"A terrible piece of bad metaphysics"? Towards a history of abstraction in nineteenth- and early twentieth-century probability theory, mathematics and logic

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CHAPTER 9


1. The complex history of nineteenth-century British algebra: algebra, geometry and abstractness

It is now well established that there were two major factors that contributed to the revitalization and reorientation of British mathematics in the early- and mid-nineteenth-century.1 Firstly, there was the external influence consisting of the dedication of the members of the anti-establishment Analytical Society to the ‘Principle of pure D-ism in opposition to the Dot-age of the University’. Charles Babbage (1791-1871), John Herschel (1792-18710) and George Peacock (1791-1858) introduced the ‘exotic’ Continental algebraic calculus of Joseph-Louis Lagrange (1736-1813) in place of the Newtonian fluxional calculus at

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Secondly, there was the internal reflection on the foundations of the symbolical approach to algebra – first put forward in Peacock’s seminal *Treatise on Algebra* of 1830 – in the writings of three ‘groups’ within the second generation of reformers of British mathematics. The first group consisted of symbolical algebraists associated with *The Cambridge Mathematical Journal (CMJ)* of which Duncan Farquharson Gregory (1813-1844), as (co-)founding editor, was the foremost representative. In brief, their aim was to generalize Peacock’s definition of algebra as a ‘science of symbols’. Second was the group of critical revisionists to which belonged, among others, Augustus De

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4 The distinction between the first and second generation of reformers of British mathematics was first introduced in Crosbie Smith and Norton Wise, *Energy & Empire. A Biographical Study of Lord Kelvin* (Cambridge, Cambridge University Press, 1989), chapter 6. A critical discussion of their characterization of the second generation is found in Lukas M. Verburgt, ‘Duncan Farquharson Gregory and Robert Leslie Ellis: second generation reformers of British mathematics’, *under review*. Given that the complex task of determining the theoretical relationship between the contributions of, for example, Gregory, De Morgan and Hamilton to algebra is beyond the scope of this paper, the distinction between three groups within the second generation is here presented without any detailed justification. The plausibility of the distinction will, hopefully, become apparent in what follows.

Morgan (1806-1871)\(^6\) and George Boole (1815-1864),\(^7\) that was able to construct non-commutative algebra out of symbolical algebra by means of its redefinition as an ‘art of reasoning’\(^8\). Third was the group consisting of abstract algebraists such as William Rowan Hamilton (1805-1865)\(^9\) and Arthur Cayley (1821-1895)\(^10\) that realized the scientific promises, in the sense of an autonomous abstract system, of symbolical algebra on the basis of a dismissal of its philosophical foundations.

The mathematicians of the first generation and the first two groups of the second generation, which together form the ‘Cambridge’ or ‘English’ school of symbolical algebra, have for a long time been considered as precursors to the modern approach to algebra as the formal study of arbitrary axioms systems.

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\(^8\) Given the argumentation of this paper, this second group of the second generation of reformers of British mathematics will not be discussed.


initiated in the work of the third group.\textsuperscript{11} But it has become clear recently that the construction of genuine algebraical systems that are not generalizations of ‘arithmetical algebra’ and that do not have a fixed interpretation in either arithmetic or geometry was not accomplished by the ‘scientific symbolical algebraists’ of the first generation, that is, Peacock, or by the first group of the second generation, that is, Gregory.\textsuperscript{12} The usual answer to the question concerning what separated the symbolical from the abstract algebraists is that their division must be sought in a fundamental difference between views of truth. Whereas Peacock and Gregory were committed to the traditional view, which has it that mathematics is concerned with an external subject matter, the second group entertained the view that mathematical truth can be defined as the consistency of an axiom system. This implies not only that for the symbolical algebraist to adopt the position of a modern mathematician would have required ‘a major change […] in the [understanding] of the nature of mathematics’,\textsuperscript{13} but also that formalism, whatever its precise meaning,\textsuperscript{14} is both the condition for the destruction of the limitations on the abstractness of mathematics and the condition for the creation of new abstract algebras. These considerations have led Crosbie Smith and Norton Wise to claim that

all of them [i.e. the ‘scientific symbolical algebraists’], although they developed algebra as a formal logic of operations, or rules of combination of symbols, nevertheless required that the [symbols and] operations be grounded in particular subject matter, in interpretation, rather than in internal consistency of the rules of combination alone.\textsuperscript{15}


\textsuperscript{12} See, for example, Daniel A. Clock, \textit{A New British Concept of Algebra}, 1825-1850 (Madison: University of Wisconsin, 1964).


\textsuperscript{15} Smith & Wise, \textit{Energy} (note 4), 171, my emphasis.
Although Richards and Smith and Wise are right about the reasons for the theoretical limits of the symbolical algebra of Peacock and Gregory, there is a twofold problem with their presentation of the development of British algebra in the 1830s-1840s in terms of a rupture between ‘pre-modern’ and ‘modern’ views, rather than a gradual transition. On the one hand, such a presentation does not recognize that Peacock formulated his symbolical approach to algebra in explicit opposition to Babbage’s radical formalism – or, more generally, that the members of the Analytical Society disagreed on the nature of mathematics. On the other hand, it is unable to account for the, somewhat ironic, fact that the first mathematician to exercise the (‘modern’) freedom of abstraction suggested by the (‘pre-modern’) symbolical algebraists, namely their abstract algebraic critic Hamilton, rejected what he perceived as the ‘formalism’ inherent in the work of the symbolical algebraists! Taken together, the statement that symbolical algebra ‘was not abstract algebra in the modern sense and never could be’ is true, but only with the provisos that symbolical algebra was never meant to be modern abstract algebra and that in so far as

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16 Verburgt, ‘Gregory and Ellis’ (note 4) argues that there is also a significant problem with the characterization of the differences between the first generation and the first group of the second generation underlying these accounts.

17 John M. Dubbey, *The Mathematical Work of Charles Babbage* (Cambridge, Cambridge University Press, 1978) established that Peacock knew of Babbage’s ‘Essays on the philosophy of analysis’ of 1821. Fisch, ‘Creative indecision’ (note 3) shows not only that the portrayal of Peacock’s *Treatise on Algebra* of 1830 as an application of (Lagrangian) formalist views, to which he was committed during the 1810s-1820s, to algebra is mistaken, but also that ‘Babbage’s construal of pure analysis differed significantly from the system [of algebra] Peacock would eventually propose’. Fisch, ‘Creative indecision’ (note 3), 155.

18 Hamilton famously dismissed both the ‘philological’ view of algebra ‘as a system of rules or else as a system of expressions’ and the ‘practical’ view of algebra ‘as an art or as a language’ and. If these two viewpoints correspond to, on the one hand, the first generation (Peacock, but, thus, especially Babbage) and the first group of the second generation and, on the other hand, the second group of the second generation, Hamilton himself approached algebra as ‘a system of truths’.

the foundations of modern abstract algebra are not abstract they are, strictly speaking, also not modern.\textsuperscript{20}

The present paper proposes to come to terms with this problem-situation by finding a ‘middle way’ that combines the strong points of the ‘gradual transition’ and ‘rupture’ explanations of the connection between the ‘scientific symbolic algebraists’ and the, equally ‘scientific’, abstract algebraists. The idea of a growth of mathematical abstractness in the contributions to algebra of Peacock, Gregory (et al.) and Hamilton (et al.), respectively, is taken up so as to show not only that a certain moderate level of abstractness was implied in the framework of symbolic algebra, but also that the high level of abstractness of the abstract algebraists was premised on their rejection of a claim that accompanied the work of the symbolic algebraists, namely that abstractness is obtained by means of abstraction. These two points can be described and related precisely with reference to the fact that the symbolic algebraists insisted that the symbols and operations are ‘grounded in particular subject matter’.\textsuperscript{21} Firstly, it was because their algebra investigated the (relations between the) abstractions of the quantities of the sciences of arithmetic and geometry that the symbolic algebraists wished to formulate the principles that would establish algebra as a demonstrative science. This meant that algebra was, in principle, limited to the treatment of abstractions from the traditional mathematico-empirical entities number and magnitude.\textsuperscript{22} And even though a process of abstraction

\textsuperscript{20} This is meant as an implicit reference to the more general problems connected to those accounts of the history of modern mathematics and logic that equate ‘modern’ with an increase in ‘abstractness’ and ‘formality’ and define ‘abstract’ and ‘formal’ in terms of its being part of the process of ‘modernization’. Compare Herbert Mehrtens, Moderne, Sprache, Mathematik: Eine Geschichte des Streits um die Grundlagen der Disziplin und des Subjekts formaler Systeme (Berlin, Surhkamp, 1990) and Jeremy Gray, Plato’s Ghost. The Modernist Transformation of Mathematics (Princeton & Oxford, Princeton University Press, 2008) to, for example, Dennis E. Hesseling, The Reception of Brouwer’s Intuitionism in the 1920s (Basel, Birkhäuser, 1991/2012). The ‘debate’ between the symbolic algebraists and the abstract algebraists does, indeed, foreshadow the complexities of the famous formalist-intuitionist debate of almost a century later – as Israel Kleiner has suggested. Israel Kleiner, A History of Abstract Algebra (Boston, Birkhäuser, 2007), 151.

\textsuperscript{21} Smith & Wise, Energy (note 4), 171.

\textsuperscript{22} For a thorough analysis of the traditional relationship between arithmetic, geometry and algebra see, for example, Henk J.M. Bos, Redefining Geometrical Exactness. Descartes’ Transformation of the Early Modern Concept of Construction (New York, Springer-Verlag, 2001), chapter 6.
could generate, say, ‘abstract quantities’, because these, in the end, referred
to empirical objects they could never be considered as free creations of the
mind. Secondly, what allowed the abstract algebraists to create new algebras
of so-called ‘quaternions’ and ‘octonions’ was their Kantian-inspired dismissal
of the (‘abstractionist’)\(^23\) view of number and space as abstractions from obser-
vational experience in favor of the Kantian definition of physical and mathem-
atical objects as being constructed from a priori ‘intuition’. \(^24\) It was because
the basic elements of algebra could be said to stand for something ‘real’, in
this idealist sense, that it became possible to investigate whether higher-order

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\(^23\) ‘Abstractionism’ is the Aristotelian-Scholastic doctrine which holds that ‘mathematical
objects are constructed out of our ordinary experience but constructed in such a way
that mathematics does not depend on specific features of the sensible world’. Douglas
of Chicago Press), 10. The doctrine was rejected by Berkeley, Kant and many other
idealist philosophers and mathematicians; see, in this context, Margaret Atherton,
‘Kant’s mathematical realism’, *The Monist*, 67 (1984), 115-134. George Elder Davie
has made a convincing argument for the presence of this doctrine – popular, as it
was, among eighteenth-century Scottish mathematicians such as Gregory’s teacher,
William Wallace (1768-1843) – in the contributions of Gregory to symbolical algebra.
George Elder Davie, *The Democratic Intellect: Scotland and Her Universities in the Nin-
teenth Century* (Edinburgh, Edinburgh University Press, 1961), part 2. The fact that
‘abstractionism’, in mathematics, was closely related to Newtonian ‘fluxional method’
– which had been popularized, among others, by the Scottish geometer Robert
Simson (1687-1768) – seems to support this argument. For the content and influence
of Newtonian fluxions see, for example, Florian Cajori, *A History of the Conceptions of
Limits and Fluxions in Great Britain from Newton to Woodhouse* (London, Open Court,
presentation of the calculus’, *Isis*, 64 (1973), 33-49. For its presence in Scotland see
Judith V. Grabiner, ‘Was Newton’s calculus a dead end? The continental influence
of Maclaurin’s *Treatise of Fluxions*, *The American Mathematical Monthly*, 104 (1997),
393-410. These considerations will briefly be reflected upon in section 4.

