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Kant's Logic Revisited

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Abstract

Kant considers his *Critique of Pure Reason* to be founded on the act of judging and the different forms of judgement, hence, take pride of place in his argumentation. The consensus view is that this aspect of the *Critique of Pure Reason* is a failure because Kant's logic is far too weak to bear such a weight. Here we show that the consensus view is mistaken and that Kant's logic should be identified with geometric logic, a fragment of intuitionistic logic of great foundational significance.

1 Preview

Below the reader will find a condensed revisionist account of Kant's so-called 'general logic', usually thought to be substandard, even when compared with the traditional logic of his day [4].[1] Ultimately our interest is in the formalisation of Kant's 'transcendental logic' (for which see [1]), but since transcendental logic takes its starting

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[1]Not to mention the scathing verdicts from the standpoint of modern logic which we take to have started with Frege and Strawson.

point in the judgement forms listed in the Table of Judgement (most of which have their origin in general logic) we must take a close look at the actual logical forms of these judgements. The result of this investigation is that Kant's general logic is not monadic, not finitary, not classical, and perhaps linear rather than intuitionistic. We will here not elaborate on the last point[2] but we will restrict ourselves to stating a completeness theorem identifying Kant's general logic with a fragment of intuitionistic logic.

2 Validity in general logic

The key to any insightful formalisation of Kant's logic is the observation that judgements in Kant's sense participate in two kinds of logics: *general* logic and *transcendental* logic. Here is how Kant introduces 'general logic' in the first *Critique* [7]:

> [G]eneral logic abstracts from all the contents of the cognition of the understanding and of the difference of its objects, and has to do with nothing (A55-6/B80) but the mere form of thinking. (A54/B78)

And later, with a slightly different emphasis:

> General logic abstracts [...] from all content of cognition, i.e. from any relation of it to the object, and considers only the logical form in the relation of cognitions to one another, i.e. the form of thinking in general. (A55/B79)

So what is the 'mere form of thinking'?

The first two paragraphs of the *Jäsche Logik* [5] marvel at the fact that all of nature, including ourselves, is bound by rules. It continues:

> Like all our powers, the understanding is bound in its actions to rules [...] Indeed, the understanding is to be regarded in general as the source and the faculty for thinking rules in general [...] [T]he understanding is the faculty for thinking, i.e. for bringing the representations of the senses under rules.

From this it derives a characterisation of logic:

> Since the understanding is the source of rules, the question is thus, according to what rules does it itself operate? [...] If we now put aside all cognition that we have to borrow from objects and merely reflect on the use just of the understanding, we discover those of its rules which are necessary without qualification, for any purpose and without regard to any particular objects, because without them we would not think at all. [...] [T]his science of the

[2]Grigori Mints was planning on studying the connection between Kant's disjunctive judgement and multiplicative linear logic.

> necessary laws of the understanding and of reason in general, or what is one and the same, of the mere form of thought as such, we call *logic*. [5, pp. 527-8] (cf. also A52/B76)

To appreciate the real import of this passage, one must resist the temptation to consider logic as consisting of a motley set of inference rules, such as modus ponens and syllogistic inferences, even though the *Jäsche Logik* will later list these too. Two definitions are pertinent here:

> §58 A rule is an assertion under a universal condition. [5, p. 615]

Here it is important to bear in mind Kant's notion of *universal representation* as 'a representation of what is common in several objects' [5, §1, p. 589]. A rule is, therefore, applicable to a domain of indefinite extension.

The second definition is that of an *inference of reason*:

> §56 An inference of reason is the cognition of the necessity of a proposition through the subsumption of its condition under a given universal rule. [5, p. 614]

At this point we will not yet provide an elaborate explanation of the notion of 'condition', but the reader is invited to take modus ponens as a concrete example. We then have the following sequence of ideas: (i) the understanding operates according to rules, (ii) the understanding's operations are necessary insofar as they pertain to the formal features of rules, and (iii) the most general formal principle is rule-application (or rule composition – as we shall see the distinction was not always made in those days). Thus Kant's logic has a general and constructive definition of validity, a consequence of the meaning of 'rule'. The *Jäsche Logik* will give concrete instances of this most general principle, such as modus ponens, but the full force of the principle will only become apparent when we come to discuss the true logical form of Kant's 'judgements'. We must note here that the general inference principle limits logic to judgements that can be seen as rules. We view Kant's emphasis on rules and their structural properties as marking the 'formal' character of his general logic. The definition of validity just given should be contrasted with the Bolzano-Tarski definition of validity: 'an argument is valid if its conclusion is true whenever its premises are' – for in this part of Kant's logic (what he calls 'general logic') there is no truth yet, there are only rules. A different kind of logic, 'transcendental logic' will introduce truth.

3 Three definitions of judgement and a Table ...

Any modern logic textbook makes a strict separation between syntax, semantics and consequence relation, and makes no reference at all to psychological processes that may be involved in a concrete case of asserting a syntactically well-formed sentence. These processes are studied in psycholinguistics, and start from the assumption that there are specific syntactic and semantic binding processes at work in the brain. For logical theorising such psycholinguistic approaches are deemed to be irrelevant. For Kant they are in fact of the essence, and his definitions of judgement also contain a cognitive component.

