical systems: see Section 7.

6.3 Temporal logic and histories of dynamical systems

Finally, while we have emphasized sparse modal languages in this paper, richer lines exist. For instance, consider the rich temporal logic of [14] for players in *iterated matrix games* responding to observed moves by others in the preceding round. There is a clear intuitive connection with social network evolution,

[17] For further examples of the surprising power of basic modal fixed-point laws in capturing essences of results in game theory or social networks, cf. [32,2].

whose precise statement goes beyond the compass of this paper. [18] Right here, our main point is just one of system choice. Temporal logics from the computational field of agency explicitly describe properties of countable histories or runs of a multi-agent process, in the format

$$\mathbf{M}, h, s \models \varphi \qquad \text{formula } \varphi \text{ is true at point } s \text{ on history } h \text{ in model } \mathbf{M}.$$

In the same manner, we could model our network evolution in temporal logic, and describe the earlier oscillation patterns in such a richer explicit formalism.

Digression. Merging perspectives What the preceding suggests is that we can use temporal logics as a sort of meta-theory for modal fixed-point logics, and represent simple notions and proofs in this richer logic. Benchmarks would be many of the simple observations in earlier sections. More ambitiously, we can also study mixtures of modal fixed-point logics and temporal logics for their computation procedures. This combination seems natural since, despite our earlier problem of defining transfer steps at limit ordinals, histories of the simplest infinite type ω fit fixed-point logics very well, witness the infinite evaluation games for the modal μ-calculus discussed in [31].

More generally, merged temporal and fixed-point logics may provide a rich reasoning style for social systems viewed at different levels.

7 Enriching the framework

Our analysis has been confined to basic logical systems that might deal with limit phenomena in social networks with update rules. We have suggested that this may be a good high-level perspective for getting qualitative insights that lie behind results obtained with the numerical models used in dynamical systems approaches to social phenomena. Of course, much more can be said about comparing qualitative logical and quantitative mathematical methods in this area, since the two methods come with different agendas. One striking difference is that numerical update rules in networks like those of the classic De Groot [8] tend to 'smoothen' values for strength of belief, whereas discrete logical approaches may create more drastic oscillations. We just note this for now, but this is obviously a point that needs much more reflection.

Next, as we said right at the start of this paper, the rules we studied are blind operations on unstructured points. What about the internal nature of the agents that make up the social network? A richer source of modeling agents than we have followed here exists in computational logic where notions from automata theory could enrich our current view (cf. [12]). This connection gets even richer when we consider the computational games associated with the logical systems that we have considered here. [19]

[18] This can be spelled out in precise detail, relating network update rules parametrized to individual points to strategy profiles for players, but we leave this for another occasion.

[19] In this connection, note also that oscillation patterns are also standard in automata theory, say with 'parity automata' for the modal μ-calculus, cf. [31]. Such patterns might be used, say, to obtain finer denotations and finer intensional notions of formula equivalence.

But even with automata in place, network dynamics remains austere. It would not distinguish between update rules for human agents, schools of fish, or neural networks. An obvious further focus then is actions that make us human, such as making observations, deliberating and deciding what to do on the basis of knowledge and beliefs about others, rather than just mechanically following our environment. [20] Moreover, human agents pursue goals connected to their preferences, while guided by intentions toward reaching these goals. All of this typically shows in their making *choices*, less or more rational.

To model real agency, we are not left with empty hands. Current dynamic epistemic logics are well up to extensions with informational actions, preferences, and acts of decision making (cf. [26] for a general treatment of logic of agency in this style – or for specific network examples, [11,6,3]). Moreover, one can draw on a flourishing literature connecting logic and game theory (cf. [27] and the references therein), giving agents positioned in networks choices as to what to do at each stage, with strategy profiles corresponding to update rules that can be studied for their long-term success in terms of achieving goals.