\(^24\) The Kantian nature of Hamilton’s work on algebra has been extensively studied – see,
for example, Anthony T. Winterbourne, ‘Algebra and pure time: Hamilton’s affinity
with Kant’, *Historia Mathematica*, 9 (1982), 195-200. In his seminal *Sir William Rowan
Hamilton*, Thomas L. Hankins has expressed his belief that Hamilton was able to
establish abstract, rather than symbolical, algebra because of his ‘constructivism’.
Hankins, *Hamilton* (note 9), 352. The fact that Hamilton and Cayley adhered, at least
to a certain degree, to Kantian epistemology – see Mathews, ‘Hamilton on analysis’
(note 9), 192 – is briefly discussed in the section with concluding remarks.
elements could be constructed out of them all the while developing algebra ‘as a Science properly so called; strict, pure, and independent’.

The major benefit of this ‘middle way’ – in which the growth of abstractness of British algebra is attributed to a change in views about abstraction – is twofold. On the one hand, it approaches neither the symbolical algebraists nor the abstract algebraists as precursors of modern abstract algebra, thereby avoiding the characterization of the work of the first group ‘as a hesitant step in the direction of a formalist approach to algebra that [it] was incapable of fully seeing through’ as well as rejecting the description of the second group as being committed to taking this formalist step. On the other hand, it is able to explicate that with respect to algebra’s status as a science the two groups were actually more allies than either appreciated. Both the symbolical algebraists and the abstract algebraists wished to show that algebra, like geometry, could be a science, but where this, in the case of the first group, went hand-in-hand with a limitation of its abstractness to abstractions from experience, in the case of the second group this meant that algebra was to be developed on the basis of a priori intuitive principles.

1.1 Between geometry, arithmetic and (new) algebra

This general explanation of the relationship between the scientific symbolical algebraists, that is, Peacock and Gregory’s associates at the CMJ, and the abstract algebraists will here be presented in relation to the particular issue of the so-called ‘imaginary quantities’ – or, more specifically, of the attempt to put them on a geometrical or algebraical basis. The fact that ideas on the justification of imaginary quantities played an important part in the development of

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25 William Rowan Hamilton, ‘Theory of conjugate functions, or algebraic couples; with a preliminary and elementary essay on algebra as the science of pure time’, Transactions of the Royal Irish Academy, 17 (1837), 293-422, 295.

26 For this observation see Fisch, ‘Creative indecision’ (note 3), 156, 145.
algebra has been elucidated in detail. But these scholarly accounts rarely if ever discuss British mathematicians and when they do, they assume, without further ado, that abstract algebra was created out of symbolical algebra. The same, by and large, holds for those, relatively few, contributions to the analysis of the influence of geometry – that is, of geometrical considerations about the representation of ‘imaginaries’ on a two-dimensional plane – on the growth of abstractness in algebra. It is because it can not only problematize such linear accounts of the history of British algebra, but also substantiate the one sketched in the foregoing sub-section that the main focus of this paper is the work of two representatives of the first group of the second generation of reformers, namely Gregory and William Walton (1823-1901), on geometry and its connection with algebra.

Peacock’s Treatise on Algebra was written as a generalization of ideas put forward by Robert Woodhouse (1773-1827). For example in a paper published in 1802 in the Philosophical Transactions Woodhouse had redefined certain aspects of algebra so as to be able to cope with negative and imaginary quanti-

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28 Rice, ‘Inexplicable?’ (note 27) is a valuable exception.

29 See, for example, Nagel, ‘Impossible numbers’ (note 27). In her seminal paper, Elaine Koppelman has not only demonstrated the importance of the so-called ‘calculus of operations’ for the formation of the abstract view of algebra, but also suggested that this calculus ‘clearly was not the only relevant factor. One that should be mentioned, and which deserves further study, is the influence of new ideas in geometry’. Elaine Koppelman, ‘The calculus of operations and the rise of abstract algebra’, Archive for History of Exact Sciences, 8 (1971), 155-242, 238. She referred to a 1939 paper of Ernest Nagel as a starting-point for this study. Ernest Nagel, ‘The formation of modern conceptions of formal logic in the development of geometry’, Osiris, 7 (1939), 142-223. It may be remarked that both papers ‘are marred by their treatment of the British works as steps toward abstract algebra in an unacceptably modern sense’. Smith & Wise, Energy (note 4), 171, f. 53.
ties. Where several other mathematicians before him had attempted to solve the problem of the meaning, definition and acceptability of these quantities by showing that they were real in so far as a geometrical interpretation could be provided for them, Woodhouse rejected this justification because it relied on geometry as the foundation of all mathematics. It was in his *Principles of Analytical Calculation* (1803) that he first formulated the idea of algebra as a symbolical language that consisted of pure ‘algebraic abstractions [and] mathematical generalizations’.

Given the postulated independence of algebra, Woodhouse claimed that ‘no ‘reality’ could be ascribed to [imaginary] numbers on the basis of a geometrical interpretation. Rather he suggested a formal basis to their use’. Although Peacock agreed with Woodhouse, he seems to have been far less dismissive of geometry. In the *Treatise on Algebra*, Peacock mentioned not only arithmetic, but also geometry as a science ‘whose operations and the general consequences of [which] should serve as the guides to the assumptions which become the foundation of symbolical Algebra’.

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33 Peacock referred to the geometrical interpretation of imaginary quantities put forward by Adrien-Quentin (Abbé) Buée (1748-1826) and John Warren (1796-1852) in the preface to his *Treatise on Algebra*. ‘The first attempt which I can find of an interpretation of the meaning of such quantities was given by M. Buée [in] a Memoir which contains some original views on the use and signification of the signs of Algebra, though presented in a very vague and unscientific form [...] At a much later period [...] the work of Mr. Warren [...] appeared [...] Mr. Warren has completely succeeded in giving an interpretation of the roots of unity, when attached to symbols which denote lines in Geometry, or any quantities which such lines may represent: in doing so however he has adhered strictly to the practice of all writers on Algebra, in making the interpretation govern the results and not the results the interpretation’. Peacock, *Treatise* (note 3), xxvii-xxviii.

the Application of Algebra to Geometry, which appears to have been published before 1830.\textsuperscript{35}

Woodhouse and Peacock’s definition of algebra as a science of symbols to which also belonged the ‘imaginaries’ played an important role in freeing mathematics from its reliance on geometrical methods. At the same time, it produced ‘a conservatism concerning [imaginary] numbers, precisely because by denying [them] anything but a formal significance, it protected the informal background’, the suggesting sciences of arithmetic and geometry, ‘from critical examination’.\textsuperscript{36} The profundness of this statement must be sought in the fact that it exposes a feature of Peacock’s symbolical algebra that can be identified in hindsight as contributing to the limitedness of its level of abstraction, namely the importance of the geometrical representation of imaginaries for the further generalization of algebra.

For Gregory and Walton it proved not only that, in so far as it had at least two models, symbolical algebra was more than a generalization of arithmetic, but also that it would be arbitrary not to use $\sqrt{-1}$ as a ‘meaningless symbol with the understanding that its geometric[al] interpretation could follow and, to an extent, vindicate its [algebraical] manipulation’.\textsuperscript{37} It was against this back-

\begin{footnotesize}
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\item Peacock seems to implicitly refer to this textbook when he spoke of ‘my system of Algebraic Geometry’ in the preface of the \textit{Treatise on Algebra}. Peacock, Treatise (note 3), xxxiv. Although it does not mention its exact date of publication, the following passage from the \textit{Report of the Council to the Thirty-Ninth Annual General Meeting of the Astronomical Society} is worth quoting: ‘[I]n 1819 Mr. Peacock was Moderator [of the Cambridge Tripos]. All the chief actors in producing [the change of the curriculum] have lived to see their work fully done, and their country in full communication with all the world after more than a century of nearly complete exclusion. Mr. Peacock subsequently published an anonymous Syllabus of Trigonometry and Algebraic Geometry [...] In 1826 he published in the \textit{Encyclopaedia Metropolitana} his historical article on Arithmetic [and in] 1830 appeared the first of his two works on Algebra’. Royal Astronomical Society, ‘Report of the Council to the Thirty-Ninth Annual General Meeting of the Astronomical Society’ (1859), 126. This historical fact suggests that the theoretical application of symbolical algebra not ‘back to’ arithmetical algebra, but to geometry in volume II of the second version of the \textit{Treatise} of 1845 could not have been such ‘a major step away from the \textit{Treatise} of fifteen years earlier’.
\item Fisch, ‘Emergency’ (note 1), 174.
\item O’Neill, ‘Formalism’ (note 32), 358.
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ground that it was proposed to extend the geometrical conceptions of plane curves to include imaginary values as branches in planes perpendicular to the $xy$ plane. Euclid’s *Elements* was accepted as the definitive treatment of the subject matter of geometry, and Gregory and Walton were, thus, confronted with the awkward task of having to represent new forms – such as the imaginary branches of plane curves – in terms of Euclidean space. Because the ideas with which geometry was said to be concerned, namely magnitude (linear, plane and solid) and direction, were to correspond to those of arithmetic, that is, could not but be of a quantitative nature, *uninterpretability* already occurred in the case of ‘a solid being extended in three dimensions’. Rather than accepting any radical changes of classical views about space that were suggested by the uninterpretable new forms, Gregory and Walton expressed their hope that an interpretation within Euclidean space would eventually become available.