But the reader trying to piece together Kant's views on logic may be forgiven a sense of bewilderment when she finds not one but three seemingly very different definitions of 'judgement', none of which specifies a syntactic form, together with a 'Table of Judgement' which specifies some syntactic forms (for example, categorical, hypothetical, disjunctive, with various other subdivisions), without an indication of how these forms relate to the three definitions. Lastly, there are the examples of judgements that Kant uses in various works, whose logical forms do not fit easily in the Table of Judgement. This looks unpromising material, but we shall show that Kant's logic is nevertheless coherent and surprisingly relevant to modern concerns.

Let us begin with the three definitions of judgement:

> A judgement is the representation of the unity of the consciousness of various representations, or the representation of their relation insofar as they constitute a concept. [5, p. 597]

> A judgement is nothing but the manner in which given cognitions are brought to the objective unity of apperception. That is the aim of the copula **is** in them: to distinguish the objective unity of given representations from the subjective [...] Only in this way does there arise from this relation a judgement, i.e. a relation that is objectively valid [...][3] (B141-2)

> Judgements, when considered merely as the condition of the unification of representations in a consciousness, are rules. (*Prol.* § 23; see [8])

Even for those unfamiliar with Kant's technical vocabulary it will be obvious that 'unity' plays a central role in all three definitions. These are different ways of saying that the expressions occurring in a judgement must be bound together so that they can be simultaneously present to consciousness. The first definition posits unity simply as a requirement. The second says that unity in a judgement is achieved if the judgement has 'relation to an object'. The third definition links unity to the meaning of a judgement. Just as an example: if for a hypothetical judgement $\varphi \rightarrow \psi$ there exists a rule transforming a proof of φ into a proof of ψ, then that judgement

[3] Where 'objectively valid' means 'having relation to an object', which is not the same as 'true of the object'.

is unified. If the hypothetical is a truth functional material implication, then antecedent and consequent are independent, hence this is not a unified representation. The presence of a notion of unity of representation raises three questions: (i) what has this got to do with formal logic?, (ii) is there a relation between the unity and the reference to objects occurring in the second definition? and (iii) what is the relation between unity and the concrete forms of judgement given in the Table of Judgement?

3.1 Objects, concepts and general logic

Categorical judgements are composed of concepts, and objects 'fall under' concepts,[4] in a sense hinted at in the following note:

> *Refl.* **3042** Judgement is a cognition of the unity of given concepts: namely, that B belongs with various other things x, y, z under the same concept A, or also: that the manifold which is under B also belongs under A, likewise that the concepts A and B can be represented through a concept B. [9, p. 58]

It appears that both concepts and objects may fall under a given concept C. The given concept is therefore *transitive* in the sense that if (concept) M belongs to C (by being a subconcept) and (object) a belongs under M, then a belongs under C. Kant uses this semantics for concepts in his 'principle for categorical inferences of reason':

> *What belongs to the mark of a thing also belongs to the thing itself.* [5, p. 617]

The next note supplies more information about these objects 'in the logical sense' (so called because they make a cameo appearance in the section 'The logical employment of the understanding' (A68-9/B93)).

> *Refl.* **4634** We know any object only through predicates that we can say or think of it. Prior to that, whatever representations are found in us are to be counted only as materials for cognition but not as cognition. Hence an object is only a something in general that we think through certain predicates that constitute its concept. In every judgment, accordingly, there are two predicates that we compare with one another, of which one, which comprises the given cognition of the object, is the logical subject, and the other, which is to be compared with the first, is called the logical predicate. If I say: a body is divisible, this means the same as: Something x, which I cognize under the predicates that together comprise the concept of a body, I also think through the predicate of divisibility. [9, p. 149]

[4] Kant also uses the phrases 'object a belongs under concept C' and 'C belongs to a'.

What this *Reflexion* tells us is that an object is generic (or most general) for the 'predicates that constitute its concept', and that the quantifier 'something x' ranges over such generic objects only.

The same idea is prominent in the section of CPR entitled 'On the logical use of the understanding in general':

> [T]he understanding can make no other use of concepts than that of judging by means of them. Since no representation pertains the object immediately except intuition alone, a concept is thus never immediately related to an object, but is always related to some other representation of it (whether that be an intuition or itself already a concept). Judgement is therefore the mediate cognition of an object, hence the representation of a representation of it. (A68/B93)

An object is therefore rather like what logicians call a *type*: i.e. a set[5] $p(x)$ of formulas containing at least the free variable x;[6] free variables not identical to x can be replaced by formal parameters representing objects, hence specified by a type. As an example, consider the predicate 'body' and the type 'x is a massive body which orbits star y' – which can be used to defined the predicate 'planet', by existential quantification over y or by replacing y by a formal parameter (representing the Sun, say). Let T be the theory of the relevant concepts. If M is a concept, we say that $M(x)$ belongs to $p(x)$ if $T, p(x) \vdash M(x)$. For example, if T contains

$$\forall x(A(x) \wedge \exists y B(x, y) \to M(x)),$$

then $p(x) = \{A(x), \exists y B(x, y)\}$ belongs to $M(x)$. It is technically convenient to introduce suitable constants witnessing a type: if $p(x)$ is a (consistent) type, let a_p be a new constant satisfying $p(a_p)$.[7] These constants correspond to the 'objects in general' that we encountered in *Reflexion* **4634**. One may then view $p(x)$ and a_p as determining the same object; and in this formal sense we have that M belongs to a_p.