This richer view of agency is realistic, but pursuing it would take us far beyond the scope and intentions stated at the beginning of this note. Moreover, there is a risk in rushing ahead, of downplaying the virtues of blind rules and automatic updates. In the cognitive life of human agents, there is a systematic switching dynamics between conscious deliberate action and automated skills or habits – because of limited attention, or for more positive reasons of saving labor. Likewise, social life would probably be impossible without some back and forth between relegating beliefs and decisions to an 'automatic pilot', versus returning them to the realm of explicit control. [21] [22]

8 Conclusion

The point of this paper is that long-term social behavior supports reasoning patterns that invite logical analysis. To do so, we must step back from fixed-points only, and see the logical structure in oscillations: cycles are not 'junk', but regular long-term behavior in its own right. We have noted a few facts and perspectives that may help us do so – suggesting that existing fixed-point logics, suitably generalized, and supplemented with dynamic and temporal logics for system evolution, may apply to many realms of limit behavior over time.

There are several ways of taking what is proposed here, that can be pursued in tandem. One is exploring new technical views of logical systems and their connections, for which we have provided a slew of suggestions. Another line is a richer description of agency, either as logical theory about agents in social

[20] A real human agent can even decide *not* to update according to some prevalent update rule in the network, thereby exercising her freedom.

[21] I thank Erik Olsson for a stimulating discussion of this point in social agency, and beyond.

[22] The purely temporal approach in this paper also neglects another dimension of the social world, that of *size*: and in particular, the interface between the individual agents and large groups. Group size in networks poses questions that are far from being exhausted by current studies of games or group knowledge (cf. [25]).

settings, or as an account of how agents reason themselves. The way I myself would like to think about the role of logic here is as providing natural levels for identifying qualitative reasoning patterns with broad sweep and simplicity. As we just noted at the end of Section 7, there may be many such levels, from automated to deliberate. [23]

Despite the technicality of this paper, I hope that its topics still connect to the challenging interface of individual agency and social life that I started with. I feel that much can be done by logicians today in understanding, and perhaps even improving, the 'thin layer' of deliberate human thinking and acting that lies so precariously in between the blind dynamics of the social systems above us and the neural networks inside us.

References

[1] Apt, K. and J. Zvesper, *Proof-theoretic analysis of rationality for strategic games*, in: *Proceedings of the 11th International Workshop on Computational Logic in Multi-Agent Systems (CLIMA XI)*, Lecture Notes in Computer Science **6245** (2010), pp. 186–199.

[2] Baltag, A., Z. Christoff, R. Rendsvig and S. Smets, *Dynamic epistemic logic for threshold models* (2010), working paper, ILLC, University of Amsterdam. Extended version of a paper presented at the ELISIEM Workshop on Epistemic Logic for Individual, Social, and Interactive Epistemology, 26th European Summer School in Logic, Language and Information, Tübingen, Germany, August 11-15, 2014.

[3] Baltag, A., F. Liu and S. Smets, *Evidence-based belief change in networks* (2014), working paper, ILLC, University of Amsterdam and Department of Philosophy, Tsinghua University.

[4] Chen, Y.-C., N. V. Long and X. Luo, *Iterated strict dominance in general games*, Games and Economic Behavior **61** (2007), pp. 299–315.

[5] Christoff, Z. and J. U. Hansen, *A logic for diffusion in social networks*, Journal of Applied Logic **13** (2015), pp. 48–77.

[6] Christoff, Z., J.-U. Hansen and C. Proietti, *Reflecting on social influence in networks* (2014), talk at the IDAS workshop on Information Dynamics in Artificial Societies (ESSLLI 2014), to appear in special issue of the Journal of Logic, Language and Information, edited by E. Lorini, L. Perrussel and R. Muehlenbernd.

[7] Dawar, A., E. Grädel and S. Kreutzer, *Inflationary fixed points in modal logic*, ACM Transactions on Computational Logic **5** (2004), pp. 282–315.

[8] de Groot, M., *Reaching a consensus*, Journal of the American Statistical Association **69** (1974), pp. 118–121.

[9] Ebbinghaus, H.-D. and J. Flum, "Finite model theory," Springer Science Publishers, 2005.

[10] Ghosh, S., B. Meijering and R. Verbrugge, *Empirical reasoning in games: Logic meets cognition*, in: T. Ågotnes, N. Alechina and B. Logan, editors, *Proceedings Third Workshop Logics for Resource Bounded Agents*, 2010, pp. 15–34.

