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Term structure extrapolation and asymptotic forward rates

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ABSTRACT

We investigate different inter- and extrapolation methods for term structures under different constraints in order to generate market-consistent estimates which describe the asymptotic behavior of forward rates. Our starting point is the method proposed by Smith and Wilson, which is used by the European insurance supervisor EIOPA. We use the characterization of the Smith–Wilson class of interpolating functions as the solution to a functional optimization problem to extend their approach in such a way that forward rates will converge to a value which is an outcome of the optimization process. Precise conditions are stated which guarantee that the optimization problems involved are well-posed on appropriately chosen function spaces. As a result, a well-defined optimal asymptotic forward rate can be derived directly from prices and cashflows of traded instruments. This allows practitioners to use raw market data to extract information about long term forward rates, as we will show in a study which analyzes historical EURIBOR swap data.

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1. Introduction

Every financial institution that uses a discount curve to estimate the current value of future cashflows has to decide how to construct such a curve from information that can be found in the fixed income markets. In such markets, cashflows are defined at a limited number of dates so curve construction will require certain interpolation methods. If the financial institution needs to assess the value of cashflows with maturities that exceed the maturities of observable market instruments, or when the market information for these longer maturities is known to be unreliable, there is also a need for extrapolation of the curve. This last problem arises naturally for many insurance companies and pension funds since by the very nature of their business their liabilities stretch out much further into the future than those of the average financial institution. At the same time regulatory frameworks such as the Solvency II proposals for European insurers stress the importance of market-consistent pricing of liabilities. This raises the question how one can systematically incorporate market information in a discount curve while at the same time recognizing that a subjective choice has to be made for maturities in between or beyond those that can be directly related to available market instruments.

Yields and forward rates provide natural coordinates to characterize discount curves and several parametrized functional forms

have therefore been proposed which describe the yield curve or the forward curve rather than the discount curve itself. Well-known examples include the approaches of Nelson and Siegel (1987) and Svensson (1994) which use given functions with three and four free parameters respectively to fit the entire yield curve. Since it is impossible to recover the exact prices of all market instruments using only a few parameters, some optimality criterion must be defined to decide which values of the parameters are chosen. In practice one often uses a weighted sum over the squares of the pricing errors for a given set of fixed income instruments.

If one insists that all observed market prices must be fitted exactly and allows the introduction of a larger number of parameters to achieve this, we face a standard interpolation problem. An important criterion to decide which interpolation methods should be applied, is the desired degree of smoothness of the resulting interpolating function. It is clear that piecewise linear functions can perfectly interpolate a given set of yields for different maturities but this leads to a yield curve which will not be differentiable and the associated forward curve will be discontinuous.

To achieve smoother discount-, yield- and forward curves, standard spline-based interpolation methods have been proposed and certain modifications based on the particular application at hand have been suggested. An overview of the many possible methods is given in a paper by Hagan and West (2006). Smoothness is important but there are other desirable properties of interpolating functions which are analyzed in that study, such as preserving monotonicity or convexity in the given dataset. The authors also investigate whether changes in the input data which are restricted to

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the vicinity of a particular maturity may influence the values of the interpolated discount curve at maturities far away from it. Strong non-local sensitivities in an interpolation method make it less suitable for risk management purposes since it complicates the use of fixed income market instruments to mitigate the interest rate risk in a portfolio of liabilities.

Similar considerations play a role when deciding on a method to extrapolate the discount curve beyond the last maturities for which reliable market information is available. Compared to the interpolation problem, the extrapolation problem for term structures has received considerably less attention in the literature until recently. The new Solvency II regulatory framework for European insurers (EIOPA, 2010) has changed this, since the prescribed method to construct the term structure from market data incorporates the assumption that forward rates must converge to an a priori specified value which was chosen to be 4.2%. The class of interpolating functions that is used is the one proposed by Smith and Wilson in earlier work (Smith and Wilson, 2000). The Smith–Wilson functions are exponential tension splines with a given tension parameter. They were introduced by Schweikert (1966) and first proposed as interpolating functions for term structures by Barzanti and Corradi (1998a,b). Andersen (2007) characterized the corresponding interpolating curves in terms of a more convenient basis of functions and showed how non-local sensitivities are controlled by the tension parameter.

Exponential tension spline functions form the solution to an explicit optimization problem for interpolation under the constraint that a set of given data points must be fitted exactly. The optimization criterion aims at making both the first order derivatives and the second order derivatives of the interpolating function small, in the sense that one minimizes the integral over all maturities of their squared values. Criteria based on second-order derivatives have often been used as a basis for term structure interpolation, see for example Lim and Xiao (2002), Adams (2001) and Adams and Van Deventer (1994). Smith and Wilson then added the constraint that the asymptotic forward rate is given a priori.

In this paper we use similar criteria to extend the interpolation problem to the case where the limit of forward rates is assumed to exist, but not given a certain value beforehand. This not only makes it possible to construct discount curves without such a restrictive assumption but it also provides information on the behavior of forward rates at high maturities in historical data, without making specific model assumptions. This provides a way to establish whether the values that are imposed in regulatory frameworks are consistent with what is found in market data at times of high market liquidity. There is already some empirical evidence that long term rates are in fact highly uncertain. This is for example the conclusion of a recent paper by Balter et al. (2014) where a Bayesian approach is used to describe parameter uncertainty in a Vasicek short rate model. At the end of this paper some further empirical evidence will be presented which suggests that long term forward rates have been rather stable before the financial crisis of 2008 but that they have decreased considerably after the crisis.

The structure of the paper is as follows. First we define the constrained term structure interpolation problem and the solution proposed by Smith and Wilson in the next two sections. We analyze its properties in Section 4 and use this to solve a related but unconstrained optimization problem in Section 5. In Section 6 another method is presented which directly describes the interpolation problem in terms of forward or yield curves instead of discount curves. This has the advantage of generating a very intuitive and simple formula for the asymptotic forward rate in terms of market yields at different maturities. In Section 7 we show some numerical results based on market data and we draw conclusions in the last section.

2. Term structure interpolation

Term structures can be represented in many equivalent ways. As usual, we use the notation $p(t, T)$ for the amount to be paid at a time $t \geq 0$ to receive a certain single unit of currency at a later time $T \geq t$, i.e. the zero-coupon bond price at time t for maturity T . The continuous-time yield $y(t, T)$ and forward rate $f(t, T)$ are then defined by

$$f(t, T) = -\frac{\partial}{\partial T} \ln p(t, T), \quad y(t, T) = -\frac{\ln p(t, T)}{T - t},$$

so

$$p(t, T) = e^{-(T-t)y(t, T)} = e^{-\int_t^T f(t, u) du}.$$

Since we do not have a continuum of bond prices available we must use interpolation methods and it is necessary to use extrapolation methods for times beyond the maximal bond maturity that is available in the market.

To do so, we will assume that at the current time $t = 0$ we know the market prices m_i of certain fixed income instruments which pay given amounts c_{ij} at given times $u_j \geq 0$. Here $i \in \mathcal{I} = \{1, 2, \dots, n_{\mathcal{I}}\}$ refers to a certain instrument and $j \in \mathcal{J} = \{1, 2, \dots, n_{\mathcal{J}}\}$ to all possible times where payments may occur. If asset i does not pay anything at time u_j , the corresponding value c_{ij} is simply set to zero. Note that we only consider the static problem in this paper, which deals with the term structure at a fixed time t that we will take to be time zero without loss of generality. This term structure may then be used as an input for dynamic models for interest rates, see for example the books by Cairns (2004) or Brigo and Mercurio (2007) for details on such models. We will also assume that prices must be fitted exactly. Alternatively one may use a least squares approach, maximum likelihood methods or a Bayesian framework (Fisher et al., 1995; Cairns, 1998; Sack, 2000; Cairns and Pritchard, 2001). We will also not require that the interpolating curve for forwards is locally monotone or convex when the inputs for the forward curve have the same property, see Hagan and West (2006) for a method based on given forward values (instead of fixed income prices) which satisfies this requirement.

An interpolating curve $\bar{p}(0, t)$ for the observed fixed income instruments must thus satisfy

$$(\forall i \in \mathcal{I}) \quad m_i = \sum_{j \in \mathcal{J}} c_{ij} \bar{p}(0, u_j). \quad (1)$$

To decide how the curve is defined in between the time points u_j , a specific criterion is needed since there are many possible ways to do this. Often a criterion is chosen which regulates the smoothness of the interpolating curve. One could for example require the first and second order derivatives of an interpolating curve g to stay small in a quadratic sense. This means that one tries to minimize

$$\mathcal{L}_{\alpha}[g] := \int_0^{\infty} [g''(s)^2 + \alpha^2 g'(s)^2] ds \quad (2)$$

over all functions g in a given function space. Notice that a parameter $\alpha > 0$ is needed for the trade-off between the possibly conflicting requirements that both slope and curvature remain small.

We thus want to define a discount curve $p(0, t)$ which is smooth and is market-consistent in the sense that it matches quoted market prices of certain fixed income instruments. The European insurance regulator also requires that forward rates converge to an a priori given value f_{∞} when the maturity goes to infinity. In the formulation of Smith and Wilson (2000) this is implemented by defining, instead of $p(0, t) = g(t)$,

$$p(0, t) = (1 + g(t))e^{-f_{\infty}t}.$$

We thus look for a sufficiently smooth function g which minimizes $\mathcal{L}_\alpha[g]$ under the constraint that $g(0) = 0$ while satisfying

$$m_i = \sum_{j \in \mathcal{J}} c_{ij}(1 + g(u_j))e^{-f_\infty u_j} \quad (3)$$

for a given $n_I \times n_{\mathcal{J}}$ matrix $\mathbf{C}$ of cashflows and an $n_I \times 1$ vector $\mathbf{m}$ of market prices. We will always assume that the rows of $\mathbf{C}$ are linearly independent vectors so there are no superfluous instruments in our set.

The solution to this problem can be written in terms of certain basis functions $W(t, u_j)$ which depend on time t and the maturities u_j . We will derive some properties of the functions W in the next section before proving that they provide a solution to the constrained optimization problem defined above.