The next question to consider is whether Kant's theory of concepts puts a bound on the complexity of concepts, i.e. the complexity of the types belonging under the concept. The $p(x)$ given in the previous paragraph can be viewed as a single *positive primitive* formula:

Definition 1. A formula is *positive primitive* if it is constructed from atomic formulas using only $\vee$, (infinite) $\bigvee, \wedge, \exists, \bot$.

[5]In our context a finite set.

[6]Relations enter Kant's logic especially in connection with the hypothetical judgement (see section 3.3.2); furthermore, as Hodges observed in [4], traditional logic allowed relations in syllogisms.

[7]The constant a_p implicitly depends on the parameters and free variables (x excluded) occurring in $p(x)$.

Suppose M, P are concepts all of whose subconcepts can be defined using positive primitive types (equivalently, formulas). The judgement 'all M are P' – or in the language of *Reflexion* **4634**: 'To everything x, to which M belongs, also P belongs – may then be expressed as

$$\bigwedge_{p \in M} \bigvee_{q \in P} \forall x(p(x) \to q(x)),$$

which is equivalent to

$$\forall x(\bigvee_{p \in M} p(x) \to \bigvee_{q \in P} q(x)),$$

and this formula satisfies the definition of a *geometric implication*:

Definition 2. A formula is *geometric* or a *geometric implication* if it is of the form $\forall \bar{x}(\theta(\bar{x}) \to \psi(\bar{x}))$, where θ and ψ positive primitive.

As it turns out, Kant's theories of concepts and of judgements contain the resources to restrict the complexity of $p(x)$ to positive primitive. The reason for this is that the complexity of the relation '$M(x)$ belongs to $p(x)$' is at most that of geometric implications. For the proof we must refer the reader to [1]; but a sketch will be given in section 4.

Geometric logic – the inferential relationships between geometric formulas – is therefore naturally suggested by Kant's theory of concepts. We will see that the logical form of Kant's own examples of judgements (in so far as they are 'objectively valid' (see section 3.2)) is that of geometric implications. As a consequence, we can show by means of 'dynamical proofs' of geometric implications that judgements can be viewed as rules:

> Judgements, when considered merely as the condition of the unification of representations in a consciousness, are rules. (*Prol.* §23; see [8])

3.2 Unity, objects and transcendental logic

The second characterisation of judgement maintains that if a judgement has a certain kind of unity (the 'objective unity of apperception') then it relates to an object – has 'objective validity' – and can express a truth or falsehood of that object; it is 'truth-apt', in modern terminology. This is the domain of *transcendental logic*, which Kant defines as follows:

> [...] a science of pure understanding and of the pure cognition of reason, by means of which we think objects completely a priori. Such a science, which would determine the origin, the domain, and the objective validity of such

> cognitions, would have to be called transcendental logic since it has to do merely with the laws of the understanding and reason, but solely insofar as they are related to objects a priori and not, as in the case of general logic, to empirical as well as pure cognitions of reason without distinction. (A57/B81-2)

For Kant, perceiving objects about which judgements can be made is an instance of what would now be called the binding problem: objects are always given as a 'manifold' of parts and features, which have to be bound together through a process of *synthesis*. What is very distinctive about Kant's treatment here is that the binding that binds expressions in judgement together at the same time binds parts and features together with a view toward constructing an object out of sensory material that relates to the judgement. Therefore the binding process, necessary to bring separately perceived parts and features together, is in the end a complex logical operation, described by transcendental logic:

> Transcendental logic is the expansion of the elements of the pure cognition of the understanding and the principles without which no object can be thought at all (which is at the same time a logic of truth). For no cognition can contradict it without at the same time losing all content, i.e. all relation to any object, hence all truth. (A62-3/B87)

In the *Critique*, transcendental logic is not recognisably presented as a logic, and it is commonly thought that it cannot be so presented. The article [1] shows otherwise, mainly by focussing on the semantics of transcendental logic. There is a vast difference between the notion of object as it occurs in first order models, and in Kant's logic. In the former, objects are mathematical entities supplied by the metatheory, usually some version of set theory. These objects have no internal structure, at least not for the purposes of the model theory. Kant's notions of object, as they occur in the semantics furnished by transcendental logic, are very different. For instance, there are 'objects of experience', somehow constructed out of sensory material; transcendental logic deals with *a priori* and completely general principles which govern the construction of such objects, and relate judgements to objects so that we may come to speak of *true* judgements.

