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Local rheology of suspensions and dry granular materials

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Abstract

The flow of dry and wet granular media is investigated in a Couette geometry using magnetic resonance imaging in order to test the applicability of the “fluidity model” for nonlocality in these materials. Local volume fraction measurements show that the systems become heterogeneous during flow. We find that the nonlocal rheology of suspensions can be correlated using the fluidity model, but the length scale that emerges is not a material property and the model cannot be used for predictive purposes. Rather, the suspension behavior is fully explained as a consequence of stress-driven particle migration and the resulting concentration gradient. The conclusion is less strong for the dry granular system, but it appears likely that the apparent nonlocal behavior is simply due to the formation of a shear band caused by granular dilatancy. © 2015 The Society of Rheology. [<http://dx.doi.org/10.1122/1.4919970>]

INTRODUCTION

Numerous rheology experiments in complex fluids have reported that under certain conditions, the local rheology appears to deviate strongly from the bulk flow curves

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[Deboeuf *et al.* (2006); Hébraud *et al.* (1997); Lauridsen *et al.* (2002); Pouliquen (2004); Varnik *et al.* (2003)]. The rheology of such materials remains a subject of much current interest, since the rheological description is a necessary input for predicting the flow behavior of these materials in different flows [Larson (1999)]. The interpretation of flow curves of such materials is not always straightforward, as a variety of confounding effects may occur. In many cases, the data can be accounted for by taking slip of the fluid close to the wall into account. However, as shown recently on several “jammed” systems, especially in strong confinement, the rheology appears to become nonlocal; i.e., there is no longer a one-to-one relation between the local stresses and shear rates, implying also that the viscosity varies spatially. Several hypotheses for the causes of these deviations have been advanced, generally associated with spatial co-operativity of rearrangements in the systems [Dauchot *et al.* (2005); Deboeuf *et al.* (2006); Henann and Kamrin (2013); Nichol *et al.* (2010); Pouliquen (2004)]. In most of the cases, however, the proposed origins rely on microscopic mechanisms that are difficult to probe experimentally. Specifically, the systems that have been studied most in this context are concentrated emulsions and suspensions of soft particles [Goyon *et al.* (2008); Ovarlez *et al.* (2006)], for which the viscosity depends extremely strongly on the volume fraction of drops or particles, so that even very small local variations in volume fraction (possibly too small to measure experimentally) potentially have large consequences for the viscosity.

A model to account for the experimentally observed nonlocality by Bocquet and coworkers [Bocquet *et al.* (2009); Goyon *et al.* (2010); Goyon *et al.* (2008)] introduces a local fluidity f , defined as the ratio between the *local* shear rate and the shear stress (the inverse of the viscosity), whose spatial variation is described as

$$f(r) = f_{\text{bulk}}(r) + \xi^2 \nabla^2 f, \quad (1)$$

where ξ is called the flow co-operativity length; model calculations for elasto-plastic systems suggest that ξ is a measure of the distance over which plastic events influence each other [Kamrin and Koval (2012)]. This model has recently been applied to different materials, from dense emulsions [Goyon *et al.* (2008)] and suspensions of soft particles to foams [Katgert *et al.* (2010); Seth *et al.* (2012)] and granular flows [Bouazid *et al.* (2013); Henann and Kamrin (2013); Kamrin and Koval (2012); Wandersman and Hecke (2014)]; the latter was done both experimentally and in simulations by treating ξ as a function of the distance from yielding in stress space. Widely differing values for ξ have been reported for these different cases, and the physical origin of the deviation from the bulk rheology has so far not been identified. Therefore, the physical meaning of ξ , which functions essentially as a fitting parameter, has remained unclear.