3. A class of interpolating functions

The functions SW introduced by [Smith and Wilson \(2000\)](#) are scaled versions of functions W that we will find more convenient to work with:

$$SW(t, u) = e^{-f_\infty(t+u)}W(t, u), \quad (4)$$

$$W(t, u) = \alpha \min(t, u) - \frac{1}{2}e^{-\alpha|t-u|} + \frac{1}{2}e^{-\alpha(t+u)},$$

for¹ $(t, u) \in \mathbb{R}^+ \times \mathbb{R}^+$ and with $\alpha > 0$ a given constant. These functions W can be characterized as follows. The proof (and other proofs in the sequel which do not follow directly after a proposition) can be found in the [Appendix](#).

Proposition 1. For a given $\alpha > 0$ consider the ordinary differential equation

$$\partial_t^2 w(t, u) = \alpha^2 w(t, u) - \alpha^3 \min(t, u) \quad (5)$$

for $(t, u) \in \mathbb{R}^+ \times \mathbb{R}^+$. For every $u > 0$ the function $W(t, u)$ is the only twice continuously differentiable solution to this equation which is zero for $t = 0$ and has a finite limit for $t \rightarrow \infty$.

We notice that $W(t, u) = W(u, t)$ and (5) shows that the function $t \rightarrow W(t, u)$ is twice differentiable in the point $t = u$ but not three times differentiable. The functions W are called *exponential tension splines* and they were introduced by Schweikert in 1966 ([Schweikert, 1966](#)). Their properties have been shown to have certain advantages which are useful for modeling yield curves and for other applications, see for example [Andersen \(2007\)](#), [Andersen and Piterbarg \(2010\)](#) and [Pruess \(1976\)](#).

In EIOPA documentation about the Smith–Wilson interpolation method ([EIOPA, 2010](#)) it is mentioned that the function $W(t, u)$ is related to the covariance function of an integrated Ornstein–Uhlenbeck process. However, a paper by [Andersson and Lindholm \(2013\)](#) shows that W does not exactly correspond to that covariance function unless some further restrictive assumptions are imposed. In the following propositions, we use their probabilistic interpretation of the functions W , to derive some properties that we will need in later proofs. We also show how functions related to W characterize the best linear predictor for the yield on a bank account and the discount factor in a Hull–White short rate model.

Proposition 2. Let Z_t be a standard Brownian Motion and L_t be an Ornstein–Uhlenbeck process $L_t = \int_0^t e^{-\alpha(t-s)} dV_s$ with V a standard Brownian Motion independent of Z and $\alpha > 0$ a given constant. Then W can be characterized as the covariance function between two Gaussian processes: $W(t, u) = \alpha \text{cov}(Z_t + L_t, Z_u - L_u)$.

Proposition 3. If I_t is an integrated Ornstein–Uhlenbeck process, so $I_t = \int_0^t L_u du$ with $L_t = \int_0^t e^{-\alpha(t-s)} dV_s$, then

$$\mathbb{E}[I_t | I_{u_1}, I_{u_2}, \dots, I_{u_n}] = \sum_j e_j(t) I_{u_j}$$

where the vector $\mathbf{e}(t) = \tilde{H}^{-1} \mathbf{b}(t)$ can be defined in terms of the matrix $\tilde{H}_{ij} := H(u_i, u_j)$ and the vector $\mathbf{b}_j(t) = H(t, u_j)$ if we choose

$$H(t, u) = W(t, u) - (1 - e^{-\alpha t})(1 - e^{-\alpha u})$$

provided the matrix $\tilde{H}$ is invertible.

This shows how H can be used to define the best linear predictor for the yield of a bank account $\frac{1}{t} \ln B_t$ in a Hull–White model (see [Hull and White, 1990](#)), since in that model $\ln B_t$ corresponds to an integrated Ornstein–Uhlenbeck process with mean reversion parameter α . The best linear predictor of a stochastic process X_t in terms of $X_{u_1}, \dots, X_{u_n}$ by definition equals $\sum_{i=1}^n \hat{w}_i X_{u_i}$ if the coefficients $(\hat{w}_1, \dots, \hat{w}_n)$ solve

$$\min_{(w_1, \dots, w_n)} \mathbb{E} \left[\left(X_t - \sum_{j=1}^n w_j X_{u_j} \right)^2 \right]. \quad (6)$$

From the proposition we find as a direct corollary that

$$\begin{aligned} \mathbb{E}[\ln B_t | \ln B_{u_1}, \ln B_{u_2}, \dots, \ln B_{u_n}] \\ = \mathbb{E}[\ln B_t] + \sum_{j=1}^n \left(\sum_{i=1}^n (\tilde{H}^{-1})_{ij} H(t, u_i) \right) (\ln B_{u_j} - \mathbb{E}[\ln B_{u_j}]). \end{aligned}$$

The conditional expectation on the left hand side is the best predictor and since it is equal to the projection on the right hand side, which is linear in $\ln B_{u_j}$, the best linear predictor is actually the best predictor in this case.

The following corollary shows that the best linear predictor of the stochastic discount factor B_t^{-1} also has a representation in terms of the function H and thus in terms of the exponential tension spline W .

Corollary 1. In a Hull–White model with mean reversion parameter α , the best linear predictor of the stochastic discount factor B_t^{-1} in terms of $B_{u_1}^{-1}, \dots, B_{u_n}^{-1}$ is given by

$$\sum_{i=1}^n \left(\sum_{j=1}^n (\tilde{H}^{-1})_{ij} \hat{H}(t, u_j) \right) B_{u_i}^{-1}, \quad (7)$$

provided that the matrix $\tilde{H}_{ij} = \hat{H}(u_i, u_j)$, $1 \leq i, j \leq n$, defined by

$$\begin{aligned} \ln \hat{H}(t, u) &= \mathbb{E}[\ln B_t^{-1}] + \mathbb{E}[\ln B_u^{-1}] \\ &\quad + \frac{1}{2} \alpha^{-3} (H(t, t) + H(u, u) + 2H(t, u)), \end{aligned}$$

is invertible.

Proposition 2 characterizes W as a covariance between two different processes. It is possible to characterize it as an (auto) covariance process of a single Gaussian process as well on any finite interval $[0, T]$, as the following result shows.

Proposition 4. For a given $T > 0$ and $\alpha > 0$ let $(z_n)_{n \in \mathbb{N}}$ be the countable infinite number of solutions to the equation

$$z^3 \tan(\alpha T) - (1 + z^2)^{\frac{3}{2}} \tanh\left((1 + z^2)^{\frac{3}{2}} \alpha T\right) - 2z^2 = 1 \quad (8)$$

¹ In our notation $\mathbb{R}^+ = [0, \infty[$ i.e. zero is included.

and define

$$\phi_n(t) = \frac{\psi_n(t)}{\sqrt{\int_0^T \psi_n^2(u) du}},$$

$$\psi_n(t) = z_n \frac{\sin(\alpha t z_n)}{\cos(\alpha T z_n)} + \sqrt{1 + z_n^2} \frac{\sinh(\alpha t \sqrt{1 + z_n^2})}{\cosh(\alpha T \sqrt{1 + z_n^2})}.$$

Then the Gaussian process

$$X_t = \frac{1}{\sqrt{\alpha}} \sum_{n=0}^{\infty} \frac{\phi_n(t) \epsilon_n}{z_n \sqrt{1 + z_n^2}}$$

with $(\epsilon_n)_{n \in \mathbb{N}}$ independent standard Gaussian stochastic variables, has the function W as covariance function on its domain $[0, T]$ so $\mathbb{E}(X_t X_u) = W(t, u)$ for all u and t in $[0, T]$.

This result may seem rather superfluous at first but we can use it to show that the matrix with entries $W_{ij} = W(u_i, u_j)$ is invertible for any sequence of maturities $0 \leq u_1 < u_2 < \dots < u_{n_g}$. This is in fact a result that is tacitly assumed in the Smith–Wilson interpolation procedure.

To prove this, let a be any vector in $\mathbb{R}^n$. Then $a^T W a$ equals the variance of the process $\sum_j a_j X_{u_j}$ which is easily seen to be strictly positive when $\|a\| \neq 0$. This shows that the matrix W is invertible for any finite value of n_g and any finite set of distinct values u_j with $j \in \mathcal{J} := \{1, 2, \dots, n_g\}$.

4. The constrained optimization problem

Now that some elementary properties of the functions W have been established, we show how these functions can be used to solve the constrained optimization problem that was proposed to find a discount curve based on interpolation and extrapolation of market data.

We define the following subset of the space $L^2(\mathbb{R}^+)$ of square-integrable functions on the positive reals:

$$\mathcal{E} = \{g \in L^2(\mathbb{R}^+) : \lim_{t \rightarrow \infty} g(t) = 0\},$$

and a subset of the space of two times differentiable functions on the same domain

$$\mathcal{F}_a = \{g \in C^2(\mathbb{R}^+) : g(0) = a, g''(0) = 0, g' \in \mathcal{E}, g'' \in \mathcal{E}\},$$

for any $a \in \mathbb{R}$. The space $\mathcal{F}_0$ turns out to be the appropriate space to find functions g which define a discount curve $\bar{p}(0, t) = (1 + g(t)) \exp(-f_{\infty} t)$ that fits given market data, has an asymptotic² forward rate f_{∞} and is as smooth as possible in the sense that it minimizes the weighted integrals over first and second order derivatives as defined by $\mathcal{L}_{\alpha}[g]$ in (2).

Problem 1 (Smoothest Discount Curve for a Given Forward Limit). Find

$$\min_{g \in \mathcal{F}_0} \mathcal{L}_{\alpha}[g] \quad (9)$$

subject to

$$\sum_{j \in \mathcal{J}} \bar{c}_{ij} g(u_j) = \bar{m}_i, \quad \text{for all } i \in \mathcal{I}, \quad (10)$$

with $\bar{c}_{ij} = c_{ij} e^{-f_{\infty} u_j}$ and $\bar{m}_i = m_i - \sum_{j \in \mathcal{J}} \bar{c}_{ij}$, to ease the notation of (3).