While the role of the geometrical interpretation of imaginaries did not influence the early symbolical algebraists Woodhouse and Peacock, it was important to the abstract algebraists. As a result of his attempt to come to terms with John Warren’s *Treatise on the Geometrical Representation of the Square Root of Negative Quantities* Hamilton searched for the three-part numbers, or ‘triple algebra’, that would do for three-dimensional space what the imaginaries of ‘double’ or ‘coordinate’ algebra had done for two-dimensional space. The extension of imaginaries to three-dimensions led Hamilton to the discovery of ‘quaternions’, numbers with one real and three imaginary parts which were con-

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structured out of the (ordinal) number-couples on which he said that the science of algebra was grounded.\textsuperscript{41} Both features were revolutionary compared to the then present state of British algebra not only because the resulting algebra was the first example of a non-commutative system, but also because it was formulated in four dimensions. Hamilton, Cayley and other abstract algebraists subsequently took up the task of constructing higher-dimensional geometries and the numbers that were needed to represent and work algebraically with them. If the publication of their work was a testimony to the freedom of algebra suggested, but never exercised by, Peacock and Gregory, its extreme abstractness was premised on a radically different form of abstraction, namely one in which, for example, imaginaries were constructed as generalizations from intuitively existing mathematical objects.

The structure of the paper is as follows. A brief biographical account of Gregory and Walton is provided in section 2. After recapitulating, in section 3, the central features of Peacock’s symbolical algebra – written, as it was, in response to the problem of negative and imaginary quantities – Gregory’s renovation of symbolical algebra is analyzed in detail in section 4. The aim of this central section of the paper is to show in what sense this algebraical work was inspired by geometrical considerations or, more specifically, how it bore upon Gregory and Walton’s contributions to ‘algebraical geometry’ and ‘geometrical algebra’.\textsuperscript{42} Section 5 returns to the historical, theoretical and methodological reflections presented in this long first section; it discusses the limits of symbolical algebra in light of the criticism of the abstract algebraists formulated, as it partly was, in response to what they perceived as the shortcomings of the views on algebra and geometry of the (younger) symbolical algebraists. The paper concludes by expressing the hope that it has been able to contribute not only to a further exploration of the ‘English trouble-shooting’\textsuperscript{43} in the period between the publication of Peacock’s \textit{Treatise on Algebra} in 1830 and Hamilton’s construction of the first non-commutative algebra in 1843, but also to the demonstration of the influence of geometrical considerations on the development of abstractness in British algebra.

\textsuperscript{41} Hamilton’s work on algebra is briefly discussed in section 5.2
\textsuperscript{42} This distinction will be explained in the introduction to section 4.
\textsuperscript{43} Fisch, ‘Emergency’ (note 1), 254.
2. Duncan Farquharson Gregory and William Walton

2.1 Gregory and Walton: remarks on their life and work

Gregory, the great-great-grandson of the famous James Gregory (1638-1675) was born in Edinburgh on April 13, 1813 as the youngest son of James Gregory (1753-1821) and Isabella Macleod’s (1772-1857) eleven children. After entering Cambridge in 1833, he ranked fifth Wrangler in 1837 and became a fellow of Trinity in 1840. All but one of his many publications on topics ranging from plane geometry, differential and integral calculus, (define) multiple integrals and, most influentially, the ‘calculus of operations’, appeared in the journal CMJ, of which Gregory together with Samuel S. Greatheed (1813-1887) and Archibald Smith (1813-1887), was a founding editor. In his role as editor, Gregory, the mentor, was important to the early academic careers of William Thomson (Lord Kelvin) (1824-1907) and Boole. Gregory was considered for the mathematics chair at the University of Edinburgh in 1838, a professorship in mathematics at the University of Toronto in 1841 and, somewhat later, for the natural philosophy chair at Glasgow University, but he was never named to a position. After having suffered a first attack of illness in 1842, Gregory left Cambridge in the spring of 1843 ‘never to return again’. He died on February 23, 1844 at the age of thirty.


47 Ellis, ‘Biographical memoir’ (note 44), xxii.
Around 1842, Gregory, while working as a Tripos moderator and tutor, had begun to write a textbook on geometry entitled *Treatise on the Application of Algebra to Solid Geometry* which, shortly after his premature death, was completed by his friend William Walton who also edited Gregory’s collected *Mathematical Writings and Examples of the Processes of the Differential and Integral Calculus*. Walton himself had come into residence as a Pensioner of Trinity College in January 1831 and was the eighth Wrangler of the 1836 Tripos and Fellow of Trinity Hall, 1868-1885. He was the (co-)editor of several numbers of the *CMJ* in 1840 and 1844-1845 to which he contributed numerous papers. During his career, in which he, in his early years, worked as a private tutor and became an Honorary Fellow of Trinity Hall and Magdalene, Walton published some five books—on theoretical and elementary mechanics, ‘plane coordinate geometry’, differential calculus and theoretical hydrostatics and hydrodynamics—and two so-called collections of problems—one concerning statics and dynamics and another on hydrodynamics and optics—designed for the ‘candidates for honours’ at Cambridge. He also contributed to the volumes

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with solutions of ‘problems and riders’ of the Senate-House examinations of 1854 and 1857.51

2.2 Memoirs versus the CMJ

The CMJ was founded, in 1837, with the aim of publishing both short papers written by upcoming mathematicians that might not otherwise find a publisher as well as translations of important papers from the memoirs of foreign academies.52 The editors considered mathematics to be a ‘progressive subject, one which was open to change, expansion, and development, rather than as a fossilized branch of certain knowledge suitable only for the Tripos examination’.53 Similarly to the first generation reformers of British mathematics – the members of the Analytical Society Babbage, Herschel and Peacock – Gregory, Greatheed and Smith and other regular contributors, such as the later editors of the CMJ, the ‘Cambridge stars’54 Robert Leslie Ellis (1817-1859) and Walton, disseminated their ideas through their journal and influenced the Cambridge curriculum in their capacity as (junior) moderators setting the ‘problem papers’.

The method for which the CMJ became famous – that of the separation of symbols of operations from symbols of quantity, also known as the ‘calculus of operations’55 – had first appeared in the pages of the Memoirs of the Analytical Society. While French mathematicians such as Lagrange, Louis François Antoine Arbogast (1759-1803) and Augustin-Louis Cauchy (1789-1857) admired the method as ‘a tool for discovery and verification’, the members of the Analytical Society considered it as a rigorous technique proof and used the method to reach solutions to analytical problems and theorems. ‘As they had proven before with their adoption of the Lagrangian, algebraic[al] calculus’, rather than Cauchy’s limit-based calculus, ‘British mathematicians accepted and developed some products of French mathematics that the French [...]
eschewed'.\textsuperscript{56} Babbage, Herschel and Peacock turned away from further developing these meta-mathematical commitments in the course of the 1820s, but, even though it 'once again left the British isolated'\textsuperscript{57} from Continental developments, the calculus of operations was revived by the new generations of British reformers associated with the \textit{CMJ}.\textsuperscript{58} The first number contained a paper written by Greatheed, who had published the first paper on the topic after Herschel’s series of articles from 1814-1822, in which the calculus of operation was taken up in a spirit critical of Peacock’s claim that it is ‘the form of the formulas which is basic […] rather [than] the rules governing their laws of combination’.\textsuperscript{59} After having read the paper, Gregory soon devoted himself to spreading ‘the gospel of the calculus of operations’.\textsuperscript{60} Between the years 1839-1841, Gregory contributed numerous papers to the \textit{CMJ} in which he defended the legitimacy of the method. The main significance of this was that the attempt to generalize its algebraical properties and to put them on a logical basis led him to give to ‘its working-power a secure and philosophical foundation’.\textsuperscript{61} The promotion of the calculus of operations took the form of an appeal for its applicability in pure and applied (or ‘mixed’) mathematics; Gregory showed that it was a useful theoretical tool for pure mathematics in so far as it allowed for the establishment of the ‘real nature’ of algebra, and Walton, Smith, Greatheed and Gregory himself applied it to problems in, for example, geometry, mechanics and theories of heat, electricity, magnetism and so on.\textsuperscript{62}

The ‘new English school’ not only distanced itself from the ‘Cambridge agenda’ of the Analytical Society as regards the utility of the ‘calculus of operations’.\textsuperscript{63} Another, more profound, difference pertained to their views on analytic and synthetic mathematics. Whereas the members of the Analytical Society

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  \item\textsuperscript{56} Despeaux, ‘“Very full of symbols”’ (note 5), 53.
  \item\textsuperscript{57} Niccolò Guicciardini, \textit{The Development of the Newtonian Calculus in Britain 1700-1800} (Cambridge, Cambridge University Press, 1989), 138.
  \item\textsuperscript{58} As Fisch, ‘Creative indecision’ (note 3) shows, the members turned away for different reasons and in strikingly different ways.
  \item\textsuperscript{59} Koppelman, ‘The calculus of operations’ (note 29), 210-211.
  \item\textsuperscript{60} Despeaux, ““Very full of symbols”’ (note 5), 56.
  \item\textsuperscript{62} See Despeaux, ““Very full of symbols”’ (note 5), 54-67.
  \item\textsuperscript{63} See Smith & Wise, \textit{Energy} (note 4), 150.
\end{itemize}
dismissed the tradition of synthetic mathematics in favour of pure analytics and, thus, considered ‘geometry and geometrical demonstration as contrary to [mathematics’] ultimate objects’,\(^{64}\) the young mathematicians of the CMJ committed themselves to the task of ‘bridging the gap between the old mathematics and the new’.\(^{65}\) More specifically, even though both generations upheld that algebra provided the route to a rigorous foundation of calculus, the fluxional notation and geometrical methods were banned in the Memoirs,\(^{66}\) but the CMJ had recourse exactly to the (quasi-analytical and non-algebraical) fluxional analysis – and this in two contexts. Firstly, Gregory, probably under the influence of the Scottish Newtonians,\(^{67}\) demonstrated that geometry could be pursued algebraically on the basis of the idea that geometrical figures are generated from the operation of transference (of a point) in one direction. Secondly, Gregory ‘never wrote explicitly on the foundations of calculus’, but from the features of his Examples in the Processes of the Differential and Integral Calculus it can be inferred that he was not in full agreement with Lagrange’s algebraical approach to calculus.\(^{68}\) In fact, given that the book was written from the perspective of the fluxional method and that Cauchy’s limit-based calculus – which, in contrast to Lagrange’s work, did not explicitly emphasize ‘the need to separate the calculus from geometry’\(^{69}\) – was formulated in terms of

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\(^{64}\) Enros, ‘Analytical society’ (note 2), 28.

\(^{65}\) Davie, Democratic Intellect (note 23), 154.


\(^{68}\) Allaire and Bradley observe that Examples ‘functioned equally well whether the student’s understanding of first principles was algebraic in flavor [or] used limits […] Indeed, someone who had learned calculus using the Newtonian doctrine of fluxions would have benefited from Gregory’s text’. Allaire & Bradley, ‘D.F. Gregory’s contribution’ (note 5), 396.

