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Umbral Moonshine and $K3$ Surfaces

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Abstract: Recently, 23 cases of umbral moonshine, relating mock modular forms and finite groups, have been discovered in the context of the 23 even unimodular Niemeier lattices. One of the 23 cases in fact coincides with the so-called Mathieu moonshine, discovered in the context of $K3$ non-linear sigma models. In this paper we establish a uniform relation between all 23 cases of umbral moonshine and $K3$ sigma models, and thereby take a first step in placing umbral moonshine into a geometric and physical context. This is achieved by relating the ADE root systems of the Niemeier lattices to the ADE du Val singularities that a $K3$ surface can develop, and the configuration of smooth rational curves in their resolutions. A geometric interpretation of our results is given in terms of the marking of $K3$ surfaces by Niemeier lattices.

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M. C. N. Cheng is on leave from CNRS, Paris.

1. Introduction and Summary

Mock modular forms are interesting functions playing an increasingly important role in various areas of mathematics and theoretical physics. The “Mathieu moonshine” phenomenon relating certain mock modular forms and the sporadic group M_{24} was surprising, and its apparent relation to non-linear sigma models of $K3$ surfaces even more so. The fundamental role played by two-dimensional supersymmetric conformal field theories and $K3$ compactifications makes this moonshine relation interesting, not just for mathematicians but also for string theorists. In 2013 it was realised that this Mathieu moonshine is but just one case out of 23 such relations, called “umbral moonshine”. The 23 cases admit a uniform construction from the 23 even unimodular positive-definite lattices of rank 24 labeled by their non-trivial root systems. While the discovery of these 23 cases of moonshine perhaps adds to the beauty of the Mathieu moonshine relation, it also adds more mystery. In particular, it was previously entirely unclear what the physical or geometrical context for these other 22 instances of umbral moonshine could be. In this paper we establish a relation between $K3$ sigma models and all 23 cases of umbral moonshine, and thereby take a first step in incorporating umbral moonshine into the realm of geometry and theoretical physics.

Background. In mathematics, the term “moonshine” is used to refer to a particular type of relation between modular objects and finite groups. It was first introduced to describe the remarkable “monstrous moonshine” phenomenon [1] relating modular functions such as the J -function discussed below and the “Fischer–Griess monster group” \mathbb{M} , the largest of the 26 sporadic groups in the classification of finite simple groups. The study of this mysterious phenomenon was initiated by the observation by J. McKay that the second coefficient in the Fourier expansion of the modular function

$$\begin{aligned} J(\tau) &= J(\tau + 1) = J(-1/\tau) \\ &= \sum_{m \geq -1} a(m) q^m = q^{-1} + 196884 q + 21493760 q^2 + 864299970 q^3 + \dots \end{aligned} \tag{1.1}$$

with $q = e^{2\pi i \tau}$ satisfies $196884 = 196883 + 1$, and 196883 is precisely the dimension of the smallest non-trivial representation of \mathbb{M} . Note that the J -function has mathematical significance as the unique holomorphic function on the upper-half plane \mathbb{H} invariant under the natural action of $PSL_2(\mathbb{Z})$ generated by $\tau \rightarrow \tau + 1$ and $\tau \rightarrow -1/\tau$, which moreover has the behaviour $J(\tau) = q^{-1} + O(q)$ near the cusp $\tau \rightarrow i\infty$. Why and how the specific modular functions and the monster group, usually thought of as belonging to two very different branches of mathematics, are related to each other, remained a puzzle until about a decade after its discovery.

The key structure that unifies the two turns out to be that of a (chiral) 2d conformal field theory (CFT), or vertex operator algebra in more mathematical terms [2, 3]. The two sides of moonshine—the modularity and the finite group symmetry—can naturally be viewed as the manifestation of two kinds of symmetries—the world-sheet and the space-time symmetries—the CFT possesses. The mathematical proof of monstrous moonshine is achieved by constructing a generalised Kac–Moody algebra based on the above chiral CFT and utilising the no-ghost theorem of string theory, which roughly corresponds to considering the full 26 dimensions, including the 2 light-cone directions of the bosonic string theory [4]. We refer to, for instance, [5] for an introduction on the theory of modular forms and to [6] or the introduction of [7] for a summary of monstrous moonshine.

In 2010, an entirely unexpected new observation, pointing towards a new type of moonshine relating “mock modular forms” and finite groups, was made in the context of the elliptic genus of $K3$ surfaces. Mock modular forms embody a novel variation of the concept of modular forms and are interesting due to their significance in number theory as well as a wide range of applications (cf. (3.3)). See, for instance, [8,9] for an expository account on mock modular forms. From a physical point of view, as demonstrated in a series of recent works, the “mockness” of mock modular forms is often related to the non-compactness of relevant spaces in the theory. See, for instance, [10–13].

As we will discuss in more detail in Sect. 4, the elliptic genus $\text{EG}(K3)$ of $K3$ surfaces enumerates the BPS states of a $K3$ non-linear sigma model, and by taking the $\mathcal{N} = 4$ superconformal symmetry of this theory into account, one arrives at a weight $1/2$ mock modular form with Fourier expansion [14–16]

$$H_1^{X=A_1^{24}}(\tau) = 2q^{-1/8}(-1 + 45q + 231q^2 + 770q^3 + O(q^4)). \tag{1.2}$$

The observation by Eguchi–Ooguri–Tachikawa then states that the numbers 45, 231, and 770 are all dimensions of certain irreducible representations of the sporadic Mathieu group M_{24} [17]. This connection has since been studied, refined, extended, and finally established in [18–28]. From a mathematical point of view, the prospect of a novel type of moonshine for mock modular forms is extremely exciting. From a physical point of view, the ubiquity of $K3$ surfaces and the importance of BPS spectra in the study of string theory makes this “Mathieu moonshine” potentially much more relevant than the previous monstrous moonshine. See [29] for a review and [30–40] for some of the explorations in string theory and $K3$ conformal field theories inspired by this connection.

In 2013, the above relation was realised to be just the tip of the iceberg, or less metaphorically just one case out of a series of such relations, called “umbral moonshine” [7,41]. As will be reviewed in more detail in Sect. 3, to each one of the 23 Niemeier lattices L^X —the 23 even unimodular positive-definite lattices of rank 24 labeled by their non-trivial root systems X —one can attach, on the one hand, a finite group G^X , and, on the other hand, a vector-valued mock modular form H^X , such that the Fourier coefficients of H^X are again suggestive of a relation to certain representations of G^X , analogous to the observation on the functions $J(\tau)$ and $H_1^{X=A_1^{24}}(\tau)$ in (1.1) and (1.2). Further evidence for this relation was provided by relating characters of the same G^X -representations to the Fourier coefficients of other mock modular forms H_g^X , for each conjugacy class $[g]$ of G^X . More precisely, it was conjectured that an infinite-dimensional G^X -module K^X reproduces the mock modular forms H_g^X as its graded g -characters. The finite group G^X is defined by considering the symmetries of the Niemeier lattice L^X , while the mock modular form is determined by its root system X . The important role played by the rank 24 root systems X suggests the importance of the corresponding 24-dimensional representation of G^X . For instance, for the Niemeier lattice with the simplest root system $X = A_1^{24}$, the mock modular form $H^{A_1^{24}}$ is simply given by the function (1.2) above, and the finite group is $G^X \cong M_{24}$. In this case, the umbral moonshine is the Mathieu moonshine first observed in the context of the $K3$ elliptic genus that we described above. Given the uniform construction of the 23 instances of umbral moonshine from the Niemeier lattices L^X , one is naturally led to the following questions: *What about the other 22 cases of umbral moonshine with $X \neq A_1^{24}$? What, if any, is the physical and geometrical relevance of umbral moonshine? Are they also related to string or conformal field theories on $K3$? What is the relation between $K3$ and the Niemeier lattices L^X ? And the group G^X ? The mock modular form H^X and the underlying G^X -module K^X ?*

Summary. In the present paper we propose a first step in answering the above questions. To discuss the relation between the mock modular form H^X and the $K3$ elliptic genus, we first take a closer look at the construction of H^X from the root system X . For any of the 23 Niemeier lattices, the root system is a union of simply-laced root systems with an ADE classification with the same Coxeter number m . As is well-known, a wide variety of elegant structures in mathematics and physics admit an ADE classification. Apart from the simply-laced root systems, another such structure that will be important for us is that of modular invariant combinations of characters of the $A_1^{(1)}$ Kac–Moody algebra at level $m - 2$ [42]. As will be reviewed in more detail in Sect. 2, this classification leads to the introduction of the so-called Cappelli–Itzykson–Zuber matrices for every ADE root system, and these matrices in turn determine the relevant mock modular properties, which uniquely determine H^X when combined with a certain analyticity condition. Hence, the Cappelli–Itzykson–Zuber matrices Ω^Φ constitute a key element in the construction of the 23 instances of umbral moonshine.

By itself, the question of the classification of certain modular invariants seems remote from any physics or geometry. However, the parafermionic description of the $\mathcal{N} = 2$ minimal models relates this classification to that of the $\mathcal{N} = 2$ minimal superconformal field theories [43–46]. Moreover, their seemingly mysterious ADE classification can be related to the ADE classification of du Val (or Kleinian, or rational) surface singularities [45,46], whose minimal resolution gives rise to smooth rational (genus 0) curves with intersection given by the corresponding ADE Dynkin diagram. A third way to think about the ADE classification is the fact that these du Val singularities are isomorphic to the quotient singularity \mathbb{C}^2/G , with G being the finite subgroup of $SU_2(\mathbb{C})$ with the corresponding ADE classification [47]. Therefore, a perhaps simple-minded but logical step towards understanding the physical and geometrical context of umbral moonshine would be to take the ADE origin of the mock modular form H^X seriously. In particular, we would like to explore if the ADE-ology in umbral moonshine can be related to that of the du Val singularities.

Recall that the du Val singularities are precisely the singularities a $K3$ surface can develop. After computing the elliptic genus of du Val singularities (see Sect. 2), one realises that the $K3$ elliptic genus can naturally be split into two parts: one is the contribution from the configuration of the singularities given by X and the other is the contribution from the mock modular form H^X . Equipped with the mock modular form H^X for the other 22 Niemeier root systems X constructed in umbral moonshine, one finds that the same splitting holds uniformly for all 23 instances of umbral moonshine (cf. (4.9)). Note that this splitting makes no reference to the $\mathcal{N} = 4$ characters, although for the special case $X = A_1^{24}$ the two considerations render the same result.

While the above fact might be surprising and suggestive, one should be careful not to claim a strong connection between umbral moonshine and $K3$ string theory too quickly: it's logically possible that the above relation is just a consequence of the fact that the space of the relevant modular objects, the Jacobi forms of weight 0 and index 1 to be more precise, is very constrained and in fact only one-dimensional. See Appendix B for more details.

To gather more evidence on the umbral moonshine—a conjecture on the existence of a G^X -module K^X , which (re)produces the mock modular forms H_g^X , $[g] \subset G^X$ as its graded characters—and the $K3$ sigma model, one should compare the way G^X acts on K^X with the way the BPS spectrum of the $K3$ CFT transforms under its finite group symmetry G , when such a non-trivial G exists. Let us first focus on the geometric

symmetries of $K3$ surfaces (as opposed to “stringy” CFT symmetries without direct geometric origins). As we will review in more detail in Sect. 5, thanks to the global Torelli theorem for $K3$, we know that a finite group G is the group of hyper-Kähler-preserving symmetries of a certain $K3$ surface M if and only if it acts on the 24-dimensional $K3$ cohomology lattice $H^*(M, \mathbb{Z})$ in a certain way. Relating this 24-dimensional representation of G to the natural 24-dimensional representation of G^X induced from its action on the root system X , this translates into a criterion for a conjugacy class $[g] \subset G^X$ to arise as $K3$ symmetries for each of the 23 G^X .

On the one hand, umbral moonshine suggests a “twined” function Z_g^X for each $[g] \subset G^X$, where $Z_g^X = \mathbf{EG}(K3)$ for the special case that $[g]$ is the identity class (cf. (4.12)). In particular, from this consideration we arrive at a conjecture for the elliptic genus of the du Val singularity twined by its symmetries given by the automorphism of the corresponding Dynkin diagram. On the other hand, whenever the CFT admits a non-trivial finite automorphism group G , one can compute the elliptic genus “twined” by any $g \in G$. These twined elliptic genera $\mathbf{EG}_g(K3)$ provide information about the Hilbert space as a representation of G . As a result, for a conjugacy class $[g] \subset G^X$ arising from $K3$ symmetries, we have two ways to attach a twined function— Z_g^X and $\mathbf{EG}_g(K3)$ —to such a “geometric” conjugacy class of $[g] \subset G^X$. It turns out that they coincide for all the geometric conjugacy classes $[g]$ of any one of the 23 G^X . This identity clearly provides non-trivial evidence that all 23 instances of umbral moonshine are related to $K3$ non-linear sigma models.

Recall that in arriving at the above relation we have interpreted the ADE root systems X as the configuration of rational curves given by the ADE singularities. The above result hence suggests that it might be fruitful to study the symmetries of different $K3$ surfaces with distinct configurations of rational curves in a different framework corresponding to the 23 cases of umbral moonshine. In fact, this has been implemented in a recent analysis of the relation between the $K3$ Picard lattice, $K3$ symplectic automorphisms, and the Niemeier lattices, through a “marking” of a $K3$ surface M by one of the L^X , such that the Dynkin diagram obtained from the smooth rational curves of M is a sub-diagram of X [48, 49]. As will be discussed in more detail in Sect. 5, through this marking by the Niemeier lattice L^X , the root system X obtains the interpretation as the “enveloping configuration of smooth rational curves” while the finite group G^X is naturally interpreted as the “enveloping symmetry group” of the $K3$ surfaces that can be marked by the given L^X . On the one hand, this provides a geometric interpretation of our results. On the other hand, one can view our results as a moonshine manifestation and extension of the geometric analysis in [48].

The organisation of the paper is as follows. In Sect. 2 we compute the elliptic genus of the ADE du Val singularities that $K3$ surfaces can develop. In Sect. 3 we review the umbral moonshine construction from 23 Niemeier lattices and introduce the necessary ingredients for later calculations. Utilising the results of Sect. 2, in Sect. 4 we establish the relation between the (twined) elliptic genus and the mock modular forms of umbral moonshine. In Sect. 5 we provide a geometric interpretation of this result. In Sect. 6 we close this paper by discussing some open questions and point to some possible future directions. In Appendix A we collect useful definitions. In Appendix B we present the calculations and proofs, and present our conjectures for the twined (or equivariant) elliptic genus for the du Val singularities. The explicit results for the twining functions are recorded in the Appendix C.

2. The Elliptic Genus of Du Val Singularities

The rational singularities in two (complex) dimensions famously admit an ADE classification. See, for instance, [50]. They are also called the du Val or Kleinian singularities and are isomorphic to the quotient singularity \mathbb{C}^2/G , with G being the finite subgroup of $SU_2(\mathbb{C})$ with the corresponding ADE classification [47]. Any such singularity has a unique minimal resolution. The so-called resolution graph, the graph of the intersections of the smooth rational (genus 0) curves of the minimal resolution, gives precisely the corresponding ADE Dynkin diagram. We will denote by Φ the corresponding simply-laced irreducible root system. In terms of hypersurfaces, it is given by $W_\Phi^0 = 0$ with

$$W_{A_{m-1}}^0 = x_1^2 + x_2^2 + x_3^m \tag{2.1}$$

$$W_{D_{m/2+1}}^0 = x_1^2 + x_2^2 x_3 + x_3^{m/2} \tag{2.2}$$

$$W_{E_6}^0 = x_1^2 + x_2^3 + x_3^4 \tag{2.3}$$

$$W_{E_7}^0 = x_1^2 + x_2^3 + x_2 x_3^3 \tag{2.4}$$

$$W_{E_8}^0 = x_1^2 + x_2^3 + x_3^5. \tag{2.5}$$

These singularities show up naturally as singularities of $K3$ surfaces and play an important role in various physical setups, such as in heterotic–type II dualities and in geometric engineering, in string theory compactifications. See, for instance, [51,52] and [53].