² The European insurance supervisor EIOPA uses the term *ultimate forward rate*. We call it an *asymptotic forward rate* because in our case its value will usually not be reached for any finite maturity.

The solution to this problem has been given by Smith and Wilson (2000). We use a slightly different notation and give a different proof (in the Appendix) since this will allow us to prove certain extensions in later sections of this paper.

Theorem 1. If a solution to Problem 1 exists, it must take the form

$$g(t) = \sum_{i \in \mathcal{I}} \zeta_i \sum_{j \in \mathcal{J}} \bar{c}_{ij} W(t, u_j) \quad (11)$$

with $(\zeta_i)_{i \in \mathcal{I}}$ a vector of weights and the functions W as defined in (4).

This result shows that the method proposed by Smith and Wilson has the interpretation of the best possible interpolation under a criterion based on the operator $\mathcal{L}_{\alpha}$, since the functions SW that they propose are simply scaled versions of the exponential tension splines W . We prefer to work with the functions W since they can be defined without referring to the asymptotic forward rate f_{∞} , a property that we will need in later sections.

The weights $(\zeta_i)_{i \in \mathcal{I}}$ in (11) can be calculated using

$$\bar{m}_i = \sum_{k \in \mathcal{J}} \bar{c}_{ik} g(u_k) = \sum_{k \in \mathcal{J}} \sum_{j \in \mathcal{J}} \sum_{i \in \mathcal{I}} \bar{c}_{ik} W(u_k, u_j) \bar{c}_{ij} \zeta_i.$$

If we define matrices $\mathbf{W}$, $\mathbf{C}$ and a vector $\mathbf{Z}$ as follows:

$$\mathbf{W}_{kj} = W(u_k, u_j), \quad \bar{\mathbf{C}}_{ij} = \bar{c}_{ij}, \quad \mathbf{Z}_i = \zeta_i,$$

then this equation reads $\bar{\mathbf{m}} = \bar{\mathbf{C}} \mathbf{W} \bar{\mathbf{C}}^T \mathbf{Z}$ so we find the parameters using³

$$\mathbf{Z} = (\bar{\mathbf{C}} \mathbf{W} \bar{\mathbf{C}}^T)^{-1} \bar{\mathbf{m}},$$

and finally, using (11),

$$p(0, t) = \left(1 + \sum_{j \in \mathcal{J}} \eta_j W(t, u_j) \right) e^{-f_{\infty} t} \quad (12)$$

with $\eta = \bar{\mathbf{C}}^T \mathbf{Z} = \bar{\mathbf{C}}^T (\bar{\mathbf{C}} \mathbf{W} \bar{\mathbf{C}}^T)^{-1} \bar{\mathbf{m}}$. Note that for square and invertible matrices $\bar{\mathbf{C}}$, such as the case where only distinct zero coupon bond prices are given, this reduces to $\eta = \mathbf{W}^{-1} \bar{\mathbf{C}}^{-1} \bar{\mathbf{m}}$.

We will show numerical examples for this interpolation method using historical swap data in a later section. But first we formulate a different version of the optimization problem, where we no longer impose to what fixed value the forward rates must converge.

5. The unconstrained optimization problem

The Smith–Wilson procedure defines a method to obtain an extrapolated discount curve based on given market data. The asymptotic forward rate is given a priori so it is an input parameter for the methodology. Smoothness, as measured by the functional $\mathcal{L}_{\alpha}$, is used as a criterion for the interpolation between maturities for which data are available. But the requirement of a given asymptotic forward rate means that the discount curve may become less smooth around the last maturity for which liquid market data are available. It would be more consistent to take the asymptotic forward rate as a free parameter, which is chosen in such a way that we get the smoothest possible discount curve for all relevant maturities. Mathematically this means that we propose to optimize the functional $\mathcal{L}_{\alpha}$ for a given value of α over all possible

³ The inverse of $\bar{\mathbf{C}} \mathbf{W} \bar{\mathbf{C}}^T$ exists by our requirement that the rows of $\mathbf{C}$, and thus of $\bar{\mathbf{C}}$, are linearly independent. Indeed, since $\mathbf{W}$ is non-singular and symmetric we can write $\mathbf{W} = \mathbf{L} \mathbf{A} \mathbf{L}^T$ for a lower triangular matrix $\mathbf{L}$ and a diagonal matrix $\mathbf{A}$ with strictly positive diagonal elements: $A_{ii} > 0$. A vector $\mathbf{x}$ satisfying $\bar{\mathbf{C}} \mathbf{W} \bar{\mathbf{C}}^T \mathbf{x} = 0$ would then imply that $0 = \mathbf{x}^T \bar{\mathbf{C}} \mathbf{W} \bar{\mathbf{C}}^T \mathbf{x} = \mathbf{y}^T \mathbf{A} \mathbf{y} = \sum_i y_i^2 A_{ii}$ for $\mathbf{y} = \mathbf{L}^T \bar{\mathbf{C}}^T \mathbf{x} \neq 0$ which is impossible since both $\mathbf{L}^T$ and $\bar{\mathbf{C}}^T$ have linearly independent columns.

functions g in a certain class but also over all possible values of f_∞ . This will not just give a smoother end result for the discount curve, but also provides an objective estimation method for the asymptotic forward rate in terms of market quotes for bond and swap data.

Problem 2 (Smoothest Discount Curve with Converging Forward Rates). Find the minimizer for f_∞ in

$$\min_{f_\infty} \min_{g \in \mathcal{H}(f_\infty)} \mathcal{L}_\alpha[g] \quad (13)$$

in which

$$\mathcal{H}(f_\infty) = \left\{ g \in \mathcal{C}^2(\mathbb{R}_+) \mid g \in \mathcal{F}_0, \right. \\ \left. \sum_{j \in \mathcal{J}} \bar{c}_{ij} g(u_j) = \bar{m}_i, \text{ for all } i \in \mathcal{I} \right\},$$

with the $\bar{c}_{ij}$ and $\bar{m}_i$ as defined in (10). $\square$

It turns out that the asymptotic forward rate can be characterized explicitly as the solution of a matrix equation. We define the matrices

$$\mathbf{W}_{ij} = W(u_i, u_j), \quad \mathbf{D}_{ij}^f = e^{-f u_j} \mathbf{1}_{i=j}, \quad \mathbf{U}_{ij} = u_j \mathbf{1}_{i=j}$$

and $\mathbf{e}$ a vector full of ones in $\mathbb{R}^{n_{\mathcal{J}}}$.

Theorem 2. For an $n_{\mathcal{I}} \times n_{\mathcal{J}}$ cashflow matrix $\mathbf{C}$ and price vector $\mathbf{m} \in \mathbb{R}^{n_{\mathcal{I}}}$ the optimized asymptotic forward rate $f = f_\infty$ of Problem 2 solves

$$(\mathbf{m} - \mathbf{C} \mathbf{D}^f \mathbf{e})^T (\mathbf{C} \mathbf{D}^f \mathbf{W} \mathbf{D}^f \mathbf{C}^T)^{-1} \mathbf{C} \mathbf{D}^f \mathbf{U} \\ \times (\mathbf{e} + \mathbf{W} \mathbf{D}^f \mathbf{C}^T (\mathbf{C} \mathbf{D}^f \mathbf{W} \mathbf{D}^f \mathbf{C}^T)^{-1} (\mathbf{m} - \mathbf{C} \mathbf{D}^f \mathbf{e})) = 0.$$

If the cashflow matrix $\mathbf{C}$ is invertible this simplifies to

$$\sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{J}} (u_j \pi_j e^{f u_j}) [\mathbf{W}^{-1}]_{jk} (\pi_k e^{f u_k} - 1) = 0$$

with $\pi = \mathbf{C}^{-1} \mathbf{m}$.

This equation can be solved rather easily using numerical algorithms since only the matrix $\mathbf{D}^f$ contains the parameter to vary over. Calculating the optimized asymptotic forward rate therefore typically takes only a fraction of a second.

6. An alternative method

The Smith–Wilson interpolation method is a transparent method which is easy to implement since it can be directly applied to bond and swap data. But it has some important disadvantages. The parameter α , which influences the speed of convergence to the asymptotic forward rate, needs to be defined a priori and for some values of α the resulting zero coupon bond prices may become negative. It is also possible that the discount function $t \rightarrow p(0, t)$ is not decreasing for certain values of t , which corresponds to the possibility of a negative forward rate. Moreover, there is the question of smoothness of forward rates. The Hull–White model (Hull and White, 1990) that we mentioned earlier is a relatively simple and popular Markovian model for stochastic interest rates, which allows a perfect fit to a given initial term structure. In that model one assumes that short rates follow a diffusion process driven by a Brownian Motion process with mean reversion at a certain speed $\kappa > 0$ towards a time-varying level $\theta(t)$. The function θ can be used to give a perfect calibration for any initial

term structure $p(0, t)$ by using the corresponding forward curve $f(0, t)$ and its derivative with respect to maturity:

$$\theta(t) = \frac{1}{\kappa} \frac{\partial}{\partial t} f(0, t) + f(0, t) + \frac{\sigma^2}{2\kappa^2} (1 - e^{-2\kappa t}).$$

If our initial term structure is modeled using the Smith–Wilson method, then $\theta(t)$ depends on $\frac{\partial}{\partial t} f(0, t) = -\frac{\partial^2}{\partial t^2} (\ln p(0, t))$ which involves a combination of second order derivatives of the functions $W(t, u_j)$. As shown in Proposition 1, these second order derivative functions are continuous but not differentiable which leads to a lack of smoothness in Hull–White calibration results in any of the maturity points u_j .

To overcome some of these problems, we now propose two alternative extrapolation methods. Where the Smith–Wilson approach takes the first and second order derivatives of the (transformed) discount curve as a criterion for smoothness, we will instead focus on the smoothness of the forward curve or the yield curve itself. More precisely, we will minimize the weighted integrals over first and second order derivatives of the forward or yield curve under the conditions that forward rates converge and that the market prices of a set of given fixed income instruments are fitted perfectly. Since $p(0, t) = \exp(-\int_0^t f(0, s) ds) = \exp(-t y(0, t))$ this will also guarantee that p , and hence all coupon bond prices, will always be strictly positive. If we apply the criterion to the yield curve, the order of differentiability of the discount curve does not change but if we apply the criterion to the forward curve, it will lead to a discount curve with a higher order of differentiability.