\(^{69}\) Craig G. Fraser, ‘Calculus as algebraic analysis: some observations on mathematical analysis in the 18th century’, Archive for History of Exact Sciences, 39 (1989), 317-335. For a detailed account of Cauchy’s approach to the calculus see, for example, Judith V. Grabiner, The Origins of Cauchy’s Rigorous Calculus (Cambridge, MA, The MIT Press, 1981).
Newtonian-style fluxions,\textsuperscript{70} Gregory does not seem to have straightforwardly opposed this approach. It is, of course, true that Gregory’s use of ‘certain algebraic techniques, referred to as ‘symbolical algebra, [which] can be traced back to Lagrange’, suggests ‘a Lagrangian conception of first principles’.\textsuperscript{71} But when it is taken into account that it was by means of geometrical operations, treated according to the fluxional method, that Gregory distinguished algebra from arithmetic in the first place, it is possible to ascertain that his presentation of symbolical algebra as the foundation for calculus was not Lagrangian \textit{per se}.\textsuperscript{72}

For the purposes of this paper, these considerations suffice as a description of the differences between the mathematicians of the Analytical Society and the mathematicians associated with the \textit{CMJ}. Their differences clearly problematize the claim that ‘by 1830 the fluxionary notation [and method] and the emphasis on geometric arguments in the calculus had […] disappeared’.\textsuperscript{73} It is interesting to observe not only that Gregory’s notion of transference (of a point) in one direction provided the context out of which the ‘new analytical’ or ‘symbolical’ geometry of the abstract algebraists emerged,\textsuperscript{74} but also that Hamilton himself directly referred to Newton’s fluxions to lend credence to


\textsuperscript{71} Allaire & Bradley, ‘The contribution of D.F. Gregory’ (note 5), 397.

\textsuperscript{72} See Davie, \textit{Democratic Intellect} (note 23), 164.

\textsuperscript{73} Koppelman, ‘The calculus of operations’ (note 29), 176.

\textsuperscript{74} For example, in his 1852 plea for the inclusion of ‘new geometry’ in the Cambridge curriculum, Ellis noted that it ‘seems to be little studied in the University yet the method of which it makes so much use, namely, the generation and transformation of figures by \textit{ideal motion}, is more natural and philosophical than the (so to speak) rigid geometry to which our attention has been confined’. Robert Leslie Ellis, ‘Evidence on mathematical studies and examinations. Answers from Robert Leslie Ellis, Esq., M.A., late Fellow of Trinity College’, in \textit{Cambridge University Commission. Report of Her Majesty’s Commissioners Appointed to Inquire Into the State, Discipline, Studies and Revenues of the University and Colleges of Cambridge; Together with the Evidence, and an Appendix} (London, W. Clowes and Sons, 1852), 222-226, 224, my emphasis. See also Richards, ‘Projective geometry’ (note 39).
his idea of time, or ‘continuous progression’, as the basis of algebra. When it comes to the influence of geometrical considerations on the ‘post-Peacockian’ developments in British algebra this seems to be yet another case in point.

3. The generalization of Peacock’s symbolical algebra

3.1 Peacock’s symbolical algebra

Peacock’s Treatise on Algebra of 1830 and his report on analysis to the British Association of 1833 were ‘jointly responsible, almost single-handedly, for the foundational debate on the nature of mathematics in general, and algebra in particular’ of the 1830s. It is well-established that the symbolical algebra developed in these works was the result of an attempt to steer between two opposed approaches to algebra; on the one hand, the ‘generalized arithmetic’ approach of Francis Maseres (1731-1824) and William Frend (1757-1841), and on the other hand Babbage’s ‘pure analysis’ approach.

The fact that Peacock responded to Maseres and Frend indicates that his system was put forward with the aim of solving the problem of negative, and hence imaginary, numbers and roots of equations that was widely discussed in the late eighteenth- and early nineteenth-century. Both writers had argued that the lack of a satisfactory definition of these numbers rendered algebraical results involving them meaningless and, thereby, ‘brought into question the very legitimacy of algebra’s standing as a science’. Maseres and Frend’s proposal was to exclude them completely and to reduce algebra to ‘Universal Arithmetick’ in which ‘symbols stood only for nonnegative numbers and signs denoted strictly arithmetical operations’. The most famous contributions to the defenses of negative and imaginary numbers as legitimate math-

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75 Hamilton wrote that Newton ‘whose revolutionary work in the higher parts of both pure and applied Algebra was founded mainly on the notion of fluxion which involves the notion of time’. Hamilton, ‘Theory of conjugate functions’ (note 25), 5-6.
77 Fisch, ‘Creative indecision’ (note 3), 140.
Mathematical entities were those of John Playfair (1748-1819), William Greenfield (?-1827), Adrien-Quentin (Abée) de Bueé (1748-1826) and Woodhouse. While Playfair, Bueé and Greenfield claimed that these meaningless numbers are ‘supported, rather by induction and analogy, than by mathematical demonstration’, Woodhouse argued that even though they cannot be demonstrated by means of ‘observations made on individual objects’, the numbers are intelligible in so far as it is possible to extend the formal rules governing the system of real numbers to ‘a system of characters of our own invention’.

Peacock admitted that the arguments of Maseres and Frend as to the inadequate justification of negative and imaginary numbers are ‘unanswerable, when advanced against the form under which the principles of algebra were exhibited in the […] works of that period, and which they have continued to retain ever since’. He, thus, agreed that what he himself called ‘arithmetical algebra’ was ‘the only form of [algebra] which is capable of strict demonstration, and which alone, therefore, [is] entitled to be considered as a science of strict and logical reasoning’. Together with the opponents of the numbers, Peacock found the root of the problematic status of algebra in the fact that it has ‘always been considered as merely such a modification of Arithmetic as arose from the use of symbolical language, and the operations of one science have been transferred to the other without any statement of an extension of their meaning and application’. At the same time, Peacock could not agree with Maseres and Frend’s conclusion to dispense with the great multitude of algebraic results and propositions, of unquestionable value and of unquestionable consistency with each other which were irreconcilable with such a system [or] not deducible from it. [This] made it necessary to consider negative and even impossible quantities

80 Woodhouse, ‘On the necessary truth’ (note 30), 90, 93
81 Peacock, ‘Report’ (note 76), 189-192.
82 Fisch, ‘Creative indecision’ (note 3), 162.
as having a real existence in algebra, however vain might be the attempt to interpret their meaning.\(^{84}\)

Given his aim of rewriting the first principles of algebra, Peacock was faced with ‘a cruel dilemma’ in the years between his preparation of the English edition of Sylvestre François Lacroix’s (1765-1843) *Traité Élémentaire de Calcul Différentiel et de Calcul Intégral* (1802) in 1816 and the publication of his *Treatise on Algebra* in 1830.\(^{85}\) On the one hand, Peacock dismissed Maseres and Frend’s ‘generalized arithmetic’ approach, which sacrificed the riches of traditional algebra ‘for the sake of preserving for mathematics a solid […] truth governed foundational system’. But, on the other hand, he also rejected his friend Babbage’s ‘pure analysis’, Lagrangian-inspired, approach which ‘drain[ed] analysis of all meaning and content for the sake of grounding mathematics anew by means of a system pertaining exclusively to pure form and process’.\(^{86}\)

Peacock’s solution was to view algebra as consisting of two antithetical parts – one, an ‘arithmetical algebra’ that corresponded to Maseres and Frend’s truth-governed ‘universal arithmetic’ and, the other, a Babbagian system of ‘symbolical algebra’ that included the negatives, imaginaries etc. The resulting problem of coming to terms with the relationship between the two algebras one again confronted Peacock with two options. Given his unwillingness to either view ‘arithmetical algebra’ as a ‘lower-level application of its symbolical counterpart’ or cut ‘symbolical algebra’ loose from such a ‘lower-level, subject-oriented’ science, Peacock opted for a twofold system in which, on the one hand, symbolical algebra was ‘shown to meaningfully depend upon, pertain to, if not actually derive from, a theory of number’ that, on the other hand, was to function as a ‘subordinate science’ the principles of which suggested those of symbolical algebra.\(^{87}\) Despite the fact that his central aim was to establish symbolical algebra as a science of ‘symbols and their combinations, constructed upon its own rules’,\(^{88}\) Peacock was forced to acknowledge the ‘watered-down,

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\(^{84}\) Peacock, ‘Report’ (note 76), 190-191, my emphasis.


\(^{86}\) Fisch, ‘Creative indecision’ (note 3), 162–163.

\(^{87}\) Fisch, ‘Creative indecision’ (note 3), 165.

\(^{88}\) Peacock, ‘Report’ (note 76), 194-195.
bottom-up dependency of symbolical on arithmetical algebra that would somehow allow to meaningfully ground the former in the latter'. More specifically, even though the construction of symbolical algebra was to be logically prior to arithmetic – such that its signs, symbols and laws of combination of operations are arbitrary and that their interpretation could follow, but does not precede, algebraic manipulation – Peacock, according to Fisch, demanded that its rules, or principles, are ‘so constructed as to ensure in advance their capacity to yield their arithmetic counterparts by interpretation’. Because of his concern for the applicability of symbolical algebra, Peacock made sure, by means of the ‘principle of the permanence of equivalent forms’, that the rules of symbolical algebra are not merely suggested by arithmetical algebra, but are ‘required a priori strictly to embody them’. It is this foundational principle of his twofold algebraical system that implied the denial of both the autonomy of symbolical algebra and the mathematical freedom to arbitrarily construct it that Peacock himself had proclaimed at the outset of his *Treatise*.

Taken together, Peacock’s symbolical algebra was nothing more and nothing less than a *generalization* of Maseres and Frend’s generalized arithmetic, namely one in which both arithmetical operations and symbols are generalized. This explains why it was by the problems it was thought to pose or, more specifically, by its formulation of the meta-mathematical issue of the relationship between arithmetical and symbolical algebra, ‘rather than by the example it was thought to set,’ that the work of Peacock provoked a younger generation to improve upon the symbolical approach to the science of algebra. What the contributions of the mathematicians associated with the *CMJ* made apparent was the possibility of generalizing Peacock’s presentation of symbolical algebra as a generalization from arithmetic by means of bringing geometry on the scene, albeit without thereby breaking symbolical algebra’s ties to these demonstrative sciences.