The 2d conformal field theory description of these (isolated) singularities was proposed in [54] to be the product of a non-compact super-coset model $\frac{SL(2,\mathbb{R})}{U(1)}$ (the Kazama–Suzuki model [55]) and an $\mathcal{N} = 2$ minimal model, followed by an orbifoldisation by the discrete group $\mathbb{Z}/m\mathbb{Z}$, where m is the Coxeter number of the corresponding simply-laced root system (cf. Table 1). In other words, we consider the super-string background that is schematically given by

$$\text{Minkowski space-time } \mathbb{R}^{5,1} \otimes \left(\mathcal{N} = 2 \text{ minimal} \otimes \mathcal{N} = 2 \frac{SL(2,\mathbb{R})}{U(1)} \text{ coset} \right) / (\mathbb{Z}/m\mathbb{Z}). \tag{2.6}$$

Recall that, when the minimal model is chosen to be the “diagonal” A_{m-1} theory, the above theory also describes the near-horizon geometry of m NS five-branes [56]. Note that this point of view plays an important role in the work of [37,57], also in the context of discussing the possible physical context of umbral moonshine.

To resolve the singularity let us consider $W_\Phi^0 = \mu$. In [54] it was proposed that the sigma model with the non-compact target space $W_\Phi^0 = \mu$ has an alternative description as the Landau–Ginsburg model with superpotential

$$\tilde{W}_\Phi = -\mu x_0^{-m} + W_\Phi^0,$$

Table 1. Simply-laced root systems, Coxeter numbers and Coxeter exponents

	A_{m-1}	$D_{1+m/2}$	E_6	E_7	E_8
Coxeter number	m	m	12	18	30
Coxeter exponents	1, 2, 3, ..., $m-1$	1, 3, 5, ..., $m-1, m/2$	1, 4, 5, 7, 8, 11	1, 5, 7, 9, 11, 13, 17	1, 7, 11, 13, 17, 19, 23, 29

where x_0 is an additional chiral superfield and m is again given by the Coxeter number of Φ .

The purpose of the rest of the section is to compute the elliptic genus of (the supersymmetric sigma model with the target space being) the du Val singularities. First let us focus on the minimal model part. The $\mathcal{N} = 2$ minimal models are known to have an ADE classification [43–46],¹ based on an ADE classification of the modular invariant combinations of chiral (holomorphic) and anti-chiral (anti-holomorphic) characters of the $A_1^{(1)}$ Kac–Moody algebra [42]. In this language, the ADE classification can be thought of as a classification of the possible ways to consistently combine left- and right-movers. To be more precise, in [42] it was found that a physically acceptable and modular invariant combination of characters of the $A_1^{(1)}$ Kac–Moody algebra at level $m - 2$ is necessarily given by a $2m \times 2m$ matrix Ω^Φ corresponding to an ADE root system Φ , where we say that a modular invariant is physically acceptable if it satisfies certain integrality, positivity and normalisation conditions. See [42] for more details. The list of these matrices is given in Table 3. The relation between Ω^Φ and the ADE root system Φ lies in the following two facts. First, Ω^Φ is a $2m \times 2m$ matrix where m is the Coxeter number of Φ . Moreover, $\Omega_{r,r}^\Phi - \Omega_{r,-r}^\Phi = \alpha_r^\Phi$ for $r = 1, \dots, m - 1$ coincides with the multiplicity of r as a Coxeter exponent of Φ (cf. Table 1). Recall that a Coxeter element $\prod_{i=1}^r r_i$ of the Weyl group of a rank- r root system is the product of reflections with respect to all simple roots (the order in which the product is taken does not change the conjugacy class of the element), and the Coxeter number is the order of such a Coxeter element.

A quantity that played an important role in the CFT/LG correspondence [60] as well as in the recent developments of mock modular form moonshine is the elliptic genus. From a physical point of view, the elliptic genus for a 2d $\mathcal{N} = (2, 2)$ superconformal field theory \mathcal{T} is defined as [61]

$$Z_{\mathcal{T}}(\tau, z) = \text{tr}_{\mathcal{H}_{\mathcal{T},RR}} \left((-1)^{F_R+F_L} y^{J_0} q^{H_L} \bar{q}^{H_R} \right) \tag{2.7}$$

where $F_{R,L}$ denotes the right- and left-moving fermion number respectively. Moreover, the left- (right-) moving Hamiltonian is given by $H_L = L_0 - c_L/24$ ($H_R = \bar{L}_0 - c_R/24$), where $J_0, L_0, \bar{J}_0, \bar{L}_0$ are the zero modes of the left- and right-moving copies of the $U(1)$ R-current and Virasoro parts of the $\mathcal{N} = 2$ superconformal algebra, respectively. $\mathcal{H}_{\mathcal{T},RR}$ denotes the space of quantum states of theory \mathcal{T} in the Ramond–Ramond sector, and c_L and c_R denote the left- and right-moving central charge of the SCFT. In the above formula, τ takes values in the upper-half plane \mathbb{H} while z takes values in the complex plane \mathbb{C} , and we have written $q = e(\tau)$ and $y = e(z)$. Throughout the paper we use $e(x) := e^{2\pi i x}$. Because of the insertion $(-1)^{F_R}$, the elliptic genus only receives contributions from left-moving states that are paired with a right-moving Ramond ground state and is therefore holomorphic, at least when the spectrum of the theory is discrete. As such, it is rigid in the sense of being invariant under any continuous deformation of the theory.

The elliptic genus of the $\mathcal{N} = 2$ minimal model can be computed in various ways. First, from the relation to the parafermion theory, we obtain that the building block of the elliptic genus is the function $\tilde{\chi}_s^r(\tau, z)$, where $|s| \leq r - 1 < m$ [44,62]. See Appendix B for the definition of $\tilde{\chi}_s^r$. From the known spectrum of the minimal model given in terms of the matrix Ω^Φ and the identity $\tilde{\chi}_s^r(\tau, 0) = \delta_{r,s} - \delta_{r,-s}$ it is straightforward to see

¹ Strictly speaking, this classification applies when one requires the presence of a spectral flow symmetry. See for instance [58,59] for a discussion on related subtleties.

that the elliptic genus of the minimal model corresponding to the ADE root system Φ is given by [63,64]

$$Z_{\text{minimal}}^\Phi(\tau, z) = \sum_{r,r' \in \mathbb{Z}/2m\mathbb{Z}} \Omega_{r,r'}^\Phi \tilde{\chi}_{r'}^r(\tau, z) = \text{Tr}(\Omega^\Phi \cdot \tilde{\chi}). \tag{2.8}$$

We again refer to Appendix B for more details.

On the other hand, the Landau–Ginzburg description facilitates a free-field computation for the elliptic genus and one obtains an infinite-product expression for $Z_{\text{minimal}}^\Phi(\tau, z)$ [61]. In terms of the Jacobi theta function (A.1), the results are [61,64]

$$Z_{\text{minimal}}^\Phi = \frac{\theta_1\left(\tau, \frac{m-1}{m}z\right)}{\theta_1\left(\tau, \frac{z}{m}\right)} \quad \text{for } \Phi = A_{m-1} \tag{2.9}$$

for the A-series where $m \geq 2$,

$$Z_{\text{minimal}}^\Phi = \frac{\theta_1\left(\tau, \frac{m-2}{m}z\right) \theta_1\left(\tau, \frac{m+2}{2m}z\right)}{\theta_1\left(\tau, \frac{2z}{m}\right) \theta_1\left(\tau, \frac{m-2}{2m}z\right)} \quad \text{for } \Phi = D_{\frac{m}{2}+1} \tag{2.10}$$

for the D series where $m \geq 6$ and even, and finally

$$Z_{\text{minimal}}^{E_6} = \frac{\theta_1\left(\tau, \frac{3}{4}z\right) \theta_1\left(\tau, \frac{2}{3}z\right)}{\theta_1\left(\tau, \frac{z}{4}\right) \theta_1\left(\tau, \frac{z}{3}\right)} \tag{2.11}$$

$$Z_{\text{minimal}}^{E_7} = \frac{\theta_1\left(\tau, \frac{7}{9}z\right) \theta_1\left(\tau, \frac{2}{3}z\right)}{\theta_1\left(\tau, \frac{2}{9}z\right) \theta_1\left(\tau, \frac{z}{3}\right)} \tag{2.12}$$

$$Z_{\text{minimal}}^{E_8} = \frac{\theta_1\left(\tau, \frac{4}{5}z\right) \theta_1\left(\tau, \frac{2}{3}z\right)}{\theta_1\left(\tau, \frac{z}{5}\right) \theta_1\left(\tau, \frac{z}{3}\right)} \tag{2.13}$$

for the E -type cases. The central charge of these minimal models are given by the Coxeter number m of the corresponding simply-laced root system by

$$\hat{c} = c/3 = 1 - \frac{2}{m}. \tag{2.14}$$

In order to obtain the elliptic genus of the isolated ADE singularities, another ingredient we need is the elliptic genus of the $\frac{SL(2, \mathbb{R})}{U(1)}$ super-coset model. The $\frac{SL(2, \mathbb{R})}{U(1)}$ super-coset model is known to describe the geometry of a semi-infinite cigar (a 2d Euclidean black hole) [65] and is mirror to the $\mathcal{N} = 2$ super Liouville theory [56,66]. The level of the super-coset model is related to the mass of the corresponding 2d black hole, and the central charge of the super Liouville theory. Here, we will consider $SL(2, \mathbb{R})$ super-current algebra of (super) level m . The central charge of the corresponding super-coset theory is

$$\hat{c} = 1 + \frac{2}{m}.$$

Due to the presence of the adjoint fermions, there is a shift between the level of the super Kac–Moody algebra $[\hat{g}]_k$ and the level of its bosonic sub-algebra $\hat{g}_{\bar{k}}$ given by the corresponding quadratic invariant as

$$\bar{k} = k - c_2(g),$$

which is given explicitly in terms of structure constants by $c_2(g)\delta_{ab} = f_a^{cd} f_{bcd}$.

The spectrum of the super-coset model and the corresponding torus conformal blocks has been discussed in [67,68], following the earlier work [69,70]. Since the model is non-compact, the spectrum not surprisingly contains both discrete and continuous states. In the geometric picture, the discrete states are those localised at the tip of the cigar while the continuous ones are those states whose wave-functions spread into the infinitely long half-cylinder and are only present above a “mass gap” $\frac{1}{4m}$ on the conformal weight [71]. The fact that the torus conformal blocks of the super-coset theory coincide with the characters of the corresponding highest weight representations of the $\mathcal{N} = 2$ superconformal algebra constitutes non-trivial evidence for its equivalence to the $\mathcal{N} = 2$ super Liouville theory. Moreover, the continuous states correspond to massive (or long) $\mathcal{N} = 2$ highest weight representations while the discrete states correspond to massless (or short) ones. As such, it is easy to see from the Hilbert space (Hamiltonian) definition (2.7) of the elliptic genus that it only receives contribution from the discrete part of the spectrum. Accepting the above argument, the building block of the elliptic genus is the Ramond character graded by $(-1)^F$

$$\text{Ch}_{\text{massless}}^{(\bar{R})}(\tau, z; s) = \frac{i\theta_1(\tau, z)}{\eta^3(\tau)} \sum_{k \in \mathbb{Z}} y^{2k} q^{mk^2} \frac{(yq^{mk})^{\frac{s-1}{m}}}{1 - yq^{mk}}$$

where $\eta(\tau) = q^{1/24} \prod_{n \geq 1} (1 - q^n)$ is the Dedekind eta function and $s/2$ is the $U(1)$ charge of the highest weight. The above formula can also be identified as $\mathcal{N} = 2$ characters extended by spectral flow. Putting them together, from the spectrum of the super-coset model it is straightforward to work out the elliptic genus of the theory

$$\begin{aligned} Z_{L_m}(\tau, z) &= \frac{1}{2} \sum_{s=1}^m \text{Ch}_{\text{massless}}^{(\bar{R})}(\tau, z; m + 2 - s) \\ &+ \text{Ch}_{\text{massless}}^{(\bar{R})}(\tau, z; s) = \frac{1}{2} \mu_{m,0}\left(\tau, \frac{z}{m}\right) \frac{i\theta_1(\tau, z)}{\eta(\tau)^3}, \end{aligned} \tag{2.15}$$

where we have used the (specialised) Appell–Lerch sum

$$\mu_{m,0}(\tau, z) = - \sum_{k \in \mathbb{Z}} q^{mk^2} y^{2km} \frac{1 + yq^k}{1 - yq^k}. \tag{2.16}$$

The above partition function has also been calculated in [11] using an alternative free-field representation of the theory. See also [72,73].

From this we can derive the elliptic genus of the super coset theory coupled to the rational theory

$$\left(\mathcal{N} = 2 \text{ minimal} \otimes \mathcal{N} = 2 \frac{SL(2, \mathbb{R})}{U(1)} \text{ coset} \right) / (\mathbb{Z}/m\mathbb{Z}),$$

describing the corresponding du Val surface singularities of type Φ , by using the orbifoldisation formula[63]

$$\begin{aligned}
 Z^{\Phi,S}(\tau, z) &= \frac{1}{m} \sum_{a,b \in \mathbb{Z}/m\mathbb{Z}} q^{a^2} y^{2a} Z_{\text{minimal}}^{\Phi}(\tau, z + a\tau + b) Z_{L_m}(\tau, z + a\tau + b) \quad (2.17) \\
 &= \frac{1}{2m} \frac{i\theta_1(\tau, z)}{\eta^3(\tau)} \sum_{a,b \in \mathbb{Z}/m\mathbb{Z}} (-1)^{a+b} q^{a^2/2} y^a Z_{\text{minimal}}^{\Phi}(\tau, z + a\tau + b) \mu_{m,0}(\tau, \frac{z + a\tau + b}{m}). \quad (2.18)
 \end{aligned}$$

Note that the above elliptic genus is not modular, as opposed to the familiar situation with elliptic genera of a supersymmetric conformal field theory. In fact, it is mock modular in the following sense [74]. Let the ‘‘completion’’ of $\mu_{m,0}(\tau, z)$ be

$$\begin{aligned}
 \hat{\mu}_{m,0}(\tau, z) &= \mu_{m,0}(\tau, z) - e(-\frac{1}{8}) \frac{1}{\sqrt{2m}} \sum_{r \in \mathbb{Z}/2m\mathbb{Z}} \theta_{m,r}(\tau, z) \\
 &\quad \times \int_{-\bar{\tau}}^{i\infty} (\tau' + \tau)^{-1/2} \overline{S_{m,r}(-\bar{\tau}')} d\tau', \quad (2.19)
 \end{aligned}$$

then $\hat{\mu}_{m,0}$ transforms like a Jacobi form of weight 1 and index m under the Jacobi group $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ but is not holomorphic. (See Appendix A for the definition of Jacobi forms.) In the above formula, $S_m = (S_{m,r})$ denotes the vector-valued cusp form for $SL_2(\mathbb{Z})$ whose components are given by the unary theta function (cf. (A.3))

$$S_{m,r}(\tau) = \sum_{k=r \pmod{2m}} k q^{k^2/4m} = \frac{1}{2\pi i} \frac{\partial}{\partial z} \theta_{m,r}(\tau, z)|_{z=0}. \quad (2.20)$$

In Appendix B.2, we will also conjecture the answer for the elliptic genera of these ADE-singularities twined by automorphisms of the corresponding Dynkin diagram, which can be thought of as permuting the smooth rational curves in the minimal resolution.