In this work, we first investigate the dry granular material—mustard grains—sheared in a Couette cell coupled with a magnetic resonance imaging (MRI) scanner, giving access to the local speeds and volume fractions. The results suggest that variations in the local volume fraction may be associated with the apparent nonlocality and therefore the fluidity, but the results are ambiguous. We then focus on a non-Brownian density-matched suspension, which is a similar but simpler material that has no yield stress [Fall *et al.* (2009)] and for which the migration is much more pronounced and well known [Fall *et al.* (2010)]. We first determine the rheology of the homogeneous density-matched suspension for different volume fractions, and subsequently apply the fluidity model to the suspensions that have undergone particle migration. We find that the spatial variation of the volume fraction alone accounts for the observed spatial variation of the viscosity.

DRY GRANULAR SYSTEMS AND THE FLUIDITY MODEL

Dry granular flows have been thoroughly studied over the past few years, and we now have a good understanding of their rheological behavior, as rheological representations have been established that account for the flow behavior in a variety of conditions and flow geometries [da Cruz *et al.* (2005)]. However, it is no longer certain that these relations continue to hold if nonlocal effects are present. Common manifestations of these nonlocalities are, for instance, dilation in shear bands [da Cruz *et al.* (2002)] or variable flow thresholds [Pouliquen *et al.* (2002)]. Kamrin and Koval (2012) recently applied the fluidity model to dry granular materials for the first time to account for the complicated shear bands in the “split bottom” geometry. Here, we compare the model to the measured velocity profiles of mustard grains sheared in a Couette Cell that are obtained while simultaneously measuring the density profiles; the spherical grains contain oil, which enables proton MRI measurement [Mueth *et al.* (2000)]. The dry granular grains are inserted in a wide-gap Couette cell (inner radius $r_i = 3$ cm, gap size $\Delta = 2$ cm, and height $H = 11$ cm) with roughened surfaces. Mustard grains have a diameter of 1.2 mm, so the gap is more than 15 grains across. The Couette cell is placed in a MRI scanner, enabling us to measure the local amount of oil, which is proportional to the number of grains and therefore proportional to the local volume fraction. The volume fraction is obtained from 3D spin density pictures (20 layers 5 mm thick in the z direction, with a spatial resolution of 0.2 mm in both x and y directions) which were performed at rest and under shear at each rotational speed. For each radial coordinate, grayscale pictures were then integrated in the circumferential direction to provide a circumferential average of the signal intensity of the oil. This value was divided by the signal intensity obtained at rest to give the radial distribution of the relative volume fraction of oil, and hence $\varphi(r)/\varphi_o$. MRI resolution is insufficient to obtain reliable results within the first few grain distances from the cylinder surfaces

Using well-established techniques, one can also access the radial variation of the local velocity in the flowing system [Bonn *et al.* (2008); Ovarlez *et al.* (2006); Raynaud *et al.* (2002); Rodts *et al.* (2005)], as shown in Fig. 1 (left). The local shear rate can be obtained from the velocity profile using [Barnes (1989)]

$$\dot{\gamma}(r) = r \frac{d}{dr} \frac{v}{r} = \frac{dv}{dr} - \frac{v}{r}. \quad (2)$$

FIG. 1. Left: Local velocity in sheared mustard grain granular samples. The symbols are the experimental results obtained by MRI, and the lines are the fits obtained with the fluidity model. Inset: Flow cooperative length used for the fluidity model. Right: Local relative concentration of the grains. The radial coordinate has been reduced so that the origin is located at the inner wall.

The local stress is given by the momentum conservation equation as

$$\sigma(r) = \frac{T}{2\pi r^2 H}, \quad (3)$$

where T is the torque necessary to rotate the inner cylinder. The *local viscosity* at every point in the material is then given by the ratio $\sigma(r)/\dot{\gamma}(r)$, and the local fluidity by $\dot{\gamma}(r)/\sigma(r)$.

The MRI velocimetry results (Fig. 1, left) show the emergence of a thin shear band; this localization of the flow is on a length scale that appears to be roughly independent of the macroscopically imposed rotational speed. The local relative volume fractions for all of our experiments, shown in Fig. 1 (right), indicate a slight tendency to a lower local volume fraction near the rotating inner cylinder and a higher volume fraction in the jammed region, but the data are very noisy because the individual seeds are as big as the MRI resolution. These data are nonetheless in agreement with the generally accepted idea that the flow is strongly correlated with the dilatancy of the grains for dry granular matter [Mills *et al.* (2000); Pouliquen *et al.* (2002)].