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89 Fisch, ‘Creative indecision’ (note 3), 166.
90 Fisch, ‘Creative indecision’ (note 3), 167.
91 Fisch, ‘Creative indecision’ (note 3), 167-168.
92 Fisch, ‘Creative indecision’ (note 3), 140.
3.2 Gregory’s generalization of Peacock’s symbolical algebra

In a paper read before the Royal Society of Edinburgh on May 7, 1838 and published in the Society’s Transactions in 1840, Gregory wrote the following:

I cannot take it on me to say that [my] views are entirely new, but at least I am not aware that any one has yet exhibited them in the same form. [T]hey appear to me to be important, as clearing up […] the obscurity which still rests on several parts of the elements of symbolical algebra. Mr. Peacock is, I believe, the only writer in this country who has attempted to write a system of algebra founded on a consideration of general principles […] Much of what follows will be found to agree with what he has laid down, as well as with what has been written by the Abbé Buee and Mr. Warren; but [I] think that the view I have taken of the subject is more general than that which they have done.\textsuperscript{93}

Although his definition of symbolical algebra as ‘the science which treats of the combination of operations defined not by their nature, that is, by what they are or what they do, but by the laws of combination to which they are subject’\textsuperscript{94} seemed strikingly similar to that of Peacock, Gregory’s reference to the combination of operations, rather than symbols, enabled him to formulate a more refined symbolical algebra that was critical of Peacock’s account.


\textsuperscript{94} Gregory, ‘Real nature’ (note 93), 208.
the symbols we use represent, we suppose the existence of classes of unknown operations subject to the same law.\textsuperscript{95}

Further developing the view that the algebraical laws of combination cannot be regarded as the results of generalizing from the laws of combination of arithmetical operations, Gregory assumed five classes of operations and defined operations as belonging to a class on the basis of ‘the laws of combination to which they are subject’. After having defined an algebraical theorem as the proof of certain relations between the different classes of operations expressed in symbols, $F$ and $f$ are put forward as the symbols – standing for operations of which the ‘nature’ is unknown – which are performed upon the symbols $(a, b, x, y$ and $h)$ for particular objects. Importantly, Gregory did not use $F$ and $f$ as notations for a function and, for him, the objects are either functions, quantities, geometrical figures or other ‘operators’. His procedure was to postulate the existence of certain (classes of) operations, to lay down the laws to which these are subject and, subsequently, to consider whether – or, actually, to show that – it is possible to find operations in branches of mathematics following the laws previously laid down.

### 3.2.1 Five classes of operations

Gregory put forward five classes of operations; circulating (or reproductive) operations, index operations, distributive and commutative operations and two unnamed classes of operations. In the first class, consisting of two classes of circulating operations, $F$ and $f$ are connected by the laws \[ 1 \] $FF(a) = F(a)$, \[ 2 \] $ff(a) = F(a)$, \[ 3 \] $Ff(a) = f(a)$, \[ 4 \] $fF(a) = f(a)$.\textsuperscript{96} An interpretation for these laws is found in the two arithmetical operations of addition and subtraction, the symbols of which are ‘+’ and ‘−’, and in two geometrical operations the first of which, namely ‘the turning of a line, or rather transferring of a point, through a circumference’, corresponds to ‘+’ and the second, ‘the transference of a point through a semicircumference’, corresponds to ‘−’. Gregory concluded that ‘whatever we are able to prove of the general symbols ‘+’ and ‘−’ from the [algebraical] laws to which they are subject, without considering the nature

\textsuperscript{95} Gregory, ‘Real nature’ (note 93), 208–209.
\textsuperscript{96} The following account draws on, and quotes from Gregory, ‘Real nature’ (note 93), 209–215.
[i.e. interpretation] of the operations they indicate, is equally true of the arithmetical operations of addition and subtraction, and of the [given] geometrical operations’. In a similar vein, the idea of their being a ‘real analogy’ between, or ‘identity of nature’ of, ‘+’ and ‘−’ in arithmetic and geometry is dismissed in favor of the view that their relation is due to ‘the fact of their being combined by the same laws’.

In the second class, namely that of index operations subject to the laws \[ f_{m+n}(a) = f_m(a)f_n(a) = f_{mn}(a) \] an interpretation is found when \( m \) and \( n \) are either integers or fractions. But, Gregory asked, 

[w]hat meaning […] is to be attached to such complex operations as \((+)^m\) or \((−)^m\)? When \( m \) is an integer […] we see […] that the operation \((+)^m\) is the same as +, but \((−)^m\) becomes alternately the same as + and as −, according as \( m \) is odd or even, whether they be the symbols of arithmetical or geometrical operations. So far there is no difficulty. But if it be fractional, what does \((+)^m\) or \((−)^m\) signify?

At this point, Gregory referred to his earlier emphasis on the difference between arithmetical and geometrical operations. For where, for example, \((−)^m\) cannot be interpreted when \( m \) is a fraction with an even denominator, ‘geometry readily furnishes us with operations which may be represented by \((+)^{1/m}\) and \((−)^{1/m}\), and which are analogous to the operations represented by ‘+’ and ‘−’. The one is the turning of a line through an angle equal to \(\frac{1}{m}\)th of four right angles, the other is the turning of a line through an angle equal to \(\frac{1}{m}\)th of two right angles’.

Gregory introduced the third class of operations, which includes operations such as differentiation and integration, by stating that they conform to, firstly, the distributive law \[ f(a) + f(b) = f(a+b) \] and, secondly, the commutative law \[ ff(a) = ff(a) \]. It is a geometrical operation of ‘transference to a distance measured in a straight line’that provides the interpretation for the distributive law; let \( x \) be the symbol for any geometrical figure and \( a(x) \) the representation of the transference of a figure, so that \( a(x) + a(y) = a(x+y) \) renders the operation \( a \) distributive. Similarly, for the compound, or complex, operation \( b(a(x)) \); let \( x \) (symbolically) represent a point, so that \( a(x) \) stands for the result of the operation of the ‘transference of a point to a rectilinear distance, or the
tracing out of a straight line’ then \( b(a(x)) \) is the ‘transferring of a line to a given distance from its original position’ in the form of a parallelogram. Because the same result is obtained, i.e. the identical parallelogram is traced out, in the case of \( a(b(x)) \), the operations are commutative: \( a(b(x)) = b(a(x)) \).

The interpretation of the (unnamed) fourth class, defined as that in which the operations are subject to the law \( f(x) + f(y) = f(xy) \), is established on the basis of its correspondence with the arithmetical logarithm in the case that \( x \) and \( y \) are numbers. In any other case, that is, when \( x \) and \( y \) are other objects, or things, the function expressing the symbolical law ‘will have a different meaning’.

The fifth, and last, class of operations consists of two operations connected by the following two laws: 

1. \( aF(x+y) = F(x)f(y) + f(x)Ff(y) \) and 
2. \( af(x+y) = f(x)ff(y) - cF(x)Ff(y) \).

Now, if these laws only function in algebra ‘as abbreviated expressions for certain complicated relations between the first three classes of operations’, about the algebraical theorem proved of this (unnamed) class Gregory noted that in arithmetical algebra the expression \( \cos x + (−)^{\frac{1}{2}} \) appearing in the theorem has no meaning since it involves \( (−)^{\frac{1}{2}} \) (Gregory’s notation for the imaginary operation \( i \)). However, in geometry this operation can be interpreted as follows:

If \( a \) represents a line, and \( acosx \) represents a line bearing a certain relation in magnitude to \( a \), and \( asinx \) a line bearing another relation in magnitude to \( a \), then \( acosx + (−)^{\frac{1}{2}}sinx \) [implies] that we […] measure a line \( acosx \) [and] another line \( asinx \) [which] in consequence of the […] operation \( (−)^{\frac{1}{2}} \) this [latter] line is to be measured not in the same direction as \( acosx \), but turned through a right angle.

3.2.2 The (new) features of Gregory’s symbolical algebra

The advance towards abstraction within the framework of symbolical algebra found in Gregory’s 1838/1840 article arose out of the criticism of, and improvement upon, Peacock’s system that was inherent in Gregory’s treatment of the five classes of operations. Where his criticism took the form of an alternative for the ‘principle of equivalent forms’, the improvement was that even though the notion of a suggesting science was upheld it was not limited to one particular ‘science’. Given that what can be proved for a class generally, holds for
all specific operations in that class, Gregory was able to maintain two things. Firstly, that in so far as a theorem is a symbolized relation between different classes of operations it holds that ‘if we can show that any operations in any science are subject to the same laws of combination as these classes, the theorems are true of these as included in the general case; provided always that the resulting combinations are all possible in the particular operation under consideration’.97 Secondly he could maintain that ‘if the combination of two operations in a science is not possible in that science, then the previous statement cannot hold, e.g. \(\sqrt{-}\) in arithmetic’.98 These two points, which together demonstrate that a theorem ‘carries from one science to another not because of any analogy between the operations nor any similarity in the operations, but because the operations involved are subject to the same laws of combination’,99 are offered as an alternative for Peacock’s ‘principle of equivalent forms’. The generalization from the presentation of arithmetical algebra as the suggesting science of symbolical algebra follows from the fact that, firstly, in the case of several classes (e.g. the first class) the operations are suggested by both arithmetic and, for example, geometry and, secondly, even though \(\sqrt{-}\) and \((-)^{\frac{1}{2}}\) are not possible in arithmetic, they can be interpreted in geometrical terms. Taken together, Davie seems justified in describing the ‘newness’ of Gregory’s symbolical algebra as follows:

What Gregory did was to set side by side geometrical operations and arithmetical operations, comparing them in respect to their likeness and their unlikeness, and finding that there are [...] correspondences between the two in the midst of [...] their differences. He then went on to claim that symbolical algebra was concerned with what arithmetic and geometry have in common, and involved an abstraction whereby one left out of account the [...] peculiarities inseparable from each, and regarded only their points of agreement.100

It must be added that the introduction of geometry or, more specifically, geometrical operations allowed Gregory to loosen symbolical algebra’s ties to arithmetic. For it was precisely the fact that there are operations which cannot

97 Gregory, ‘Real nature’ (note 93), 208.
100 Davie, Democratic Intellect (note 23), 164.
be given an arithmetical meaning, but which can be interpreted in geometrical terms, that suggested that symbolical algebra was an abstraction from both of these quantitative sciences.

Because ‘general symbolical Algebra’ resulted from the attempt to treat ‘the [algebraical] symbols of operation like those of quantity’, Gregory acknowledged that it appeared ‘as if [they] were drawn from analogy’.\(^\text{101}\) However, he insisted that this ‘is not really the case, [since] the reasoning on which we proceed is perfectly strict and logical’.\(^\text{102}\) In his ‘On the impossible logarithms of quantities’, Gregory also warned his readers to ‘avoid the confusion which [is] caused by the introduction into general [...] Algebra of symbols limited in their signification’,\(^\text{103}\) that is, symbols that recall arithmetical notions. Before the publication of the paper ‘On a difficulty in the theory of algebra’,\(^\text{104}\) Gregory sometimes seems to have slipped, characterizing the operations of arithmetic as the ‘natural’, ‘ordinary’ or ‘usual’ subject of algebra. For example, in ‘On the elementary principles of the application of algebraical symbols to geometry’, Gregory characterized his ‘On the real nature of symbolical algebra’ as a discussion of ‘the principles on which certain symbols of operation become subject to the same rules of combination as the symbols of numbers, which are those usually handled in Algebra’.\(^\text{105}\) Gregory explicitly admitted his earlier ‘Peacockian doubts’ about abandoning symbolical algebra’s ties to arithmetic in a paper of 1843. Here, he wrote that

I have in previous paper held, in common I believe with every other writer, an opinion which a more attentive consideration induces me to think erroneous [i.e.] that the symbols ‘+’ and ‘−’ signify primarily addition and subtraction, and that any other meanings which we may attach to them must be derived from the fundamental [arithmetical] significations. The theory which I have now to maintain is [...] that the


\(^{102}\) Gregory, ‘Linear differential equations’ (note 101), 30.