This absence of the usual modularity can be attributed to the fact that the target space of the theory is non-compact and hence the spectrum contains a continuous part [11]. This is however seemingly in contradiction with the expectation that a path integral formulation of the elliptic genus should render a function transforming nicely under $SL_2(\mathbb{Z})$, corresponding to the $SL_2(\mathbb{Z})$ mapping class group of the world-sheet torus underlying the path integral formulation. This issue has been recently addressed in [11], and further refined in [75, 76], for the cigar theory. These authors found that a path integral computation indeed renders an answer that is modular but non-holomorphic, and the breakdown of holomorphicity is attributed to the imperfect cancellation between contributions of the bosonic and fermionic states to the elliptic genus (2.7) in the continuous part of the spectrum. Analogously, we expect the path integral formulation of the elliptic genus of the ADE singularities will render as the answer the real Jacobi form

$$\begin{aligned}
 \hat{Z}^{\Phi,S}(\tau, z) &= \frac{1}{2m} \frac{i\theta_1(\tau, z)}{\eta^3(\tau)} \\
 &\quad \times \sum_{a,b \in \mathbb{Z}/m\mathbb{Z}} (-1)^{a+b} q^{a^2/2} y^a Z_{\text{minimal}}^{\Phi}(\tau, z + a\tau + b) \hat{\mu}_{m,0}(\tau, \frac{z + a\tau + b}{m}). \quad (2.21)
 \end{aligned}$$

Finally, we note that there is a different definition of elliptic genus that is purely geometric. For a compact complex manifold M with $\dim_{\mathbb{C}} M = d_0$, the elliptic genus is

defined as the character-valued Euler characteristic of the formal vector bundle [61, 77–80]

$$\mathbf{E}_{q,y} = y^{d/2} \bigwedge_{-y^{-1}} T_M^* \otimes_{n \geq 1} \bigwedge_{-y^{-1}q^n} T_M^* \otimes_{n \geq 1} \bigwedge_{-yq^n} T_M \otimes_{n \geq 0} S_{q^n}(T_M \oplus T_M^*),$$

where T_M and T_M^* are the holomorphic tangent bundle and its dual, and we adopt the notation

$$\bigwedge_q V = 1 + qV + q^2 \bigwedge^2 V + \dots, \quad S_q V = 1 + qV + q^2 S^2 V + \dots,$$

with $S^k V$ denoting the k -th symmetric power of V . In other words, we have

$$\mathbf{EG}(\tau, z; M) = \int_M ch(\mathbf{E}_{q,y}) Td(M) \tag{2.22}$$

where $Td(M)$ is the Todd class of T_M . For M a (compact) Calabi–Yau manifold, the above geometric definition and the conformal field theory definition, when the CFT is taken to be the 2d non-linear sigma model of M , are believed to give the same function [78, 81]. The fact that the CFT elliptic genus is rigid corresponds to the geometric fact that $\mathbf{EG}(\tau, z; M)$ is a topological invariant. Note that the above definition is manifestly holomorphic. We expect that a suitable generalisation of the above definition which handles non-compact geometries will lead to the geometric elliptic genus $\mathbf{EG}(\tau, z; \Phi) = Z^{\Phi,S}(\tau, z)$ of the du Val singularity. In this paper we will simply refer to $Z^{\Phi,S}(\tau, z)$ as the elliptic genus of the ADE singularity of type Φ .

3. Umbral Moonshine and Niemeier Lattices

In this section we will briefly review the umbral moonshine conjecture and its construction from the 23 Niemeier lattices [41]. The readers are referred to [41] for more details. Let us start by recalling what the Niemeier lattices are. Consider positive-definite lattices of rank 24, we would like to know which of them are even and unimodular. In string theory, one is often interested in even, unimodular lattices due to the modular invariance of their theta functions. In the classification of positive-definite even unimodular lattices, a special role will be played by the root system of the lattice L , given by $\Delta(L) = \{v \in L \mid \langle v, v \rangle = 2\}$.

The even unimodular positive-definite lattices of rank 24 were classified by Niemeier [82]. There are 24 of them (up to isomorphisms). The Leech lattice is the unique even, unimodular, positive-definite lattice of rank 24 with no roots [83], discovered shortly before the classification of Niemeier [84, 85]. Apart from the Leech lattice, there are 23 other inequivalent even unimodular lattices of rank 24. They are uniquely determined by their root systems $\Delta(L)$, that are all unions of the simply-laced root systems. Moreover, the 23 root systems of the 23 Niemeier lattices are precisely the 23 unions of ADE root systems satisfying the following two simple conditions: first, all of the irreducible components have the same Coxeter numbers; second, the total rank is 24. They are listed in Table 2, where n denotes $\mathbb{Z}/n\mathbb{Z}$. Here and in the rest of the paper we will adopt the shorthand notation $A_{m-1}^{d_A} D_{m/2+1}^{d_D} (E^{(m)})^{d_E}$ for the direct sum of d_A copies of A_{m-1} , d_D copies of $D_{m/2+1}$ and d_E copies of

$$E^{(m)} = \begin{cases} E_6, E_7, E_8 & \text{for } m = 12, 18, 30 \\ \emptyset & \text{otherwise} \end{cases} \tag{3.1}$$

Table 2. Umbral groups

X	A_1^{24}	A_2^{12}	A_3^8	A_4^6	$A_5^4 D_4$	A_6^4	$A_7^2 D_5^2$
G^X	M_{24}	$2.M_{12}$	$2.AGL_3(2)$	$GL_2(5)/2$	$GL_2(3)$	$SL_2(3)$	Dih_4
\bar{G}^X	M_{24}	M_{12}	$AGL_3(2)$	$PGL_2(5)$	$PGL_2(3)$	$PSL_2(3)$	2^2
X	A_8^3	$A_9^2 D_6$	$A_{11} D_7 E_6$	A_{12}^2	$A_{15} D_9$	$A_{17} E_7$	A_{24}
G^X	Dih_6	4	2	4	2	2	2
\bar{G}^X	Sym_3	2	1	2	1	1	1
X	D_4^6	D_6^4	D_8^3	$D_{10} E_7^2$	D_{12}^2	$D_{16} E_8$	D_{24}
G^X	$3.Sym_6$	Sym_4	Sym_3	2	2	1	1
\bar{G}^X	Sym_6	Sym_4	Sym_3	2	2	1	1
X	E_6^4	E_8^3					
G^X	$GL_2(3)$	Sym_3					
\bar{G}^X	$PGL_2(3)$	Sym_3					

Let X be one of the 23 root systems listed above, and denote by L^X the unique (up to isomorphism) Niemeier lattice with root system X . For each of these 23 L^X we will have an instance of umbral moonshine as we will explain now. First, we need to define the finite group relevant for this new type of moonshine. Let us consider the automorphism group $\text{Aut}(L^X)$ of the lattice L^X . Clearly, any element of the Weyl group $\text{Weyl}(X)$ generated by reflections with respect to any root vector leaves the lattice invariant. In fact, $\text{Weyl}(X)$ is a normal subgroup of $\text{Aut}(L^X)$ and we define the ‘‘umbral group’’ G^X to be the corresponding quotient

$$G^X = \text{Aut}(L^X)/\text{Weyl}(X). \tag{3.2}$$

The list of the 23 G^X is given in Table 2.

After defining the relevant finite group G^X , we will now define the relevant (vector-valued) mock modular forms $H_g^X, [g] \subset G^X$, for the umbral moonshine. As explained in Sect. 2, the ADE classification of the modular invariant combinations of $\hat{A}_1^{(1)}$ characters is given by a symmetric matrix Ω^Φ of size $2m$, where m denotes the Coxeter number of Φ , for every simply-laced root system Φ . As we have seen, the Cappelli–Itzykson–Zuber matrix Ω^Φ also controls the spectrum and hence the elliptic genus (2.9) of the 2d minimal model of type Φ . Now consider any one of the 23 Niemeier root systems X listed above. Since they are unions $X = \cup_i \Phi_i$ of simply-laced root systems Φ_i with the same Coxeter number, we can extend the definition of the Ω -matrix to $\Omega^X = \sum_i \Omega^{\Phi_i}$. Using these Ω -matrices we can then define for each Niemeier lattice L^X the vector-valued weight $3/2$ cusp form

$$S^X = \Omega^X S_m = (S_r^X), \quad r \in \mathbb{Z}/2m\mathbb{Z}$$

with the r -th component given by

$$S_r^X = \sum_{r' \in \mathbb{Z}/2m\mathbb{Z}} \Omega_{r,r'}^X S_{m,r'}$$

in terms of the unary theta function (2.20). From $(\Omega^X)_{r,r'} = (\Omega^X)_{-r,-r'}$ and $S_{m,r} = -S_{m,-r}$ it is easy to see that $S_r^X = -S_{-r}^X$.

Given the cusp form S^X , we can now specify the mock modular form H^X by the following two conditions. First we specify its mock modular property: we require H^X to be a weight $1/2$ vector-valued mock modular form whose shadow is given by S^X . More precisely, let

$$\hat{H}_r^X(\tau) = H_r^X(\tau) + e(-\frac{1}{8}) \frac{1}{\sqrt{2m}} \int_{-\bar{\tau}}^{i\infty} (\tau' + \tau)^{-\frac{1}{2}} \overline{S_r^X(-\bar{\tau}')} d\tau',$$

then

$$\sum_{r \in \mathbb{Z}/2m\mathbb{Z}} \hat{H}_r^X(\tau) \theta_{m,r}(\tau, z)$$

transforms as a Jacobi form of weight 1 and index m under the Jacobi group $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$. Recall that the shadow $s(\tau)$ of a mock modular form $f(\tau)$ of weight w is the function, a modular form of weight $2 - w$ itself for the same $\Gamma < SL_2(\mathbb{R})$, whose integral gives the non-holomorphic completion

$$\hat{f}(\tau) = f(\tau) + e(\frac{w-1}{4}) \int_{-\bar{\tau}}^{i\infty} (\tau + \tau')^{-w} \overline{s(-\bar{\tau}')} d\tau' \tag{3.3}$$

of f which transforms as a weight w modular form. This definition has a straightforward generalisation to the vector-valued case which we have employed above.

After specifying the mock modularity, we impose the following analyticity condition : we require its growth near the cusp to be

$$q^{1/4m} H_r^X(\tau) = O(1) \text{ as } \tau \rightarrow i\infty \tag{3.4}$$

for every element $r \in \mathbb{Z}/2m\mathbb{Z}$. The above two conditions turn out to be sufficient to determine H^X uniquely (up to a rescaling), as shown in [12,41,86]. We also fix the scaling by requiring $q^{1/4m} H_1^X(\tau) = -2 + O(q)$.

For instance, when considering the Niemeier lattice with the simplest root system, $X = A_1^{24}$, the unique mock modular form determined by the above condition reads

$$H_1^{X=A_1^{24}}(\tau) = -H_{-1}^{X=A_1^{24}}(\tau) = \frac{-2E_2(\tau) + 48F_2^{(2)}(\tau)}{\eta(\tau)^3} \tag{3.5}$$

$$= 2q^{-1/8}(-1 + 45q + 231q^2 + 770q^3 + O(q^4)). \tag{3.6}$$

where $E_2(\tau)$ stands for the weight 2 Eisenstein series and

$$F_2^{(2)}(\tau) = \sum_{\substack{r>s>0 \\ r-s=1 \pmod 2}} (-1)^r s q^{rs/2} = q + q^2 - q^3 + q^4 + \dots$$

As mentioned in Sect. 1, the first observation that led to the recent development in the moonshine phenomenon for mock modular forms is the fact that the above numbers 45, 231, 770 coincide with the dimensions of certain irreducible representations of the corresponding umbral group $G^X \cong M_{24}$ for $X = A_1^{24}$.

Note that without the non-holomorphic completion, the function

$$\sum_{r \in \mathbb{Z}/2m\mathbb{Z}} H_r^X \theta_{m,r}$$

does not transform nicely under the modular group; it is a mock Jacobi form according to the definition given in [12]. In [41], following [12], this mock Jacobi form is interpreted as the finite part of a meromorphic (as a function of z) Jacobi form with simple poles at m -torsion points. For later convenience, we will define another mock Jacobi form

$$\phi^X(\tau, z) = \frac{i\theta_1(\tau, mz)\theta_1(\tau, (m-1)z)}{\eta^3(\tau)\theta_1(\tau, z)} \sum_{r \in \mathbb{Z}/2m\mathbb{Z}} H_r^X(\tau) \theta_{m,r}(\tau, z) \tag{3.7}$$

which contains exactly the same information as the vector-valued mock modular form H^X .

In order to relate such functions to representations of the finite group G^X that we have constructed, we need as many vector-valued functions similar to H^X as the number of conjugacy classes of G^X to encode the characters of the underlying representation. Hence, for every Niemeier lattice X , and for every conjugacy class $[g] \subset G^X$ we would like to define a vector-valued mock modular form H_g^X . As before, first we need to specify their mock modular properties. The relevant congruence subgroup $\Gamma_0(n_g) \subseteq SL_2(\mathbb{Z})$ (see (A.6)), is determined by n_g , the order group element g . This is similar to the situation both in monstrous moonshine [1] and, not unrelatedly, 2-dimensional CFT.

We need two more pieces of data to completely specify the mock modularity of H_g^X . The first one is the shadow. By studying the action of $\langle g \rangle$, the cyclic group generated by g , we can analogously define a $2m \times 2m$ matrix Ω_g^X and the corresponding cusp form $S_g^X = \Omega_g^X S_m = S_{g,r}^X$. See §5.1 of [41] for the list of Ω_g^X . The second piece of data we need is the multiplier system on $\Gamma_0(n_g)$, namely a projective representation $\nu_g : \Gamma_0(n_g) \rightarrow GL_{2m}(\mathbb{C})$ of the congruence subgroup $\Gamma_0(n_g)$. In the case where the specified shadow S_g^X does not vanish, the definition of the shadow stipulates the multiplier of the mock modular form to be the inverse of the shadow. As a result, this second piece of data is implied by the first. If however $S_g^X = 0$, namely when the mock modular form H_g^X is in fact modular, one needs to specify the multiplier system independently. It turns out that ν_g is identical to the inverse of the multiplier of S^X on a group $\Gamma_0(n_g h_g) < \Gamma_0(n_g)$ for certain integral $h_g > 1$. See [41] for more details. In particular, let

$$\hat{H}_{g,r}^X(\tau) = H_{g,r}^X(\tau) + e(-\frac{1}{8}) \frac{1}{\sqrt{2m}} \int_{-\bar{\tau}}^{i\infty} (\tau' + \tau)^{-1/2} \overline{S_{g,r}^X(-\bar{\tau}')} d\tau',$$

then

$$\sum_{r \in \mathbb{Z}/2m\mathbb{Z}} \hat{H}_{g,r}^X(\tau) \theta_{m,r}(\tau, z)$$

transforms like a Jacobi form of weight 1 and index m under the group $\Gamma_0(n_g h_g) \rtimes \mathbb{Z}^2$. By the same token, the function $\sum_{r \in \mathbb{Z}/2m\mathbb{Z}} H_{g,r}^X \theta_{m,r}$ is a mock Jacobi form of weight 1 and index m under $\Gamma_0(n_g h_g) \rtimes \mathbb{Z}^2$.