We formalize the new optimization problem as follows.

Problem 3 (Smoothest Forward or Yield Curve with Converging Forward Rates). Find

$$\min_{g \in \mathcal{H}^f} \mathcal{L}_{\bar{\alpha}}[g], \quad \min_{g \in \mathcal{H}^y} \mathcal{L}_\alpha[g] \quad (14)$$

in which

$$\mathcal{H}^f = \left\{ g \in \mathcal{C}^2(\mathbb{R}_+) \mid g \in \mathcal{F}_a, \sum_{j=1}^{n_{\mathcal{J}}} c_{ij} e^{-\int_0^{u_j} g(s) ds} = m_i, \right. \\ \left. \text{for } i = 1, \dots, n_{\mathcal{I}} \right\} \\ \mathcal{H}^y = \left\{ g \in \mathcal{C}^2(\mathbb{R}_+) \mid g \in \mathcal{F}_a, \sum_{j=1}^{n_{\mathcal{J}}} c_{ij} e^{-u_j g(u_j)} = m_i, \right. \\ \left. \text{for } i = 1, \dots, n_{\mathcal{I}} \right\}.$$

Note that $g(0) = a$ for functions g in $\mathcal{H}^f$ or $\mathcal{H}^y$ so the initial short rate r_0 equals a given constant a . This implies that we assume that the short rate is given. If this is not the case, we may determine its value by including it in the optimization procedure. We will show later in this section how to do this.

This problem can be solved explicitly in terms of the earlier defined functions W and a different class of interpolating and extrapolating functions:

$$\bar{W}(t, u) = 1 - e^{-\bar{\alpha} t} \frac{\cosh(\bar{\alpha} u) - 1}{\frac{1}{2} \bar{\alpha}^2 u^2} \\ + \mathbf{1}_{t \leq u} \left(\frac{\cosh(\bar{\alpha}(u - t)) - 1 - \frac{1}{2} \bar{\alpha}^2 (u - t)^2}{\frac{1}{2} \bar{\alpha}^2 u^2} \right). \quad (15)$$

These functions can be written as affine combinations of integrals of Smith–Wilson functions and are therefore smoother than the Smith–Wilson functions themselves, see Fig. 1.

Fig. 1. Scaled functions $-W(t, u)/(\alpha u)$ and $\bar{W}(t, u)$ for $\alpha = \bar{\alpha} = \frac{1}{10}$ and $-W(-t, u)/(\alpha u)$ and $\bar{W}(-t, u)$ for $\alpha = \bar{\alpha} = \frac{1}{2}$. W in red (bottom) and $\bar{W}$ in blue (top).

Theorem 3. If solutions for Problem 3 exist, they must be of the form

$$g^f(t) = g(0) + \sum_{i \in \mathcal{I}} \zeta_i^f \sum_{j \in \mathcal{J}} \pi_j^f c_{ij} u_j^2 \bar{W}(t, u_j), \quad (16)$$

$$g^y(t) = g(0) + \sum_{i \in \mathcal{I}} \zeta_i^y \sum_{j \in \mathcal{J}} \pi_j^y c_{ij} u_j W(t, u_j)$$

with the functions $\bar{W}$ as defined in (15) and where the $(\zeta_i^f)_{i \in \mathcal{I}}$ and $(\pi_j^f)_{j \in \mathcal{J}}$ solve the equations

$$(\forall i \in \mathcal{I}) \quad m_i = \sum_{j \in \mathcal{J}} c_{ij} \pi_j^f \quad (17)$$

$$(\forall k \in \mathcal{J}) \quad -(\ln \pi_k^f)/u_k = g(0) + \sum_{i \in \mathcal{I}} \zeta_i^f \sum_{j \in \mathcal{J}} \pi_j^f c_{ij} u_j^2 \times \frac{1}{u_k} \int_0^{u_k} \bar{W}(s, u_j) ds \quad (18)$$

while the $(\zeta_i^y)_{i \in \mathcal{I}}$ and $(\pi_j^y)_{j \in \mathcal{J}}$ should solve

$$(\forall i \in \mathcal{I}) \quad m_i = \sum_{j \in \mathcal{J}} c_{ij} \pi_j^y \quad (19)$$

$$(\forall k \in \mathcal{J}) \quad -(\ln \pi_k^y)/u_k = g(0) + \sum_{i \in \mathcal{I}} \zeta_i^y \sum_{j \in \mathcal{J}} \pi_j^y c_{ij} u_j \times W(u_k, u_j). \quad (20)$$

For every $u > 0$ we have that $\lim_{t \rightarrow \infty} \bar{W}(t, u) = 1$ and $\bar{W}(0, u) = 0$ and the functions $t \rightarrow \bar{W}(t, u)$ become linear for very small positive values of t in the sense that $\partial_t^2 \bar{W}(0, u) = 0$. Moreover, if α and $\bar{\alpha}$ go to infinity, the scaled interpolating functions $t \rightarrow W(t, u)/(\alpha u)$ and $t \rightarrow \bar{W}(t, u)$ will converge to 1 for $t > u$. In the limit, the interpolating functions $g(t)$, $g^y(t)$ and $g^f(t)$ are thus all constant after the last maturity point u_n which is considered in the interpolation.⁴

We can use this to derive expressions for the asymptotic forward rate under both methods. Since $\lim_{t \rightarrow \infty} \bar{W}(t, u) = 1$ and

$\lim_{t \rightarrow \infty} W(t, u) = \alpha u$ for all $u > 0$, this follows from (16):

$$f_\infty^f = \lim_{t \rightarrow \infty} g^f(t) = g(0) + \sum_{i \in \mathcal{I}} \zeta_i^f \sum_{j \in \mathcal{J}} \pi_j^f c_{ij} u_j^2 \quad (21)$$

$$f_\infty^y = \lim_{t \rightarrow \infty} g^y(t) = g(0) + \sum_{i \in \mathcal{I}} \zeta_i^y \sum_{j \in \mathcal{J}} \pi_j^y c_{ij} \alpha u_j^2. \quad (22)$$

We will now show that we can write these two estimated asymptotic forward rates in terms of a linear combination of yields at different maturities. Let

$$y_k = y(u_k) = -\ln p(0, u_k)/u_k$$

denote the yield for maturity u_k when $1 \leq k \leq n$. We define $u_0 := 0$ in order to write $y_0 = f(0, 0) = y(u_0)$ for the short rate.

Corollary 2. The optimized asymptotic forward rates can be written as a linear combination of the yields:

$$f_\infty = \sum_{k=0}^n v_k y_k.$$

The weights $\{v_k, k = 1..n\}$ equal $v_k = \sum_{j=1}^n [G^{-1}]_{jk}$ and $v_0 = 1 - \sum_{k=1}^n v_k$, where the matrices $\mathbf{G} = \mathbf{G}^f$ and $\mathbf{G} = \mathbf{G}^y$ for the forward and yield curve based methods are:

$$G_{kj}^f = \frac{1}{u_k} \int_0^{u_k} \bar{W}(s, u_j) ds, \quad G_{kj}^y = \frac{1}{\alpha u_j} W(u_k, u_j).$$

If the matrix with cashflows $\mathbf{C}_{ij}$ is invertible, we can observe the yields directly since $\pi = \mathbf{C}^{-1} \mathbf{m}$ and $y_k = -(\ln \pi_k)/u_k$. In that case we can calculate f_∞^f and f_∞^y as a linear combination of directly observable quantities in the markets, with weights that are fixed and can be determined beforehand for a given set of maturities $(u_i)_{i \in \mathcal{I}}$.

If the short rate is not given a priori, it can be chosen as a free parameter in the optimization problems. An explicit formula for this value of the short rate which gives the smoothest curve for small maturities can be derived when the cashflow matrix is invertible.

Corollary 3. If $n_{\mathcal{I}} = n_{\mathcal{J}} = n$, $(c_{ij})_{i \in \mathcal{I}, j \in \mathcal{J}}$ is an invertible matrix and the initial short rate is not known, then the previous formula holds if we substitute for the unknown short rate the optimized value

$$y_0 = \frac{\sum_{j=1}^n \frac{1}{u_j} \sum_{k=1}^n G_{jk}^{-1} \frac{y(u_j) + y(u_k)}{2}}{\sum_{j=1}^n \frac{1}{u_j} \sum_{k=1}^n G_{jk}^{-1}}$$

where $\mathbf{G}$ equals $\mathbf{G}^f$ or $\mathbf{G}^y$ as defined above.

7. Estimates of asymptotic forward rates

In this section we present the results of an empirical study of the extrapolation methods that we introduced earlier. Input data consisted of historical EURIBOR swap rates on all trading days in the last ten years that were obtained from Datastream. All rates were middle rates and quotes were given with an accuracy that varied from 0.50 basis point (in the beginning of the dataset) up to 0.01 basis point (for the most recent data). Data for swaps with maturities 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 15, 20, 30, 40 and 50 were downloaded and we always used all swap rates up to and including maturity 20 but sometimes included maturity 30 as well and sometimes included all rates, as will be indicated below.

An example of inter- and extrapolated yield and forward curves is shown in Fig. 2 which shows results for the market data of March

⁴ Before EIOPA announced the ultimate forward rate methodology some central banks, such as the Dutch Central Bank, used an extrapolation method where the forward rate for the last liquid maturity was taken to be the value for all forward rates beyond that maturity.

Fig. 2. Inter- and extrapolation of Datastream swap data up to maturity 20 on March 29, 2013. Blue lines show yield curves, red lines forward curves. Dashed lines correspond to the EIOPA forward rate limit of 4.2% while solid curves are based on forward rates with a limit that is not specified a priori. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

29, 2013. We used all swap data up to and including maturity 20 years and took $\alpha = \frac{1}{10}$, the value used in the original proposal of EIOPA.