\(^{105}\) Duncan Farquharson Gregory, ‘On the elementary principles of the application of algebraical symbols to geometry’, *Cambridge Mathematical Journal*, 2 (1839), 1-9, 1.
symbols + and – do not represent the arithmetical operations of addition and subtraction; and that though they were originally intended to bear these meanings, they have become really the representatives of very different operations.\textsuperscript{106}

And it is indeed the case, as Koppelman noted, that the ‘contrast between the work of Peacock and Gregory is well illustrated by their treatment of the symbols + and −’.\textsuperscript{107} Gregory’s central argument was that the four (algebraical) laws of combination of the symbols + and −, namely [1] \(+a = +a\), [2] \(-a = -a\), [3] \(+a = -a\), and [4] \(-a = +a\), do not correspond to or represent the operations of addition and subtraction. One of the examples that Gregory offered is that these two arithmetical operations are ‘inverse operations, whereas the second and third of the preceding [algebraical] laws are inconsistent with the idea that + and − are inverse symbols, the character of which is, that the one undoes what the other does’.\textsuperscript{108} As to the objection ‘that we do actually define + and − to be the symbols representing addition and subtraction, and therefore that they must represent these operations’, Gregory, referring to a work of Robert Murphy (1806-1843), responded by introducing wholly ‘new symbols to represent the operations of addition and subtraction’ so as to show ‘how the laws of combination of the operation of addition may be represented […] without the aid of a subsidiary symbol such as +’.\textsuperscript{109} ‘Our conceptions would be much clearer and our minds more free from prejudice’, Gregory concluded,

if we never used in the general science symbols to which definite meanings had been appropriated in the particular science. [But] practice has […] so wedded us to the use of the symbols + and − that we find it difficult to dispense with them, and still more difficult [to] avoid being misled by ideas drawn from arithmetical.\textsuperscript{110}

\textsuperscript{106} Gregory, ‘On a difficulty’ (note 104), 153.
\textsuperscript{107} Koppelman, ‘The calculus of operations’ (note 29), 216.
\textsuperscript{108} Gregory, ‘On a difficulty’ (note 104), 154.
\textsuperscript{110} Gregory, ‘On a difficulty’ (note 104), 156.
This particular ‘practice’ has prevented mathematicians not only from giving to a symbol such as ‘+’ an algebraical individuality as a symbol subject to certain laws of combination which [are] not those [of] the operation of addition’, but, thereby, also from recognizing that ‘+’ ‘can receive [an]other interpretation than that which was originally assigned to [it]’.\footnote{111} Importantly, it was at this point that Gregory put forward the statement that

\begin{quote}
there is no doubt that we can give these symbols [...] a geometrical interpretation, and it is the possibility of doing so which has occasioned the difficulties of the Theory of Algebra considered as something more general than Arithmetic, and which has led to the more extended views which in recent years have been taken of the subject.\footnote{112}
\end{quote}

Three observations must be made in light of this crucial statement. Firstly, it was the introduction of geometry into the debate about the nature of symbolical algebra that allowed not only for the questioning of Peacock’s view of the ties between algebra and arithmetic but also for the development of algebra as something more than a generalization of arithmetic. Secondly, because geometry was itself understood as the science of measurable extension in three-dimensional Euclidean space these ties were not broken, but merely loosened such that algebra was considered an abstraction from the similarities between arithmetical and geometrical operations and not as an abstract system. These (abstractionist- and fluxional-inspired) considerations, which formed the core of ‘algebraical geometry’, are discussed in section 4.1. Thirdly, when these two points are combined it becomes possible to grasp the central motivation for Gregory and Walton’s work on ‘geometrical algebra’; as Gregory wrote, once ‘we admit anything beyond [...] positive values of the variables, that is, [...] values wholly independent of the [arithmetical] symbol + [i.e. of addition] we must be prepared to consider the curve as existing in several planes’.\footnote{113} The fact that even though ‘symbolical geometry’ was suggested by the geometry-inspired generalization of symbolical algebra, in so far as this generalization itself resulted from an abstraction from the relationship between the sciences of

\footnotesize
\begin{itemize}
\item[\footnote{111}]{Gregory, ‘On a difficulty’ (note 104), 157.}
\item[\footnote{112}]{Gregory, ‘On a difficulty’ (note 104), 155, my emphases.}
\item[\footnote{113}]{Duncan Farquharson Gregory, ‘On the existence of branches of curves in several planes,’ \textit{Cambridge Mathematical Journal}, 1 (1839), 259-266.}
\end{itemize}
arithmetic and geometry Gregory and Walton did not develop it is reflected upon in section 4.2.

4. ‘Algebraical geometry’ and ‘geometrical algebra’

Before discussing their contributions to these two topics, it is worthwhile to draw explicitly distinguished the Cartesian method of algebraically expressing the known geometrical forms (‘algebraical geometry’) from the ‘inverse’ Cartesian method of geometrically interpreting, in the form of representations, algebraical equations (‘geometrical algebra’). This important, but not often noted distinction demands some clarification for it is somewhat confusing — and that for the following reason. In many of his papers written prior to 1852, Walton gave the topic of the (‘indirect’) geometrical signification of equations the name ‘algebraic(al) geometry’. Yet, De Morgan, in ‘On the signs + and – in geometry (continued), and on the interpretation of the equation of a curve’ of that particular year, proposed to call the use of geometry ‘in aid of algebra to assist in gaining representation of functions’ by the name of ‘geometrical algebra’ and reserved ‘algebraic geometry’ for the use of algebra ‘in aid of geometry to assist in gaining knowledge of forms’. Given that De Morgan’s reversal of Gregory and Walton’s earlier distinction was thereafter accepted by others, it will be retained in what follows.

114 See, for example, William Walton, ‘On the general interpretation of equations between two variables in algebraic geometry’, Cambridge Mathematical Journal, 2 (1840), 103-113.
115 Augustus De Morgan, ‘On the signs + and – in geometry (continued), and on the interpretation of the equation of a curve’, Cambridge and Dublin Mathematical Journal, 7 (1852), 242-251, 243.
116 See Richards, Mathematical Visions (note 38), 53-54.
4.1 The application of symbolical algebra to geometry: analytical or algebraical geometry

It was in an anonymously published paper of 1838, entitled ‘On some elementary principles in the application of algebra to geometry’ and signed ‘D’,\(^{117}\) that Gregory, thereby having recourse to traditional fluxional analysis,\(^{118}\) defined the ‘object’ to be represented by the algebraical symbols ‘+’ and ‘−’ as extension combined with direction, that is, as ‘fluent quantity’. In his own words, ‘the subject represented by the algebraical symbols is not geometrical extension, but rather extension combined with direction. It is not the distance between two points, but the distance of one point rightwards or leftwards of another’. The reason for this is that quantities of the [...] pure geometrical kind have no opposite relation of plus or minus to their two extremities; subtraction [leads to] the expunging of a line, and when the magnitude has once been entirely expunged, the operation can be carried on no longer. The negative sign cannot be applied to distance alone, but to distance or progress in a given direction [...] We say, therefore, that the quantities we have selected for the application of algebraical reasoning are essentially positive and negative, independently of any rules of addition and subtraction.\(^{119}\)

Where in ‘pure algebra’ it is the case that even though the negative sign of affection sign ‘represents something not implied in the primary arithmetical definition of subtraction’, it is ‘intimately connected with that operation and can have no meaning distinct from [it]’, in ‘applied algebra’ the case is different. Here, there are two meanings of the negative sign; firstly, ‘quantities of

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\(^{117}\) Duncan Farquharson Gregory, ‘On some elementary principles in the application of algebra to geometry’, *Cambridge Mathematical Journal*, 1 (1838), 74-77.

\(^{118}\) See Davie, *Democratic Intellect* (note 23), 154, 161, 166. Smith & Wise, *Energy* (note 4), 184. The Newtonian character of Gregory’s application of the calculus of operations to geometry must be sought in his treatment of the relation between process and operation or, more specific, of lines as being generated by moving points. Newton himself employed geometrical fluxional analysis precisely to treat curves as arising from the motion of a point and to formulate them in terms of quantities changing with time (‘fluents’). See note 23 for references to accounts of Newton’s mathematical method.

\(^{119}\) Gregory, ‘On some elementary principles’ (note 117), 75.
specific natures’ and, secondly, ‘operations’ the rules for the performance of which ‘shall not be limited, as in arithmetic, by the particular relations of the quantities, but shall be applicable in all cases’. This seems to suggest that it is applied algebra – in which ‘the independent sign of affection is the foundation, and not, as in Algebra, the consequence of the rule of operation’ – that allows, in principle, for the search for rules that are independent of the relations between arithmetical quantities.

Gregory’s ‘On the elementary principles of the application of algebraical symbols to geometry’ of 1839 opened with the statement that his ‘On the real nature of symbolical algebra’ was written with the goal of formulating ‘the principles on which various branches of science may be symbolized – that is to say, on which their study is facilitated by expressing the operations by means of symbols [...] Among [these] sciences [...] that of Geometry is the most important’. The paper further pursued the theory of the representation of direction by means of, what are now called, the symbols instead of the quantities or signs of affection ‘+’ and ‘−’ and introduced the ‘obscure’ and ‘little attended to’ theory of the representation by numbers of (linear, plane and solid) magnitude. Gregory, once again echoing fluxional ideas of his Scottish predecessors, proposed that the three geometrical magnitudes (lines, areas and solids) could be conceived of as the result of specific combinations of the operation of ‘transference in one constant direction’ (or ‘a’) of a ‘(subject)-point’ (or '.') in space – such that the compound symbol $a(\cdot)$ represents a straight line. Because ‘we have only to consider the combination of symbols of operation – the subject, being always the same, may be understood, and the symbol for it omitted’, $a$ represents a line of a given length and combined with another symbol $b$ representing transference of a point in another direction a parallelogram is obtained as soon as $b(a)$ is interpreted. Similarly, a ‘parallelopiped’ can be shown to result from the establishment of the meaning of $c(a(b))$.