As before, after specifying the mock modular property we also need to fix the analyticity property of H_g^X . For $\Gamma_0(n_g)$ with $n_g > 1$, there is more than one cusp (representative), namely more than one $\Gamma_0(n_g)$ -orbit among $\mathbb{Q} \cup i\infty$. For the cusp (representative) located at $\tau \rightarrow i\infty$ we require the same growth condition

$$q^{1/4m} H_{g,r}^X(\tau) = O(1) \text{ as } \tau \rightarrow i\infty. \tag{3.8}$$

for every $r \in \mathbb{Z}/2m\mathbb{Z}$. Moreover we require the function to be bounded

$$H_{g,r}^X(\tau) = O(1) \text{ as } \tau \rightarrow \alpha \in \mathbb{Q}, \quad \alpha \notin \Gamma_0(n_g)\infty. \tag{3.9}$$

at all other cusps.

After specifying the shadow S_g^X , the multiplier system ν_g and the behaviour at the cusps, a vector-valued mock modular form H_g^X of weight $1/2$ for $\Gamma_0(n_g)$ was then given in [41] for every $[g] \subset G^X$ and for all 23 Niemeier lattices L^X . See [41] for explicit Fourier coefficients of the q -expansions of $H_{g,r}^X$. Finally, it was conjectured in [41] that H_g^X is the unique (up to rescaling) vector-valued mock modular form with the above mock modularity and poles. For later convenience, we will also define

$$\phi_g^X(\tau, z) = \frac{i\theta_1(\tau, mz)\theta_1(\tau, (m-1)z)}{\eta^3(\tau)\theta_1(\tau, z)} \sum_{r \in \mathbb{Z}/2m\mathbb{Z}} H_{g,r}^X(\tau) \theta_{m,r}(\tau, z) \tag{3.10}$$

Note that we recover H^X and ϕ^X by putting $[g]$ to be the identity class in the above discussions on H_g^X and ϕ_g^X .

After constructing the finite group G^X and the set of vector-valued mock modular forms $H_g^X = (H_{g,r}^X)$ for each Niemeier lattice L^X , we can now formulate the umbral moonshine conjecture [41]. This conjecture states that for every Niemeier lattice X , for every $1 \leq r \leq m-1$ we have an infinite-dimensional \mathbb{Z} -graded module $K_r^X = \bigoplus_D K_{r,D}^X$ for G^X such that $H_{g,r}^X$ is essentially given by the graded characters $\sum_{D=-r^2 \pmod{4m}, D>0} q^{D/4m} \text{Tr}_{K_{r,D}^X} g$, up to the possible inclusion of a polar term $-2q^{-1/4m}$ and a constant term (as well as an additional factor of 3 in the case $X = A_8^3$). See §6.1 of [41] for the precise statement of the conjecture. In summary, umbral moonshine conjectures for each of the 23 Niemeier lattices the existence of a special module K^X of the finite group G^X , which underlies the special mock Jacobi forms ϕ_g^X . This conjecture has so far been proven for the case $X = A_1^{24}$ [28], and explicitly verified till the first hundred terms in the q -expansion for the other 22 cases. In the following section we will demonstrate the relation between the mock Jacobi forms ϕ_g^X and the elliptic genus of $K3$ surfaces. Subsequently we will explore the relation between the Niemeier lattices L^X , the finite group G^X , the (conjectural) G^X -module K^X , and the (stringy) symmetry of $K3$ surfaces.

4. Umbral Moonshine and the (Twined) $K3$ Elliptic Genus

In Sect. 2 we have computed the elliptic genus of Du Val singularities a $K3$ surface can develop. In Sect. 3 we have briefly reviewed the umbral moonshine conjecture relating a finite group G^X and a set of mock Jacobi forms ϕ_g^X for every Niemeier lattice L^X via an underlying G^X -module K^X . In this section we will see how these two separate topics meet in the framework of (twined) elliptic genera for $K3$ surfaces.

Let's start by briefly reviewing the relation between the elliptic genus of $K3$ surfaces and the Mathieu group M_{24} , which is also the umbral group G^X for the Niemeier lattice with root system $X = A_1^{24}$. The 2d non-linear sigma model of a $K3$ surface is a CFT with central charge $c = 6$ and with a (small) $\mathcal{N} = 4$ superconformal symmetry. As explained in Sect. 2, the elliptic genus (2.7) is the same for different $K3$ sigma models

and coincides with the geometric elliptic genus of $K3$. It is computed to be (cf. (A.1)) [16]

$$\mathbf{EG}(\tau, z; K3) = 8 \sum_{i=2,3,4} \left(\frac{\theta_i(\tau, z)}{\theta_i(\tau, 0)} \right)^2. \tag{4.1}$$

The $\mathcal{N} = 4$ superconformal symmetry of the theory implies that the spectrum is composed of irreducible representations (“multiplets”) of the $\mathcal{N} = 4$ superconformal algebra, and the elliptic genus permits a decomposition into their characters.

Recall that the $\mathcal{N} = 4$ superconformal algebra contains subalgebras isomorphic to the affine Lie algebra $\hat{\mathfrak{sl}}_2$ and the Virasoro algebra, and in a unitary representation the former of these acts with level $m - 1$ and the latter with central charge $c = 6(m - 1)$ for some integer $m > 1$. The unitary irreducible highest weight representations $v_{m;h,j}$ are labelled by the two quantum numbers h and j which are the eigenvalues of L_0 and $\frac{1}{2}J_0^3$ of the highest weight state, respectively, when acting on the highest weight state [15,87]. (We adopt a normalisation of the $SU(2)$ current J^3 such that the zero mode J_0^3 has integer eigenvalues. The shift by -1 in the central charge and the level of the current algebra is due to the -1 difference between the level and the index of the theta functions underlying the characters, as we will see below.) The algebra has two types of highest weight representations: the *short* (or *BPS*, *supersymmetric*) ones and the *long* (or *non-BPS*, *non-supersymmetric*) ones. In the Ramond sector, the former has $h = \frac{c}{24} = \frac{m-1}{4}$ and $j \in \{0, \frac{1}{2}, \dots, \frac{m-1}{2}\}$, while the latter has $h > \frac{m-1}{4}$ and $j \in \{\frac{1}{2}, 1, \dots, \frac{m-1}{2}\}$. Their (Ramond) graded characters, defined as

$$\text{ch}_{m;h,j}(\tau, z) = \text{tr}_{v_{m;h,j}} \left((-1)^{J_0^3} y^{J_0^3} q^{L_0 - c/24} \right), \tag{4.2}$$

are given by

$$\text{ch}_{m;h,j}(\tau, z) = \frac{i \theta_1(\tau, z)^2}{\eta^3(\tau)\theta_1(\tau, 2z)} \mu_{m,j}(\tau, z) \tag{4.3}$$

and

$$\text{ch}_{m;h,j}(\tau, z) = \frac{i \theta_1(\tau, z)^2}{\eta^3(\tau)\theta_1(\tau, 2z)} q^{h - \frac{c}{24} - \frac{j^2}{m}} (\theta_{m,2j}(\tau, z) - \theta_{m,-2j}(\tau, z)) \tag{4.4}$$

in the short and long cases, respectively [15]. In the above formulas, $\mu_{m,j}$ is given by $\mu_{m,0}$ (2.16) and the identity

$$\mu_{m,\frac{r}{2}} = (-1)^r (r + 1) \mu_{m,0} + (-1)^{r-n+1} \sum_{n=1}^r n q^{-\frac{(r-n+1)^2}{4m}} (\theta_{m,r-n+1} - \theta_{m,-(r-n+1)}).$$

When applying the above formula to the $K3$ sigma models which have $c = 6 (m = 2)$, we obtain the following rewriting of the function in (4.1):

$$\begin{aligned} \mathbf{EG}(\tau, z; K3) &= 20 \text{ch}_{2;\frac{1}{4},0} - 2 \text{ch}_{2;\frac{1}{4},\frac{1}{2}} + (90 \text{ch}_{2;\frac{5}{4},\frac{1}{2}} \\ &\quad + 462 \text{ch}_{2;\frac{9}{4},\frac{1}{2}} + 1540 \text{ch}_{2;\frac{13}{4},\frac{1}{2}} + \dots) \end{aligned} \tag{4.5}$$

$$= \frac{i \theta_1(\tau, z)^2}{\eta^3(\tau)\theta_1(\tau, 2z)} \left\{ 24 \mu_{2,0}(\tau, z) + (\theta_{2,-1}(\tau, z) - \theta_{2,1}(\tau, z)) \right. \\ \left. \times (-2q^{-1/8} + 90q^{7/8} + 462q^{15/8} + 1540q^{23/8} + \dots) \right\} \quad (4.6)$$

where \dots corresponds to terms in $\mathbf{EG}(\tau, z; K3)$ of the form $\frac{i \theta_1(\tau, z)^2}{\eta^3(\tau)\theta_1(\tau, 2z)} q^\alpha y^\beta$ with $\alpha - \beta^2/8 > 3$. Note that the q -series in the last line is nothing but the umbral mock modular form (3.5) corresponding to the Niemeier lattice with root system $X = A_1^{24}$ that we introduced in the previous section. As mentioned in Sect. 1, it was precisely in this context of decomposing the $K3$ elliptic genus into $\mathcal{N} = 4$ characters that the first case of moonshine for mock modular forms was observed [17].

From the above discussion, we see that the two contributions to $\mathbf{EG}(\tau, z; K3)$, given by

$$24 \mu_{2,0}(\tau, z)$$

and

$$- \sum_{r \in \mathbb{Z}/4\mathbb{Z}} H_r^{X=A_1^{24}}(\tau)\theta_{2,r}(\tau, z),$$

in the $\{\}$ bracket, can roughly be thought of as the contributions from the BPS and non-BPS $\mathcal{N} = 4$ multiplets respectively.²

However, there is a possible alternative interpretation, thanks to the identity between the short $\mathcal{N} = 4$ characters and the elliptic genus of an $\Phi = A_1$ singularity:

$$Z^{A_1, S}(\tau, z) = \text{ch}_{2; \frac{1}{4}, 0}(\tau, z), \quad (4.7)$$

which follows from the identity

$$\frac{1}{2} \sum_{a,b=0}^1 q^{a^2} y^{2a} \theta_1(\tau, z + a\tau + b) \mu_{2,0}(\tau, \frac{z + a\tau + b}{2}) = \frac{\theta_1(\tau, z)^2}{i\theta_1(\tau, 2z)} \mu_{2,0}(\tau, z).$$

In other words, we can re-express the elliptic genus of $K3$ as

$$\mathbf{EG}(\tau, z; K3) = 24Z^{A_1, S}(\tau, z) - \frac{i \theta_1(\tau, z)^2}{\eta^3(\tau)\theta_1(\tau, 2z)} \sum_{r \in \mathbb{Z}/4\mathbb{Z}} H_r^{A_1^{24}} \theta_{2,r}(\tau, z). \quad (4.8)$$

Using the identity

$$\theta_{2,1}(\tau, z) - \theta_{2,-1}(\tau, z) = -i\theta_1(\tau, 2z),$$

and

$$-q^{1/2}y \theta_1(\tau, z + \tau) = \theta_1(\tau, z)$$

² Strictly speaking, the polar term “ $-2q^{-1/8}$ ” of $H_1^{X=A_1^{24}}$ also corresponds to the contributions from BPS multiplets, while all the infinitely many other terms are contributions from non-BPS multiplets.

we can rewrite the above expression as

$$\mathbf{EG}(\tau, z; K3) = Z^{X,S}(\tau, z) + \frac{1}{2m} \sum_{a,b \in \mathbb{Z}/m\mathbb{Z}} q^{a^2} y^{2a} \phi^X\left(\tau, \frac{z + a\tau + b}{m}\right) \quad (4.9)$$

for $X = A_1^{24}$, where ϕ^X is the function defined in (3.7) that encodes the umbral moonshine mock modular form H^X . In the above, for a root system X that is the union of simply-laced root systems with the same Coxeter number m (cf. (3.1))

$$X = A_{m-1}^{d_A} D_{m/2+1}^{d_D} (E^{(m)})^{d_E},$$

we write

$$Z^{X,S} = d_A Z^{A_{m-1}} + d_D Z^{D_{m/2+1}} + d_E Z^{E^{(m)}}, \quad (4.10)$$

corresponding to a collection of non-interacting ADE theories with the total Hilbert space given by the direct sum of the Hilbert spaces of the component theories.

In other words, instead of interpreting the two contributions to the $K3$ elliptic genus as that of the BPS and that of the non-BPS $\mathcal{N} = 4$ multiplets, one might interpret them as the contribution from the 24 copies of A_1 -type surface singularities and the ‘‘umbral moonshine’’ contribution given by the umbral moonshine mock modular forms H^X with $X = A_1^{24}$.

The first surprise we encounter is that such an interpretation actually holds for all 23 cases of umbral moonshine. In particular, the equality (4.9) is valid not only for the case $X = A_1^{24}$ but also for all other 22 cases corresponding to all the 23 Niemeier lattices L^X . The detailed proof will be supplied in Appendix B. Put differently, corresponding to the 23 Niemeier lattices L^X we have 23 different ways of separating $\mathbf{EG}(K3)$ into two parts. On the one hand, by replacing the Niemeier root system X with the corresponding configuration of singularities, we obtain a contribution to the $K3$ elliptic genus by the singularities. On the other hand, the umbral moonshine construction attaches a mock Jacobi form ϕ^X to every L^X , which gives the rest of $\mathbf{EG}(K3)$ after a summation procedure reminiscent of the ‘‘orbifoldisation’’ formula for the elliptic genus of orbifold SCFTs [63].

Recall that in umbral moonshine for a given Niemeier lattice L^X , the mock Jacobi form ϕ^X is conjectured to encode the graded dimension of an infinite-dimensional module K^X of the umbral finite group G^X . The existence of such a module is supported by the construction of the other mock Jacobi forms ϕ_g^X for the other (non-identity) conjugacy classes $[g]$ of the umbral group G^X (cf. (3.2)), that are conjectured to encode the graded characters of K^X . Given the above relation between the $K3$ elliptic genus and the mock modular form $H^X = H_g^X$ for $[g]$ being the identity class, a natural question is whether a $K3$ interpretation also exists for other mock modular forms H_g^X corresponding to other conjugacy classes of the group G^X .

