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Boonen, T.J.; Tan, K.S.; Zhuang, S.C.

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Pricing in Reinsurance Bargaining with Comonotonic Additive Utility Functions

Tim J. Boonen*, Ken Seng Tan†, Sheng Chao Zhuang‡,†

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Abstract

Optimal contracts have widely been studied in the literature, yet the bargaining for optimal prices has remained relatively unexplored. Therefore the key objective of this paper is to analyze the price of reinsurance contracts. We use a novel way to model the bargaining powers of the insurer and reinsurer, which allows us to generalize the contracts according to the Nash bargaining solution, indifference pricing, and the equilibrium contracts. We illustrate these pricing functions by means of inverse-S shaped distortion functions of the insurer and the Value-at-Risk for the reinsurer.

1 Introduction

This paper analyzes optimal reinsurance design and its pricing when firms are endowed with comonotonic additive utility functions. Comonotonic additive preferences are such that the utilities are additive for comonotonic risks. It gained particular interest after Schmeidler (1986) characterized a class of comonotonic additive preferences as Choquet integrals. As a special case, we focus on dual utilities (Yaari, 1987), that which is equivalent with minimizing distortion risk measures (Wang et al., 1997). Broadly speaking there are two recent streams of literature that consider risk sharing with dual utility functions. Both streams study roughly the same objective function in mathematical terms, but with different motivations. Firstly, several authors study optimal risk sharing and Pareto equilibria (see, e.g.,

*Amsterdam School of Economics, University of Amsterdam, Roetersstraat 11, 1018 WB Amsterdam, The Netherlands

†Department of Statistics and Actuarial Science, University of Waterloo, Waterloo, Ontario, N2L 3G1, Canada

‡Corresponding Author

Heath and Ku, 2004; Barrieu and El Karoui, 2005; Filipović and Kupper, 2008; Jouini et al., 2008; Ludkovski and Young, 2009; Boonen, 2015). Secondly, there is a stream in the literature that studies optimal (re)insurance contract design with a given premium principle (see, e.g., Asimit et al., 2013; Cui et al., 2013; Chi and Tan, 2011; Chi and Meng, 2014; Assa, 2015; Boonen et al., 2015; Cheung and Lo, 2015). The problem is often formulated from the point of view of the insurer by optimizing its own utility given the fact that the reinsurance premium is given by a distortion premium principle and does not impose any Pareto optimality condition. Instead, a moral hazard constraint is typically included that states that reinsurance contracts are increasing, but not more increasing than the underlying losses. This paper combines both settings in the sense that we use a bargaining approach for optimal risk sharing in the context of optimal reinsurance contract design. We model the preferences of the insurer and reinsurer by a distortion risk measure under the Pareto optimality framework under a moral hazard constraint. To the best of our knowledge, we are the first to explicitly combine both streams of literature.

Pricing of insurance and reinsurance contracts is typically done by assuming indifference. In other words, the price is set such that the reinsurer or insurer is indifferent to selling the contract or not. In this way, one determines the zero-utility premium. A second approach is introduced by Zhou et al. (2015a) focussing on a competitive equilibrium approach (also called a tâtonnement approach). If the reinsurer uses an additive utility function, this method yields indifference prices. A feature that is common to both approaches is that both firms benefit from trading. Moreover, there is a stream in the literature that focuses on empirical data on insurance prices, and try to derive the implied pricing functions. The problem with such approach is that the number of transactions in reinsurance is typically limited. Our approach is different from these three approaches. We determine the prices via a cooperative bargaining process.

Kihlstrom and Roth (1982), Schlesinger (1984), and Quiggin and Chambers (2009) all use the Nash bargaining solution for an insurance contract between a client and an insurer. Moreover, Aase (2009) uses the Nash bargaining problem to price reinsurance risk as well. Specifically for longevity risk, Boonen et al. (2012) and Zhou et al. (2015b) use Nash bargaining solutions to price longevity-linked Over-The-Counter contracts. All these authors focus on firms that maximize Von Neumann-Morgenstern expected utility. We use a cooperative bargaining approach to derive optimal reinsurance contracts and their corresponding prices via the Nash bargaining solutions. Moreover, we let firms maximize a comonotonic additive utility function. In contrast to indifference pricing, bargaining solutions allow us to share the benefits from trading, leading to profits for both parties. We provide a unique mechanism that allows us to generalize optimal contracts even if there is asymmetric bargaining power such as for the asymmetric Nash bargaining solution (Kalai, 1977). This mechanism includes indifference pricing as well, which leads to the extreme case that one firm is indifferent from trading, and the other firm gains maximally. This assumption is popular in the economic and actuarial literature, dating back from the concept of Bertrand equilibria (Bertrand, 1883).

This paper contributes to the literature in the following ways. We characterize the optimal hedge benefits (alternative interpreted as welfare gains) from bilateral bargaining for reinsurance. In the special case in which the preferences are given by a distortion risk measure, we derive a simple expression of the hedge benefits. Moreover, we derive bounds on the individual rational prices of a specific Pareto optimal contract, and provide to any price a corresponding bargaining power for the asymmetric Nash bargaining solution. To highlight our results, we illustrate the construction of the premium principle under the special case that the insurer is endowed with preferences given by an inverse-S shaped distortion risk measure, and the reinsurer optimizes a trade-off between the expected value and the Value-at-Risk (VaR). This leads to a discontinuous pricing function. Inverse-S shaped distortion risk measures are getting more popular to use as preferences since Quiggin (1982, 1991, 1992) and Tversky and Kahneman (1992).

Risk sharing and optimal reinsurance with expected utilities is discussed by, e.g., Borch (1960, 1962), Wilson (1968), Raviv (1979), Lemaire (1990), Taylor (1992a,b), and Aase (1993a,b, 2002). This paper focuses on preferences that are monotone and comonotonic additive. This includes convex risk measures (Föllmer and Schied, 2002; Frittelli and Rossaza-Gianin, 2002) that are comonotonic additive. Risk sharing under convex risk measures is first studied by Barrieu and El Karoui (2005), and later extended by, e.g., Burgert and Rüschendorf (2006), Filipović and Kupper (2008), and Jouini et al. (2008). In the more specific context of coherent risk measures, this problem is studied by, e.g., Heath and Ku (2004) and Burgert and Rüschendorf (2008). In this paper, we study bargaining for optimal reinsurance contracts. We derive conditions on Pareto optima in the context of reinsurance contract design, where a contract is an indemnity function on the insurer's risk and a price. The use of comonotonic additive preferences helps us to disentangle the characterization of the indemnity function and a method to determine the price. A special case of our preferences are distortion risk measures, as introduced and characterized by Wang et al. (1997). Distortion risk measures are related to coherent risk measures (see Wang et al., 1997; Artzner et al., 1999), as well as ambiguity aversion. Based on Schmeidler (1989), ambiguity is typically modeled by distorted probabilities (see, e.g., Chateauneuf et al., 2000; Werner, 2001; Tsanakas and Christofides, 2006; De Castro and Chateauneuf, 2011).

This paper is set out as follows. Section 2 provides all general results on bargaining with comonotonic additive utility functions. Section 3 shows how these results translate to preferences given by a distortion risk measure. Section 4 provides important insight on the premium principle for the class of inverse-S shaped distorted preferences and the well-known Value-at-Risk. Section 5 concludes the paper.