The solid lines represent the smoothest extension of the discount curve (top), yield curve (bottom left) and forward curve (bottom right) as measured by our operator $\mathcal{L}$, whereas the dashed lines correspond to extrapolation to the ultimate forward rate of 4.2% specified by EIOPA. We see that the restriction that the forward rate must converge to that asymptotic value causes the forward curve to show a kink around maturity 20, i.e. at the last maturity for which market data was included in the construction of the curves. The solid lines in the three figures show that in all the three methods proposed in this paper, where such a restriction is no longer imposed, the kink disappears. As a result we find asymptotic values for forward rates that are much lower than 4.2%. Moreover, the asymptotic values found seem to be reasonably consistent across the three methods.

Fig. 3 can be used to examine this consistency between the three proposed methods, as well as the influence of the maturities that are included to construct the curves. It shows monthly updates for the estimated asymptotic forward rate generated by the methods which smooth the discount curve (top graph), the yield curve (middle graph) and the forward curve (bottom graph). Estimates are displayed for the cases where instruments are interpolated up to 20 years (red line), 30 years (blue line) and 50 years (purple line). The three graphs show that the first two methods provide estimates which are more stable over time than the last method. Indeed, the third method only gives good results when the last swap with maturity 50 is included; if this not the case then the estimates are too volatile to be of practical use. We also note that before the financial crisis of 2008 asymptotic rates were close to the assumed value of 4.2% but they have dropped substantially.

After 2012 the first two methods show values which do not differ much when market data beyond maturity 20 are included. Between 2009 and 2012 there is a more marked effect but the difference between results up to and beyond maturity 30 is smaller

than 25 basis points at all times for the second method, which is based on yield curve smoothing. We conclude that extrapolation methods which implement an explicit trade-off between first and second order derivatives of the yield curve give stable estimates when data up to maturity 20 are used. But inclusion of the data for maturity 30 seems preferable since this does not lead to more volatile estimates, and swap rates of maturities up to 30 or up to 50 give roughly the same estimates.

The effects discussed above are even more pronounced in Fig. 4, in which the same estimates of the asymptotic forward rate are plotted but now for a daily update frequency. For the last method only results for maturity 50 are shown since the estimates are too volatile when earlier maturities are taken as the last liquid point. This suggests that extrapolation of yields is possible beyond maturity 20 but that market-implied forward rates are not smooth enough to allow extrapolation if information from the far end of the curve is not used. We notice again that if extrapolation is applied to the discount curve or the yield curve, then the choice of the last used maturity only mildly affects the asymptotic estimates that are found after 2011. This supports the use of such extrapolation methods to determine a proxy for long-term yields.

8. Conclusions

We have investigated inter- and extrapolation techniques that can be used to create discount curves from observed market data under different assumptions on asymptotic forward rates. We have shown that Smith–Wilson extrapolation, which requires the a priori specification of the asymptotic value, can be extended in such a way that this limit is implied by market data. Moreover, we show how the optimal smoothness criterion of the original problem can also be applied to the forward or yield curve instead of the discount curve. This leads to a market-consistent asymptotic forward rate which can be written as a weighted combination of yields at earlier maturities, with weights that can be calculated beforehand. This provides an intuitive characterization of asymptotic forward rates in terms of well understood market information.

Fig. 3. Inter- and extrapolation based on Datastream swap data up to maturity $T = 20, 30, 50$ when applying our objective functional $\mathcal{L}$ to the discount curve (top figure), yield curve (middle figure) and forward curve (bottom figure). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

On days where reliable market information is available for high maturities these quotes can be easily incorporated into the extrapolation method, while at times of reduced liquidity for the highest maturities the same method can be used with a restricted set of maturities. We find that since 2012 the use of market data up to maturities 20, 30 or 50 gives similar results when extrapolation is based on yield curves. Before 2012 it seems best to use all available maturities up to 50 since the inclusion of later maturities does not lead to an increase in the volatility of the estimates. This suggests that the market information at

higher maturities is reliable enough to be of use in the estimation process.

In this paper we always assumed that market prices must be fitted exactly. If one allows some mispricing of market instruments in return for smoother curves, a different optimization problem must be solved. One could keep the same optimization function $\mathcal{L}_\alpha$, but one would have to impose, for example, that the weighted sum of pricing errors does not exceed a certain threshold value. Taking that value equal to zero will give back our old solutions but by varying it we can implement a tradeoff between smoothness

Fig. 4. The same quantities are shown as in the previous figure but now on a daily basis. The forward curve method in the bottom graph gave very volatile results after 2009 when the last included maturity was taken to be 20 or 30, so these results have been omitted.

and pricing accuracy. Such problems have been studied for the case where there are no constraints on the asymptotic behavior of forward rates and when forward rates are observed directly (see for example Andersen and Piterbarg, 2010). Finding the solution under our additional requirements is an interesting topic for further research.

Appendix

Proof of Proposition 1. Assume that $W(t, u)$ solves this differential equation and satisfies the conditions stated for a certain $u > 0$.

We consider the differential equations for $t > u$ and $t < u$ separately. In both cases we have a linear equation is easily seen to have αu and αt as possible solution respectively while the homogeneous equation is solved by linear combinations of $e^{\alpha t}$ and $e^{-\alpha t}$. Therefore

$$t < u : W(t, u) = \alpha t + c_1 e^{\alpha t} + c_2 e^{-\alpha t}$$

$$t > u : W(t, u) = \alpha u + c_3 e^{\alpha t} + c_4 e^{-\alpha t}.$$

A finite limit for $t \rightarrow \infty$ is only possible if $c_3 = 0$, the solution being zero at $t = 0$ implies that $c_1 = -c_2$ so we can actually reduce

the solution to

$$\begin{aligned} t < u : W(t, u) &= \alpha t + c_1(e^{\alpha t} - e^{-\alpha t}) \\ t > u : W(t, u) &= \alpha u + c_4 e^{-\alpha t}. \end{aligned}$$

Matching value and derivative at $t = u$ gives $c_1 e^{\alpha u} = (c_1 + c_4) e^{-\alpha u}$ and $\alpha(1 + c_1 e^{\alpha u} + c_1 e^{-\alpha u}) = -\alpha c_4 e^{-\alpha u}$. This gives $c_1 = \frac{1}{2} e^{-\alpha u}$ and $c_4 = \frac{1}{2} (e^{\alpha u} - e^{-\alpha u})$ which proves the result. $\square$

Proof of Proposition 2. By independence $\alpha \text{cov}(Z_t + L_t, Z_u - L_u) = \alpha \text{cov}(Z_t, Z_u) - \alpha \text{cov}(L_t, L_u)$ so the result follows immediately from

$$\begin{aligned} \text{cov}(L_t, L_u) &= \int_0^{\min(t, u)} e^{-\alpha(t-s)} e^{-\alpha(u-s)} ds \\ &= \frac{1}{2\alpha} e^{-\alpha(t+u)} (e^{2\alpha \min(t, u)} - 1) \\ &= \frac{1}{2\alpha} e^{-\alpha(\max(t, u) - \min(t, u))} - \frac{1}{2\alpha} e^{-\alpha(t+u)} \\ &= \frac{1}{2\alpha} e^{-\alpha|t-u|} - \frac{1}{2\alpha} e^{-\alpha(t+u)}. \quad \square \end{aligned}$$

Proof of Proposition 3. Since I is Gaussian with mean zero, the conditional expectation $\mathbb{E}[I_t | I_{u_1}, I_{u_2}, \dots, I_{u_n}]$ equals $\sum_j e_j(t) I_{u_j}$ with $\mathbf{e}(t) = \Sigma^{-1} \mathbf{b}(t)$ for $\Sigma_{ij} = \mathbb{E} I_{u_i} I_{u_j}$ and $b_j(t) = \mathbb{E} I_t I_{u_j}$. But since $dL = dV - \alpha L dt = dV - \alpha I dt$, we have $I_t = \alpha^{-1} (V_t - L_t) = \alpha^{-1} \int_0^t (1 - e^{-\alpha(t-s)}) dV_s$ so we find

$$\begin{aligned} \mathbb{E} I_t I_u &= \alpha^{-2} \int_0^{\min(t, u)} (1 - e^{-\alpha(t-s)}) (1 - e^{-\alpha(u-s)}) ds \\ &= \alpha^{-2} \left[\min(t, u) + \text{cov}(L_t, L_u) \right. \\ &\quad \left. - (e^{-\alpha t} + e^{-\alpha u}) \int_0^{\min(t, u)} e^{\alpha s} ds \right] \\ &= \alpha^{-3} \left[\alpha \min(t, u) + \frac{1}{2} e^{-\alpha|t-u|} - \frac{1}{2} e^{-\alpha(t+u)} \right. \\ &\quad \left. + (e^{-\alpha t} + e^{-\alpha u}) (1 - e^{\alpha \min(t, u)}) \right] \\ &= \alpha^{-3} \left[\alpha \min(t, u) - \frac{1}{2} e^{-\alpha|t-u|} + \frac{1}{2} e^{-\alpha(t+u)} \right. \\ &\quad \left. + (-e^{-\alpha(t+u)} + e^{-\alpha t} + e^{-\alpha u} - 1) \right] \\ &= \alpha^{-3} (W(t, u) - (1 - e^{-\alpha t})(1 - e^{-\alpha u})). \end{aligned}$$

This gives the result by defining $H_{ij} = \alpha^3 \Sigma_{ij}$. $\square$

Proof of Corollary 1. Differentiating Eq. (6) we obtain, for $1 \leq i \leq n$,

$$0 = -\mathbb{E} \left[2 \left(B_t^{-1} - \sum_{j=1}^n \hat{w}_j B_{u_j}^{-1} \right) B_{u_i}^{-1} \right].$$

Rearranging terms yields

$$\begin{aligned} &\begin{pmatrix} \mathbb{E}[B_{u_1}^{-1} B_{u_1}^{-1}] & \cdots & \mathbb{E}[B_{u_1}^{-1} B_{u_n}^{-1}] \\ \vdots & & \vdots \\ \mathbb{E}[B_{u_n}^{-1} B_{u_1}^{-1}] & \cdots & \mathbb{E}[B_{u_n}^{-1} B_{u_n}^{-1}] \end{pmatrix} \begin{pmatrix} \hat{w}_1 \\ \vdots \\ \hat{w}_n \end{pmatrix} \\ &= \begin{pmatrix} \mathbb{E}[B_t^{-1} B_{u_1}^{-1}] \\ \vdots \\ \mathbb{E}[B_t^{-1} B_{u_n}^{-1}] \end{pmatrix}. \end{aligned} \quad (23)$$