To discuss the relation between the graded characters in umbral moonshine and the elliptic genus of $K3$, let us first discuss how the equality (4.9) might be ‘‘twined’’ in the presence of a non-trivial group element. On the left-hand side (the $K3$ side) of the equation is the elliptic genus, defined in terms of the Ramond-Ramond Hilbert space $\mathcal{H}_{\mathcal{T},RR}$ of the underlying supersymmetric sigma model \mathcal{T} as in (2.7). In the event that every Hilbert subspace $\mathcal{H}_{h,j;\mathcal{T},RR} \subset \mathcal{H}_{\mathcal{T},RR}$, consisting of states with the same L_0, J_0 eigenvalues h and j , is a representation of the cyclic group generated by g , or that g acts

on the theory and commutes with the superconformal algebra in other words, we can define the so-called “twisted elliptic genus” as the graded character

$$\mathbf{EG}_g(\tau, z; K3) = \text{tr}_{\mathcal{H}_{T,RR}} \left(g (-1)^{F_R+F_L} y^{J_0} q^{H_L} \bar{q}^{H_R} \right). \tag{4.11}$$

Let us now turn to the right-hand side (the umbral moonshine side) of the equation. Assuming the (linear) relevance of the umbral moonshine module K^X for the calculation of $\mathbf{EG}(K3)$, the unique way to twine the second term

$$\sum_{a,b \in \mathbb{Z}/m\mathbb{Z}} q^{a^2} y^{2a} \phi^X \left(\tau, \frac{z + a\tau + b}{m} \right)$$

is to replace it with

$$\sum_{a,b \in \mathbb{Z}/m\mathbb{Z}} q^{a^2} y^{2a} \phi_g^X \left(\tau, \frac{z + a\tau + b}{m} \right)$$

where ϕ_g^X is defined in (3.10). This is equivalent to replacing the graded dimension of the module K^X with its graded character. What remains to be twined is the first term in (4.9), the contribution from the configuration of singularities stipulated by the root system X of the Niemeier lattice L^X . For an element g of the umbral group G^X , consider its action on the rank 24 root system X . In the case that g simply permutes the different irreducible components of its root system, it is easy to write down the twining of the singularity part $Z^{X,S}$ of $\mathbf{EG}(K3)$: $Z_g^{X,S}$ is simply given by the contribution from the irreducible components of X that are left invariant by the action of g . For instance, for $X = A_1^{24}$, consider the order 2 element g of the umbral group $G^X = M_{24}$ whose action on L^X is to exchange 8 pairs of A_1 root systems and leave the other 8 copies of A_1 invariant when restricted to the root vectors of L^X . In this case the twined singularity part of the elliptic genus is simply $Z_g^{X,S} = 8Z^{A_1,S}$. It can also happen that g also involves a non-trivial automorphism of the individual irreducible components of the root system, such as the $\mathbb{Z}/2\mathbb{Z}$ symmetry of the $A_n, n > 1$ Dynkin diagram and the $\mathbb{Z}/3\mathbb{Z}$ symmetry of the D_4 Dynkin diagram. In this case the computation for $Z_g^{X,S}$ is more involved and will be discussed in Appendix B.2. Combining the two parts, we can now define the twining for the right-hand side (the umbral moonshine side) (4.9) which we denote by

$$Z_g^X(\tau, z) = Z_g^{X,S}(\tau, z) + \frac{1}{2m} \sum_{a,b \in \mathbb{Z}/m\mathbb{Z}} q^{a^2} y^{2a} \phi_g^X \left(\tau, \frac{z + a\tau + b}{m} \right). \tag{4.12}$$

The second surprise is that these twining functions given by umbral moonshine precisely reproduce the elliptic genus twined by a geometric symmetry of the underlying $K3$ surface whenever the latter interpretation is available, a fact we will now explain. The symmetries of a $K3$ surface M that are of interest for the purpose of studying the elliptic genus are the so-called finite symplectic automorphisms of M , as we need to require the symmetry to preserve the hyper-Kähler structure in order for it to commute with the $\mathcal{N} = 4$ superconformal algebra. As we will discuss in Sect. 5, a necessary condition for a subgroup $G \subseteq G^X$ to admit such an interpretation as the group of finite symplectic automorphisms of a certain $K3$ surface is that it has at least 5 orbits and 1 fixed point on the 24-dimensional representation of G^X . See [88] for a proof by S. Kondō utilising the previous results by Nikulin [89,90], and [48] for a more refined analysis.

For convenience, above and in the rest of the paper we will simply refer to the 24-dimensional representation that encodes the action of G^X on X as “the 24-dimensional representation” of G^X . As above and in Sect. 5, this representation is also the relevant one when describing the action of various subgroups of G^X on the $K3$ cohomology lattice, via the embedding of its sub-lattice into L^X . The action of an element $g \in G^X$ on the 24-dimensional representation is encoded in the 24 eigenvalues, or equivalently its “24-dimensional cycle shape”

$$\Pi_g^X = \prod_i^k \ell_i^{m_i}, \text{ where } m_i \in \mathbb{Z}_{>0}, 0 < \ell_1 < \dots < \ell_k \text{ and } \sum_i^k m_i \ell_i = 24, \quad (4.13)$$

where the relation between the cycle shape and the eigenvalues $\lambda_1, \dots, \lambda_{24}$ is given by

$$\prod_i^k (x^{\ell_i} - 1)^{m_i} = (x - \lambda_1) \dots (x - \lambda_{24}). \quad (4.14)$$

We will say that an element $g \in G^X$ satisfies the “geometric condition” if it satisfies the criterium of Mukai, namely when it has at least 5 orbits ($\sum_i^k m_i \geq 5$) and one fixed point ($\ell_1 = 1$) on the 24-dimensional representation.

Moreover, this implies that G must be (isomorphic to) a subgroup of one of the 11 maximal subgroups of M_{23} listed in [91] and provides an alternative proof of Mukai’s theorem [91]. Conversely, given any G^X among the 23 umbral groups and for any element $g \in G^X$ satisfying the geometric condition, there exists a $K3$ surface M whose finite group of symplectic automorphisms has a subgroup isomorphic to $\langle g \rangle$. This can be shown using the global Torelli theorem and in fact holds not just for the Abelian groups but also for all 11 maximal subgroups of M_{23} . See the Appendix by Mukai in [88].

As a result, for any of the 23 G^X for any element $g \in G^X$ satisfying the geometric condition, one can compute $\mathbf{EG}_g(K3)$ geometrically by considering the supersymmetric sigma model on a $K3$ surface with $\langle g \rangle$ symmetry. Note that the latter is well-defined because of the uniqueness of the $\langle g \rangle$ action. To be more precise, it was shown in [92] that if $G_i \cong \langle g \rangle$ acts on a $K3$ surface M_i faithfully and symplectically ($i = 1, 2$), then there exists a lattice isomorphism $\alpha : H^2(M_1, \mathbb{Z}) \rightarrow H^2(M_2, \mathbb{Z})$ preserving the intersection forms such that $\alpha \cdot G_1 \cdot \alpha^{-1} = G_2$ in $H^2(M_2, \mathbb{Z})$ (see [93] for a generalisation of this result to many non-Abelian groups). Together with the global Torelli theorem, which states that any lattice isomorphism $\varphi^* : H^2(M, \mathbb{Z}) \rightarrow H^2(M', \mathbb{Z})$ between the second cohomology groups of two $K3$ surfaces that preserves the Hodge structure and the effectiveness of the cycles is induced by a unique isomorphism $\varphi : M \rightarrow M'$, this shows the uniqueness of the symplectic action of $\langle g \rangle$ on $K3$ and thereby that of $\mathbf{EG}_g(K3)$.

On the other hand, using the prescription of umbral moonshine (4.12) one can compute Z_g^X . The first non-trivial fact is that, whenever $g_1 \in G^{X_1}$ and $g_2 \in G^{X_2}$ both satisfy the geometric condition and moreover have the same 24-dimensional cycle shape $\Pi_{g_1}^{X_1} = \Pi_{g_2}^{X_2}$, we obtain

$$Z_{g_1}^{X_1} = Z_{g_2}^{X_2} \quad (4.15)$$

despite the fact that they are defined in a very different way and each consists of two very different contributions (cf. (4.12)). Second, the result also coincides with the geometrically twined elliptic genus for a $K3$ admitting $\langle g \rangle$ -symmetry

$$Z_g^X = \mathbf{EG}_g(K3) \quad (4.16)$$

whose induced action on 24-dimensional representation is isomorphic to that of $g \in G^X$.

For the conjugacy classes $g \in G^X$ that do not satisfy the geometric condition, the interpretation of the function Z_g^X is much less clear, similar to the situation in the M_{24} -moonshine. Just like the more familiar case when $X = A_1^{24}$ [32], some of them correspond to SCA-preserving symmetries of certain SCFT \mathcal{T} in the same moduli space as that of $K3$ sigma model, while some of them don't. We will discuss their interpretation in Sect. 6. The explicit formulas for the Z_g^X for all the conjugacy classes $[g] \subset G^X$ for all 23 X can be found in Appendix C.

5. Geometric Interpretation

The result of the previous section suggests that it can be fruitful to study the symmetries of (the non-linear sigma models on) different $K3$ surfaces with different configurations of rational curves in a different framework corresponding to the 23 different cases of umbral moonshine. In this section we will see how, on the geometric side, this has in fact been implemented in a recent analysis of the relation between the $K3$ Picard lattice, $K3$ symplectic automorphisms, and the Niemeier lattices [48, 49]. On the one hand, this provides a geometric interpretation of the results in this paper. On the other hand, one can view our results as a moonshine manifestation and extension of the geometric analysis in [48].

To discuss this interpretation, let us first briefly review the result in [48], in which Nikulin advocates a more refined study of the geometric and arithmetic properties of $K3$ surfaces by introducing an additional *marking* using Niemeier lattices. Usually, to specify a “marking” of a $K3$ surface M is to specify an isomorphism between the rank 22 lattice $H^2(M, \mathbb{Z})$ and the unique (up to isomorphism) even unimodular lattice $\Gamma_{3,19} \cong 2E_8(-1) \oplus 3U$ of signature $(3,19)$, where U is the hyperbolic lattice $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.³ To introduce an additional marking by Niemeier lattices, on top of the marking described above, an important ingredient is the Picard lattice

$$\text{Pic}(M) = H^2(M, \mathbb{Z}) \cap H^{1,1}(M)$$

of M . The real space $H^{1,1}(M, \mathbb{R})$ has signature $(1, 19)$ and the Picard lattice is either: (a) negative definite with $0 \leq \text{rk}(\text{Pic}(M)) \leq 19$; (b) hyperbolic of signature $(1, \text{rk}(\text{Pic}(M)) - 1)$ and with $1 \leq \text{rk}(\text{Pic}(M)) \leq 20$; (c) semi-negative definite with a null direction and with $1 \leq \text{rk}(\text{Pic}(M)) \leq 19$. The condition (b) holds if and only if M is algebraic. On the other hand, a generic non-algebraic $K3$ surface satisfies the first condition. Unless differently stated, we will focus on these two, the “generic” (a.) and the “algebraic” (b.), cases.

To obtain an additional marking of M by a Niemeier lattice, consider the maximal negative definite sublattice of the Picard lattice, denoted by $S_M(-1) \subseteq \text{Pic}(M)$. To be more explicit, in the generic case we have simply $S_M(-1) = \text{Pic}(M)$, while in the algebraic case $S_M(-1) = h_{\text{Pic}(M)}^\perp$ is the orthogonal complement in the Picard lattice of the one-dimensional sublattice generated by the primitive $h \in \text{Pic}(M)$ with $h^2 > 0$ corresponding to a nef divisor on M . Using the properties of the Torelli period map, one can show that a lattice S_M may arise in the above way from a $K3$ surface M if and

³ The “(-1)” means that we multiply the lattice bilinear form by a factor of -1 . This (-1) comes from the fact that the signature of the $K3$ cohomology lattice is mostly negative while the usual convention for the signature of the simply-laced root system and hence the Niemeier lattices is positive definite. The same goes for the (-1) factor in the definition of S_M below.

only if $S_M(-1)$ admits a primitive embedding into $\Gamma_{3,19}$, a condition that can be further translated into more concrete terms using the lattice embedding results in [89].

We say $\iota_{M,X}$ is a marking of the $K3$ surface M by the Niemeier lattice L^X if $\iota_{M,X} : S_M \rightarrow L^X$ is a primitive embedding of S_M into L^X . The first result of [48] states that every $K3$ surface admits a marking by (at least) one of the 23 Niemeier lattices. This can be shown using the fact that $S_M(-1)$ admits a primitive embedding into $\Gamma_{3,19}$ and the embedding theorem in [89].⁴ We will denote by \tilde{S}_M the image of S_M , and $(\tilde{S}_M)_{L^X}^\perp$ by its orthonormal complement in L^X .

The second result, demonstrating the importance of *all* 23 Niemeier lattices for the study $K3$ surfaces, proves that for every L^X with the exception of $X = A_{24}$ and $X = A_{12}^2$, there exists a $K3$ surface that can only be marked using L^X and not by any other Niemeier lattice. It was also conjectured in [48] that the same statement also holds for $X = A_{24}$ and $X = A_{12}^2$. In particular, from this point of view the case $X = A_1^{24}$ is not more special than any other of the 22 cases. The third result on the additional Niemeier marking states that, for any L^X , any primitive sublattice of L^X which can be primitively embedded into $\Gamma_{3,19}(-1)$ arises from the Picard lattice $\text{Pic}(M)$ in the way described above for a certain $K3$ surface M .

The above three results show that the additional marking of $K3$ lattices is general and universally applicable. Now we will see that such an extra marking is also useful. In [48], two applications of the Niemeier marking are discussed. As we will see, both are crucial for the geometric interpretation of our results. The first application is to use the Niemeier marking to constrain the configuration of smooth rational curves in a $K3$ surface: for the generic cases, a $K3$ surface M that can be marked by L^X has the configuration of all smooth rational curves given by $X \cap S_M$; for the algebraic cases, this holds modulo multiples of the primitive nef element. In particular, if one thinks of the rational curves as arising from the minimal resolutions of the du Val singularities, then the singularities have to be given by a sub-diagram of the Dynkin diagram corresponding to X . The second application involves studying the symmetries of $K3$. If M is a $K3$ surface of the generic or the algebraic type and M admits a marking by L^X , then the finite symplectic automorphism group G_M of M is a subgroup of G^X . More precisely, we have

$$G_M = \{g \in G^X \mid gv = v \text{ for all } v \in (\tilde{S}_M)_{L^X}^\perp\}.$$

In the other direction, $G \subset G^X$ is the finite symplectic automorphism group of some $K3$ surface if the orthonormal complement $(L^X)_G \subset L^X$ of the fixed point lattice $\{v \in L^X \mid gv = v \text{ for all } g \in G\}$ can be primitively embedded into $\Gamma_{3,19}(-1)$. Such $G \subset G^X$ that arise from $K3$ symmetries have been computed in [48] for all 23 L^X . In particular, it is easy to see that they indeed satisfy the geometric condition mentioned in Sect. 4: they must have at least 5 orbits on the 24-dimensional representation and at least 1 fixed point.

From the above two applications, we see that the marking by Niemeier lattices facilitates a more refined study of $K3$ geometry by labelling a $K3$ surface by one of the Niemeier lattices L^X via marking. This labelling is, as explained above, sometimes unique and sometimes not. It tends to be unique when the $K3$ surface has very large symmetry—the type of $K3$ surfaces especially of interest to us. In the above two applications, the two most important pieces of data associated to the Niemeier lattice L^X for

⁴ The trick of considering $S_M \oplus A_1$, also used in [88] to prove Mukai’s theorem, is employed here to exclude the Leech lattice.

the construction of umbral moonshine—the root system X and the umbral group G^X —acquire the meaning of the “enveloping smooth rational curve configuration” and the “enveloping symmetry group” respectively, for all the $K3$ surfaces that can be labelled by L^X . Employing this obvious interpretation for X and G^X , the contribution from the ADE singularities to the (twined) $K3$ elliptic genus (cf. (4.9) and (4.12)) acquires the interpretation of the contribution from the “enveloping smooth rational curve configuration” of the (class of) $K3$ surface, while the twining given by umbral moonshine is to be interpreted as encoding the action of the “enveloping symmetry group” on the non-linear sigma model.