3.3 The Table of Judgement (A70/B95)

The three definitions describe judgement either in terms of certain cognitive operations ('unity of representations') or in terms of a function that a judgement has to perform (establishing 'relation to an object'). There is no hint of a specific form of judgement here. We find such hints in the Table of Judgement, but there we do not find a comparison with definitions of judgement; e.g. the *Critique*'s definition occurs only at (B141-2), way after the Table of Judgement is introduced. This raises the problem of how we know that the forms proposed in the Table satisfy the three definitions, and conversely, how for instance the functional characterisation given at (B141-2) leads to specific forms of judgement.

We now turn to the forms of judgement listed in the Table of Judgement, and we discuss (some of) the inferences in which these judgements participate, in part to emphasise the many differences between Kant's logic and modern logic[8] We will also comment on the relation between the Table of Judgement and the Table of Categories (A80/B106), although a full treatment is beyond the scope of this paper.

We will begin our discussion with the title 'Relation' (A70/B95), where we find

Relation
Categorical
Hypothetical
Disjunctive

3.3.1 Categorical judgements

These are judgements in subject-predicate form, combined with quantifiers and optional negation, which can occur on the copula and on the concepts occurring in the judgement. The Table of Judgement further specifies categorical judgements with regard to Quantity and Quality:

Quantity
Universal
Particular
Singular

In the Table of Categories we find a corresponding list of 'pure concepts of the understanding':

Of Quantity
Unity
Plurality
Totality

The precise correspondence between judgement forms and Categories is a matter of controversy. Here we argue on logical grounds that Kant intended a correspondence between the universal judgement and Unity, between the particular judgement and Plurality, and between the singular judgement and Totality.[9]

[8]See note 1.

[9]See Frede and Krüger [3] for a different correspondence linking the singular judgement and Unity.

As explained in section 3.1, the universal judgement 'all M are P', or as Kant would have it 'To everything x to which M belongs, also P belongs', should not be interpreted as the classical $\forall x(M(x) \to P(x))$, but as

$$\forall x(\bigvee_{p \in M} p(x) \to \bigvee_{q \in P} q(x));$$

and because the subject is maintained 'assertorically', not 'problematically', we require that the types in M do not contain $\bot$. These types are therefore satisfiable – meaning that the (nonempty) collection of M's is given as that which the judgement is about, and the quantifier 'To everything x' is restricted to M, not to some universe of discourse.

The association 'universality – unity' is motivated by the fact that in the universal judgement 'all M are P' the predicate P makes no distinctions among the things falling under the subject M. Relative to P, M can hence be taken as a unit.

The things falling under M form a plurality that is not a unity (with respect to the predicate P) if there are true particular judgements 'some M are P' and 'some M are not P'.

In an unpublished note about the relation between universal and singular judgement, Kant writes:

> *Refl.* **3068** In the universal concept the sphere [=extension] of a concept is entirely enclosed in the sphere of another concept; [. . .] in the singular judgement, a concept that has no sphere at all is consequently merely enclosed as a part under the sphere of another concept. Thus singular judgements are to be valued equally with the universal ones, and conversely, a universal judgement is to be considered a singular judgement with regard to the sphere, much as if it were only one by itself. [9, p. 62]

Now consider (B111), where we read 'Thus **allness** (totality) is nothing other than plurality considered as a unity [. . .]'

Taking a plurality M to be a totality involves considering M as a unity, which means that a pair of judgements 'some M are P' and 'some M are not P' is replaced by one of 'all M are P' and 'all M are not P'. M is thus totally determined with respect to the available predicates. Since M cannot be divided using a predicate, this means that the concept M is used singularly, and hence a universal judgement 'all M are P' can equivalently be regarded as the singular judgement 'M is P', whence the correspondence between the singular judgement and totality.

Quality
Affirmative
Negative
Infinite

There is no need for our present purposes to dwell extensively on this Category, except to say that Kant makes a distinction between sentence negation as in the negative particular judgement 'some A are not B' and predicate negation, represented by the infinite judgement 'some A are non-B', which is affirmative but requires infinitary logic for its formalisation: $\bigvee_{B \cap C = \emptyset}$ (some A are C). Hence Kant's logic is not finitary. The difference with classical first order logic will only increase as we go on.

3.3.2 Hypothetical judgements

It would be a mistake to identify Kant's hypothetical judgements with a propositional conditional $p \to q$, let alone material implication as defined by its truth table: a material implication need not have any rule-like connection between antecedent and consequent. Here is the definition in the *Jäsche Logik*:

> The matter of hypothetical judgements consists of two judgements that are connected to each other as ground and consequence. One of these judgements, which contains the ground, is the *antecedent*, the other, which is related to it as consequence, is the *consequent*, and the representation of this kind of connection of two judgements to one another for the unity of consciousness is called the *consequentia* which constitutes the form of hypothetical judgements. [5, p. 601, par. 59][10]

This definition seems to say that the hypothetical is a propositional connective, and some of Kant's examples fall into this category:

> If there is perfect justice, then obstinate evil will be punished. (A73/B98)

However, other examples exhibit a more complex structure, involving relations, variables and binding. In the context of a discussion of the possible temporal relations between cause and effect Kant writes in *CPR*:

> If I consider a ball that lies on a stuffed pillow and makes a dent in it as a cause, it is simultaneous with its effect. (A203/B246)

The hypothetical that can be distilled from this passage is:

> If a ball lies on a stuffed pillow, it makes a dent in that pillow.