2 Model formulation

We consider a one-period model involving two firms, with one firm representing an insurer (I) and the other firm representing the reinsurer (R). Let $(\Omega, F, \mathbb{P})$ be a probability space, and $L^\infty(\Omega, F, \mathbb{P})$ be the class of bounded random variables on it. When there is no confusion, we simply write $L^\infty = L^\infty(\Omega, F, \mathbb{P})$. The total insurance liabilities that the insurer faces is given by the non-negative, bounded risk $X \in L^\infty$. Here we assume that the insurer is interested in transferring a part of this risk to a reinsurer. Let us denote $M = \operatorname{esssup} X = \inf\{a \in R : \mathbb{P}(X > a) = 0\}$. The reinsurance contract is given by the tuple (f, π), where $f(X)$ is the indemnity paid by the reinsurer to the insurer and $\pi \in \mathbb{R}$ is the price (or premium) paid by the insurer to the reinsurer. It is natural to assume that $f \in \mathcal{F}$, where

$$\mathcal{F} = \{f : \mathbb{R}_+ \to \mathbb{R}_+ \,|0 \le f(x) - f(y) \le x - y, \forall x \ge y \ge 0, f(0) = 0\}, \quad (1)$$

i.e., we assume that the indemnity $f \in \mathcal{F}$ is non-decreasing and 1-Lipschitz. The assumption that $f \in \mathcal{F}$ is often used in the literature on reinsurance contract design and its importance is highlighted in Chi and Tan (2011). More specifically, using the criterion of minimizing the VaR of the total risk of the insurer, Chi and Tan (2011) demonstrate that if we do not impose non-decreasing constraint on the indemnity f, the truncated stop-loss reinsurance is optimal. This form of reinsurance has the peculiar property that if losses exceed a certain threshold, the amount that is indemnified from the reinsurer to insurer is reduced to zero. Reinsurance treaty with such structure is perceived to be undesirable in that it encourages the insurer to under report its losses. See also Denuit and Vermandele, 1998; Young, 1999; Asimit et al., 2013; Chi and Meng, 2014; Assa, 2015; Xu et al., 2015. On the other hand, if f were to increase more rapidly than losses increase, then the insurer would have an incentive to create incremental losses. Both of these cases trigger the so-called moral hazard in the sense they create opportunity for the insurer to mis-report its actual losses to the reinsurer. The assumption that $f \in \mathcal{F}$ also makes sure that the indemnity function is continuous.

By denoting W_k as the deterministic initial wealth for firm k, where $k \in \{I, R\}$, and π_I as the premium received by the insurer for accepting risk X, then without the reinsurance the wealth at a pre-determined future time for the insurer and reinsurer are $W_I + \pi_I - X$ and W_R, respectively. If the insurer were to transfer part of its risk to a reinsurer using $f(X)$ with corresponding price π, then the wealth at a pre-determined future time for the insurer becomes

$$W_I + \pi_I - X + f(X) - \pi. \tag{2}$$

Similarly, the wealth for the reinsurer changes to

$$W_R - f(X) + \pi. \tag{3}$$

To assess if there should be a risk transfer between both firms, we need to make additional assumption on how firms evaluate such preference. In particular, we next define the preferences that we discuss in this paper.

Definition 2.1 *The preference relations $V_k, k \in \{I, R\}$ are such that*

- *it is monotone with respect to the order of L^∞;*
- *it satisfies the normalization conditions $V_k(0) = 0$ and $V_k(1) = 1$;*
- *it is comonotonic additive, i.e., $V_k(-Y) = V_k(-Y + f(Y)) + V_k(-f(Y))$ for all $Y \in L^\infty$ and all $f \in \mathcal{F}$.*

Note that the normalizations and comonotonic additivity imply that V_k has the cash-invariance property, i.e. $V_k(X+a) = V_k(X)+a$ for every $X \in L^\infty$ and $a \in \mathbb{R}$. Note that comonotonic additivity and the monotonicity assumption on V_k together with a regularity assumption on continuity imply the Choquet representation of V_k (Schmeidler, 1986; Wang et al., 1997). Providing an example of a Choquet representation is tedious, so we relegate the constructive examples to Sections 3 and 4, where we consider more specific preferences. It is well-known that the initial wealth and π_I are irrelevant for preferences given by cash-invariant utility functions, and therefore we set without loss of generality $W_I = \pi_I = W_R = 0$. Since we consider non-decreasing 1-Lipschitz indemnities, we have that $-X + f(X)$ and $-f(X)$ are comonotonic for all $f \in \mathcal{F}$. By focussing on contracts $(f, \pi) \in \mathcal{F} \times \mathbb{R}$, the comonotonic additivity of V_k implies that the utility function V_k is additive (and hence concave) on the subdomain. Note that Jouini et al. (2008) and Filipović and Kupper (2008) use monetary utility functions that are, in addition to the properties in Definition 2.1, concave. For instance, the risk measure Value-at-Risk that we will formally define in Section 4 is not concave in general, and so not a convex risk measure. It is however comonotonic additive, and so concave (in fact additive) on the subdomain of comonotonic risks.

We call a reinsurance contract $(f, \pi) \in \mathcal{F} \times \mathbb{R}$ Pareto optimal if there does not exist a contract $(\hat{f}, \hat{\pi}) \in \mathcal{F} \times \mathbb{R}$ such that $V_I(-X + \hat{f}(X) - \hat{\pi}) \geq V_I(-X + f(X) - \pi)$ and $V_R(-\hat{f}(X) + \hat{\pi}) \geq V_R(-f(X) + \pi)$, with at least one strict inequality.

The problem of finding an optimal contracts $(f, \pi) \in \mathcal{F} \times \mathbb{R}$ is analogous to the problem of finding optimal comonotonic risk sharing contracts. Jouini et al. (2008, Theorem 3.1 therein) characterize Pareto optimal risk sharing contracts for the class that contains also non-comonotonic risk sharing contracts. We extend this to the case where we restrict $f \in \mathcal{F}$.

Proposition 2.2 *Let $V_k, k \in \{I, R\}$ as in Definition 2.1. It holds that $(f, \pi) \in \mathcal{F} \times \mathbb{R}$ is Pareto optimal if and only if f is an element of:*

$$\arg\max_{f \in \mathcal{F}} V_I(-X + f(X)) + V_R(-f(X)), \tag{4}$$

where $\mathcal{F}$ is defined in (1)*.*

Proof First, we prove for "only if" part. We suppose that $(f, \pi) \in \mathcal{F} \times \mathbb{R}$ is Pareto optimal, but f is not an element of the set (4). Then, there exists an $\hat{f} \in \mathcal{F}$ such that $V_I(-X + f(X)) + V_R(-f(X)) < V_I(-X + \hat{f}(X)) + V_R(-\hat{f}(X))$. By defining $\hat{\pi} := V_R(-f(X) + \pi) - V_R(-\hat{f}(X))$ and the cash-invariance property of V_R, we have $V_R(-f(X) + \pi) = V_R(-\hat{f}(X) + \hat{\pi})$. Note that $V_I(-X + f(X) - \pi) + V_R(-f(X) + \pi) < V_I(-X + \hat{f}(X) - \hat{\pi}) + V_R(-\hat{f}(X) + \hat{\pi})$, as π and $\hat{\pi}$ will cancel out due to cash-invariance of V_I and V_R. Therefore, it follows that $V_I(-X + f(X) - \pi) < V_I(-X + \hat{f}(X) - \hat{\pi})$ as $V_R(-f(X) + \pi) = V_R(-\hat{f}(X) + \hat{\pi})$, which is a contradiction with $(f, \pi) \in \mathcal{F} \times \mathbb{R}$ being Pareto optimal. Hence, f is an element of the set (4).

For the "if" part, it also follows easily from the cash-invariance property of V_I and V_R. The proposition is thus proved.

Note that π does not appear in the above objective function due to the cash invariance property of $V_k, k \in \{I, R\}$: the π's cancel out each other. Proposition 2.2 asserts that the Pareto optimality of a reinsurance contract (f, π) depends only on the indemnity contract $f \in \mathcal{F}$. It is important to note that this property holds under the assumption of comonotonic additive utility functions, and it does not necessary apply for general utility functions, with notable exception is the exponential utility (Bühlmann and Jewell, 1979; Gerber and Pafumi, 1998).

The following proposition asserts the existence of Pareto optimal reinsurance contracts.