If we assume that dilatancy is the *only* way that granular flow can occur, then without a mechanism to induce dilatancy the viscosity is infinite. In Eq. (1), $f_{bulk}(r)$, the fluidity that the granular medium would have without any nonlocalities, must then be taken equal to zero. Equation (1) is a second order differential equation and thus requires two boundary conditions, for which we will use the fluidity at the points near the inner and outer cylinders where the MRI data are reliable. ξ is therefore the only unknown parameter here, and it can be determined from the data by using a classical least squares method. We can finally deduce the velocity from the fluidity $f(r)$ by setting $\dot{\gamma}(r) = \sigma(r)f(r)$ and integrating Eq. (2). The results for the velocity profile are shown in Fig. 1 and are in very good agreement with the experimental data, i.e., the fluidity model does well in accounting for the local rheology of the system. It is worth noting that, unlike most fluidity model fit attempts on granular materials that use a pressure weighted inverse viscosity, we use here the more classical definition of the fluidity, the inverse of viscosity, as is common for foams or emulsions.

The inset to Fig. 1 (left) also shows that the values of ξ that give the optimal fits are on the order of 4 mm, which is roughly the diameter of 3 beads and the size of the shear band. This scale suggests that dilatancy is the cause of the observed heterogeneity, since the length scale that follows from the fluidity model is very similar to the width of the shear band. However, in spite of the fact that these fits work quite well, the variation of the volume fraction is so small and the noise in that measurement so large that it is difficult to assess the physical origin of the apparent nonlocality. Figure 1 (right) suggests the dilation in the shear band and a small compaction outside of the shear band, but the data are quite noisy. Using the fluidity model on a well-known system presenting nonlocalities would be a good way to see what physical meaning ξ carries. Because of this, we now investigate a system that presents strong and long-range particle migration, namely, dense non-Brownian suspensions.

NON-BROWNIAN SUSPENSIONS

Non-Brownian neutrally buoyant suspensions have been studied extensively [Boyer *et al.* (2011); Fall *et al.* (2009); Fall *et al.* (2010); Lauridsen *et al.* (2002); Mills and Snabre (2009); Ovarlez *et al.* (2006); Pouliquen (2004); Trulsson *et al.* (2012); Zarraga

et al. (2001)]. They display two distinct phenomena: Particle migration across curved streamlines and shear thickening. The shear thickening happens only for relatively high shear rates. On the other hand, particle migration is always present when the stress is heterogeneous: An initially homogeneous suspension subjected to a shear rate gradient will see the particles move from high to low shear rate regions, leading to inhomogeneity of the suspensions. Since the viscosity depends strongly on the volume fraction, this migration directly leads to a spatially varying viscosity. For low shear rates, Huang and Bonn (2007) and Ovarlez *et al.* (2006) showed that the local rheology can be mapped onto the global flow curve simply by taking the variation of the volume fraction into account. Here, we probe higher shear rates, for which shear thickening occurs, and compare the local rheology to the predictions of the fluidity model.

We analyze experiments on a $\varphi = 0.59$ volume fraction suspension of $40\ \mu\text{m}$ polystyrene beads in a neutrally buoyant water + NaI solution by Fall *et al.* (2010) in two different Couette cells: A large (inner radius $r_i = 4.1\ \text{cm}$, gap size $\Delta = 1.9\ \text{cm}$) and a small one (gap size $\Delta = 1\ \text{cm}$). We obtain the local volume fractions and speed in the same way as for the mustard grains, except that in this case the MRI signal gives the volume fraction of water, or one minus the volume fraction of beads. The velocity profiles for the $1.9\ \text{cm}$ gap are shown in Fig. 2.