Before closing the section, let us give a few examples to illustrate the above discussion. Consider a $K3$ surface M with 16 smooth rational curves giving the root system A_1^{16} , generating a primitive sublattice Π_K of $\text{Pic}(M)$. It is known that such a $K3$ surface is a Kummer surface, i.e. a resolution of T^4/\mathbb{Z}_2 by replacing the 16 A_1 du Val singularities with 16 rational curves [94]. Note that the $K3$ is not necessarily algebraic since the T^4 can be non-algebraic. From the above discussion we see that M can only be marked by the Niemeier lattice L^X with $X = A_1^{24}$ and hence its finite symplectic automorphism group is a subgroup of M_{24} . More precisely, it is a subgroup of $\{g \in M_{24} | g(\Pi_K) = \Pi_K\}$. Similarly, let’s consider as the second example a $K3$ surface M with 18 smooth rational curves giving the root system A_2^9 . It can arise in the Kummer-type construction, where we consider the minimal resolution of the nine A_2 type singularities of T^4/\mathbb{Z}_3 (for a certain type of T^4 and a certain \mathbb{Z}_3). Similarly, M can only be marked by the Niemeier lattice L^X with $X = A_2^{12}$ and hence its finite symplectic automorphism group is a subgroup of $G^X \cong 2.M_{12}$. For a certain T^4/\mathbb{Z}_6 model, by resolving the singularities of type $A_5 \oplus A_2^4 \oplus A_1^5$ we obtain a $K3$ surface that can be marked by L^X with $X = A_7^2 D_5^2$. See [95,96] for the detailed description of these $K3$ at the orbifold limit. From the above analysis the symmetry of this $K3$ lies in $G^X \cong Dih_4$.

6. Discussion

In this paper we established a relation between umbral moonshine and the $K3$ elliptic genus, thereby taking a first step in placing umbral moonshine into a geometric and physical context. However, many questions remain unanswered and much work still needs to be done before one can solve the mystery of umbral moonshine. In this section we discuss some of the open questions and future directions.

- In Sect. 5 we have provided an interpretation of the umbral group G^X as the “enveloping symmetry group” of the (sigma model of) $K3$ surfaces that can be marked by the given Niemeier lattice L^X . It would be interesting to investigate to what extent this general idea of “enveloping symmetry group” can be made precise and can be confirmed by combining geometric symmetries at different points in the moduli space, similar to the idea explored in [35]. Abstractly, it seems rather clear that varying the moduli induces a varying primitive embedding of S_M into L^X and can generate a subgroup of G^X that doesn’t necessarily admit an interpretation as a group of geometric symmetries of any specific $K3$ surface. As a concrete example, one family of $K3$ surfaces that might be amenable to an explicit analysis is the torus orbifold T^4/\mathbb{Z}_3 , where one can easily vary the moduli of the T^4 . As discussed in Sect. 5, the umbral group relevant for this family is $G^X \cong 2.M_{12}$ with $X = A_2^{12}$, analogous to the M_{24} case for the torus orbifold T^4/\mathbb{Z}_2 studied in [35].

- Another obvious possible interpretation for the conjugacy classes $[g]$ that do not admit a geometric interpretation in the present context is as stringy symmetries of certain $K3$ sigma models preserving the $\mathcal{N} = (4, 4)$ superconformal symmetries that have no counterpart in classical geometry. Note that they must have at least 4 orbits in the 24-dimensional representation in order for this interpretation to be possible [32,97]. As a result, it is clear that not all conjugacy classes of all of the 23 G^X admit such a possible interpretation. When a conjugacy class $[g]$ does have at least 4 orbits, often the resulting umbral moonshine twining Z_g^X is observed to coincide with a known elliptic genus $\mathbf{EG}_{g'}(K3)$ twined by a certain symmetry g' of the non-linear sigma model whose induced action on the 24-dimensional representation is isomorphic to that of g , i.e., they have the same cycle shape. However, we have not been able to match all Z_g^X with some known CFT twining results for all $[g] \subset G^X$ with at least 4 orbits. Moreover, for non-geometric classes g the twining Z_g^X is not uniquely determined by the cycle shape Π_g^X and it can occur that $Z_g^X \neq Z_{g'}^{X'}$ even when $\Pi_g^X = \Pi_{g'}^{X'}$. See the following point for a closely-related discussion.

Curiously, various twining functions Z_g^X coincide with those obtained in the work of x[40]. It will be interesting to better understand the relation of the two analyses.

- It seems possible and natural to generalise the analysis in Sect. 5 beyond the realm of geometric symmetries to include the CFT symmetries. To do so, one should consider the “quantum Picard lattice” $\text{Pic}(M) \oplus U$ instead of $\text{Pic}(M)$ and consider its embedding into $\Gamma_{4,20} = \Gamma_{3,19} \oplus U$ instead of $\Gamma_{3,19}$. The relevant symmetry groups are again subgroups of G^X , now with at least 4 orbits on the 24-dimensional representation. The analysis should amount to a combination of that in [48] and in [32]. However, a lack of a Torelli type theorem means some of the very strong results in [48] will not necessarily hold for the CFT generalisation. Finally, given a fixed Niemeier marking, one may also generalise the “symmetry surfing” analysis (see above) into the realm of CFT symmetries.
- It would be illuminating to provide the CFT underpinning of the separation of $\mathbf{EG}(K3)$ into the contribution from the singularities and the rest (4.9), by, for instance, analysing the twisted and untwisted fields in the orbifold $K3$ models.
- It would be interesting to extend the geometrical definition of elliptic genus (2.22) to non-compact spaces and obtain a geometric derivation of the CFT result (2.17). Similarly, one should compute the geometrical twined (or equivariant) elliptic genera and compare them with the conjecture in Appendix B.2.
- The map (4.12) from the umbral moonshine function H_g^X (or equivalently ϕ_g^X) to the weak Jacobi form Z_g^X is a projection: the summing over the torsion points projects out terms that would have corresponded to states with fractional $U(1)$ charges. In particular, determining a G^X -module for the set of weak Jacobi forms Z_g^X is in general not sufficient to construct the G^X -module K^X underlying H_g^X . It is hence important to gain a better understanding about the physical origin of this projection. Its form is very reminiscent of the Landau–Ginzburg description of the non-linear sigma model and we are currently investigating the relation between umbral moonshine and Landau–Ginzburg type theories.
- The above fact suggests that the full content of umbral moonshine might go well beyond the realm of $K3$ sigma models, and to explain the origin of umbral moonshine we might need to go beyond CFT. It has been suggested that Mathieu moonshine has imprints in a variety of string theory setups (see for instance [18,26,34,36,37,57]).

Analogously, for all 23 cases of umbral moonshine, it would be interesting to explore the possible string theoretic extension of the current result.

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A. Special Functions

First, we define the *Jacobi theta functions* $\theta_i(\tau, z)$ as follows.

$$\begin{aligned}
 \theta_1(\tau, z) &= -iq^{1/8}y^{1/2} \prod_{n=1}^{\infty} (1 - q^n)(1 - yq^n)(1 - y^{-1}q^{n-1}) \\
 \theta_2(\tau, z) &= q^{1/8}y^{1/2} \prod_{n=1}^{\infty} (1 - q^n)(1 + yq^n)(1 + y^{-1}q^{n-1}) \\
 \theta_3(\tau, z) &= \prod_{n=1}^{\infty} (1 - q^n)(1 + yq^{n-1/2})(1 + y^{-1}q^{n-1/2}) \\
 \theta_4(\tau, z) &= \prod_{n=1}^{\infty} (1 - q^n)(1 - yq^{n-1/2})(1 - y^{-1}q^{n-1/2})
 \end{aligned} \tag{A.1}$$

In particular we will use the transformation of θ_1 under the Jacobi group

$$\begin{aligned}
 \theta_1(\tau, z) &= -\theta_1(\tau, -z) \\
 &= e(-\frac{1}{2}\frac{z^2}{\tau})(i\tau)^{-1/2}\theta_1(-\frac{1}{\tau}, \frac{z}{\tau}) \\
 &= e(-1/8)\theta_1(\tau + 1, z) \\
 &= (-1)^{\lambda+\mu}e(\frac{1}{2}(\lambda^2\tau + 2\lambda z))\theta_1(\tau, z + \lambda\tau + \mu).
 \end{aligned} \tag{A.2}$$

Second, we introduce the theta functions

$$\theta_{m,r}(\tau, z) = \sum_{k \equiv r \pmod{2m}} q^{k^2/4m} y^k. \tag{A.3}$$

for $m \in \mathbb{Z}_{>0}$ which satisfy

$$\theta_{m,r}(\tau, z) = \theta_{m,r+2m}(\tau, z) = \theta_{m,-r}(\tau, -z).$$

The theta function $\theta_m = (\theta_{m,r})$, $r \in \mathbb{Z}/2m\mathbb{Z}$, is a vector-valued Jacobi form of weight $1/2$ and index m satisfying

$$\theta_m(\tau, z) = \sqrt{\frac{1}{2m}} \sqrt{\frac{i}{\tau}} e(-\frac{m}{\tau}z^2) \mathcal{S}_\theta.\theta_m(-\frac{1}{\tau}, \frac{z}{\tau})$$

$$\begin{aligned} &= \mathcal{T}_\theta.\theta_m(\tau + 1, z) \\ &= \theta_m(\tau, z + 1) = e(m(\tau + 2z))\theta_m(\tau, z + \tau), \end{aligned} \tag{A.4}$$

where the \mathcal{S}_θ and \mathcal{T}_θ matrices are $2m \times 2m$ matrices with entries

$$(\mathcal{S}_\theta)_{r,r'} = e\left(\frac{rr'}{2m}\right) e\left(\frac{-r+r'}{2}\right), \quad (\mathcal{T}_\theta)_{r,r'} = e\left(-\frac{r^2}{4m}\right) \delta_{r,r'}. \tag{A.5}$$

For later use we also introduce some weight two modular forms for the Hecke congruence subgroups

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ cN & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}, ad - bcN = 1, \right\}. \tag{A.6}$$

including $\Lambda_N \in M_2(\Gamma_0(N))$ for all $N \in \mathbb{Z}_{>0}$

$$\begin{aligned} \Lambda_N(\tau) &= Nq\partial_q \log\left(\frac{\eta(N\tau)}{\eta(\tau)}\right) \\ &= \frac{N(N-1)}{24} \left(1 + \frac{24}{N-1} \sum_{k>0} \sigma(k)(q^k - Nq^{Nk})\right), \end{aligned} \tag{A.7}$$

where $\sigma(k)$ is the divisor function $\sigma(k) = \sum_{d|k} d$. For $N = 44$ we will need the unique weight two newform

$$f_{new}^{44} = q + q^3 - 3q^5 + 2q^7 - 2q^9 - q^{11} - 4q^{13} - 3q^{15} + 6q^{17} + \dots$$

Finally we discuss Jacobi forms following [98]. For every pair of integers k and m , we say a holomorphic function $\phi : \mathbb{H} \times \mathbb{C} \rightarrow \mathbb{C}$ is an (*unrestricted*) *Jacobi form* of weight k and index m for the Jacobi group $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ if it satisfies

$$\phi(\tau, z) = e(m(\lambda^2\tau + 2\lambda z)) \phi(\tau, z + \lambda\tau + \mu) \tag{A.8}$$

$$= e\left(-m\frac{cz^2}{c\tau+d}\right) (c\tau + d)^{-k} \phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right). \tag{A.9}$$

The invariance of $\phi(\tau, z)$ under $\tau \rightarrow \tau + 1$ and $z \rightarrow z + 1$ implies a Fourier expansion

$$\phi(\tau, z) = \sum_{n,r \in \mathbb{Z}} c(n, r) q^n y^r \tag{A.10}$$

for $q = e(\tau)$ and $y = e(z)$, and the elliptic transformation can be used to show that $c(n, r)$ depends only on the *discriminant* $D = r^2 - 4mn$ and on $r \pmod{2m}$. An unrestricted Jacobi form is called a *weak Jacobi form* when the Fourier coefficients satisfy $c(n, r) = 0$ whenever $n < 0$. See, for instance, [41] for an introduction of Jacobi forms following [98].

B. Calculations and Proofs

B.1. Proof of (4.9). The aim of this subsection is to provide more details on the elliptic genus computed in Sect. 2 and to prove the identity (4.9) for all 23 Niemeier lattices L^X .

As we mentioned in the main text, the Cappelli–Itzykson–Zuber matrices govern the spectrum of $\mathcal{N} = 2$ minimal models as well as the mock modularity of mock modular forms featuring in umbral moonshine. Explicitly, the matrices Ω^Φ labelled by the root system Φ is given in Table 3, where we have introduced for each divisor n of m the following matrices

$$\Omega_m(n)_{r,r'} = \begin{cases} 1 & \text{if } r + r' = 0 \pmod{2n} \text{ and } r - r' = 0 \pmod{2m/n}, \\ 0 & \text{otherwise,} \end{cases} \tag{B.1}$$

One significance of the Cappelli–Itzykson–Zuber matrices in our context is that it captures the action of the so-called Eichler–Zagier operator $\mathcal{W}_m(n)$, defined for every divisor n of m acting on a function $f : \mathbb{H} \times \mathbb{C} \rightarrow \mathbb{C}$ as [98]

$$(f|\mathcal{W}_m(n))(\tau, z) = \frac{1}{n} \sum_{a,b=0}^{n-1} e\left(m\left(\frac{a^2}{n^2}\tau + 2\frac{a}{n}z + \frac{ab}{n^2}\right)\right) f\left(\tau, z + \frac{a}{n}\tau + \frac{b}{n}\right). \tag{B.2}$$

To be more precise, acting on the theta function (A.3) it satisfies

$$\theta_m|\mathcal{W}_m(n) = \Omega_m(n) \cdot \theta_m. \tag{B.3}$$

In order to exploit this equality in the calculation, we define the operator \mathcal{W}^Φ by replacing $\Omega_m(n)$ with $\mathcal{W}_m(n)$ in the definition of Ω^Φ (cf. Table 3), with the understanding that $f|\sum_i \mathcal{W}_m(n_i) = \sum_i f|\mathcal{W}_m(n_i)$. Similarly, we define $\mathcal{W}^{\Phi'} = \sum_i \mathcal{W}^{\Phi_i}$ for a union of the simply-laced root systems $\Phi' = \cup_i \Phi_i$ where all Φ_i have the same Coxeter number.

For later convenience, analogous to (3.7) we will also define

$$\phi^{\Phi',P}(\tau, z) = \frac{-i\theta_1(\tau, mz)\theta_1(\tau, (m-1)z)}{\eta^3(\tau)\theta_1(\tau, z)}(\mu_{m,0}|\mathcal{W}^{\Phi'}(\tau, z)) \tag{B.4}$$

where m denotes the Coxeter number of Φ as usual.

In [41] a meromorphic function

$$\psi^{X,P} = \mu_{m,0}|\mathcal{W}^X$$

was defined for every Niemeier root system X , where $\mu_{m,0}$ is given by the Appell–Lerch sum as in (2.16) and \mathcal{W}^X is defined as above. Note that ψ^X as a function of z has in

Table 3. The ADE matrices Ω of Cappelli–Itzykson–Zuber [42]

Φ	Ω^Φ
A_{m-1}	$\Omega_m(1)$
$D_{m/2+1}$	$\Omega_m(1) + \Omega_m(m/2)$
E_6	$\Omega_{12}(1) + \Omega_{12}(4) + \Omega_{12}(6)$
E_7	$\Omega_{18}(1) + \Omega_{18}(6) + \Omega_{18}(9)$
E_8	$\Omega_{30}(1) + \Omega_{30}(6) + \Omega_{30}(10) + \Omega_{30}(15)$

general poles at $z \in \frac{\mathbb{Z}}{m} + \frac{\mathbb{Z}}{m} \tau$. In [41], following [12] this meromorphic function has the interpretation as the polar part of the meromorphic Jacobi form

$$\psi^X = \mu_{m,0} |W^X - \sum_{r \in \mathbb{Z}/2m\mathbb{Z}} H_r^X \theta_{m,r}$$

of weight 1 and index m .