Proposition 2.3 *Let $V_k, k \in \{I, R\}$ as in Definition 2.1. Under the assumption that $V_k(X) < \infty, k \in \{I, R\}$, there exists a Pareto optimal reinsurance contract $(f, \pi) \in \mathcal{F} \times \mathbb{R}$, i.e. there exists an f solving* (4)*.*

Proof First, it follows from $V_k(X) < \infty$ for $k \in \{I, R\}$ that problem (4) is well-posed, i.e., the supremum of the objective in (4) is finite. Functions in $f \in \mathcal{F}$ are 1-Lipschitz, and therefore any sequence $f_n \in \mathcal{F}$ for $n \in \mathbb{N}$ is equicontinuous. By defining the norm $d(f^1, f^2) = \max_{t\in[0,M]} |f^1(t) - f^2(t)|$, for any $f^1, f^2 \in \mathcal{F}$, then the set $\mathcal{F}$ is compact under this norm d by Arzela-Ascoli's theorem.

Next, we show that $V_k, k \in \{I, R\}$, are 1-Lipschitz continuous in f under the norm d. Let $\varepsilon > 0$, and $d(\hat{f}, f) \le \varepsilon$. Then, we have $f(t) - \varepsilon \le \hat{f}(t) \le f(t) + \varepsilon$ for any $t \in [0, M]$. From this, we get $-\varepsilon = V_k(f(X) - \varepsilon) - V_k(f(X)) \le V_k(\hat{f}(X)) - V_k(f(X)) \le V_k(f(X) + \varepsilon) - V_k(f(X)) = \varepsilon$ holds by monotonicity and cash-invariance property. This implies that $|V_k(\hat{f}(X)) - V_k(f(X))| \le \varepsilon$. Hence, by the Weierstrass extreme value theorem, the optimal solution to (4) exists.

We now consider the benefits of reinsurance to both firms. Recall that without reinsurance, the utility of the insurer for insuring risk X is $V_I(-X)$ and the utility of the reinsurer is simply 0. If both firms agree to an indemnity function $f(X)$ with corresponding price π, then the resulting utility of the insurer changes to $V_I(-X + f(X) - \pi)$ so that the difference

$$V_I(-X + f(X) - \pi) - V_I(-X) = -V_I(-f(X)) - \pi \tag{5}$$

can be interpreted as the hedge benefit to the insurer using the indemnity $f \in \mathcal{F}$. The right hand side of the above equation follows from comonotonic additivity and cash invariance of V_I. Similarly, from the perspective of the reinsurer its hedge benefit is

$$V_R(-f(X) + \pi) - V_R(0) = V_R(-f(X)) + \pi. \tag{6}$$

Positive differences imply that there are incentives for reinsurance due to the gains in (monetary) utility. By denoting $HB(f)$ as the aggregate hedge benefits or the aggregate utility gains in the market for exercising the indemnity $f \in \mathcal{F}$, then we have

$$HB(f) = V_R(-f(X)) - V_I(-f(X)).$$

Note that $HB(f)$ is simply the sum of the hedge benefit of both insurer and reinsurer and hence for brevity we refer $HB(f)$ as the (aggregate) hedge benefit for a given indemnity $f \in \mathcal{F}$. Note also that $HB(f)$ can be positive, negative, or zero, depending on $f(X)$ and the heterogeneous preferences of insurer and reinsurer. Since the utility functions are cash-invariant, the hedge benefit $HB(f)$ is expressed in monetary terms.

If the contract (f^*, π) is Pareto optimal, then this implies the achievable hedge benefit of the market is maximal. By setting $HB^* \equiv HB(f^*)$, we have

$$HB^* = HB(f^*) = V_R(-f^*(X)) - V_I(-f^*(X)) \ge 0. \tag{7}$$

Note that the maximum achievable hedge benefit cannot be negative since $f(X) = 0$ is a feasible strategy in $\mathcal{F}$. If $V_I = V_R$, then the comonotonic additivity of V_k

leads to $HB^* = 0$; i.e. there is no gain in welfare in the market regardless of the indemnity $f \in \mathcal{F}$.

Depending on the market conditions, the hedge benefit HB^* will be shared among both firms. Particularly, we require that the following two conditions are satisfied:

- Pareto optimality,
- individual rationality, or both firms are weakly better off than when they do not trade: $V_I(-X + f(X) - \pi) \geq V_I(-X)$ and $V_R(-f(X) + \pi) \geq 0$.

Recall that Pareto optimality of the contract $(f, \pi) \in \mathcal{F} \times \mathbb{R}$ does not depend on the price π. The following proposition establishes the lower and upper bounds of individual rational prices corresponding to f. The key to deriving these bounds is based on the minimum acceptable price that a reinsurer is willing to accept the risk from an insurer and the maximum price that an insurer is willing to pay to transfer its risk to a reinsurer.

Proposition 2.4 *Let $V_k, k \in \{I, R\}$ as in Definition 2.1, and let $(f, \pi) \in \mathcal{F} \times \mathbb{R}$ be Pareto optimal and individual rational. Then, f solves* (4), *and*

$$\pi \in [-V_R(-f(X)), -V_I(-f(X))].$$

Proof For any $\pi \in \mathbb{R}$, the solution f solving (4) is Pareto optimal (see Proposition 2.2). Note that due to the cash-invariance of V, we obtain that $V_I(-X+f(X)-\pi)$ is strictly decreasing and continuous in π, and $V_R(-f(X)+\pi)$ is strictly increasing and continuous in π. Hence, the set of individual rational pricing is given by an interval, where the lower bound is such that $V_R(-f(X) + \pi) = V_R(0)$, and the upper bound is such that $V_I(-X+f(X)-\pi) = V_I(-X)$. The lower bound follows directly from cash-invariance and $V_R(0) = 0$, and the upper bound follows directly from cash-invariance, comonotonic additivity, and the fact that $-X + f(X)$ and $-f(X)$ are comonotonic. Finally, $-V_R(-f(X)) \leq -V_I(-f(X))$ follows from (7). This concludes the proof.

Note that for any Pareto optimal and individual rational contract (f, π), we have $\pi \geq 0$.

For a given indemnity function f, we now define a pricing principle. Given that both firms are individual rational, the hedge benefit $HB(f)$ is to be allotted among both firms. Then, the problem writes as a problem to "share a dollar" (see, e.g, Osborne and Rubinstein, 1990; Binmore, 1998), i.e., we aim to allocate a monetary amount among two firms. Define $\alpha \in [0, 1]$ as the proportion of the hedge benefit that is allocated to the insurer; i.e. $\alpha HB(f)$ hedge benefit is assigned to the insurer and the remaining $(1-\alpha)HB(f)$ hedge benefit to the reinsurer. Corresponding to the hedge benefit allocation α and the indemnity function f, it is of our interest to determine the resulting price of the reinsurance contract. To do this, it is useful to interpret the price π as a function of both α and f, so that $\pi \equiv \pi(\alpha, f)$ represents the price of indemnity $f(X)$ where the insurer receives $\alpha HB(f)$ hedge benefit and reinsurer receives the remaining $(1-\alpha)HB(f)$ hedge benefit.

Definition 2.5 *Let $V_k, k \in \{I, R\}$ as in Definition 2.1. For a given $\alpha \in [0,1]$ and $f \in \mathcal{F}$, the price $\pi(\alpha, f)$ is defined as the unique solution to*

$$V_I(-X + f(X) - \pi(\alpha, f)) = V_I(-X) + \alpha \cdot HB(f).$$

The above relation, together with the cash invariance property of V_I, lead to

$$\pi(\alpha, f) = V_I(-X + f(X)) - V_I(-X) - \alpha \cdot HB(f). \tag{8}$$

Moreover, the posterior utility of the reinsurer is given by $(1-\alpha)HB(f)$.

For a given $f \in \mathcal{F}$, the function $V_I(-X + f(X) - \pi)$ is continuous and strictly decreasing in π, and the function $V_R(-f(X) + \pi)$ is continuous and strictly increasing in π. Therefore, the pricing function for a given α and $f \in \mathcal{F}$ can be defined as the solution to the following optimization problem:

$$\pi(\alpha, f) = \arg\max_{\pi} V_I(-X + f(X) - \pi) \tag{9}$$

$$\text{s.t. } V_R(-f(X) + \pi) \geq (1-\alpha)HB(f). \tag{10}$$

The following proposition explicitly provides an expression for determining the price $\pi(\alpha, f)$.