Figure 3 displays the evolution of the local solid volume fraction in the Couette cell upon shearing. These data are of higher quality than those for the seeds because the MRI averages over a region that is five times the particle diameter. As the rotational velocity is increased above 4 rpm, which constitutes the bulk of the data, particle migration is observed on the experimental time scale; the particles migrate radially away from the moving inner cylinder and toward the stationary outer cylinder. This migration state is found to be irreversible when the rotational speed is subsequently decreased below 4 rpm [Fall *et al.* (2010)], as shown in the inset of Fig. 3. Because of this migration, macroscopic rheology is not able to provide reliable results on the flow behavior of the suspensions, since all rheometry analyses assume that the sample is homogeneous.

The flow curves extracted from the local viscosity data for a range of cylinder speeds are shown in Fig. 4 (left). One clearly observes two regimes: Newtonian behavior at low shear rates and a shear thickening regime for higher rates. These flow curves appear to be parallel, with a transition between regimes at a shear stress of about $0.3\ \text{Pa}$. The shear rate $\dot{\gamma}_c$ at which the transition occurs, which is obtained from the intersection of straight-line fits to the data on log-log coordinates, is linear in volume fraction and

FIG. 2. Local velocity profiles in an initially homogeneous $\varphi = 0.59$ volume fraction suspension in a $1.9\ \text{cm}$ gap. The lines are the fits obtained from the fluidity model.

FIG. 3. Local volume fraction obtained by MRI in a 1 cm gap Couette cell in an initially homogeneous neutrally buoyant $\varphi = 0.59$ volume fraction suspension for an increasing shear rate ramp, followed by a decreasing shear rate ramp. The behavior is the same for the larger Couette cell (data not shown).

follows a relation $\dot{\gamma}_c = A(\varphi_{max} - \varphi)$, with $A = 65.0 \text{ s}^{-1}$ and $\varphi_{max} = 0.604$. It is shown in the Appendix that this linearly varying shear rate implies that the shear stress diverges near the maximum volume fraction following a $(\varphi_{max} - \varphi)^{-1}$ law in the Newtonian regime, and a $(\varphi_{max} - \varphi)^{-2}$ law in the shear thickening regime, and the data then superpose onto a single master curve when the shear stress is plotted versus $\dot{\gamma}/\dot{\gamma}_c$, as shown in Fig. 4 (right). As shown in the Appendix, the shear stress is proportional to $\dot{\gamma}^2$ in the shear thickening regime. It is notable that the shear thickening in these data is continuous, whereas recent simulations of shear thickening in non-Brownian suspensions predict discontinuous shear thickening in this concentration range [Mari *et al.* (2014); Seto *et al.* (2013)].

This suspension is a perfect candidate for analysis with the fluidity model because the particle migration causes a position-dependent viscosity. Thus, if the fits work, we could link the fluidity length ξ to a physical mechanism, and so test the predictive power of the model on a sample that is well understood. In Eq. (1), f_{bulk} is the fluidity of the sample

FIG. 4. Left: True shear stress versus shear rate in a dense non-Brownian suspension for different local volume fractions, constructed from local measurements. Right: True shear stress versus rescaled shear rate in dense non Brownian suspensions. All the data collapse onto a master curve.

without spatial co-operativity and is therefore taken to be the fluidity of a homogeneous 0.59 volume fraction suspension, as given in the Appendix.

In order to test the validity of the fluidity model on the non-Brownian suspensions, we follow these steps:

- We calculate the experimental fluidity from the local velocity and the macroscopic torque by using Eqs. (2) and (3).
- Then, we also calculate f_{bulk} , the fluidity that an ideal homogeneous suspension with $\varphi = 0.59$ would have at each location in the gap, using the local shear rate.
- Finally, we determine the fluidity length ξ for which the fluidity model best fits the experimental data in a least squares sense. As with the mustard grains, we use the experimental fluidity near the two walls as the two boundary conditions.