From this we see that (i) the antecedent and consequent need not be closed judgements but may contain variables, and (ii) antecedent and consequent may contain relations and existential quantifiers.

[10] Here it is of interest to observe that in the same paragraph *consequentia* is also used to refer to an inference.

We now give an extended quote from the *Prolegomena* §29 [8] which provides another example of a hypothetical judgement whose logical structure likewise exhibits the features listed in (i) and (ii) above:

> It is, however, possible that in perception a rule of relation will be found, which says this: that a certain appearance is constantly followed by another (though not the reverse); and this is a case for me to use a hypothetical judgement and, e.g., to say: If a body is illuminated by the sun for long enough, it becomes warm. Here there is of course not yet the necessity of connection, hence not yet the concept of cause. But I continue on, and say: if the above proposition, which is merely a subjective connection of perceptions, is to be a proposition of experience, then it must be regarded as necessarily and universally valid. But a proposition of this sort would be: The sun through its light is the cause of the warmth. The foregoing empirical rule is now regarded as a law, and indeed as valid not merely of appearances, but of them on behalf of a possible experience, which requires universally and therefore necessarily valid rules [...] the concept of a cause indicates a condition that in no way attaches to things, but only to experience, namely that experience can be an objectively valid cognition of appearances and their sequence in time only insofar as the antecedent appearance can be connected with the subsequent one according to the rule of hypothetical judgements. [8, p. 105]

The logical form of the first hypothetical (a 'judgement of perception') is something like:

> If x is illuminated by y between time t and time s and $s - t > d$ and the temperature of x at t is v, then there exists a $w > 0$ such that the temperature of x at s is $v + w$ and $v + w > c$,

where d is the criterion value for 'long enough' and c a criterion value for 'warm'. We find all the ingredients of polyadic logic here: relations and quantifier alterations. The causal connection which transforms the judgement into a 'judgement of experience' arises when the existential quantifiers are replaced by explicitly definable functions.

We now move on to the logical properties of the hypothetical judgement. Here it is of some importance to note that the term *consequentia*, characterising the logical form of the hypothetical, is also used to describe the inferences from the hypothetical:

> The *consequentia* from the ground to the grounded, and from the negation of the grounded to the negation of the ground, is valid. [5, p. 623]

Furthermore, the negation of a hypothetical is not defined.[11] This strongly suggests that the hypothetical judgement is really a license for inferences. Indeed, in the

[11]Note that the negation of a categorical judgement is defined, although its properties do seem

Jäsche Logik Kant characterises inferences such as modus ponens and modus tollens as immediate inferences and as such needing only one premise, not two premises [5, p. 623]. Modern proof systems conceive of modus ponens as a two-premise inference, p implies q and p, therefore q. But Kant does not think of it in this way. He thinks of it as an inference with premise p, conclusion q, which is governed by a license for inference. This strongly suggests that Kant does not have a single entailment relation, as in modern logic,[12] but only local entailment relations defined by specific inferences. We end this discussion of the hypothetical judgement with a further twist: its logical properties change when it is considered in a causal context, i.e. in transcendental logic:

> When the cause has been posited, the effect is posited ‹*posita causa ponitur effectus*› already flows from the above. But when the cause has been cancelled, the effect is cancelled ‹*sublata causa tollitur effectus*› is just as certain; when the effect has been cancelled, the cause is cancelled ‹*sublato effectu tollitur causa*› is not certain, but rather the causality of the cause is cancelled ‹*tollitur causalitas causae*›. [6, p.336-7]

3.3.3 Disjunctive judgements

These are again not what one would think, judgements of the form $p \vee q$. The *Jäsche Logik* provides the following definition:

> A judgement is disjunctive if the parts of the sphere of a given concept determine one another in the whole or toward a whole as complements [...] [A]ll disjunctive judgements represent various judgements as in the community of a sphere [...] [O]ne member determines every other here only insofar as they stand together in community as parts of a whole sphere of cognition, outside of which, in a certain relation, nothing may be thought.(*Jäsche Logik*, §27, 28) [5, pp. 602-3]

As examples Kant provides:

> Every triangle is either right-angled or not right-angled.
> A learned man is learned either historically, or in matters of reason.

Thus the logical form is something like $\forall x(C(x) \to A(x) \vee B(x))$, where C represents the whole, A, B its parts; here it is not immediately clear whether the parts can be taken to exist outside the context of the whole. But actually the situation is much

to be weaker than classical negation: 'some A are not B' is the negation of 'All A are B', but it is a moot point whether the negative particular judgement has existential import. Its infinitive counterpart does have existential import.