Proposition 2.6 *Let $V_k, k \in \{I, R\}$ as in Definition 2.1. For all $f \in \mathcal{F}$ and $\alpha \in [0,1]$, we have*

$$\pi(\alpha, f) = -(1-\alpha)V_I(-f(X)) - \alpha V_R(-f(X)), \tag{11}$$

where $\pi(\alpha, f)$ is defined in Definition 2.5.

Proof It follows from (8) that

$$\begin{aligned}
&\pi(\alpha, f) \\
&= V_I(-X + f(X)) - V_I(-X) - \alpha HB(f) \\
&= V_I(-X + f(X)) - V_I(-X) - \alpha[-V_I(-f(X)) + V_R(-f(X))] \\
&= V_I(-X) - V_I(-f(X)) - V_I(-X) - \alpha[-V_I(-f(X)) + V_R(-f(X))]
\end{aligned}$$

$$= -(1-\alpha)V_I(-f(X)) - \alpha V_R(-f(X)).$$

Here, the second last equation follows from comonotonic additivity of V_I and the fact that $-X + f(X)$ and $-f(X)$ are comonotonic since $f \in \mathcal{F}$. This concludes the proof.

Note that $HB(f)$ can be negative, but it is bounded from above by $HB(f^*)$ with f^* solving (4). Moreover, by using the property that V_k is monotone and $V_k(0) = 0$, we can see that $\pi(\alpha, f)$ is always non-negative even when $HB(f)$ is negative. Note that if $HB(f)$ is negative, any contract (f, π) is not individual rational.

A more interesting situation to analyze is the Pareto optimal case with (f, π), where f is the optimal solution to (4). Recall that the resulting HB^* gives the highest attainable hedge benefit among the insurer and the reinsurer and that α captures the proportion of HB^* that is assigned to insurer. As α increases from 0 to 1, the portion of the hedge benefit that is allocated to the insurer increases until $\alpha = 1$ with the insurer receives the entire hedge benefit. Consequently the parameter α measures the bargaining power of the insurer; the higher the α, the greater the bargaining power of the insurer. The extreme cases $\alpha = 0, 1$ reflect cases of indifference pricing: all hedge benefits in the market are shifted to one party. It is easy to show that the competitive equilibrium outcome, also called tâtonnement outcome (see, e.g., Zhou et al., 2015a), corresponds to $\alpha = 1$ when the reinsurer uses a comonotonic additive utility function.

Next, we provide a characterization of our mechanism which is well-studied in the classical economic literature. In particular, the asymmetric Nash bargaining solution (Kalai, 1977) with asymmetry parameter α is given by:

$$\underset{(f,\pi)\in\mathcal{F}\times\mathbb{R}_+}{\arg\max}\ [V_I(-X + f(X) - \pi) - V_I(-X)]^{\alpha} V_R(-f(X) + \pi)^{1-\alpha} \tag{12}$$

$$\text{s.t. } V_I(-X + f(X) - \pi) \geq V_I(-X), \tag{13}$$

$$V_R(-f(X) + \pi) \geq 0. \tag{14}$$

Kalai (1977) characterizes this rule for convex bargaining problems as introduced by Nash (1950). A convex bargaining problem is given a convex and compact set $A \subset \mathbb{R}^2$ of feasible utility levels, and a disagreement point $d \in \mathbb{R}^2$ that is in our case the vector $d = (V_I(-X), 0)$. According to Proposition 2.2, we can write the set of all utility levels corresponding to the Pareto optimal and individually rational reinsurance contracts $(f, \pi) \in \mathcal{F} \times \mathbb{R}$ as $A = \{(V_I(-X + f^*(X) - \pi), V_R(-f^*(X) + \pi)) : \pi \in \mathbb{R}\} \cap \{x \in \mathbb{R}^2 : x \geq d\}$ with f^* solving (4). This set is a line in $\mathbb{R}^2$, and so it is convex. As the asymmetric Nash bargaining solution yields Pareto optimal and individually rational contracts (Kalai, 1977), we can restrict the feasible set in (12) to the utility levels in the set A.

Then, the asymmetric Nash bargaining rule is characterized by means of two more properties by Kalai (1977). One property is independent of equivalent utility representatives, which implies that affine transformations of the utility functions do not affect the outcome. The other property is independent of irrelevant alternatives (IIA). This property resembles a gradual elimination of other contracts from the feasible set, where eliminated contracts have no effect on the bargaining solution. Non-cooperative characterizations of the asymmetric Nash bargaining solution are provided by, e.g., Britz et al. (2010) and Miyakawa (2012). The Nash bargaining solution gained popularity in pricing reinsurance risk as well (see, e.g., Aase, 2009; Boonen et al., 2012; Zhou et al., 2015b; Boonen, 2016).

Proposition 2.7 *Let $V_k, k \in \{I, R\}$ as in Definition 2.1. If $HB^* > 0$ and $f \in \mathcal{F}$ solves* (4)*, then the price $\pi(\alpha, f)$ with $\alpha \in (0, 1)$, defined in Definition 2.5, coincides one-to-one with the asymmetric Nash bargaining solution with asymmetry parameter α, as defined in* (12)-(14).

Proof It is well-known that the asymmetric Nash bargaining solution is Pareto optimal (see, e.g., Kalai, 1977).

For any Pareto optimal contract (f, π), Proposition 2.2 implies that $f \in \mathcal{F}$ solves (4). It follows from (5) and (6) that the objective in (12), with constraints (13) and (14), can be reformulated as

$$\arg\max_{a\in[0,1]} [aHB^*]^\alpha \cdot [(1-a)HB^*]^{1-\alpha}. \tag{15}$$

Obviously, the solution of (15) does not depend on $HB^* > 0$. Then, we derive straightforwardly that the solution a is given by $a = \alpha$. Finally, $\pi = \pi(\alpha, f)$ follows by definition. This concludes the proof.

The case $\alpha = \frac{1}{2}$ corresponds to equal sharing of the hedge benefits, and corresponds to the Nash bargaining solution (Nash, 1950). The following proposition characterizes the pricing rule $\pi(\alpha, f)$ in a way that is commonly used in economic theory as well.

Proposition 2.8 *Let $V_k, k \in \{I, R\}$ as in Definition 2.1. For every $f^* \in \mathcal{F}$ solving* (4)*, π as defined in Definition 2.5 and $\alpha \in [0, 1]$, we have that the reinsurance contract $(f^*, \pi(\alpha, f^*))$ is an element of*

$$\left.\begin{array}{rl} \arg\max\limits_{(f,\pi)\in\mathcal{F}\times\mathbb{R}_+} & V_I(-X + f(X) - \pi) \\ s.t. & V_R(-f(X) + \pi) \geq (1-\alpha)\cdot HB^*, \end{array}\right\} \tag{16}$$

where HB^* is fixed, and given by (7).

Proof First, we show that every solution to (16) is Pareto optimal. Let $(f, \pi) \in \mathcal{F} \times \mathbb{R}_+$ solve (16) and suppose that it is not Pareto optimal. Then, there exist $(\hat{f}, \hat{\pi})$ such that $V_I(-X + \hat{f}(X) - \hat{\pi}) \geq V_I(-X + f(X) - \pi)$ and $V_R(-\hat{f}(X) + \hat{\pi}) \geq V_R(-f(X) + \pi)$ with one strict inequality. Since (f, π) solve (16), we get that $V_I(-X + \hat{f}(X) - \hat{\pi}) = V_I(-X + f(X) - \pi)$, and, hence, $V_R(-\hat{f}(X) + \hat{\pi}) > V_R(-f(X) + \pi)$. The function $V_I(-X + f(X) - \pi)$ is continuous and strictly decreasing in π, and the function $V_R(-f(X) + \pi)$ is continuous and strictly increasing in π. Hence, there exist $(\hat{f}, \widetilde{\pi})$, with $\widetilde{\pi} < \hat{\pi}$ such that $V_I(-X + \hat{f}(X) - \widetilde{\pi}) > V_I(-X + f(X) - \pi)$ and $V_R(-\widetilde{f}(X) + \hat{\pi}) > V_R(-f(X) + \pi)$. This is a contradiction. Hence, (f, π) is Pareto optimal. Hence, according to Proposition 2.2, f solves (4). For all Pareto optimal $(f, \pi) \in \mathcal{F} \times \mathbb{R}$, we get due to Proposition 2.2 and cash-invariance that $V_I(-X + f(X) - \pi)V_R(-f(X) + \pi)$ is the same. So, if (f, π) solves (16), then for every $\hat{f}$ solving (4) there exists a $\hat{\pi} \in \mathbb{R}$ such that $(\hat{f}, \hat{\pi})$ solves (16) as well.