First, we would like to prove

$$Z^{\Phi,S}(\tau, z) = \frac{1}{2m} \sum_{a,b \in \mathbb{Z}/m\mathbb{Z}} q^{a^2} y^{2a} \phi^{\Phi,P}(\tau, \frac{z+a\tau+b}{m}). \tag{B.5}$$

We will start by providing more details on the expression (2.8) of the minimal model elliptic genus, which is a building block of the elliptic genus of the ADE singularities (2.17).

Fix m and let $\bar{m} = m - 2$. The \hat{A}_1 string functions (chiral parafermion partition function times $\eta(\tau)$, see [44]) are given by $c'_s = 0$ if $r = s \pmod{2}$ and otherwise

$$c'_s(\tau) = \frac{1}{\eta^3(\tau)} \sum_{\substack{-|\alpha| < \beta \leq |\alpha| \\ (\alpha, \beta) \text{ or } (\frac{1}{2}-\alpha, \frac{1}{2}+\beta) = (\frac{r}{2\bar{m}}, \frac{s}{2\bar{m}}) \pmod{\mathbb{Z}^2}}} \text{sgn}(\alpha) q^{m\alpha^2 - \bar{m}\beta^2}$$

Note that we have shifted r by one compared to the convention in, for instance, [44, 63]. Clearly, $r \in \mathbb{Z}/2m\mathbb{Z}$ and $s \in \mathbb{Z}/2\bar{m}\mathbb{Z}$, and $c'_s(\tau) = -c'_{-s}(\tau) = c'_{-s}(\tau)$. They can also be defined through the branching relation

$$\sum_{s \in \mathbb{Z}/2\bar{m}\mathbb{Z}} c'_s \theta_{\bar{m},s} = \frac{\theta_{m,r} - \theta_{m,-r}}{\theta_{2,1} - \theta_{2,-1}},$$

where we have used the theta function defined in (A.3). Define

$$\chi_{s,\epsilon}^r(\tau, z) = \sum_{k \in \mathbb{Z}/\bar{m}\mathbb{Z}} c'_{s-\epsilon+4k}(\tau) \theta_{2m\bar{m}, 2s+(4k-\epsilon)m}(\tau, \frac{z}{2m}).$$

We have $\epsilon \in \mathbb{Z}/4\mathbb{Z}$, from which $\epsilon = 0, 2$ correspond to the NS and $\epsilon = 1, 3$ to the Ramond sector. Note that now both r and s in $\chi_{s,\epsilon}^r$ take value in $\mathbb{Z}/2m\mathbb{Z}$.

Now let

$$\tilde{\chi}_s^r(\tau, z) = \chi_{s,1}^r(\tau, z) - \chi_{s,-1}^r(\tau, z).$$

It is easy to check that it transforms under the elliptic transformation as

$$\tilde{\chi}_s^r(\tau, z + a\tau + b) = (-1)^{a+b} e(\frac{sb}{m}) e(-\frac{\hat{c}}{2}(a^2\tau + 2az)) \tilde{\chi}_{s-2a}^r(\tau, z). \tag{B.6}$$

They are the Ramond sector superconformal blocks relevant for the $\mathcal{N} = 2$ minimal models with $c = 3\frac{m-2}{m}$.

Using these building blocks, the elliptic genus of the minimal model corresponding to the simply-laced root system Φ is then given by

$$Z_{\text{minimal}}^\Phi(\tau, z) = \frac{1}{2} \sum_{0 < r, r' < m} (\Omega_{r,r'}^\Phi - \Omega_{r,-r'}^\Phi) \sum_{s \in \mathbb{Z}/2m\mathbb{Z}} \tilde{\chi}_s^r(\tau, z) \tilde{\chi}_s^{r'}(\bar{\tau}, 0).$$

Using $\Omega_{r,r'}^\Phi = \Omega_{-r,-r'}^\Phi$, $\tilde{\chi}_{r'}^r(\tau, z) = -\tilde{\chi}_{r'}^{-r}(\tau, z)$ and $\tilde{\chi}_s^r(\tau, 0) = \delta_{r,s} - \delta_{r,-s}$, we arrive at

$$Z_{\text{minimal}}^\Phi(\tau, z) = \frac{1}{2} \sum_{r,r' \in \mathbb{Z}/2m\mathbb{Z}} \Omega_{r,r'}^\Phi \tilde{\chi}_{r'}^r(\tau, z) = \frac{1}{2} \text{Tr}(\Omega^\Phi \cdot \tilde{\chi}).$$

Now we define for any $n\tilde{n} = m, n, \tilde{n} \in \mathbb{Z}$ and operator acting on a function $f : \mathbb{H} \times \mathbb{C} \rightarrow \mathbb{C}$ as

$$f|\tilde{\mathcal{W}}_m(n)(\tau, z) = \frac{1}{n} \sum_{\substack{a,b \in \mathbb{Z}/m\mathbb{Z} \\ a,b=0(\tilde{n})}} (-1)^{a+b+ab} e\left(\frac{m-2}{2m}(a^2\tau + 2az + ab)\right) f(\tau, z + a\tau + b)$$

Using (B.6) it is easy to check that

$$\tilde{\chi}_s^r|\tilde{\mathcal{W}}_m(n) = (\Omega_m(n) \cdot \tilde{\chi})_s^r = \sum_{s' \in \mathbb{Z}/2m\mathbb{Z}} \delta_{s-s',0} (2\tilde{n}) \delta_{s+s',0} (2n) \tilde{\chi}_{s'}^r$$

Finally, one can verify that

$$\begin{aligned} & \sum_{\alpha,\beta=0}^{m-1} (-1)^{\alpha+\beta} q^{\alpha^2/2} y^\alpha (\tilde{\chi}_s^r|\tilde{\mathcal{W}}_m(n))(\tau, z + \alpha\tau + \beta) \mu\left(\tau, \frac{z + \alpha\tau + \beta}{m}\right) \\ &= \sum_{\alpha,\beta=0}^{m-1} (-1)^{\alpha+\beta} q^{\alpha^2/2} y^\alpha \tilde{\chi}_s^r(\tau, z + \alpha\tau + \beta) (\mu|\mathcal{W}_m(n))\left(\tau, \frac{z + \alpha\tau + \beta}{m}\right). \end{aligned}$$

Subsequently, the identity (B.5) follows from the above equality and

$$Z_{\text{minimal}}^{A_{m-1}}(\tau, z) = \frac{1}{2} \text{Tr}\tilde{\chi} = \frac{\theta_1(t, z/m)}{\theta_1(t, z(m-1)/m)}.$$

Finally we are ready to prove (4.9), which can be re-expressed as

$$\mathbf{EG}(\tau, z; K3) = \frac{1}{2m} \sum_{a,b \in \mathbb{Z}/m\mathbb{Z}} q^{a^2} y^{2a} \phi^{X,T}\left(\tau, \frac{z + a\tau + b}{m}\right) \tag{B.7}$$

when combined with the identity (B.5) that we just verified and when we use the definition

$$\phi^{X,T}(\tau, z) = (\phi^{X,P} + \phi^X)(\tau, z) = \frac{-i\theta_1(\tau, mz)\theta_1(\tau, (m-1)z)}{\eta^3(\tau)\theta_1(\tau, z)} \psi^X(\tau, z).$$

From the fact that ψ^X transforms as a weight 1, index m Jacobi form and using the transformation (A.2) of the Jacobi theta function, it is straightforward to show that the RHS of (B.7) transforms as a weight 0, index 1 Jacobi form. Moreover, the poles of ψ^X at m -torsion points are combined with the zeros of $\theta_1(\tau, mz)$ and as a result $\phi^{X,T}$ is a holomorphic function on $\mathbb{H} \times \mathbb{C}$ admitting a double-expansion in powers of q and y . In order to show that the RHS of (B.7) is a weight 0, index 1 weak Jacobi form, we need to prove that there is no term in its Fourier expansion with $q^n, n < 0$. This can be shown by using the explicit formulas involving $\mu_{m,0}$ and θ_1 , combining with the fact that $H_r^X = O(q^{-r^2/4m})$ and the fact that the sum over b projects out all terms with fractional powers of y . After showing that both sides of (B.7) are weight 0, index 1 weak Jacobi forms, using the fact that the space of such functions is one-dimensional, the equality is proven by comparing both sides at, say, $z = 0$.

B.2. Computing Z_g^X . In this subsection we compute the twining function Z_g^X in (4.12). The results of the computation are recorded in Appendix C. In particular, we will give the details of the computation of $Z_g^{X,S}$. As a part of the computation, we also make conjectures for the elliptic genus $Z_h^{\Phi,S}$ of du Val singularities twined by certain automorphisms $\langle h \rangle$ of the corresponding Dynkin diagram Φ .

From the action of $g \in G^X$ on the Niemeier root lattice X , we can divide the conjugacy classes $[g]$ into the following two types. In the first type, there exists an element in the conjugacy class that only permutes the irreducible components of X . More precisely, there exists an element g in the class that descends from an element in $\bar{G}^X \subseteq G^X$, where \bar{G}^X is a quotient of G^X and is defined by

$$\bar{G}^X = \text{Aut}(L^X)/\hat{W}^X,$$

where $\hat{W}^X < \text{Aut}(L^X)$ is the subgroup of lattice automorphisms that stabilize the irreducible components of X . See Table 2 for the list of \bar{G}^X . In the second type, the action of an element in $[g]$ necessarily involves certain non-trivial automorphisms of some of the irreducible components in X . See [41] for a more detailed discussion.

As mentioned in Sect. 4, the twined function $Z_g^{X,S}$ for a conjugacy class $[g]$ of the first type, point-wise fixing a (not necessarily non-empty) union $X_g = \cup_i \Phi_i \subset X$ of the irreducible components Φ_i , is simply given by

$$Z_g^{X,S} = \sum_i Z^{\Phi_i,S}.$$

In order to compute the twined function $Z_g^{X,S}$ for $[g]$ for the second type of conjugacy classes, we need to twine the elliptic genus of the (irreducible) ADE singularities by

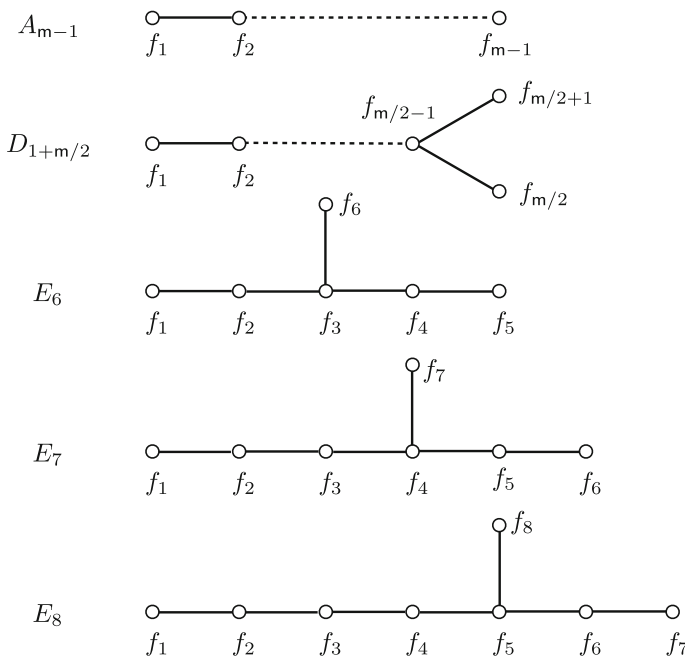


Fig. 1. The ADE Dynkin diagrams

symmetries corresponding to the automorphisms of the Dynkin diagram Φ . In the rest of this appendix we will propose a conjectural answer.

For the A_{m-1} singularity with $m > 2$ we have the \mathbb{Z}_2 automorphism exchanging the simple root f_i with f_{m-i} , in the notation shown in Fig. 1. We conjecture that the corresponding twined elliptic genus is

$$Z_{\mathbb{Z}_2}^{\Phi,S} = Z^{\Phi,S}|\mathcal{W}^{(-)}, \quad \Phi = A_{m-1},$$

where we have defined the operator acting on a function $f : \mathbb{H} \times \mathbb{C} \rightarrow \mathbb{C}$ as

$$f|\mathcal{W}^{(-)}(\tau, z) = -f(\tau, z + \frac{1}{2}).$$

In fact, the above expression for $Z_{\mathbb{Z}_2}^{\Phi,S}$ can be deduced from the action of \mathbb{Z}_2 on the eigenvectors of the appropriate Coxeter element, and similarly for the twined elliptic genus of the D- and E-type singularities discussed below.

For later use we also define the operators

$$f|\mathcal{W}^{(3)}(\tau, z) = \frac{1}{3} \sum_{a=0}^2 f(\tau, z + \frac{a}{3})$$

$$f|\mathcal{W}^{(6)}(\tau, z) = \frac{1}{6} \sum_{a=0}^5 f(\tau, z + \frac{a}{6}).$$

We remark that the above conjecture, if proven, provides a geometrical explanation of the following interesting property of the G^X -module K^X . It was observed and conjectured in [7, 41] that in the cases where X has only A-type components (i.e. when $m - 1|24$), the G^X -module K_r^X underlying the even components of the mock modular form $H_{g,r}^X$ (r even), are composed of irreducible faithful representations of G^X . On the other hand, the module K_r^X underlying the odd components of the mock modular form $H_{g,r}^X$ (r odd), are composed of G^X -representations that factor through \tilde{G}^X . Similar considerations also apply to the cases when X contains also D- and E-type components.