[11] Girard, P., F. Liu and J. Seligman, *Logical dynamics of belief change in the community*, Synthese **191** (2014), pp. 2403–2431.

[12] Grädel, A., E. Thomas and T. Wilke, editors, "Automata, Logics, and Infinite Games," Lecture Notes in Computer Science, Springer Verlag, Berlin, 2002.

[13] Gupta, A., *Truth and paradox*, Journal of Philosophical Logic **11** (1982), pp. 1–60.

[23] This style of thinking is connected to finding 'natural logics' of agency (cf. [28] on extending linguistic monotonicity inferences) where we identify simple pervasive concepts and reasoning patterns in natural language that enable us to function in a complex world of communication.

[14] Gutierrez, J., P. Harrenstein and M. Wooldridge, *Reasoning about equilibria in game-like concurrent systems*, in: C. Baral, G. De Giacomo and T. Eiter, editors, *Proceedings of the Fourteenth International Conference on Principles of Knowledge Representation and Reasoning (KR-2014)*, 2014, Vienna, Austria.

[15] Heifetz, A. and D. Samet, *Knowledge spaces with arbitrarily high rank*, Games and Economic Behavior **22** (1998), pp. 260–273.

[16] Herzberger, H., *Notes on naive semantics*, Journal of Philosophical Logic **11** (1982), pp. 61–102.

[17] Isaac, A., J. Szymanik and R. Verbrugge, *Logic and complexity in cognitive science*, in: A. Baltag and S. Smets, editors, *Johan van Benthem on Logic and Information Dynamics*, Springer, Dordrecht, 2014 pp. 787–833.

[18] Kremer, P. and G. Mints, *Dynamic topological logic*, in: M. Aiello, I. Pratt-Harman and J. van Benthem, editors, *Handbook of Spatial Logics*, Springer Verlag, Dordrecht, 2007 pp. 565–606.

[19] Kreutzer, S., *Partial fixed-point logic on infinite structures*, in: *Proceedings of the 16th International Workshop and 11th Annual Conference of the EACSL on Computer Science Logic (CSL'02)*, Springer, 2002, pp. 337–351.

[20] Kreutzer, S., *Expressive equivalence of least and inflationary fixed-point logic*, Annals of Pure and Applied Logic **130** (2004), pp. 61–78.

[21] Kripke, S., *Outline of a theory of truth*, Journal of Philosophy **72** (1975), pp. 690–716.

[22] Lipman, B., *A note on the implications of common knowledge of rationality*, Games and Economic Behavior **6** (1994), pp. 114–129.

[23] Miller, J. and L. Moss, *The undecidability of iterated modal relativization*, Studia Logica **79** (2005), pp. 373–407.

[24] Pacuit, E. and S. Salame, *Majority logic*, in: *Proceedings of the Fourth International Conference on Principles of Knowledge Representation and Reasoning (KR 2004)*, 2004, pp. 598–605.

[25] Ramanujam, R., *Logical player types for a theory of play*, in: A. Baltag and S. Smets, editors, *Johan van Benthem on Logic and Information Dynamics*, Springer, 2014 pp. 509–528.

[26] van Benthem, J., "Logical Dynamics of Information and Interaction," Cambridge University Press, Cambridge, UK, 2011.

[27] van Benthem, J., "Logic in Games," MIT Press, Cambridge, MA, 2014.

[28] van Benthem, J., *Natural language and logic of agency*, Journal of Logic, Language and Information **23** (2014), pp. 367–382.

[29] van Benthem, J. and D. Sarenac, *The geometry of knowledge*, in: J.-Y. Béziau, A. Costa-Leite and A. Facchini, editors, *Aspects of Universal Logic*, Centre de Recherches Sémiologiques, Université de Neuchatel, 2005 pp. 1–31.

[30] van Benthem, J., J. van Eijck and B. Kooi, *Logics of communication and change*, Information and Computation **204** (2006), pp. 1620–1662.

[31] Venema, Y., *Lectures on the modal μ-calculus* (2007), lecture notes, ILLC, University of Amsterdam.

[32] Zvesper, J., "Playing with Information," Ph.D. thesis, ILLC, University of Amsterdam (2010).