The result that $\pi = \pi(\alpha, f)$ follows from the (9)-(10). This concludes the proof.

In Proposition 2.8, we characterize a pricing rule such that the insurer maximizes its profit under a participation constraint. This participation constraint might incorporate a reservation utility, which is non-negative. This approach coincides with, e.g., the approach Filipović et al. (2015) to price insurance arrangements under limited liability.

The preference relations V_k is called law-invariant when we have $V_k(X) = V_k(Y)$ for all $X, Y \in L^\infty$ that have the same distribution with respect to $\mathbb{P}$. To conclude this section, we point out that if the reinsurance contracts f are allowed to have any shape, i.e., $f : [0, M] \to [0, M]$ with $f(0) = 0$, and not restricted to the set $\mathcal{F}$, and if the preferences are concave and law-invariant, then Proposition 2.2, Proposition 2.3, and Proposition 2.7 still hold. It is important to note that Landsberger and Meilijson (1994) and Ludkovski and Rüschendorf (2008) show in risk sharing that there exists a Pareto optimal contract (f, π) such that $f \in \mathcal{F}$.

3 Distortion risk measures

In this section, we assume that insurer I and reinsurer R are endowed with a particular type of comonotonic additive utility function. More specifically, their preferences are given by a distortion risk measure. Wang et al. (1997) show that distortion risk measures satisfy the properties of Section 2, and are in addition law-invariant and satisfy a regularity condition on continuity.

Definition 3.1 *Preference relation $V_k, k \in \{I, R\}$ is given by a distortion risk* *measure when we have for every random variable* $W \in L^\infty$ *that*

$$V_k(W) := -E^{g_k}[-W] = \int_{-\infty}^{0} [1 - g_k(S_{-W}(z))]\, dz - \int_{0}^{\infty} g_k(S_{-W}(z))\, dz, \qquad (17)$$

where $k \in \{I, R\}$*,* $S_{-W}(z) = 1 - F_{-W}(z)$ *is the survival function of* $-W$*, and* $g_k : [0,1] \to [0,1]$ *is a non-decreasing function such that* $g_k(0) = 0$ *and* $g_k(1) = 1$*.*

A non-decreasing function $g : [0,1] \to [0,1]$ with $g(0) = 0$ and $g(1) = 1$ is called a distortion function. As a special case, when $Y \geq 0$, we have

$$E^{g_k}[Y] = \int_{0}^{\infty} g_k(S_Y(z))\, dz. \qquad (18)$$

Moreover, distortion risk measures are convex if the distortion functions are concave. Distortion risk measures are popular as it is related to the dual theory (Yaari, 1987) and the coherent risk measures (Artzner et al., 2001). Maximizing dual utility is equivalent to minimizing a distortion risk measure. Risk-aversion for distortion risk measures is equivalent to using a concave distortion function (Yaari, 1987). The Value-at-Risk (VaR) and all coherent risk measures satisfying law-invariance and comonotonic additivity are distortion risk measures (see Wang et al., 1997). Also, maximizers of a risk-reward trade-off $V_k(W) = (1-\gamma)E[W] - \gamma\rho_k(W), \gamma \in [0,1]$, are captured by this preference relation if ρ_k is a distortion risk measure. Here, γ reflects the aversion towards risk (see, e.g., De Giorgi and Post, 2008). Note that we do not require the distortion functions $g_k, k \in \{I, R\}$ to be concave or continuous. For instance, the distortion risk measure might be Value-at-Risk, which we will specify later in Section 4.

Pareto optimal reinsurance contracts $(f^*, \pi) \in \mathcal{F} \times \mathbb{R}$ for distortion risk measures follow from Cui et al. (2013) and Assa (2015). Every optimal indemnity function f^*, so that solves (4), can be shown to satisfy the following relationship:

$$f^{*\prime}(z) = \begin{cases} 1 & \text{if } g_I(S_X(z)) > g_R(S_X(z)), \\ \beta(z) & \text{if } g_I(S_X(z)) = g_R(S_X(z)), \\ 0 & \text{otherwise,} \end{cases} \qquad (19)$$

for all $z \geq 0$ almost surely, where $\beta(z) \in [0,1]$. Note that the indemnities $f \in \mathcal{F}$ are 1-Lipschitz, and therefore absolutely continuous. Hence the derivative of f exists almost everywhere. The indemnities in (19) are given by specific tranching of the insurer's risk. Every tranche is allocated to the firm that is endowed with the smallest distortion function on this quantile. This is interpreted as the locally least risk-averse firm (Boonen, 2015).

It is here important to remark that the contract in (19) is analogous to Pareto optimal risk sharing contracts with distortion risk measures. In the risk sharing context, Ludkovski and Young (2009, Theorem 2) characterize the same structure of Pareto optimal risk allocations for concave distortions. Our class of distortion risk measures allow also for non-concave distortion functions. Ludkovski and Young (2009) explicitly require the admissible set of risk sharing contracts to be such that the risks are comonotonic with the aggregate risk. This condition is analogous to requiring reinsurance indemnities to be non-decreasing and 1-Lipschitz as we impose in $\mathcal{F}$, and the price be any element of $\mathbb{R}$. In risk sharing, assuming contracts to be comonotonic is unrealistic if there might not exist Pareto optimal risk sharing contracts that are comonotonic when distortion functions are not concave (see Ludkovski and Rüschendorf, 2008). In reinsurance contract design, however, focussing on non-decreasing and 1-Lipschitz indemnities contracts is popular.

An interesting consequence of using the distortion risk measure to capture the comonotonic additive utility function is that the hedge benefit HB^* can be determined without knowing the Pareto optimal contract (f, π). This is asserted in the following proposition, of which the proof is trivial.

Proposition 3.2 *Suppose V_I and V_R are both distortion risk measures, defined in Definition 3.1. Then, it holds that*

$$HB^* = \int_0^\infty \Delta g_+(S_X(z))\, dz,$$

where $\Delta g_+ = (g_I - g_R)_+$, $(y)_+ = \max\{y, 0\}$, and where HB^* is defined in (7)

Corollary 3.3 *Let V_I and V_R are both distortion risk measures, defined in Definition 3.1, and let X have a compact support $[0, M]$. It holds that $HB^* = 0$ if and only if the Lebesgue measure of the set $\{z \in [0, M] : g_I(S_X(z)) - g_R(S_X(z)) > 0\}$ is zero. Furthermore, if X has a positive density on its support $[0, M]$, then it holds that $HB^* = 0$ if and only if the Lebesgue measure of the set $\{z \in [0, M] : g_I(z) - g_R(z) > 0\}$ is zero.*

Recall that $HB^* = 0$ signifies the situation that both firms are not able to strictly benefit from risk sharing. Corollary 3.3 provides an explicit characterization for this situation. In the economic literature on risk sharing, this situation is also called no-trade (De Castro and Chateauneuf, 2011).

We next derive a pricing function associated with the distortion risk measures.

Proposition 3.4 *Suppose V_I and V_R are both distortion risk measures, defined in Definition 3.1. For any $\alpha \in [0, 1]$ and $f \in \mathcal{F}$ satisfying* (19)*, we obtain*

$$\pi(\alpha, f) = E^{(1-\alpha)g_I + \alpha g_R}[f(X)].$$

Proof Let $f \in \mathcal{F}$ solve (4). The fact that $f \in \mathcal{F}$ implies $f(X) \geq 0$. If $V_k, k \in \{I, R\}$, are distortion risk measures, then by substituting (18) in Proposition 2.6, we obtain

$$\begin{aligned}\pi(\alpha, f) &= (1-\alpha)E^{g_I}(f(X)) + \alpha E^{g_R}(f(X)) \\ &= (1-\alpha)\int_0^\infty g_I(S_{f(X)}(z))\,dz + \alpha\int_0^\infty g_R(S_{f(X)}(z))]\,dz \\ &= \int_0^\infty [(1-\alpha)g_I(S_{f(X)}(z)) + \alpha g_R(S_{f(X)}(z))]dz \\ &= E^{(1-\alpha)g_I+\alpha g_R}[f(X)].\end{aligned}$$

This concludes the proof.