Fluidity profiles for the dense suspensions in the 1.9 cm gap are shown in Fig. 5, together with fits obtained from the model. The fits describe the experimental data of the heterogeneous system very well. The values of ξ are shown in the inset of Fig. 5; for large rotation rates, where the migration is fully developed, we find that ξ is slightly less than half the gap size ($\xi = 0.45\Delta$) and two orders of magnitude larger than the bead size.

To determine the possible influence of the geometry on the fluidity profile, we performed the same experiment with a gap of 1 cm, but this time, in contrast to the experiments with the larger gap, we fit the fluidity and speed profiles for all rotation speeds only after the rotational speed had first been increased beyond the rate at which irreversible particle migration occurs. The results are shown in Fig. 6.

The values of ξ found are almost constant over the entire rotation speed range and are again slightly less than half the gap width ($\xi = 0.46\Delta$), and the sharp increase seen in the previous data is no longer present since the migration is complete. We observe a slight deviation of the model from the experimental fluidity data, which is also visible on the velocity profiles, but overall the results are in good agreement with the experimental data.

DISCUSSION

Our main observation is that, for the suspensions and possibly for the granular systems, while the rheology appears to be nonlocal, the apparent nonlocality is not a

FIG. 5. Local fluidity in an initially homogeneous 0.59 volume fraction suspension. The symbol is the experimental data obtained from the MRI, and the lines are the fits obtained from the fluidity model. Inset: Values of ξ used to perform the fluidity model fit.

FIG. 6. Local fluidity (left) and velocity (right) profiles in an initially homogeneous 0.59 volume fraction suspension. The symbol is the experimental data obtained from the MRI and the lines are the fits obtained from the fluidity model. Inset: Values of ξ used to perform the fluidity model fit. The velocities have been rescaled with the inner cylinder velocity.

consequence of the fluidity theory that leads to Eq. (1) and there is no need to invoke collective rearrangements in our system; all the data can be accounted for by considering a concentration gradient resulting from particle migration. For the non-Brownian suspension, the migration happens over the entire gap, and accordingly, we find that the value of ξ is macroscopic; indeed, the constant value of ξ/Δ for gaps differing by a factor of two indicates that ξ is simply reflecting the interpolation of the fluidity that results from the concentration gradient across the gap. Of particular importance is the fact that ξ is not a material parameter, since its value depends on geometry as well as on the properties of the medium; hence it cannot be used to predict the behavior of the suspension in any geometry other than for flow between concentric cylinders. This behavior for the suspension appears to fit within the framework of the work of Kamrin and Koval (2014), who showed that the effect of nonlocality decreased for granular media as interparticle friction decreased, leading to the expectation that a fully lubricated interaction would be free of nonlocality. The simulations of Seto *et al.* (2013) and Mari *et al.* (2014) for similar dense suspensions indicate that strong interparticle frictional effects would be expected in the concentration range of the present experiments, leading to discontinuous shear thickening, whereas the data collapse shown in Fig. 4 (right) suggests a more continuous shear thickening in our system. The origin of this difference is not clear.

For the dry granular system, the particles are in contact at rest and the particle migration under shear is less pronounced. In this case, we find values of ξ that are on the order of a few grain diameters. In fact, the relevant length scale in the system seems to be the width of the shear band, which is indeed of the same order. Hennan and Kamrin (2013) applied the nonlocality model to dry granular flow experiments of Fenistein and Van Hecke (2003) [Deboeuf *et al.* (2006); Varnik *et al.* (2003)] and found that the velocity profiles can be adequately described by a ξ of roughly two particle diameters, similar to what is found here. Table I also shows that for other systems with a yield stress, ξ usually does not exceed the size of a few colloidal objects (bubbles, beads, and droplets). The small values of ξ for different complex fluids suggest that a similar mechanism as the one for dry granular materials may be behind the apparent nonlocality for the foams or dense emulsions. However, it should be noted that Goyon *et al.* [Goyon *et al.* (2010); Goyon *et al.* (2008)] and Katgert *et al.* (2010) did not detect a spatial change in volume fraction in their experiments.