The random variables $\ln B_{u_i} = I_{u_i}$, $1 \leq i \leq n$, follow a multivariate Gaussian law with a moment generating function that can be expressed in terms of its mean and covariance matrix. Using the results obtained in the proof of Proposition 3 we find that, for all $1 \leq i, j \leq n$,

$$\begin{aligned} \ln \mathbb{E} [B_{u_i}^{-1} B_{u_j}^{-1}] &= \ln \mathbb{E} [e^{-\ln B_{u_i} - \ln B_{u_j}}] = \ln \mathbb{E} [e^{-I_{u_i} - I_{u_j}}] \\ &= -\mathbb{E} [I_{u_i}] - \mathbb{E} [I_{u_j}] + \frac{1}{2} \text{Var}(I_{u_i}) + \frac{1}{2} \text{Var}(I_{u_j}) \\ &\quad + \text{Cov}(I_{u_i}, I_{u_j}) \\ &= -\mathbb{E} [I_{u_i}] - \mathbb{E} [I_{u_j}] \\ &\quad + \frac{1}{2} \alpha^{-3} (H(u_i, u_i) + H(u_j, u_j) + 2H(u_i, u_j)). \end{aligned}$$

Hence Eq. (23) can be written as $\bar{H} \hat{w} = \hat{H}(t, u)$ and from the assumed invertibility of $\bar{H}$ we obtain $\hat{w}_i = \sum_{j \in \mathcal{J}} (\bar{H}^{-1})_{ij} H(t, u_j)$. $\square$

Proof of Proposition 4. We first solve the eigenvalue problem $\lambda \phi(t) = \int_0^T W(t, u) \phi(u) du$ by writing

$$\begin{aligned} \lambda \phi(t) &= \int_0^t \phi(u) \left[\alpha u - \frac{1}{2} e^{-\alpha(t-u)} + \frac{1}{2} e^{-\alpha(t+u)} \right] du \\ &\quad + \int_t^T \phi(u) \left[\alpha t - \frac{1}{2} e^{\alpha(t-u)} + \frac{1}{2} e^{-\alpha(t+u)} \right] du \\ &= \int_0^t \phi(u) [\alpha u - e^{-\alpha t} \sinh(\alpha u)] du \\ &\quad + \int_t^T \phi(u) [\alpha t - e^{-\alpha u} \sinh(\alpha t)] du \end{aligned}$$

so in particular $\phi(0) = 0$ and

$$\lambda \phi(T) = \int_0^T \phi(u) [\alpha u - e^{-\alpha T} \sinh(\alpha u)] du.$$

Differentiating gives

$$\begin{aligned} \lambda \phi'(t) &= \alpha \int_0^t \phi(u) e^{-\alpha t} \sinh(\alpha u) du \\ &\quad + \alpha \int_t^T \phi(u) [1 - e^{-\alpha u} \cosh(\alpha t)] du \end{aligned}$$

and in particular

$$\lambda \phi'(T) = \alpha e^{-\alpha T} \int_0^T \phi(u) \sinh(\alpha u) du.$$

Differentiating twice gives

$$\begin{aligned} \lambda \phi''(t) &= \alpha \phi(t) e^{-\alpha t} \sinh(\alpha t) - \alpha \phi(t) (1 - e^{-\alpha t} \cosh(\alpha t)) \\ &\quad - \alpha^2 \int_0^t \phi(u) e^{-\alpha t} \sinh(\alpha u) du \\ &\quad + \alpha^2 \int_t^T \phi(u) [-e^{-\alpha u} \sinh(\alpha t)] du \\ &= -\alpha^2 \int_0^t \phi(u) e^{-\alpha t} \sinh(\alpha u) du \\ &\quad + \alpha^2 \int_t^T \phi(u) [-e^{-\alpha u} \sinh(\alpha t)] du \\ &= \alpha^2 (\lambda \phi(t) - \alpha \int_0^t u \phi(u) du - \alpha t \int_t^T \phi(u) du) \end{aligned}$$

which shows that $\phi''(0) = \alpha^2 \lambda \phi(0) = 0$ and

$$\begin{aligned}\lambda\phi''(T) &= \alpha^2(\lambda\phi(T) - \alpha \int_0^T u\phi(u)du) \\ &= -\alpha^2 \int_0^T \phi(u)e^{-\alpha T} \sinh(\alpha u)du = -\alpha\lambda\phi'(T)\end{aligned}$$

so $\phi''(T) = -\alpha\phi'(T)$. Differentiating once more we find

$$\lambda\phi'''(t) - \lambda\alpha^2\phi'(t) = -\alpha^3 \int_t^T \phi(u)du$$

$$\lambda\phi''''(t) - \lambda\alpha^2\phi''(t) = \alpha^3\phi(t)$$

and $\phi'''(T) = \alpha^2\phi'(T)$. Substitution of $\phi(t) = \exp(mt)$ in the last equation gives $\lambda(m^2)^2 - \lambda\alpha^2m^2 - \alpha^3 = 0$ so $m^2 = \frac{1}{2}\alpha(\alpha \pm \sqrt{\alpha^2 + 4\alpha/\lambda})$. Define $z^2 = \frac{1}{2}(\sqrt{1 + 4/(\alpha\lambda)} - 1)$ then $m^2 = -z^2\alpha^2$ or $m^2 = (1 + z^2)\alpha^2$ and $\alpha z^2(1 + z^2) = 1/\lambda$. Solutions must therefore take the form

$$\begin{aligned}\phi(t) &= c_1 \sin(tz\alpha) + c_2 \cos(tz\alpha) + c_3 \sinh(t\alpha\sqrt{1 + z^2}) \\ &\quad + c_4 \cosh(t\alpha\sqrt{1 + z^2}).\end{aligned}$$

Since $\phi(0) = \phi'(0) = 0$ we must have $c_2 = c_4 = 0$. The boundary condition $\phi'''(T) - \alpha^2\phi'(T) = 0$ shows that

$$0 = c_1(-z^3\alpha^3 - \alpha^2\alpha z) \cos(Tz\alpha) + c_3((1 + z^2)^{\frac{3}{2}}\alpha^3 - \alpha^2\alpha\sqrt{1 + z^2}) \cosh(T\sqrt{1 + z^2}\alpha)$$

$$0 = -c_1 z(z^2 + 1) \cos(Tz\alpha) + c_3 z^2 \sqrt{1 + z^2} \cosh(T\sqrt{1 + z^2}\alpha)$$

$$0 = -c_1 \sqrt{1 + z^2} \cos(Tz\alpha) + c_3 z \cosh(T\sqrt{1 + z^2}\alpha).$$

This shows that $\phi(t)$ must be of the form

$$\phi(t)/c = z \frac{\sin(tz\alpha)}{\cos(Tz\alpha)} + \sqrt{1 + z^2} \frac{\sinh(t\alpha\sqrt{1 + z^2})}{\cosh(T\alpha\sqrt{1 + z^2})}$$

for some c which is not zero. The last boundary condition $\phi''(T) = -\alpha\phi'(T)$ then gives the equation for the z_n . We see that there are countably infinite many solutions since for all $k \in \mathbb{N}$ there is at least one solution on the interval $[(k - \frac{1}{2})\pi/(\alpha T), (k + \frac{1}{2})\pi/(\alpha T)]$. This is because the left hand side of (8) is continuous in z on that interval and goes to minus infinity for $z \downarrow (k - \frac{1}{2})\pi/(\alpha T)$ and to plus infinity for $z \uparrow (k + \frac{1}{2})\pi/(\alpha T)$. The ϕ_n are orthogonal on $[0, T]$ since

$$\begin{aligned}\lambda_n \int_0^T \phi_n(s)\phi_m(s)ds &= \int_0^T \int_0^T W(s, u)\phi_n(u)du\phi_m(s)ds \\ &= \lambda_m \int_0^T \phi_n(u)\phi_m(u)du\end{aligned}$$

and $\lambda_m \neq \lambda_n$. This makes $\phi_n(t)$ an orthonormal set of functions and our definition of X then gives

$$\mathbb{E}(X_t X_u) = \frac{1}{\alpha} \sum_{n=0}^{\infty} \frac{\phi_n(t)\phi_n(u)}{z_n^2(1 + z_n^2)} = \sum_{n=0}^{\infty} \lambda_n \phi_n(t)\phi_n(u)$$

so for all $m \in \mathbb{N}$ we find

$$\int_0^T [W(t, u) - \mathbb{E}(X_t X_u)]\phi_m(u)du = \lambda_m \phi_m(t) - \lambda_m \phi_m(t) = 0.$$

Since the $(\phi_m)_{m \in \mathbb{N}}$ form an orthonormal basis of $L^2[0, T]$ by Mercer's theorem, this proves the result. $\square$

Proof of Theorem 1. We introduce Lagrange multipliers $\alpha^3 \zeta_i$ to get the unconstrained optimization problem

$$\begin{aligned}\tilde{\mathcal{L}}_\alpha[g] &= \int_0^\infty [g''(s)^2 + \alpha^2 g'(s)^2]ds \\ &\quad + \alpha^3 \sum_{i \in \mathcal{I}} \zeta_i \left(m_i - \sum_{j \in \mathcal{J}} \bar{c}_{ij} g(u_j) \right).\end{aligned}$$