Suppose the state space is finite; let $\Omega = \{\omega_1, \ldots, \omega_p\}$, $\mathbb{P}(\omega) > 0$ for all $\omega \in \Omega$, and $X(\omega_1) > \cdots > X(\omega_p)$. Then, we get by direct calculations for $g := (1-\alpha)g_I + \alpha g_R$ and $Y := f(X)$ that

$$\begin{aligned}E^g(Y) &= \sum_{\omega\in\Omega} Y(\omega)[g(\mathbb{P}(X \geq X(\omega))) - g(\mathbb{P}(X > X(\omega)))] \\ &= \sum_{k=1}^{p-1}[Y(\omega_k) - Y(\omega_{k+1})]g(\mathbb{P}(X \geq X(\omega_k))) + Y(\omega_p),\end{aligned} \tag{20}$$

where we interpret $g(\mathbb{P}(X \geq X(\omega))) - g(\mathbb{P}(X > X(\omega))), \omega \in \Omega$ as state prices that we characterized in Proposition 3.4. If the function g is strictly increasing, the state prices are positive. Because $f \in \mathcal{F}$, we get that $Y(\omega_k) \geq Y(\omega_{k+1}) \geq 0$ for all k. So, if $Y(\omega_1) > 0$, we get that $E^g(Y) \geq Y(\omega_1)g(\mathbb{P}(\omega_1)) > 0$ if the function g is strictly increasing. Hence, we then get $Y(\omega_1) > 0$ if and only if $E^g(Y) > 0$, i.e., a non-negative risk with positive realizations has a positive price, and so the prices do not allow arbitrage opportunities. Note that we here do not need to restrict the pricing function g to be continuous.

By assuming that g is absolutely continuous and the state space is continuous, we have $E^g(Y) = E^g(f(X)) = \int_0^M f(z)g'(1-F_X(z))dF_X(z) = E[f(X)g'(1-F_X(X))] = E[Yg'(1-F_X(X))]$, where we used the fact that $f \in \mathcal{F}$. Here, $g'(1-F_X(X))$ is the pricing kernel that we characterized in Proposition 3.4. So, the prices reflect the preferences of a "representative" agent; namely the preferences of an agent who is endowed with distortion function g. In risk sharing with expected utilities, a representative agent model is also characterized by, e.g., Aase (1993).

Proposition 3.4 establishes that the pricing function is a distortion premium principle. The use of distortion premium principles to price risk has gained popularity in the actuarial literature (see, e.g., De Waegenaere et al., 2003; Cui et al., 2013; Assa, 2015). Note that these authors all assume that the distortion premium principles are given, whereas we derive it from cooperative bargaining.

To conclude this section, we point out that suppose the insurer is a risk-averse distortion risk measure minimizer (i.e., g_I is concave), and the reinsurer is risk-neutral, then it is optimal to reinsure all risk to the reinsurer, i.e., $f(X) = X$. This follows from $E^{g_I}[Y] \geq E[Y]$ for all $Y \in L^\infty$ (see, e.g., Boonen, 2015). However, the reinsurer might ask for a mark-up above the expected value premium if $\alpha < 1$, i.e., when the reinsurer has some bargaining power. Therefore, the premium is always larger than the expected value of a risk, which is also typically observed in reinsurance.

4 An illustration: inverse-S shaped distorted preferences and VaR

The objective of this section is to provide explicitly the pricing function under some additional assumptions on the preferences of insurer and reinsurer. We assume that the utility function for the insurer is dictated by an inverse-S shaped distorted function while the reinsurer relies on the value-at-risk (VaR). We consider inverse-S shaped distortion risk measures because of their desirable properties in modeling human behavior and their popularity in recent years (Quiggin 1982, 1991, 1992; Tversky and Kahneman 1992; Tversky and Fox 1995; Wu and Gonzalez, 1999; Abdellaoui, 2000; Rieger and Wang, 2006; Jin and Zhou, 2008; He and Zhou, 2011; Xu and Zhou, 2013; Bernard et al., 2015). Similarly, we adopt VaR in our example because it is a prominent measure of risk among financial institutions and insurance companies. It is also a regulatory risk measure adopted by the Solvency II regulations for insurance companies in European Union.

We first focus on the insurer's utility function and then followed by the reinsurer's. For the insurer, we additionally assume that the adopted distortion function g_I is continuously differentiable so that for every random variable $W \in L^\infty$, (17) can be written as

$$V_I(W) = \int_0^1 F_W^{-1}(s) g_I'(s) ds, \tag{21}$$

where $F_W^{-1}(s) = \inf\{z \in \mathbb{R} : F_W(z) \geq s\}, s \in [0, 1]$. The above representation demonstrates the role of the shape of the distortion function on evaluating wealth. If the function g_I is strictly concave; i.e. $g_I'(0) > 1$ and $g_I'(1) < 1$, then (21) implies that the good outcomes receive higher weights and the bad ones get smaller weights. If the function g_I is strictly convex; i.e. $g_I'(0) < 1$ and $g_I'(1) > 1$, then the good outcomes get smaller weights and the bad ones receive higher weights. On the other hand, inverse-S shaped preferences ($g_I'(0) > 1$ and $g_I'(1) > 1$) are such that both bad outcomes and good outcomes are heavily weighted. This is consistent with numerous psychological experiments conducted to study individual's risk aversion (Tversky and Kahneman, 1992; Tversky and Fox, 1995). Therefore, we consider an inverse-S shaped function in our example.

We now formally provide the definition of an inverse-S shaped distortion function.

Definition 4.1 *A distortion function g is called inverse-S shaped if:*

- *it is continuously differentiable;*
- *there exists $b \in (0, 1)$ such that g is strictly concave on the domain $(0, b)$ and strictly convex on the domain $(b, 1)$;*
- *it holds that $g'(0) = \lim_{s\downarrow 0} g'(s) > 1$ and $g'(1) = \lim_{s\uparrow 1} g'(s) > 1$.*

The point b in the above definition is the inflection point such that the g changes from locally concave to locally convex. Many distortion functions used in the literature are examples of inverse-S shaped. For example, let us consider the function proposed by Tversky and Kahneman (1992), which is parameterized by:

$$g^{\zeta}(s) = \frac{s^{\zeta}}{(s^{\zeta} + (1-s)^{\zeta})^{\frac{1}{\zeta}}} \qquad \text{for all } s \in [0, 1], \tag{22}$$

where $\zeta > 0$. Figure 1 plots (22) using $\zeta = 0.5$. Rieger and Wang (2006) point out that (22) is increasing and inverse-S shaped for $\zeta \in (0.279, 1)$.

Recall that Proposition 3.4 formally establishes that the distortion function in premium principle is given by a convex combination of both functions g_I and g_R. If the distortion functions g_I and g_R are inverse-S shaped, the pricing function is a distortion premium principle with an increasing distortion function. Any shape can be generated by a choice of α, g_I and g_R.

Remark If the functions g_I and g_R are concave, then π is a concave distortion premium principle.[1] If the functions g_I and g_R are convex, then π is a convex distortion premium principle. If the functions g_I and g_R are inverse-S shaped and the inflection points of g_I and g_R are the same, then π is an inverse-S shaped distortion premium principle.

[1] A concave distortion premium principle follows from bargaining between two firms that both use a concave distortion risk measure. Concave distortion risk measures resemble risk-averse preferences (Yaari, 1987).