[12] See Hodges [4] for relevant discussion.

more complicated. The *Jäsche Logik* equivocates between concepts and judgements making up the whole, and this is intentional, as we read in the *Vienna Logic*:

> The disjunctive judgment contains the relation of different judgment insofar as they are equal, as *membra dividentia*, to the *sphaera* of a *cognitio divisa*. E.g., All triangles, as to their angles, are either right-angled or acute or obtuse. I represent the different members as they are opposed to one another and as, taken together, they constitute the whole *sphaera* of the *cognitio divisa*. This is in fact nothing other than a logical division, only in the division there does not need to be a *conceptus divisus*; instead, it can be a *cognitio divisa*. E.g., If this is not the best world, then God was not able or did not want to create a better one. This is the division of the *sphaera* of the cognition that is given to me. [5, p. 374-5]

So it is not just concepts that can be divided in the familiar way, also cognitions (*Erkenntnisse*), including judgements, can be so divided. What this means for the complexity of Kant's logic can be seen if we look at the expanded example in the *Dohna-Wundlacken Logic*:

> If this world is not the best, then God either was unfamiliar with a better [one] or did not wish to create it or could not create [it], etc. Together these constitute the whole *sphaera*. [5, p. 498]

It will be instructive to formalise this example. Let w_0 be the actual world, G a constant denoting God, let $\mathsf{B}(w_0, w)$ represent 'w is a better world than w_0, and let $Uf(\mathsf{G}, w)$, $Uw(\mathsf{G}, w)$, $Uc(\mathsf{G}, w)$ represent: 'God was unfamiliar with w', 'God was unwilling to create w' and 'God was unable to create w', respectively. We then get the combined hypothetical-disjunctive judgement:

$$\exists w \mathsf{B}(w_0, w) \to \forall w(\mathsf{B}(w_0, w) \to (Uf(\mathsf{G}, w) \vee Uw(\mathsf{G}, w) \vee Uc(\mathsf{G}, w))).$$

It is to be noted that this hypothetical-disjunctive judgement consists entirely of relations, and that the division is formulated in terms of singular judgements containing a parameter ('God') and a variable. As in the case of the hypothetical judgement, the negation for a disjunctive judgement is not defined, which suggests that it is actually a license for inferences, using quantified forms of the disjunctive syllogism, for example:

1. Starting from the premise 'God is familiar with a better world' (which is taken to imply $\exists w(\mathsf{B}(w_0, w) \wedge \neg Uf(\mathsf{G}, w)))$ now introduces the positive primitive formula $\exists w(\mathsf{B}(w_0, w) \vee (Uw(\mathsf{G}, w) \vee Ua(\mathsf{G}, w)))$.

2. Similarly the premise 'God is familiar with all better worlds' yields the formula $\forall w(\mathsf{B}(w_0, w) \to (Uw(\mathsf{G}, w) \vee Ua(\mathsf{G}, w)))$.

Kant evidently believes these inferences are perfectly proper cases of the disjunctive syllogism, but the present-day reader may well ask whether his general logic has the resources to break down these inferences in smaller steps. But if the hypothetical and the disjunctive judgement are licenses for inferences, this means that they can be taken as given as far as general logic is concerned (much like a Prolog program is taken as given and is used only to derive atomic facts). This somewhat eases the burden on general logic, in the sense that it need not have the resources to prove hypothetical and disjunctive judgements.

As we did for the hypothetical judgement, we will also look at the intended transcendental use of the disjunctive judgement:

> The same procedure of the understanding when it represents to itself the sphere of a divided concept, it also observes in thinking of a thing as divisible; and just as in the first case the members of the division exclude each other, and yet are connected in one sphere, so in the latter case the understanding represents to itself the parts of the latter as being such that existence pertains to each of them (as substances) exclusively of the others, even while they are combined together in one whole. (B113)

The disjunctive judgement is said to involve the cognitive act of dividing a thing, while keeping the resulting parts simultaneously active in one representation. Here we are concerned with the logical principles that Kant's disjunction satisfies. Kant gives as inferences valid for a disjunctive judgement $\mathsf{C} \to \mathsf{A} \vee \mathsf{B}$, the two halves of the so-called disjunctive syllogism:

C and $\neg\mathsf{A}$ implies B
C and A implies $\neg\mathsf{B}$.

These inference rules are considerably weaker than those that are valid for the classical or intuitionistic disjunction, and remind one of the multiplicative disjunction of linear logic. Can one impose stronger inference rules on the disjunction? That is doubtful. For example, the standard right disjunction rule in sequent calculus:

$$\frac{\Gamma \Rightarrow \mathsf{A}, \Delta}{\Gamma \Rightarrow \mathsf{A} \vee \mathsf{B}, \Delta}$$

is invalid for Kant, because it allows the addition of an arbitrary B to A, without the guarantee that A, B constitute a whole.