Local minima must satisfy for every h in $\mathcal{F}_0$:

$$\begin{aligned}0 &= \frac{1}{2} \frac{d}{d\epsilon} \tilde{\mathcal{L}}_\alpha[g + \epsilon h] \Big|_{\epsilon=0} \\ &= \int_0^\infty [g''(s)h''(s) + \alpha^2 g'(s)h'(s)]ds \\ &\quad - \frac{1}{2} \alpha^3 \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \zeta_i \bar{c}_{ij} h(u_j) \\ &= \int_0^\infty h'(s) \left[-g'''(s) \right. \\ &\quad \left. + \alpha^2 g'(s) - \frac{1}{2} \alpha^3 \sum_{i \in \mathcal{I}} \zeta_i \sum_{j \in \mathcal{J}} \bar{c}_{ij} \mathbf{1}_{s \leq u_j} \right] ds\end{aligned}$$

where we have performed a partial integration on the function $h'(t)g''(t)$ and used that $h, g \in \mathcal{F}_0$ which implies that $h(0) = 0$ and that $g''(t)$ vanishes for $t = 0$ and $t \rightarrow \infty$. This must hold for all $h \in \mathcal{F}_0$ so we can thus conclude that for all $s > 0$

$$-g'''(s) + \alpha^2 g'(s) - \frac{1}{2} \alpha^3 \sum_{i \in \mathcal{I}} \zeta_i \sum_{j \in \mathcal{J}} \bar{c}_{ij} \mathbf{1}_{s \leq u_j} = 0. \quad (24)$$

This ODE has the solution (for real constants a_i)

$$\begin{aligned}g'(t) &= a_1 e^{\alpha t} + a_2 e^{-\alpha t} + \frac{1}{4} \alpha^2 \int_0^t (e^{\alpha(s-t)} - e^{-\alpha(s-t)}) \\ &\quad \times \sum_{j \in \mathcal{J}} e_j \mathbf{1}_{s \leq u_j} ds\end{aligned}$$

where we have lightened notation by writing $e_j = \sum_{i \in \mathcal{I}} \zeta_i \bar{c}_{ij}$. The last term equals

$$\begin{aligned}\frac{1}{4} \alpha^2 \sum_{j \in \mathcal{J}} e_j \int_0^{\min(t, u_j)} (e^{\alpha(s-t)} - e^{-\alpha(s-t)}) ds \\ = \frac{1}{4} \alpha \sum_{j \in \mathcal{J}} e_j (e^{\alpha(\min(t, u_j)-t)} + e^{-\alpha(\min(t, u_j)-t)}) \\ = \frac{1}{2} \alpha \sum_{j \in \mathcal{J}} e_j (\mathbf{1}_{t < u_j} + \cosh(\alpha(t - u_j)) \mathbf{1}_{t > u_j}).\end{aligned}$$

Integrating $g'(t)$ and using that $g(0) = 0$ thus shows that the optimal solution must have the form

$$\begin{aligned}g(t) &= a_3 \sinh(\alpha t) + a_4 (1 - e^{-\alpha t}) + \frac{1}{2} \sum_{j \in \mathcal{J}} e_j (\alpha \min(t, u_j) \\ &\quad + \sinh(\alpha(t - u_j)) \mathbf{1}_{t > u_j})\end{aligned}$$

for certain real constants a_3 and a_4 . For $t \rightarrow \infty$ the value of $g(t)$ will involve a term $e^{\alpha t}$ times $(\frac{1}{2}a_3 + \frac{1}{4} \sum_{j \in \mathcal{J}} e_j e^{-\alpha u_j})$ so we must choose $a_3 = -\frac{1}{2} \sum_{j \in \mathcal{J}} e_j e^{-\alpha u_j}$ to make this term vanish. We then get

$$\begin{aligned}g(t) &= a_4 (1 - e^{-\alpha t}) + \sum_{j \in \mathcal{J}} e_j \left(\alpha \min(t, u_j) + \frac{1}{2} e^{\alpha(t-u_j)} \right. \\ &\quad \left. \times (\mathbf{1}_{t > u_j} - 1) - \frac{1}{2} e^{-\alpha(t-u_j)} \mathbf{1}_{t > u_j} + \frac{1}{2} e^{-\alpha(t+u_j)} \right) \\ &= a_4 (1 - e^{-\alpha t}) + \sum_{j \in \mathcal{J}} e_j \left(\alpha \min(t, u_j) \right. \\ &\quad \left. - \frac{1}{2} e^{-\alpha|t-u_j|} + \frac{1}{2} e^{-\alpha(t+u_j)} \right) \\ &= a_4 (1 - e^{-\alpha t}) + \sum_{j \in \mathcal{J}} \sum_{i \in \mathcal{I}} \zeta_i \bar{c}_{ij} W(t, u_j).\end{aligned}$$

From Proposition 1 it follows that $g''(0) = -\alpha^2 a_4$. Due to the requirement $g''(0) = 0$ we thus have $a_4 = 0$, the result follows. $\square$

Proof of Theorem 2. From Eq. (12) and the remarks below it we know that $g(t) = \sum_j \eta_j W(t, u_j)$ with $\eta = \bar{\mathbf{C}}^T (\bar{\mathbf{C}} \mathbf{W} \bar{\mathbf{C}}^T)^{-1} \bar{\mathbf{m}}$. We now remark that for any $g \in \mathcal{F}_0 \supseteq \mathcal{H}(f_\infty)$ we have that

$$\begin{aligned} \mathcal{L}_\alpha[g] &= \int_0^\infty (g''(s)^2 + \alpha^2 g'(s)^2) ds \\ &= [g'(s)g''(s)]_{s=0}^{s=\infty} - \int_0^\infty g'(s)g'''(s) ds \\ &\quad + \alpha^2 \int_0^\infty g'(s)^2 ds \\ &= \int_0^\infty g'(s) (-g'''(s) + \alpha^2 g'(s)) ds. \end{aligned} \quad (25)$$

But optimality conditions for g in the proof of Theorem 1, see (24), imply that $-g'''(s)^2 + \alpha^2 g'(s) = \frac{1}{2}\alpha^3 \sum_j \eta_j \mathbf{1}_{s \leq u_j}$ and $g(0) = 0$ so

$$\begin{aligned} \mathcal{L}_\alpha[g] &= \frac{1}{2}\alpha^3 \sum_j \eta_j \int_0^{u_j} g'(s) ds \\ &= \frac{1}{2}\alpha^3 \sum_j \eta_j (g(u_j) - g(0)) = \frac{1}{2}\alpha^3 \sum_j \eta_j \sum_k \eta_k \mathbf{W}_{jk} \\ &= \frac{1}{2}\alpha^3 \cdot \bar{\mathbf{m}}^T (\bar{\mathbf{C}} \mathbf{W} \bar{\mathbf{C}}^T)^{-1} \bar{\mathbf{C}} \mathbf{W} \bar{\mathbf{C}}^T (\bar{\mathbf{C}} \mathbf{W} \bar{\mathbf{C}}^T)^{-1} \bar{\mathbf{m}} \\ &= \frac{1}{2}\alpha^3 \cdot \bar{\mathbf{m}}^T (\bar{\mathbf{C}} \mathbf{W} \bar{\mathbf{C}}^T)^{-1} \bar{\mathbf{m}}. \end{aligned}$$

We denote $\bar{\mathbf{C}} \mathbf{W} \bar{\mathbf{C}}^T$ by $\mathbf{A}$ to lighten notation. Setting the derivative with respect to the asymptotic forward rate f equal to zero gives

$$0 = \frac{\partial}{\partial f} (\bar{\mathbf{m}}^T \mathbf{A}^{-1} \bar{\mathbf{m}}) = \bar{\mathbf{m}}^T \left(2\mathbf{A}^{-1} \frac{\partial \bar{\mathbf{m}}}{\partial f} - \mathbf{A}^{-1} \frac{\partial \mathbf{A}}{\partial f} \mathbf{A}^{-1} \bar{\mathbf{m}} \right).$$

Since $\frac{\partial \bar{\mathbf{m}}}{\partial f} = \frac{\partial}{\partial f} (\mathbf{m} - \mathbf{C} \mathbf{D}^f \mathbf{e}) = -\mathbf{C} \mathbf{D}^f \mathbf{U} \mathbf{e} = -\bar{\mathbf{C}} \mathbf{U} \mathbf{e}$ and

$$\frac{\partial \mathbf{A}}{\partial f} = \frac{\partial}{\partial f} (\mathbf{C} \mathbf{D}^f \mathbf{W} \mathbf{D}^f \mathbf{C}^T) = 2\mathbf{C} \mathbf{D}^f \mathbf{U} \mathbf{W} (\mathbf{D}^f)^T \mathbf{C}^T = 2\bar{\mathbf{C}} \mathbf{U} \mathbf{W} \bar{\mathbf{C}}^T$$

we must have that

$$0 = -2\bar{\mathbf{m}}^T \mathbf{A}^{-1} \bar{\mathbf{C}} \mathbf{U} (\mathbf{e} + \mathbf{W} \bar{\mathbf{C}}^T \mathbf{A}^{-1} \bar{\mathbf{m}})$$

which gives the first result after substitution of $\bar{\mathbf{C}} = \mathbf{C} \mathbf{D}^f$ and $\bar{\mathbf{m}} = \mathbf{m} - \mathbf{C} \mathbf{D}^f \mathbf{e}$ to write everything in terms of the original cashflow matrix $\mathbf{C}$ and price vector $\mathbf{m}$. If $\mathbf{C}$ is invertible, we can write $\pi = \mathbf{C}^{-1} \mathbf{m}$ and reduce the equation to the form $(\mathbf{D}^{-f} \pi - \mathbf{e})^T \mathbf{W}^{-1} \mathbf{U} (\mathbf{D}^{-f} \pi) = 0$ which gives the second result. $\square$

Proof of Theorem 3. Suppose that $g^f = \hat{g}$ is the solution to Problem 3 for the forward curve, then $\hat{g}$ must be a local optimum of the following functional with Lagrangian parameters ζ_i for the constraints:

$$\begin{aligned} \tilde{\mathcal{L}}_\alpha[g] &= \frac{1}{2} \int_0^\infty g''(s)^2 + \alpha^2 (g'(s))^2 ds + 2\bar{\alpha}^2 \\ &\quad \times \sum_{i \in \mathcal{I}} \zeta_i \left(-m_i + \sum_{j \in \mathcal{J}} c_{ij} e^{-\int_0^{u_j} g(s) ds} \right) \end{aligned}$$

and under the additional requirements $g''(0) = \lim_{t \rightarrow \infty} g''(t) = 0$ and for a given $g(0)$ but free limit $\lim_{t \rightarrow \infty} g(t)$. This implies that for every h satisfying $h''(0) = \lim_{t \rightarrow \infty} h''(t) = h(0) = 0$, the perturbed solution $g := \hat{g} + \varepsilon h$ must satisfy $0 = (\frac{d}{d\varepsilon} \tilde{\mathcal{L}}_\alpha[\hat{g} + \varepsilon h])_{\varepsilon=0}$.