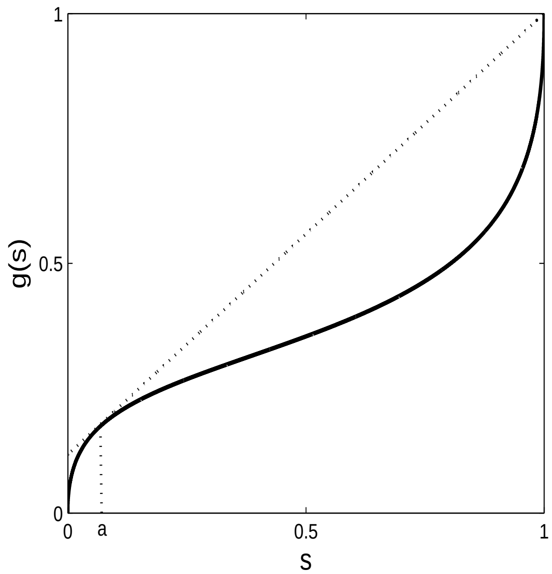

Figure 1: The solid line is the inverse-S shaped distortion function g^{ζ}, defined in (22), with $\zeta = 0.5$. The point a will be explained later in Lemma 4.3.

We now discuss the reinsurer's utility function, which is based on the Value-at-Risk (VaR). Formally, the VaR of the random variable $W \in L^{\infty}$ at a confidence level $\beta \in (0, 1)$ is given by $VaR_{\beta}(W) = E^{g}(-W)$ with $g(s) = \mathbf{1}_{s>\beta}$. It is popular in insurance industry due to regulations (see, e.g., Pritsker, 1997). This risk measure is connected to the quantile function via $VaR_{\beta}(W) = -F_{W}^{-1}(\beta)$. We assume that the reinsurer is risk-neutral, but bears the costs of holding capital. We further assume that the amount of capital to withhold is captured by the VaR risk measure, with the corresponding cost given by $CoC \cdot (VaR_{\beta}(W) - E[-W])$, where $CoC \in [0, 1]$ is the cost of holding capital and $\beta \in (0, 1)$. Then, maximizing the expected value of future wealth is equivalent to maximizing $E[W] - CoC \cdot (VaR_{\beta}(W) - E[-W])$, which in turn leads to the following preferences of the reinsurer:

$$V_R(W) = \gamma E[W] - (1 - \gamma) VaR_{\beta}(W), \tag{23}$$

for all $W \in L^{\infty}$, where $\gamma := 1 - CoC \in [0, 1]$, and $\beta \in (0, 1)$. The above representation implies that firms optimize a trade-off between expected return and risk, where risk is measured by VaR.[2] By construction, the price incorporates an expected value as well. In the literature, this is also called the risk-adjusted

[2]We emphasize that the preferences are similar to mean-variance preferences of Markowitz (1952), but with the risk captured by VaR, instead of variance.

value of the liabilities (for more detailed information, see Chi, 2012; Chi and Weng, 2013; Cheung and Lo, 2015).

The preference relation (23) is a distortion risk measure, which corresponds to setting the distortion function g_R as a weighted average of $g(s) = s$ and $g(s) = \mathbf{1}_{s>\beta}$:

$$g_R(s) = \begin{cases} \gamma s & \text{if } s \leq \beta, \\ \gamma s + (1-\gamma) & \text{if } s > \beta. \end{cases}$$

The VaR is criticized for being discontinuous in the sense that small changes in the risk leads to disproportional changes in the VaR. Therefore, it leads to undesirable outcomes when used for risk sharing. In this paper we focus on non-decreasing, 1-Lipschitz indemnities f, as defined in (1). We next show that VaR_β is continuous in f on the domain $\mathcal{F}$ under the norm $d(X,Y) = \|X-Y\|_1 := E[|X-Y|]$.[3].

Proposition 4.2 *The preference relation V_R is continuous in f on the domain $\mathcal{F}$ under the norm $d(X,Y) = \|X-Y\|_1 := E[|X-Y|]$, where V_R is defined in* (23) *with a given $\beta, \gamma \in (0,1) \times [0,1]$.*

Proof We show that for all $\varepsilon > 0$, there exists a $\delta > 0$, such that for any $f_1, f_2 \in \mathcal{F}$ satisfying $E[|f_1(X) - f_2(X)|] < \delta$, we have $|V_R(f_1(X)) - V_R(f_2(X))| < \varepsilon$.

We first show this result for $V_R = VaR_\beta$, i.e., when $\gamma = 0$. If $VaR_\beta(X) = 0$, then the results holds trivially as $|VaR_\beta(f_1(X)) - VaR_\beta(f_2(X))| = |f_1(VaR_\beta(X)) - f_2(VaR_\beta(X))| = 0$. Therefore, we assume that $VaR_\beta(X) > 0$. For all $\varepsilon > 0$, we denote by $A_\varepsilon := \mathbb{P}(F_X^{-1}(\beta) - \frac{\varepsilon}{2} \leq X \leq F_X^{-1}(\beta) + \frac{\varepsilon}{2}) \geq F_X(F_X^{-1}(\beta) + \frac{\varepsilon}{2}) - F_X(F_X^{-1}(\beta) - \frac{\varepsilon}{2}) \geq F_X(F_X^{-1}(\beta)) - F_X(F_X^{-1}(\beta) - \frac{\varepsilon}{2}) \geq \beta - F_X(F_X^{-1}(\beta) - \frac{\varepsilon}{2}) > 0$, where we use the facts that $F_X(F_X^{-1}(\beta)) \geq \beta$ and $\beta > F_X(F_X^{-1}(\beta) - \frac{\varepsilon}{2})$. Let $\varepsilon > 0$, and let $E[|f_1(X) - f_2(X)|] < \delta := \frac{\varepsilon}{2} A_\varepsilon$. Suppose we have $|VaR_\beta(f_1(X)) - VaR_\beta(f_2(X))| \geq \varepsilon$. Then, due to the fact that f_1 and f_2 are non-decreasing, we get $|VaR_\beta(f_1(X)) - VaR_\beta(f_2(X))| := |-F^{-1}_{f_1(X)}(\beta) - (-F^{-1}_{f_2(X)}(\beta))| = |f_2(F_X^{-1}(\beta)) - f_1(F_X^{-1}(\beta))|$. From $|f_2(F_X^{-1}(\beta)) - f_1(F_X^{-1}(\beta))| \geq \varepsilon$ and 1-Lipschitz continuity of f_1 and f_2, we get $|f_1\big(F_X^{-1}(\beta) + z\big) - f_2\big(F_X^{-1}(\beta) + z\big)| \geq \frac{\varepsilon}{2}$ for all $z \in [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$. Therefore, we have $E[|f_1(X) - f_2(X)|] > \frac{\varepsilon}{2}\mathbb{P}(F_X^{-1}(\beta) - \frac{\varepsilon}{2} \leq X \leq F_X^{-1}(\beta) + \frac{\varepsilon}{2}) = \frac{\varepsilon}{2} A_\varepsilon = \delta$. This is a contradiction. Hence, we have $|VaR_\beta(f_1(X)) - VaR_\beta(f_2(X))| < \varepsilon$. This concludes the result that VaR_β is continuous for f on the domain $\mathcal{F}$.

It follows from $|E[f_1(X)] - E[f_2(X)]| \leq E[|f_1(X) - f_2(X)|]$ that $E[f(X)]$ is continuous in f. Hence, it follows directly that V_R, defined in (23), is continuous for f on the domain $\mathcal{F}$.

[3]Note that continuity under the norm $\|X-Y\|_1$ is much stronger than the regularity condition on continuity by Schmeidler (1986), who requires continuity under the norm $d(X,Y) = \|X - Y\|_\infty := \sup_{\omega\in\Omega} |X(\omega) - Y(\omega)|$

Next, we show the Pareto optimal reinsurance contracts. It turns out that for the construction of the optimal indemnity function f, the following function for inverse-S shaped distortion functions plays a crucial rule:

$$p(s) = \frac{1 - g_I(s)}{1 - s}, \text{ for all } s \in [0, 1). \tag{24}$$

This function is introduced by Bernard et al. (2015). The following lemma follows from Xu et al. (2015).