An additional consideration is the connection with divisibility; here the parts must be present simultaneously, which is what the rule just given expresses. This formulation lends some credibility to Kant's association of the disjunctive judgement with the category of simultaneity in the third Analogy of Experience. However, the new formulation raises the issue of what one should say if A and B are identical. Kant makes an important distinction between two kinds of identity in 'On the amphiboly of concepts of reflection':

> If an object is presented to us several times, but always with the same inner determinations, then it is always exactly the same if it counts as an object of pure understanding, not many but only one thing; but if it is appearance, then [...] however identical everything may be in regard to [concepts], the difference of the places of these appearances at the same time is still an adequate ground for the numerical difference of the object (of the senses) itself. Thus, in the case of two drops of water one can completely abstract from all inner difference (of quality and quantity), and it is enough that they be intuited in different places at the same time for them to be held to be numerically different. (A263-4/B319-20)

Suppose one has a 'whole' that is divided into spatially distinct parts that have 'the same inner determinations'. This hypothetical situation suggests that a logic for Kant's disjunction does not include a rule for (right) contraction:

$$\frac{\Gamma \Rightarrow A, A, \Delta}{\Gamma \Rightarrow A, \Delta}$$

But in that case also the standard rule for left disjunction introduction:

$$\frac{\Gamma, A \Rightarrow \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta}$$

must be dropped because otherwise right contraction becomes derivable. Instead, one would have a rule like:

$$\frac{\Gamma, A \Rightarrow \Delta \qquad \Gamma, B \Rightarrow \Delta'}{\Gamma, A \vee B \Rightarrow \Delta, \Delta'}$$

3.4 Logical form of judgements

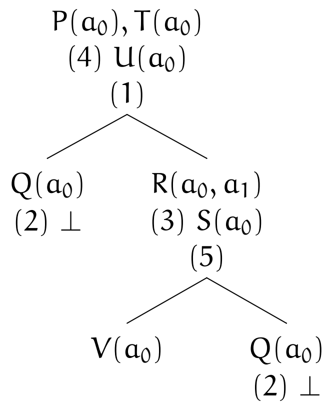
Looking back at our examples we see that, with one exception (the negative particular judgement, which, as discussed in [1] was meant by Kant to be purely negative), they are all geometric judgements. Geometric logic, i.e. the logic of geometric formulas, plays an important role in several branches of mathematics, Euclidean geometry being one but not the only example. More germane to our purposes is a result in [1], which shows that all objectively valid judgements in the sense of (B141-2) must be finite conjunctions of geometric implications.

3.5 'Functions of unity in judgements': dynamical proofs

In a dynamical proof one takes a geometric theory[13] as defining a consequence relation holding between two sets of facts. An example, taken from Coquand [2], illustrates the idea. The theory is:[14]

1. $P(x) \wedge U(x) \rightarrow Q(x) \vee \exists y R(x, y)$
2. $P(x) \wedge Q(x) \rightarrow \perp$
3. $P(x) \wedge R(x, y) \rightarrow S(x)$
4. $P(x) \wedge T(x) \rightarrow U(x)$
5. $U(x) \wedge S(x) \rightarrow V(x) \vee Q(x)$

And here is an example of a derivation of $V(a_0)$ from $P(a_0), T(a_0)$:



We give some comments on the derivation. The dynamical proof just given can also be taken to prove $\forall x(P(x) \wedge T(x) \rightarrow V(x))$, where the proof is the link between antecedent and consequent, hence a 'function of unity'. Furthermore, the geometric theory defines the consequence relation, hence the geometric implications occurring in it can be seen as inference rules. Disjunctions lead to branching of the tree, as we see in (1) and (5). The existential quantifier in formula (1) introduces a new term in the proof, here a_1, which appears in the right branch of (1). This constant is the 'object in general' of *Reflexion* **4634**. Lastly, a fact is derivable if it appears on

[13] We assume the geometric implications in the theory have antecedents consisting of conjunctions of atomic formulas only.

[14] We omit the universal quantifiers.

every branch not marked by $\perp$, which leaves $V(a_0)$. If X is a collection of facts whose terms are collected in I, F a fact with terms in I, and T a geometric theory, then there exists a dynamical proof of F from X if and only if $T, X \vdash F$ in intuitionistic logic.

It is clear how a dynamical proof of a geometric implication from a geometric theory proceeds: if T is the geometric theory and $\forall\bar{x}(\tau(\bar{x}) \to \theta(\bar{x}))$ the geometric implication (τ is a conjunction of atomic formulas, and for simplicity take θ an existentially quantified conjunction θ' of atomic formulas; we interpret θ' as a set), choose new terms not occurring in either T or $\forall\bar{x}(\tau(\bar{x}) \to \theta(\bar{x}))$, plug these terms into τ and construct a dynamical proof tree with the sets θ' at the leaves. There may occur terms in θ' not in τ; these have to be quantified existentially. Introduce any other existential quantifiers on θ' as required by θ. The result is an intuitionistic derivation of $\forall\bar{x}(\tau(\bar{x}) \to \theta(\bar{x}))$ from T. Conversely, if there is an intuitionistic derivation of $\forall\bar{x}(\tau(\bar{x}) \to \theta(\bar{x}))$ from T, then there exists a dynamical proof in the sense just sketched.