For the D-type singularity different from D_4 , we have the \mathbb{Z}_2 automorphism exchanging the simple root $f_{m/2}$ with $f_{1+m/2}$, in the notation shown in Fig. 1. We conjecture that the corresponding twined elliptic genus is

$$Z_{\mathbb{Z}_2}^{\Phi,S}(\tau, z) = \frac{1}{m} \sum_{a,b \in \mathbb{Z}/m\mathbb{Z}} q^{a^2} y^{2a} \phi_{\mathbb{Z}_2}^{\Phi,P}(\tau, \frac{z+a\tau+b}{m})$$

for $D_{1+m/2}$ for $m \neq 6$, where

$$\phi_{\mathbb{Z}_2}^{\Phi,P}(\tau, z) = \frac{-i\theta_1(\tau, mz)\theta_1(\tau, (m-1)z)}{\eta^3(\tau)\theta_1(\tau, z)} (-\mu_{m,0}|\mathcal{W}_m(m/2) + \mu_{m,0}|\mathcal{W}^{(-)})(\tau, z). \tag{B.8}$$

The $\Phi = D_4$ Dynkin diagram permits a S_3 symmetry on the roots $\{f_1, f_3, f_4\}$. We conjecture that the corresponding twined elliptic genera are given by

$$Z_{\mathbb{Z}_2}^{\Phi,S}(\tau, z) = \frac{1}{6} \sum_{a,b \in \mathbb{Z}/6\mathbb{Z}} q^{a^2} y^{2a} \phi_{\mathbb{Z}_2}^{\Phi,P}(\tau, \frac{z+a\tau+b}{6})$$

$$Z_{\mathbb{Z}_3}^{\Phi,S}(\tau, z) = \frac{1}{6} \sum_{a,b \in \mathbb{Z}/6\mathbb{Z}} q^{a^2} y^{2a} \phi_{\mathbb{Z}_3}^{\Phi,P}\left(\tau, \frac{z + a\tau + b}{6}\right)$$

where

$$\begin{aligned} \phi_{\mathbb{Z}_2}^{\Phi,P}(\tau, z) &= \frac{-i\theta_1(\tau, 6z)\theta_1(\tau, 5z)}{\eta^3(\tau)\theta_1(\tau, z)} \mu_{6,0}|\mathcal{W}_{D_4, \mathbb{Z}_2}(\tau, z) \\ \phi_{\mathbb{Z}_3}^{\Phi,P}(\tau, z) &= \frac{-i\theta_1(\tau, 6z)\theta_1(\tau, 5z)}{\eta^3(\tau)\theta_1(\tau, z)} \mu_{6,0}|\mathcal{W}_{D_4, \mathbb{Z}_3}(\tau, z) \end{aligned}$$

and

$$\begin{aligned} \mathcal{W}_{D_4, \mathbb{Z}_2} &= -\mathcal{W}_6(3) + \mathcal{W}^{(-)} + 2\mathcal{W}^{(6)} \\ \mathcal{W}_{D_4, \mathbb{Z}_3} &= \mathcal{W}^{D_4} - 3\mathcal{W}^{(3)}. \end{aligned}$$

The only E-type diagram with non-trivial automorphism is the \mathbb{Z}_2 generated by the action $f_i \mapsto f_{6-i}$ of E_6 , for $1 \leq i \leq 5$. We conjecture that corresponding twined elliptic genus is

$$Z_{\mathbb{Z}_2}^{E_6,S}(\tau, z) = \frac{1}{12} \sum_{a,b \in \mathbb{Z}/12\mathbb{Z}} q^{a^2} y^{2a} \phi_{\mathbb{Z}_2}^{E_6,P}\left(\tau, \frac{z + a\tau + b}{12}\right)$$

where

$$\begin{aligned} \phi_{\mathbb{Z}_2}^{E_6,P}(\tau, z) &= \frac{-i\theta_1(\tau, 12z)\theta_1(\tau, 11z)}{\eta^3(\tau)\theta_1(\tau, z)} (\mu_{12,0}|\mathcal{W}_{12}(6) + \mu_{12,0}|\mathcal{W}^{(-)} \\ &\quad + \mu_{12,0}|\mathcal{W}_{12}(4)|\mathcal{W}^{(-)})(\tau, z). \end{aligned} \tag{B.9}$$

After giving the conjectural answer for the building blocks $Z_h^{\Phi,S}$ of the twining of $Z_g^{X,S}$, we need to know how such a $g \in G^X$ acts on the Niemeier root system X . This is encoded in the twisted Euler characters $\bar{\chi}^{X_A}, \chi^{X_A}, \bar{\chi}^{X_D}, \chi^{X_D} \dots$ attached to the A-, D-, and E-components of each X . See §2.4 of [41] for details and see Appendix B.2 of the same reference for the values of such twisted Euler characters for all 23 X . Combining these ingredients leads to the answer for $Z_g^{X,S}$ for all conjugacy classes $[g]$ for all the umbral groups G^X . This completes our computation of Z_g^X (4.12).

C. The Twining Functions

In this appendix we provide the expression of $Z_g^X(\tau, z)$ (cf. (4.12)) in terms of the function h_g^X :

$$Z_g^X(\tau, z) = \frac{i\theta_1(\tau, z)^2}{\theta_1(\tau, 2z)\eta^3(\tau)} \left\{ c_g^X \mu_{2,0}(t, z) + h_g^X(\tau)(\theta_{2,-1}(\tau, z) - \theta_{2,1}(\tau, z)) \right\}$$

where c_g^X is the number of fixed point in the 24-dimensional representation of G^X . In other words, for the cycle shape Π_g^X defined in (4.13), we have $c_g^X = m_1$ if $\ell_1 = 1$ and $c_g^X = 0$ otherwise. For instance, for $X = A_1^{24}$ and $[g]$ the identity class, the above formula gives the $\mathcal{N} = 4$ character decomposition of $\mathbf{EG}(K3)$ in (4.6).

For $X = A_1^{24}$, the functions $h_g^X(\tau)$ for all $[g] \subset G^X \cong M_{24}$ have been worked out in [18–21]. We refer to these papers, or the summary in [7, 29]. For convenience we will denote $h_g^{A_1^{24}}$ simply by H_g for $[g] \subset M_{24}$. Recall that H_g is nothing but the function discussed in (1.2) when $[g] = 1A$ is the identity class of M_{24} . There are two cases, corresponding to $X = D_{24}$ and $X = D_{16}E_8$, with trivial G^X . As a result they are not included in the present appendix.

When h_g^X coincides with $H_{g'}$ for a certain g' , we will simply use this identity to define h_g^X . When there does not exist such a $[g'] \subset M_{24}$, we write

$$h_g^X(\tau) = \frac{c_g^X}{24} H_{1A}(\tau) - \frac{\tilde{T}_g^X(\tau)}{\eta(\tau)^3} \tag{C.1}$$

and we will give the explicit expression for \tilde{T}_g^X in the following tables using the functions given in Appendix A. We also use the short hand notation $(n)^k := \eta^k(n\tau)$ (Tables 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23).

Table 4. $X = A_2^{12}$

$[g]$	Π_g^X	h_g^X
1A	1^{24}	H_{1A}
2A	2^{12}	H_{2B}
4A	4^6	H_{4C}
2B	$1^8 2^8$	H_{2A}
2C	2^{12}	H_{2B}
3A	$1^6 3^6$	H_{3A}
6A	$2^3 6^3$	$\tilde{T}_{6A}^X = 3\Lambda_2 + 2\Lambda_3 - \Lambda_4 - 3\Lambda_6 + \Lambda_{12}$
3B	3^8	$\tilde{T}_{3B}^X = 2(-4\Lambda_3 + \Lambda_9 - (1)^6/(3)^2)$
6B	6^4	$\tilde{T}_{6B}^X = 2 \frac{(1)^5(3)}{(2)(6)}$
4B	$2^4 4^4$	H_{4A}
4C	$1^4 2^2 4^4$	H_{4B}
5A	$1^4 5^4$	H_{5A}
10A	$2^2 10^2$	H_{10A}
12A	12^2	$\tilde{T}_{12A}^X = 2 \frac{(1)(2)^5(3)}{(4)^2(6)}$
6C	$1^2 2^2 3^2 6^2$	H_{6A}
6D	$2^3 6^3$	$h_{6D}^X = h_{6A}^X$
8AB	$4^2 8^2$	$\tilde{T}_{8AB}^X = 2(2)^4(4)^2/(8)^2$
8CD	$1^2 2^1 4^1 8^2$	H_{8A}
20AB	$4^1 20^1$	$\tilde{T}_{20AB}^X = 2 \frac{(2)^7(5)}{(1)(4)^2(10)}$
11AB	$1^2 11^2$	$\tilde{T}_{11AB}^X = (2\Lambda_{11}(\tau) + 33(1)^2(11)^2)/5$
22AB	$2^1 22^1$	$\tilde{T}_{22AB}^X = (3\Lambda_2 - \Lambda_4 + 2\Lambda_{11} - 3\Lambda_{22} + \Lambda_{44})/15 - \frac{22}{3} f_{new}^{44} + \frac{11}{5} (1)^2(11)^2 + \frac{44}{5} (2)^2(22)^2 + \frac{88}{5} (4)^2(44)^2$

Table 5. $X = A_3^8$

$[g]$	Π_g^X	h_g^X
1A	1^{24}	H_{1A}
2A	$1^8 2^8$	H_{2A}
2B	2^{12}	H_{2B}
4A	$2^4 4^4$	H_{4A}
4B	4^6	$\tilde{T}_{4B}^X = 2\Lambda_4 - 3\Lambda_8 + \Lambda_{16} - 2(1)^4(2)^2/(4)^2$
2C	$1^8 2^8$	H_{2A}
3A	$1^6 3^6$	H_{3A}
6A	$1^2 2^2 3^2 6^2$	H_{6A}
6BC	$2^3 6^3$	$h_{6BC}^X = h_{6A}^Y, Y = A_2^{12}$
8A	$4^2 8^2$	$\tilde{T}_{8A}^X = (2\Lambda_8 - 3\Lambda_{16} + \Lambda_{32})/8 + 8(4)^4(16)^4/(8)^4 - 8(4)^2(8)^2$
4C	$1^4 2^2 4^4$	H_{4B}
7AB	$1^3 7^3$	H_{7AB}
14AB	$1^1 2^1 7^1 14^1$	$\tilde{T}_{14AB}^X = (-\Lambda_2 - \Lambda_7 + \Lambda_{14} + 28(1)(2)(7)(14))/3$

Table 6. $X = A_4^6$

$[g]$	Π_g^X	h_g^X
1A	1^{24}	H_{1A}
2A	2^{12}	H_{2B}
2B	2^{12}	H_{2B}
2C	$1^8 2^8$	H_{2A}
3A	3^8	H_{3B}
6A	6^4	H_{6B}
5A	$1^4 5^4$	H_{5A}
10A	$2^2 10^2$	$\tilde{T}_{10A}^X = (3\Lambda_2 - \Lambda_4 + 2\Lambda_5 - 3\Lambda_{10} + \Lambda_{20} + 40(2)^2(10)^2)/3$
4AB	4^6	$h_{4AB}^X = h_{4B}^Y, Y = A_3^8$
4CD	$1^4 2^2 4^4$	H_{4B}
12AB	12^2	$\tilde{T}_{12AB}^X = 2 \frac{(2)^2(6)^4(1)^2}{(3)^2(12)^2}$

Table 7. $X = A_5^4 D_4$

$[g]$	Π_g^X	h_g^X
1A	1^{24}	H_{1A}
2A	$1^8 2^8$	H_{2A}
2B	$1^8 2^8$	H_{2A}
4A	$1^4 2^2 4^4$	H_{4B}
3A	$1^6 3^6$	H_{3A}
6A	$1^2 2^2 3^2 6^2$	H_{6A}
8AB	$1^2 2^1 4^1 8^2$	H_{8A}

Table 8. $X = D_4^6$

$[g]$	Π_g^X	h_g^X
1A	1^{24}	H_{1A}
3A	$1^6 3^6$	H_{3A}
2A	$1^8 2^8$	H_{2A}
6A	$1^2 2^2 3^2 6^2$	H_{6A}
3B	$1^6 3^6$	H_{3A}
3C	3^8	$h_{3C}^X = h_{3B}^Y, Y = A_2^{12}$
4A	$2^4 4^4$	H_{4A}
12A	$2^1 4^1 6^1 12^1$	$\tilde{T}_{12A}^X = (-2\Lambda_2 + 3\Lambda_4 + 2\Lambda_6 - \Lambda_8 - 3\Lambda_{12} + \Lambda_{24})/4 + 18(2)(4)(6)(12)$
5A	$1^4 5^4$	H_{5A}
15AB	$1^1 3^1 5^1 15^1$	$\tilde{T}_{15AB}^X = (-\Lambda_3 - \Lambda_5 + \Lambda_{15} + 45(1)(3)(5)(15))/4$
2B	$1^8 2^8$	H_{2A}
2C	2^{12}	H_{2B}
4B	$1^4 2^2 4^4$	H_{4B}
6B	$1^2 2^2 3^2 6^2$	H_{6A}
6C	6^4	$h_{6C}^X = h_{6B}^Y, Y = A_2^{12}$

Table 9. $X = A_6^4$

$[g]$	Π_g^X	h_g^X
1A	1^{24}	H_{1A}
2A	2^{12}	H_{2B}
4A	4^6	H_{4C}
3AB	$1^6 3^6$	H_{3A}
6AB	$2^3 6^3$	$h_{6AB}^X = h_{6A}^Y = h_{6BC}^Z, Y = A_2^{12}, Z = A_3^8$

Table 10. $X = A_7^2 D_5^2$

$[g]$	Π_g^X	h_g^X
1A	1^{24}	H_{1A}
2A	$1^8 2^8$	H_{2A}
2B	$1^8 2^8$	H_{2A}
2C	$1^8 2^8$	H_{2A}
4A	$2^4 4^4$	H_{4A}

Table 11. $X = A_8^3$

$[g]$	Π_g^X	h_g^X
1A	1^{24}	H_{1A}
2A	2^{12}	H_{2B}
2B	$1^8 2^8$	H_{2A}
2C	2^{12}	H_{2B}
3A	3^8	$h_{3A}^X = h_{3B}^Y, Y = A_2^{12}$
6A	6^4	$\tilde{T}_{6A}^X = 2 \frac{(1)^5(3)}{(2)(6)} + 24(6)^4$

Table 12. $X = A_9^2 D_6$

$[g]$	Π_g^X	h_g^X
1A	1^{24}	H_{1A}
2A	$1^8 2^8$	H_{2A}
4AB	$1^4 2^2 4^4$	H_{4B}

Table 13. $X = D_6^4$

$[g]$	Π_g^X	h_g^X
1A	1^{24}	H_{1A}
2A	2^{12}	H_{2B}
3A	$1^6 3^6$	H_{3A}
2B	$1^8 2^8$	H_{2A}
4A	4^6	H_{4C}

Table 14. $X = A_{11} D_7 E_6$

$[g]$	Π_g^X	h_g^X
1A	1^{24}	H_{1A}
2A	$1^8 2^8$	H_{2A}

Table 15. $X = E_6^4$

$[g]$	Π_g^X	h_g^X
1A	1^{24}	H_{1A}
2A	$1^8 2^8$	H_{2A}
2B	$1^8 2^8$	H_{2A}
4A	$2^4 4^4$	H_{4A}
3A	$1^6 3^6$	H_{3A}
6A	$1^2 2^2 3^2 6^2$	H_{6A}
8AB	$4^2 8^2$	$\tilde{T}_{8AB}^X = (2\Lambda_8 - 3\Lambda_{16} + \Lambda_{32})/8 + 24(4)^2(8)^2 + 8(4)^4(16)^4/(8)^4$

Table 16. $X = A_{12}^2$

$[g]$	Π_g^X	h_g^X
1A	1^{24}	H_{1A}
2A	2^{12}	H_{2B}
4AB	4^6	$h_{4AB}^X = h_{4B}^Y = h_{4AB}^Z, Y = A_3^8, Z = A_4^6$

Table 17. $X = D_8^3$

$[g]$	Π_g^X	h_g^X
1A	1^{24}	H_{1A}
2A	$1^8 2^8$	H_{2A}
3A	3^8	H_{3B}

Table 18. $X = A_{15}D_9$

$[g]$	Π_g^X	h_g^X
1A	1^{24}	H_{1A}
2A	$1^8 2^8$	H_{2A}

Table 19. $X = A_{17}E_7$

$[g]$	Π_g^X	h_g^X
1A	1^{24}	H_{1A}
2A	$1^8 2^8$	H_{2A}

Table 20. $X = D_{10}E_7^2$

$[g]$	Π_g^X	h_g^X
1A	1^{24}	H_{1A}
2A	$1^8 2^8$	H_{2A}

Table 21. $X = D_{12}^2$

$[g]$	Π_g^X	h_g^X
1A	1^{24}	H_{1A}
2A	2^{12}	H_{2B}

Table 22. $X = A_{24}$

$[g]$	Π_g^X	h_g^X
1A	1^{24}	H_{1A}
2A	2^{12}	H_{2B}

Table 23. $X = E_8^3$

$[g]$	$\tilde{\Gamma}_g$	Z_g
1A	1^{24}	H_{1A}
2A	$1^8 2^8$	H_{2A}
3A	3^8	$h_{3B}^X = h_{3B}^Y, Y = A_2^{12}$

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