Lemma 4.3 *The function p, defined in* (24)*, is continuous. Moreover, there exists $a \in (0, b)$ such that p is strictly decreasing on $[0, a]$ and strictly increasing on $[a, 1)$.*

The point a in the above lemma is illustrated in Figure 1. While it is difficult to provide an economic interpretation of a, we will see shortly that the inflection point a is useful for differentiating different cases of the indemnity contract. If $\gamma < 1$, there are five cases to consider for the Pareto optimal insurance contracts given by (19). These five cases are illustrated in Figure 2.

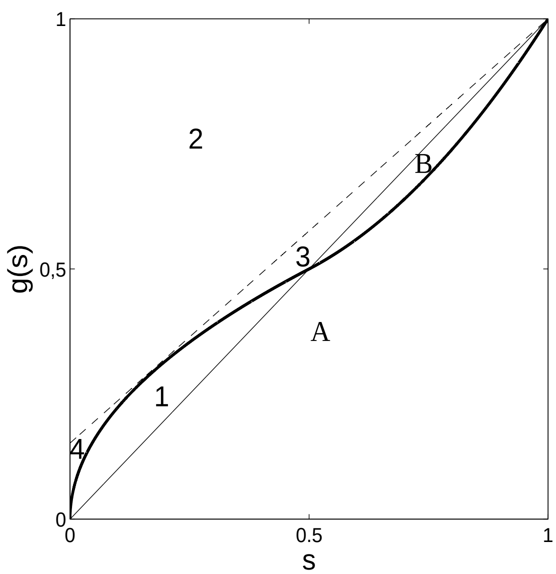

Figure 2: The solid line is an inverse-S shaped distortion function g_I. The five different cases are indicated via the areas $A, B, 1, 2, 3, 4$.

Case 4.1 In Figure 2, we have $(\beta, \gamma\beta) \in A$ and $(\beta, \gamma\beta + 1 - \gamma) \in 1$, i.e., $g_I(\beta) \geq \gamma\beta + 1 - \gamma$. Then, there exists $c \in [\beta, 1)$ such that $g_R(s) < g_I(s)$ for $s \in (0, c)$ and

$g_R(s) > g_I(s)$ for $s \in (c, 1)$. The optimal solution in (19) is given by

$$f^*(z) = \begin{cases} 0 & \text{if } 0 \le z \le VaR_c(-X), \\ z - VaR_c(-X) & \text{if } z > VaR_c(-X), \end{cases} \tag{25}$$

or, equivalently, $f^*(X) = (X - VaR_c(-X))_+$.

Case 4.2 In Figure 2, we have $(\beta, \gamma\beta) \in A$ and $(\beta, \gamma\beta + 1 - \gamma) \in 2$, i.e., $\gamma\beta < g_I(\beta) < \gamma\beta + 1 - \gamma$, $\gamma > p(a)$ and $\beta \ge a$. Then, we have $g_R(s) < g_I(s)$ for $s \in (0, \beta)$ and $g_R(s) > g_I(s)$ for $s \in (\beta, 1)$. The optimal solution in (19) is given by $f^*(X) = (X - VaR_\beta(-X))_+$.

Case 4.3 In Figure 2, we have $(\beta, \gamma\beta) \in A$ and $(\beta, \gamma\beta + 1 - \gamma) \in 3$, i.e., $\gamma\beta < g_I(\beta) < \gamma\beta + 1 - \gamma$ and $\gamma \le p(a)$. The optimal solution coincides with the solution of Case 4.2.

Case 4.4 In Figure 2, we have $(\beta, \gamma\beta) \in A$ and $(\beta, \gamma\beta + 1 - \gamma) \in 4$, i.e., $\gamma\beta < g_I(\beta) < \gamma\beta + 1 - \gamma$, $\gamma \le p(a)$ and $\beta \ge a$. Then, there exist two points $c \in (0, a)$ and $d \in (a, 1)$ such that $g_R(s) < g_I(s)$ for $s \in (0, \beta)$, $g_R(s) > g_I(s)$ for $s \in (\beta, c)$, $g_R(s) < g_I(s)$ for $s \in (c, d)$ and $g_R(s) > g_I(s)$ for $s \in (d, 1)$. The optimal solution in (19) is given by

$$f^*(z) = \begin{cases} 0 & \text{if } 0 \le z \le VaR_d(-X), \\ z - VaR_d(-X) & \text{if } VaR_d(-X) < z \le VaR_c(-X), \\ VaR_c(-X) - VaR_d(-X) & \text{if } VaR_c(-X) < z \le VaR_\beta(-X), \\ z - VaR_\beta(-X) + VaR_c(-X) - VaR_d(-X) & \text{if } z > VaR_\beta(-X), \end{cases}$$

or, equivalently, $f^*(X) = \min\{(X - VaR_d(-X))_+, VaR_c(-X) - VaR_d(-X)\} + (X - VaR_\beta(-X))_+$.

Case 4.5 In Figure 2, we have $(\beta, \gamma\beta) \in B$, i.e., $g_I(\beta) \le \gamma\beta$. Then, there exists $e \in [\beta, 1)$ such that $g_R(s) < g_I(s)$ for $s \in (0, e)$ and $g_R(s) > g_I(s)$ for $s \in (e, 1)$. The optimal solution in (19) is given by $f^*(X) = (X - VaR_e(-X))_+$.

The case $\gamma = 1$ is analogue to Case 4.1, where, without loss of generality, we set $\beta = 0$.

In Figure 3, we graphically illustrate the pricing premium principle of Proposition 3.4 for Case 4.1. We observe that the pricing distortion function is discontinuous and piecewise concave. This function is then used for the optimal stop-loss reinsurance contract given in (25). If the state space is countable, we get from (20) that the state price at the quantile β of the VaR is very high due to this discontinuity.

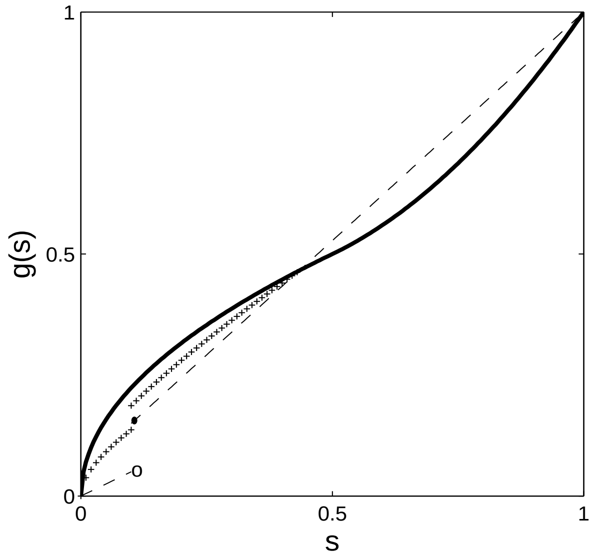

Figure 3: We graphically display the premium principle. The preferences of the insurer are inverse-S shaped (the solid line) and the preferences of the reinsurer are given in (23), with $\gamma = 0.9$ and $\beta = 0.1$ (the dotted line). The optimal reinsurance contract is derived from Case 4.1, and is given by $f^*(X) = (X - VaR_c(-X))_+$ for $c \approx 0.45$ in this figure. The line of crosses is the distortion function that serves as a premium principle after bargaining with $\alpha = 0.5$.

5 Conclusion

This paper studies bargaining for optimal reinsurance contracts with comonotonic additive preferences. In classical economics, one often assume that all profits from trading in economic markets are borne by one party. This leads to no profits for the other parties. Very few papers in the literature consider the fact in Over-The-Counter trades, benefits from sharing risk are shared between the two parties. Some exceptions are Kihlstrom and Roth (1982), Schlesinger (1984), Aase (2009), Boonen et al. (2012), Zhou et al. (2015b), and Boonen (2016). All these authors consider the Nash bargaining solution.

If firms have comonotonic additive preferences and use the Nash bargaining solution, the profits are shared equally between the two parties. We generalize this concept by parameterizing the share of the hedge benefits that are assigned to the insurer. We derive an implicit premium principle, which is analogue to a comonotonic additive utility function.

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