Dynamical proofs as a semantics for geometric implications can explain Kant's characterisation of judgements as rules, as well as 'a unity of the consciousness of various representations'; after all, the diagram represents 'unity' as a single spatial representation. What remains to be done is to situate a judgement's 'objective validity' relative to its other properties.

4 Completeness of the Table of Judgement

In [1] it is argued that (i) Kant's implied semantics for logic is radically different from that of classical first order logic, (ii) the implied semantics, centered around Kant's three different notions of object, can be given a precise mathematical expression, thus leading to a formalised transcendental logic, and (iii) on the proposed semantics, Kant's formal logic turns out to be geometric logic.

It is not appropriate to repeat the technical exposition here, so we will follow a different strategy starting from Kant's most fundamental characterisation of judgement:

> A judgement is nothing but the manner in which given cognitions are brought to the objective unity of apperception. (B141)

A judgement is the act of binding together mental representations; this is what the term 'unity' refers to. The aim of judgement is indicated by means of the word 'objective', which is Kant's terminology for 'having relation to an object'. But for Kant, objects are not found in experience, but they are constructed ('synthesised') from sensory matter under the guidance of the Categories, which are defined as 'concepts of an object in general, by means of which the intuition of an object is regarded as determined in respect of one of the logical functions of judgement' (B128). It is here that judgement plays an all-important role, since Kant's idea is that objects are synthesised through the act of making judgements about them.

Technically, these acts of synthesis are modelled as a kind of possible worlds structure (an 'inverse system'), where the possible worlds are finite first order models whose elements are partially synthesised objects, except for the unique top-world (the 'inverse limit') which represents (the idea of) fully synthesised objects. Bringing a (formal) judgement φ to the 'objective unity of apperception' is now characterised by the property: for any such possible worlds structure, if φ is true on all worlds, then φ is also true on the top-world. That is to say, if φ is true for all stages of synthesis of an object, then φ is true of some fully synthesised object. Kant calls judgements φ satisfying this conditional property 'objectively valid.' It turns out that the objectively valid formulas are exactly the geometric formulas. It follows that no judgement whose logical form is more complex than that allowed by the Table of Judgement can be objectively valid, i.e. this Table is complete.

It is of some interest that the key idea in the proof sheds light on Kant's logical reinterpretation of the Categories of Quantity as constraints on concepts (B113-6):

> In every cognition of an object there is, namely, **unity** of the concept, which one can call **qualitative unity** insofar as by that only the unity of the comprehension of the manifold of cognition is thought, as, say, the unity of the theme in a play, a speech, or a fable. Second, truth in respect of the consequences. The more true consequences from a given concept, the more indication of its objective reality. One could call this the **qualitative plurality** of the marks that belong to a concept as a common ground ... Third, finally, **perfection**, which consists in plurality conversely being traced back to the unity of the concept, and agreeing completely with this one and no other one, which one can call **qualitative completeness** (totality).

The phrase 'unity of the theme in a play' is probably a reference to Aristotle's 'unity of action' in tragedy, where

> the structural union of the parts [must be] such that, if any one of them is displaced or removed, the whole will be disjointed and disturbed. For a thing whose presence or absence makes no visible difference, is not an organic part of the whole (*Poetics*, VIII).

Hence we read 'qualitative unity' as the requirement that the concept under consideration is integrated with other concepts by means of a theory, and is invariant under structure-preserving mappings (homomorphisms). The latter requirement forces all subconcepts of the given concept to have the same logical complexity. We are now in a position to spell out the logical meaning of B113-6 in formal terms.

Let C be a concept which satisfies 'qualitative unity' and let T be the first order theory witnessing 'qualitative unity'. Define a 'qualitative plurality' Σ by

$$\Sigma(x) = \{\theta(x) \mid \mathsf{T} \models \forall x(\mathsf{C}(x) \to \theta(x)), \theta \text{ pos. prim.}\}.$$

Because we may have, for each θ, 'some θ aren't C', for all we know Σ could be a proper plurality. But 'qualitative completeness' now becomes provable:

$$\Sigma(x), \mathsf{T} \models \mathsf{C}(x),$$

hence by compactness there is positive primitive $\tau(x)$ such that

$$\mathsf{T} \models \forall x(\tau(x) \leftrightarrow \mathsf{C}(x)).$$

It follows that, as announced in section 3.1, universal judgements 'all M are P' can be expressed as geometric implications, provided the concepts M, P satisfy 'qualitative unity'.

In summary, we have shown that after formalisation, Kant's general logic turns out to be at least as rich as geometric logic, while it coincides with it when taking into account the semantics of judgements dictated by 'transcendental logic'.[15] This latter result is but one example of interesting metalogical theorems that may be proved about Kant's logic; B113-6, formally reinterpreted as a theorem about definability of concepts, is another.

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