After discussing ‘the means of representing symbolically the geometrical ideas of magnitude’, that is, of ‘solid Geometry’, Gregory continued by considering

120 Gregory, ‘On some elementary principles’ (note 117), 81–82.
121 Gregory, ‘On some elementary principles’ (note 117), 74-75.
122 Gregory, ‘On the elementary principles’ (note 105), 1.
123 See Davie, Democratic Intellect (note 23), 161.
124 Gregory, ‘On the elementary principles’ (note 105), 2.
the symbolization of the subject-matter of ‘plane Geometry’, namely direction, by means of so-called ‘rectilinear angles’

which affords us an easy means of symbolizing [for] by supposing a straight line to revolve round a point situate within it, we can make it generate any given angle. This [is] the operation which we shall express by a symbol, and the laws of which we are to investigate.\textsuperscript{125}

Taking the ‘complete revolution of the line, or revolution through four right angles’\textsuperscript{126} (\(\lambda\)) as the standard angle resulting from the operation to be symbolized, Gregory showed how ‘(sub)-multiples of this angle can be produced ‘by performing the operation a certain number of times or by performing a certain part of the operation’, that is, by attaching ‘integer’ and ‘fractional’ indices to \(\lambda\). Gregory concludes that ‘by the use of the simple algebraical notation of indices, joined to the geometrical operation of turning a line through a given angle, we are able to express the operation of turning a line through any angle whatsoever, and so to express all relations of directions between lines situate in a plane’. Although it is ‘not necessary that it should admit of any other geometrical interpretation’, \(\lambda\) can also be interpreted such that it represents the direction of a (two-dimensional) plane, rather than the direction of a (one-dimensional) line. The fundamental reason for this is that

the operation of turning the area completely round is \textit{subject to the same law} as that of turning the line. Hence it follows, that these two operations may be represented by the same symbol so that if in any process of Analytical Geometry we find the symbol […] which was originally applied to the symbol for a line, ultimately applied to the symbol for an area, we are able to interpret it.

Gregory himself explicitly acknowledged that his presentation of the ‘obscure’ theory of the algebraical representation or symbolization of geometry in terms of a one-dimensional operation is of necessity unable to account for three-dimensional (‘solid’) figures:

\textsuperscript{125} Gregory, ‘On the elementary principles’ (note 105), 5.
\textsuperscript{126} The following draws on Gregory, ‘On the elementary principles’ (note 105), 5-7.
If we combine more symbols than these [three], we find no geometrical interpretation for the result. In fact, it may be looked on as an impossible geometrical operation; just as \(\sqrt{-1}\) is an impossible, arithmetical one. For a solid, having equal relations to the three dimensions of space, cannot have any relation with one particular direction, which refers only to one dimension, and direction is essentially involved in the operation we have been considering.

Gregory did not reflect on the limits of his algebra of two-dimensional space of coordinate geometry and its specific inability to ‘mathematize’ the three-dimensional world. The reason for this was not only that the sole purpose of his paper was to symbolically express the known figures of Euclidean geometry, but also that the symbols representing them were formulated such that they embodied or ‘coincide algebraically’ with the rules of combination of the symbols for numbers. Because it is ‘solely from the previous knowledge which we have of the combinations of arithmetical symbols’ that ‘the study of Geometry may be facilitated by having its operations symbolized’, Gregory devoted the bulk of the paper to demonstrating that the symbols of algebraical geometry ‘are subject to the two laws of combination which characterize the symbols of number, [...] viz. the commutative law and the distributive law’. Following his traditional ideas on geometry, Gregory’s ‘algebraical geometry’ was aimed at the reduction of ‘geometrical investigations to processes of arithmetical calculation’ – i.e. it aimed to show that geometrical ideas, all of which resulted from the (‘fluxional’) operations governing points moving in space, could by represented by arithmetical symbols expressing the laws of combination of symbolized arithmetical operations. This did not mean, of course, that geometry could be reduced to arithmetic, but rather that ‘algebraical geometry’ was both premised on and concerned with what these two demonstrative sciences had in common.

Following the communis opinio among the members of his generation of mathematicians associated with the CMJ,\(^{127}\) Gregory emphasized that if the application of algebra to any other science consists of symbolizing its operations in terms of algebraical symbols, the interpretation of the results ‘is out of

the province of Algebra and belongs to the science [itself]'\textsuperscript{128} Several points must be made about the operations which are said to govern the two parts of geometry, namely, in the case of ‘solid geometry’, ‘transference in one direction of a point’ and, in the case of ‘plane geometry’, ‘a straight line revolving round a point’. Firstly, even though Gregory introduced the question of the determination of the ‘algebraical nature’ of the symbols representing these operations as a separate topic, it is clear that these symbols are chosen so that they coincide algebraically with the symbols for numbers. For example, Gregory himself admitted that ‘[w]e might […] use symbols representing different kinds of transference [but] having done so, we should derive no assistance from any previous labours’\textsuperscript{129} concerning arithmetic. This suggests, secondly, that the limitation that accompanied his choice for the specific operations actually implied the geometrical uninterpretability of the combination of (more than) three symbols. For example, ‘expressions as $+abc$ or $−abc$ are […] uninterpretable consistently with the geometrical meaning we attach to the symbols $+$ and $−$’.\textsuperscript{130} Thirdly, the possibility of geometrically interpreting expressions currently uninterpretable was upheld such that it followed, on the one hand, that algebraical results were not accepted as such but connected to the farther progress of geometry and, on the other hand, that the idea of choosing symbols for geometrical operations that did not ‘coincide algebraically’ with those of arithmetic was hinted at, but not pursued.

### 4.2 The geometrical interpretation of algebraical equations – including impossibles: ‘geometrical algebra’

Gregory commenced his ‘On the existence of branches of curves in several planes’ with the following statement:

\[\text{In tracing a curve expressed by an equation [...] it is customary to make use of negative as well as of positive values of the variables, but to reject those which are usually called impossible or imaginary. This practice was allowable so long as it was supposed that impossible quantities had no meaning in geometry; but if we once admit the possibility of interpret-}\]

\textsuperscript{128} Gregory, ‘On the elementary principles’ (note 105), 2.
\textsuperscript{129} Gregory, ‘On the elementary principles’ (note 105), 5.
\textsuperscript{130} Gregory, ‘On the elementary principles’ (note 105), 9.
Referring to Buée’s paper of 1806 in which the author had attempted to explain the use of imaginary quantities in geometry, that is, to provide the ‘general interpretation of formulae in Analytical Geometry’, Gregory here drew attention to, by and large, four points. Firstly, in, what he came to call, the ‘geometry of Descartes’ (i.e., coordinate geometry) the geometrical equation involving two variables, \(x\) and \(y\), defining a (plane) curve, namely \(f(x,y)\), can only have positive values as long as it is considered as an arithmetical equation: ‘If we agree that the values of \(x\) are to represent lines measured from \(O\) along \(Ox\), and values of \(y\) lines measured from \(O\) along \(Oy\), we can by means of the arithmetical values alone of \(x\) and \(y\) determine the positions of all points within the angle \(xOy\)’. Secondly, it is clear that the equation, as a matter of arbitrary convention, can also be made to express a negative variable which, albeit being uninterpretable in arithmetic, can be interpreted, as Gregory himself had done, along the lines of the original definition of the symbols. Consequently, the aim of completely tracing a curve on a given plane includes that of extending the interpretation of the symbols so as to be able to let the equation \(f(x,y) = 0\) express the position of points in ‘all parts of the plane in which the axes \(Ox\) and \(Oy\) lie’. Thirdly, when, in turn, ‘−’ is not interpreted as the amount of geometrical motion, i.e. the measurement of the ‘length \(a\) in a direction opposite to that originally taken’, but, more generally, as meaning that \(-a\) means that the line \(a\) is to be turned round through two right angles, we are led to the [...] interpretation of such an expression as \((\pm a^n)^{\frac{1}{n}}\), viz. that the line \(a\) is to be turned round through the \(n\)th part of four right angles. This gives [...] a farther extension of the use of the equation \(f(x,y) = 0\); for, as the turning of a line through a given angle is not confined to any one plane, we are enabled to express by the equation to the curve the position of a point situated in any part of space.
This new interpretation is the result of an extension of the negative values of the variables $x$ and $y$ or, for that matter, of the fact that ‘once [we] admit anything beyond what are called positive values of the variables, that is, pure arithmetical values wholly independent of the symbol ‘+’, there is no [‘algebraical’] reason why we should confine ourselves to $-\text{ or } +\frac{1}{2}$’. The new interpretation led Gregory to consider curves having existence for ‘imaginary’ values and, in turn, defining (i.e. interpreting) these as those curves leaving the actual ‘plane of the axes [or] plane of reference’. At the same time, it was the recognition of the existence of branches of curves in several planes that inclined him to renounce the word ‘imaginary’ in the first place. For, as Gregory had it, ‘the word imaginary [has] so frequently been applied to the symbol $\sqrt{-1}$, that it has made ‘persons unwilling to believe that it could possibly admit of any interpretation’. Given that there is a geometrical interpretation of expressions such as $(-a^2)^{\frac{1}{2}}$, Gregory proposed to replace the notion of ‘imaginary’ quantities or operation for ‘impossible’ ones and to define these as being ‘uninterpretable in arithmetic’ – so that, in effect, his extending of the ‘geometrical conceptions of plane curves to include imaginary values as branches in planes perpendicular to the $xy$ plane’ was intimately connected to the recognition of the possibility of giving the symbols ‘+’ and ‘−’ a meaning irreducible to that of the arithmetical operations of addition and subtraction.