Define $\pi_j = \exp(-\int_0^{u_j} \hat{g}(s) ds)$, then

$$0 = \int_0^\infty [\hat{g}''(s)h''(s) + \bar{\alpha}^2 \hat{g}'(s)h'(s)] ds - 2\bar{\alpha}^2$$

$$\begin{aligned} &\times \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \zeta_i c_{ij} \pi_j \int_0^{u_j} h(v) dv \\ &= [h'(s)g''(s)]_{s=0}^{s=\infty} + \int_0^\infty h'(s) [-\hat{g}'''(s) + \bar{\alpha}^2 \hat{g}'(s)] ds \\ &\quad - 2\bar{\alpha}^2 \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \zeta_i c_{ij} \pi_j \left([h(t)t]_0^{u_j} - \int_{s=0}^{s=u_j} h(s) s ds \right) \\ &= \int_0^\infty h'(s) \left[-\hat{g}'''(s) + \bar{\alpha}^2 \hat{g}'(s) - 2\bar{\alpha}^2 \sum_{i \in \mathcal{I}} \zeta_i \right. \\ &\quad \left. \times \sum_{j \in \mathcal{J}} c_{ij} \pi_j (u_j - s) \mathbf{1}_{s \leq u_j} \right] ds. \end{aligned}$$

In the last step we used that $u_j h(u_j) = u_j \int_0^{u_j} h'(s) ds$ since $h(0) = 0$. Variation over h gives

$$-\hat{g}'''(s) + \bar{\alpha}^2 \hat{g}'(s) = 2\bar{\alpha}^2 \sum_{j \in \mathcal{J}} e_j^f (u_j - s) \mathbf{1}_{s \leq u_j} \quad (26)$$

where $e_j^f = \pi_j \sum_i \zeta_i c_{ij}$. The solution to this differential equation can be written in terms of the functions $\bar{W}(\cdot, u)$ which are three times differentiable in all points and satisfy for all $u > 0$

$$\partial_1^2 \bar{W}(0, u) = \lim_{t \rightarrow \infty} \partial_1^2 \bar{W}(t, u) = 0, \quad \lim_{t \rightarrow \infty} \bar{W}(t, u) = 1,$$

$$\bar{W}(0, u) = 0$$

while

$$-\partial_1^3 \bar{W}(t, u) + \bar{\alpha}^2 \partial_1 \bar{W}(t, u) = \frac{2\bar{\alpha}^2}{u^2} (u - t) \mathbf{1}_{t \leq u} \quad (27)$$

as may be shown by direct differentiation. The solution to the ODE (26) must thus be of the form

$$g(s) = a_0 + a_1 e^{-\bar{\alpha} s} + a_2 e^{\bar{\alpha} s} + \sum_{j \in \mathcal{J}} \eta_j^f \bar{W}(s, u_j)$$

with $\eta_j^f = u_j^2 e_j^f = u_j^2 \pi_j \sum_i \zeta_i c_{ij}$ and for certain constants a_0, a_1 and a_2 in $\mathbb{R}$. The constraint that $g''(0) = \lim_{t \rightarrow \infty} g''(t) = 0$ gives $a_1 = a_2 = 0$ and we have $a_0 = g(0)$. Substituting the expression for g in the definition of $(\pi_j)_{j \in \mathcal{J}}$ then gives the result (17)–(18) for the forward curve.

For the problem of finding the best yield curve $g = g^y$, we want to minimize

$$\begin{aligned} \tilde{\mathcal{L}}_\alpha[g] &= \frac{1}{2} \int_0^\infty [g''(s)^2 + \alpha^2 g'(s)^2] ds - \alpha^3 \\ &\quad \times \sum_{i \in \mathcal{I}} \zeta_i \left(m_i - \sum_{j \in \mathcal{J}} c_{ij} e^{-u_j g(u_j)} \right) \end{aligned}$$

under the constraints $g''(0) = \lim_{t \rightarrow \infty} g''(t) = 0$ and for a given $g(0)$ but free limit $\lim_{t \rightarrow \infty} g(t)$. We set $0 = (\frac{d}{d\varepsilon} \tilde{\mathcal{L}}_\alpha[g + \varepsilon h])_{\varepsilon=0}$ with $h''(0) = \lim_{t \rightarrow \infty} h''(t) = h(0) = 0$:

$$\begin{aligned} 0 &= \int_0^\infty [g''(s)h''(s) + \alpha^2 g'(s)h'(s)] ds \\ &\quad - \alpha^3 \sum_i \zeta_i \sum_j u_j c_{ij} e^{-u_j g(u_j)} h(u_j) \\ &= \int_0^\infty h'(s) \left[-g'''(s) + \alpha^2 g'(s) - \alpha^3 \sum_i \zeta_i \right. \\ &\quad \left. \times \sum_j u_j c_{ij} e^{-u_j g(u_j)} \mathbf{1}_{s \leq u_j} \right] ds \end{aligned}$$

so

$$0 = -g'''(s) + \alpha^2 g'(s) - \alpha^3 \sum_j e_j^y u_j \mathbf{1}_{s \leq u_j},$$

in which $e_j^y = \pi_j \sum_i \zeta_i c_{ij}$, $\pi_j = e^{-u_j g(u_j)}$.

We know from our reasoning after Eq. (24) that the solution of this ODE for a given value of $g(0)$ and under the constraints that $g''(0) = \lim_{t \rightarrow 0} g''(t) = 0$ equals

$$\begin{aligned} g(s) &= g(0) + \sum_j \eta_j^y W(s, u_j) \\ &= g(0) + \sum_j u_j \pi_j \sum_i \zeta_i c_{ij} W(s, u_j) \end{aligned}$$

with $\eta_j^y = u_j e_j^y$. Therefore, if we can find $(\pi_j)_{j \in \mathcal{J}}$ and $(\zeta_i)_{i \in \mathcal{I}}$ such that

$$-(\ln \pi_k)/u_k = g(0) + \sum_i \sum_j \zeta_i c_{ij} u_j \pi_j W(u_k, u_j)$$

$$m_i = \sum_j c_{ij} \pi_j$$

then all optimality conditions are fulfilled. $\square$

Proof of Corollary 2. Define as before $\eta_j^f = \sum_i \zeta_i^f c_{ij} \pi_j^f u_j^2$ and $\eta_j^y = \sum_i \zeta_i^y c_{ij} \pi_j^y u_j$. Then we have by (17)–(22) that

$$\begin{aligned} y_k &= g(0) + \sum_j \eta_j^f G_{kj}^f, \quad f_\infty^f = g(0) + \sum_j \eta_j^f \\ y_k &= g(0) + \sum_j \eta_j^y W_{kj}^y, \quad f_\infty^y = g(0) + \sum_j \eta_j^y (\alpha u_j) \\ &= g(0) + \sum_j (\eta_j^y \alpha u_j) G_{kj}^y \end{aligned}$$

so

$$\begin{aligned} f_\infty^f &= g(0) + \sum_j \sum_k [G^f]_{jk}^{-1} (y_k - g(0)), \\ f_\infty^y &= g(0) + \sum_j \sum_k [G^y]_{jk}^{-1} (y_k - g(0)) \end{aligned}$$

which gives the result after rearranging terms. $\square$

Proof of Corollary 3. We first note that for any g in $\mathcal{H}^f$ or $\mathcal{H}^y$ and any $\beta \in \mathbb{R}$ we have by (25) that

$$\mathcal{L}_\beta[g] = \int_0^\infty g'(s) (-g'''(s) + \beta^2 g'(s)) ds. \quad (28)$$

For $g(s) = g^f(s)$ we found $g(s) = g(0) + \sum_j \eta_j^f \bar{W}(s, u_j)$ and in the proof of Corollary 2 we established $\eta_j^f = \sum_k [G^f]_{jk}^{-1} (y_k - g(0))$ so because of (26) and (27) this gives

$$\begin{aligned} \mathcal{L}_{\bar{\alpha}}[g] &= 2\bar{\alpha}^2 \sum_j \eta_j^f u_j^{-2} \int_0^{u_j} g'(s) (u_j - s) ds \\ &= 2\bar{\alpha}^2 \sum_j \eta_j^f u_j^{-2} \int_0^{u_j} (g(s) - g(0)) ds \\ &= 2\bar{\alpha}^2 \sum_j \eta_j^f u_j^{-1} (y_j - g(0)) \\ &= 2\bar{\alpha}^2 \sum_j u_j^{-1} \sum_k [G^f]_{jk}^{-1} (y_k - g(0)) (y_j - g(0)). \end{aligned}$$

Setting the derivative with respect to the free parameter $g(0)$ equal to zero and rewriting then establishes the result for the forward curve optimization.

For the result of the yield curve optimization $g(s) = g^y(s) = g(0) + \sum_j \eta_j^y W(s, u_j)$ and $\eta_j^y = \frac{1}{\alpha u_j} \sum_k [G^y]_{jk}^{-1} (y_k - g(0))$. Then (28) gives

$$\begin{aligned} \mathcal{L}_\alpha[g] &= \int_0^\infty g'(s) \alpha^3 \sum_j \eta_j^y \mathbf{1}_{\{s \leq u_j\}} ds \\ &= \alpha^3 \sum_j \eta_j^y (g(u_j) - g(0)) = \alpha^3 \sum_j \eta_j^y (y_j - g(0)) \\ &= \alpha^3 \sum_j \frac{1}{\alpha u_j} \sum_k [G^y]_{jk}^{-1} (y_k - g(0)) (y_j - g(0)) \end{aligned}$$

so we find the same result but with a different matrix $\mathbf{G}$. $\square$

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