TABLE I. Compilations of the values of the flow co-operativity length found in our experiments and comparison with previous studies.

Experiment	ξ value found (mm)	Gap size Δ (mm)	Particle size d (mm)	ξ/d	ξ/Δ
Dense suspensions 1	8.7	19	40×10^{-3}	217.5	0.45
Dense suspensions 2	4.6	10	40×10^{-3}	115	0.46
Dry mustard beads	4.0	20	1.2	3.33	0.20
Dense emulsions [Goyon <i>et al.</i> , (2010)]	$10\text{--}40 \times 10^{-3}$	varying	6×10^{-3}	$\sim 1.5\text{--}6$	Gap independent
Soft particles suspensions [Seth <i>et al.</i> (2012)]	125×10^{-3}	750×10^{-3}	0.22×10^{-3}	570	0.17
Dry granular flows [Henann and Kamrin (2013)]	2.28	105	1.2	1.9 (if chosen constant)	0.021
Foams [Katgert <i>et al.</i> (2010)]	9	52	~ 3	3	0.17

The only other notable exception (besides our suspension measurements) to a value of ξ that is of the order of the elements of the discrete objects is the results of Seth *et al.* on suspensions of soft particles [Seth *et al.* (2012)], who report that these soft particles shrink when sheared. This would lead to smaller particles and hence perhaps a lower volume fraction where the shear rate is high, suggesting a similar mechanism to the one described here. Performing the same experiment with a different gap to see if ξ varies or remains unchanged would provide useful insights into this matter.

In summary, the nonlocal rheology of suspensions can be correlated using the fluidity model, but the length scale that emerges is not a material property and the model cannot be used for predictive purposes. Rather, the suspension behavior is fully explained as a consequence of stress-driven particle migration and the resulting concentration gradient. The conclusion is less strong for the dry granular system, but it appears likely that the apparent nonlocal behavior is similarly due to the formation of a shear band caused by granular dilatancy.

APPENDIX: ANALYTICAL EXPRESSION FOR THE FLOW CURVES OF THE DENSE NON-BROWNIAN SUSPENSIONS

In this Appendix, we will exhibit an analytical expression for the homogeneous flow curves of the suspensions displayed in Fig. 4 of this paper. This expression will be then used in the fluidity model for f_{bulk} . As noted in the body of the paper, there are two different regimes observed in the homogeneous flow curves: Newtonian behavior at low shear rates and a shear thickening regime for higher rates where the shear stress is proportional to $\dot{\gamma}^2$.

If we define $\dot{\gamma}_c$ as the shear rate where the transition between the Newtonian and the shear thickening regime occurs, we can write

$$\sigma = \eta_0(\phi)\dot{\gamma} \quad \text{if } \dot{\gamma} < \dot{\gamma}_c, \quad (\text{A1})$$

$$\sigma = A(\phi)\dot{\gamma}^2 \quad \text{if } \dot{\gamma} > \dot{\gamma}_c, \quad (\text{A2})$$

where η_0 is the viscosity of the suspension in the Newtonian regime. The crossover between the two regimes occurs when

$$\dot{\gamma} = \dot{\gamma}_c = \frac{\eta_0(\phi)}{A(\phi)}. \quad (\text{A3})$$

If we assume that η_0 and A follow power laws with respect to $(\Phi_{max} - \Phi)$, as in [Fall *et al.* (2010); Lemaître *et al.* (2009)]

$$\eta_0 = C(\Phi_{max} - \Phi)^a \quad \text{and} \quad A = K(\Phi_{max} - \Phi)^b \quad (\text{A4})$$

then using these relations in Eq. (A3) gives a condition on the scaling of $\dot{\gamma}_c$ with $(\Phi_{max} - \Phi)$

$$\dot{\gamma}_c = A(\Phi_{max} - \Phi)^{a-b}. \quad (\text{A5})$$

Fall *et al.* (2010) already noticed that the critical shear rate for dense suspensions varies linearly with $(\Phi_{max} - \Phi)$; Fig. 7 shows that for our system $\dot{\gamma}_c = A(\Phi_{max} - \Phi)$ with $A = 65.0 \text{ s}^{-1}$ and $\Phi_{max} = 0.604$. A critical shear rate varying linearly with volume fraction then implies that $a = b + 1$.

Now, in order to complete our analytical expression for the flow curve, we need to determine a and b . A great deal of work on dry granular media has already been done. In the case of frictional contacts, the exponent b is well documented and has been found to be equal to -2 , in agreement with numerical simulations [da Cruz *et al.* (2005)] and simple models [Henderson (1975); Luding and Santos (2004)] as well as our data. Equation (A5) also implies that if $\dot{\gamma}_c$ varies linearly with volume fraction, and if $b = -2$, then a should equal -1 . Once again, plotting the viscosity in the Newtonian regime versus the volume fraction and performing a power law fit can confirm this hypothesis, as is done in Fig. 8.

It is interesting to see that one can collapse all the shear stress versus shear rate curves for different local volume fractions onto a single master curve. Taking Eqs. (A1) and (A2) and combining them with Eq. (A4), one obtains

$$\sigma = C(\Phi_{max} - \Phi)^{-1} \dot{\gamma} \quad \text{if } \dot{\gamma} < \dot{\gamma}_c, \quad (\text{A6})$$

$$\sigma = K(\Phi_{max} - \Phi)^{-2} \dot{\gamma}^2 \quad \text{if } \dot{\gamma} > \dot{\gamma}_c. \quad (\text{A7})$$

FIG. 7. Critical shear rate at which the suspension is subject to a change from the Newtonian to the shear thickening regime as a function of the volume fraction. The red line is a linear fit $y = A(\phi - \phi_{max})$ with $A = 65.0 \text{ s}^{-1}$ and $\phi_{max} = 0.604$.

FIG. 8. Left: Viscosity of the suspension η_0 in the linear regime as a function of the volume fraction. The dashed line is a power law fit $\eta_0 = C(\phi_{max} - \phi)^{-1}$ with $\phi_{max} = 0.604$ and $C = 4.5 \times 10^{-3}$ Pa.s, the full line is a Krieger–Dougherty fit. Right: Slope of σ vs $\dot{\gamma}^2$ in the shear thickening suspension as a function of the volume fraction. The line is a power law fit $A = K(\phi_{max} - \phi)^{-2}$ with $\phi_{max} = 0.604$ and $K = 6.8 \times 10^{-5}$ Pa.s².

Collapse of all the flow curves implies removing the $(\Phi_{max} - \Phi)$ dependence, which is done by introducing $\bar{\gamma} = \dot{\gamma} / \dot{\gamma}_c$, leading to

$$\sigma = CA \bar{\gamma} \quad \text{if } \bar{\gamma} < 1, \quad (\text{A8})$$

$$\sigma = KA^2 \bar{\gamma}^2 \quad \text{if } \bar{\gamma} > 1. \quad (\text{A9})$$

Figure 4 (right) of the main article shows the extremely good collapse we obtain and tends once again to confirm the viability of our analytical expression. Furthermore, a t-test has been performed on the rescaled shear thickening regime to verify statistically the hypothesis of the quadratic evolution, and the result is positive (at confidence level of 95%).

A final thing one should notice is that regardless of the volume fraction, the transition between the Newtonian and the shear thickening regime occurs at the same critical shear stress $\sigma_c = C(\Phi_{max} - \Phi)^{-1} \dot{\gamma}_c = K(\Phi_{max} - \Phi)^{-2} \dot{\gamma}_c^2 = CA = KA^2$. A pertinent check on the validity of our treatment is to verify that $\sigma_c = CA = KA^2$. We find that $KA^2 = 0.29$ Pa and $CA = 0.28$ Pa.

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