In order to trace a curve from its algebraical equation, Gregory’s general procedure was as follows; he solved the equation ‘with respect to one or other of the variables, if the solution be in a form which enables us to determine readily its value for different values of the other variable’, then assigned to ‘$x$ all positive values from $0$ to $\infty$’ and marked ‘those which make $y = 0$ [giving the points where the curve cuts the axis of $x$], $y = \infty$ [giving the infinite branches], or $y$ impossible [showing where the curve quits the plane of reference and giving the limits of the curve in that plane]’. His ‘On the existence of branches of curves in several planes’ was, essentially, dedicated to solving the equations of the ellipse and parabola and showing that these include an ‘infinite number of curves with infinite branches passing through the extremities of the axes’. Gregory also

135 Gregory, ‘On the existence of branches’ (note 113), 257.
136 Gregory, ‘On the existence of branches’ (note 113), 259.
137 Gregory, ‘Impossible logarithms’ (note 103), 229.
138 Richards, Mathematical Visions (note 38), 51.
139 Gregory & Walton, Treatise on Solid Geometry (note 48), 177.
briefly touched upon the more complicated curves of sines and cosines and the logarithmic curve so as to expose the ‘considerable interest attached to them in a geometrical point of view’.\(^\text{140}\) About the imaginary branches of curves existing outside the plane of reference he merely remarked that ‘[p]ractically [i.e. ‘geometrically’], little attention will be paid to [them], since the curves themselves do not come sufficiently \textit{under our eye} to attract much interest’. As may be expected from the considerations put forward in section 4.1, Gregory concluded that they ‘ derive their chief value from their bearing on the General Theory of the Science of Symbols’.\(^\text{141}\)

4.2.1 \textit{Walton’s contributions to geometrical algebra: 1840-1841}

It was Walton, the ‘diehard geometer’,\(^\text{142}\) who, in several papers published between 1840 and 1841, further pursued the topic first suggested by Gregory, namely that of the geometrical signification of algebraical equations involving two or three variables generally interpreted, that is, of three-dimensional forms resulting from equations including imaginary values for the variables. Walton himself summarized the central aim of his writings by remarking that even though it is ‘to a certain extent arbitrary what interpretation we give to our algebraical equations’, it is the case that ‘the greatest advantage is gained when we adopt the most general methods, and when every algebraical symbol has its appropriate geometrical representation’.\(^\text{143}\) Reflecting in somewhat more detail on the objective of attaining generality in geometrical algebra, he writes that the symbol ‘−’ may have been left uninterpreted, but that it must be clear how much generality might be gained by interpreting the line \(-a\) as of equal length, but opposite direction, to the line \(+a\); and no curve is now considered as completely traced unless the negative, as well as the positive, values of the variables be taken into account [...] [Although] [t]his system of interpretation (viz. that given by the theory of impossibles [...] is quite as legitimate an extension as that of the negative values

\(^{140}\) Gregory, ‘On the existence of branches’ (note 113), 262.

\(^{141}\) Gregory, ‘On the existence of branches’ (note 113), 265-266, my emphasis.


of the variables, and is as necessary to the thorough understanding the
course of a curve [...] is merely a matter of convention [...].\textsuperscript{144}

This much is reflected in Walton’s own writings on geometrical algebra in
which he attempted to formulate a further generalization of the tracing of a
curve from its equation, namely one in which Gregory’s settling for the charac-
terization of ‘imaginary curves’ as those leaving the plane of reference ‘without
much troubling himself to inquire where [they] went to’, was corrected and
improved upon. It may be noted that this emphasis on generality, typical as it
was for so many of the junior Cambridge mathematicians contributing to the
promotion of the ‘calculus of operations’ in the pages of the \textit{CMJ}, enabled
Walton to further ‘release the power of symbolic[al] representation from the
cumbrous constrictions of particular interpretations’ – and this by question-
ing the restriction of geometry to ‘our previous geometrical knowledge\textsuperscript{145}
of Euclidean space and, thereby, by letting new algebraical results further push
the limits of geometrical \textit{interpretation}.

Thus, in his ‘On the general interpretation of equations between two varia-
bles’, Walton generalized the results put forward by Gregory.\textsuperscript{146} This he did
by inferring from ‘the laws of algebraical combination and the first principles
of geometrical interpretation’\textsuperscript{147} – that is, from the fact that while in the equa-
tions of algebraical geometry all traces are lost of any peculiar meaning it holds
that there must always exist geometrical meanings \textit{for} the equations – that the
equation $x^2 + y^2 = a^2$, according to one of an infinite number of [...] impossible
axes, represents ‘any one of an infinite number of curves of which the locus is
a surface of the fourth degree’.\textsuperscript{148} In both this article as well as in, for instance,
‘On the general theory of multiple points’ and ‘On the existence of possible

\begin{footnotes}
\footnotetext[144]{George Salmon, \textit{A Treatise on the Higher Plane Curves. Intended as a Sequel to A Treatise
on Conic Sections} (Dublin, Hodges and Smith, 1852), 302.}
\footnotetext[145]{Salmon, \textit{Treatise on Higher Plane Curves} (note 144), 303.}
\footnotetext[146]{In his articles Walton mostly referred to Gregory, ‘Linear differential equations’ (note
principles’ (note 105).}
\footnotetext[147]{William Walton, ‘On the general interpretation of equations between two variables
in algebraic geometry’, \textit{Cambridge Mathematical Journal}, 2 (1840), 103-113, 103.}
\footnotetext[148]{Walton, ‘On the doctrine of impossibles’ (note 143), 238.}
\end{footnotes}
asymptotes to impossible branches of curves’, Walton’s method involved translating, or ‘transforming’, expressions for curves and surfaces ‘from the [so-called] affectional equation between [...] two variables x and y to equivalent quantitive [sic] equations between three variables x, y, and z.’ and showing that, in so far as these equations are equivalent, it is possible, without thereby affecting the locus of the ‘affectional’ equation, to arbitrarily select from an ‘infinite variety’ of curves which one ‘we [...] regard as the companion of the circle’. This, to be sure, was considered by Walton to be the advantage of his approach to geometrical algebra at large; namely, its ‘natural connection’ with the system of interpretation of geometrical equations, which exhibits an entire correspondence between the degree of the equation in x and y and the geometrical characters of its locus’.

5. **Gregory, Walton and the rise of abstract algebra and geometry**

From hindsight, the fundamental problem of Gregory and Walton’s ‘algebraical geometry’ and ‘geometrical algebra’ was that their specific definition of the connection between algebra and geometry could yield neither the idea of an algebraical representation of three-dimensional space nor a geometry that was not concerned with the description of the features of the empirical world. It was precisely the formulation and search for a solution of these two related issues that enabled younger contributors to the *CMJ* and, from 1845/1846 on, contributors to the *Cambridge and Dublin Mathematical Journal* such as Hamilton and Cayley to criticize the symbolical approach to algebra in favor of abstract algebra. What became apparent in their work was the thoroughgoing relationship between geometries of higher-dimensions and the construction of algebras that were genuinely independent of arithmetic. The goal of this concluding section is to briefly reflect on Hamilton and Cayley’s geometry-inspired crit-

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150 Walton, ‘On the general interpretation’ (note 147), 104.


152 Walton, ‘On the general theory’ (note 149), 165.
icism of symbolical algebra so as to come to terms with the limits of Gregory and Walton’s investigations and those of ‘scientific’ symbolical algebra at large.

5.1 The last group of symbolical algebraists

Before doing so it is important to remark that if it is true that the last representatives of the Cambridge-school of symbolical algebraists, Boole and De Morgan, created a non-quantitative algebra both were committed to the view of symbolical algebra as an art of reasoning, rather than a science. For example, in a series of four papers ‘On the foundation of algebra’,153 Peacock’s student De Morgan opposed the symbolical or ‘technical’ algebra of ‘rules without meaning’ to ‘logical’, ‘double’ or ‘complete’ algebra defined as ‘the science which investigates the method of giving meaning to the primary symbols and of interpreting all subsequent symbolic results’.154 During the 1840s, De Morgan was of the opinion that, as an ‘art’ or ‘method of operation’, symbolical algebra was a non-mathematical step from universal arithmetic to a meaningful algebra that was scientific in so far as it would ‘enable us to give a meaning to every symbol and combination of symbols before it is used, and consequently to dispense, first, with all unintelligible combination, secondly with all search after interpretation of combinations subsequently to their first appearance’.155 Gregory’s ‘pupil’ Boole, who ‘did not write explicitly on the formalization of algebra, but expressed his view on the nature of algebra and mathematics in […] logical works’156 such as The Mathematical Analysis of Logic of 1847,157 was able to free symbolical algebra from its ties to arithmetic by putting forward the ‘laws of thought’, that is, the operations of the mind, as its broad suggesting science. This allowed for a further step in the process of the growing abstractness of algebra – one that, interestingly, resembled Hamilton’s radical contributions in terms of its ‘extra-algebraical’ motivation158 –, but it was a far cry from the attempt to turn algebra into a demonstrative ‘science of symbols’.

154 Pycior, ‘De Morgan’s algebraic work’ (note 6), 222.
155 De Morgan, ‘On the foundation’ (note 153), 173.
156 Koppelman, ‘The calculus of operations’ (note 29), 220.
5.2 Hamilton criticism of symbolical algebra

Although Hamilton developed his work in isolation from the mathematicians of the Cambridge school of symbolical algebra, his Kantian-inspired ‘constructivist’ or ‘intuitionist’ revision of algebra was written with the explicit hope of rejecting those ‘practical’ and ‘philological’ views\textsuperscript{159} ‘which regard Algebra as an Art, or as a Language; as a System of Rules, or else as a System of Expressions, but not as a System of Truths, or Results having any other validity than what they may derive from their practical usefulness, or their logical […] coherence’.\textsuperscript{160} Hamilton’s central aim was to demonstrate, on the one hand, that ‘if algebra is to be regarded as a science, then it must be the science of pure time’\textsuperscript{161} and, on the other hand, that algebra, in so far as it is a science, consists of propositions and theorems with meaning or truth-values.

It was in the preface to his \textit{Lectures on Quaternions} that Hamilton, in a spirit critical of fluxional analysis, wrote the following about his position vis-à-vis the meaning of imaginary numbers:

While agreeing with those who had contended that negatives and imaginaries were not properly quantities, I still felt dissatisfied with any view which should not give to them from the outset a clear interpretation and meaning; and wished that this should be done, for square roots of negatives, without introducing considerations so expressly geometrical as those which involved the conception of an angle […] It appeared to me that these ends might be attained by our […] regard[ing] Algebra […] as the Science of Order in Progression. [T]he successive states of such a progression might (no doubt) be represented by points upon a line yet I thought that their simple successiveness was better conceived by comparing them with moments of time divested [of] all reference to cause and effect so that ‘time’ […] might be said to be abstract, ideal or pure.\textsuperscript{162}

\begin{itemize}
  \item \textsuperscript{159} These two views may be said to correspond to the ‘non-scientific’ symbolical algebraists and ‘scientific’ symbolical algebraists, respectively.
  \item \textsuperscript{160} Hamilton, ‘Theory of conjugate functions’ (note 25), 5.
  \item \textsuperscript{161} Ohrstrom, ‘Hamilton’s view of algebra’ (note 9), 46.
  \item \textsuperscript{162} William Rowan Hamilton, \textit{Lectures on Quaternions} (Dublin, Hodges and Smith, 1953), 1-64, 2.
\end{itemize}
5.2.1 Hamilton’s ‘higher-dimensional algebra’

This statement was reflective of the first part of Hamilton’s original threefold argument for algebra as the science of pure time – as found in his ‘Theory of conjugate functions, or algebraic couples; with a preliminary and elementary essay on algebra as the science of pure time’, 163. The first part consisted of the claim that ‘the notion of Time is connected with existing Algebra’ – for example with that of Newton and Lagrange;

[t]he Newtonian method [...] regards the curve and line not as already formed and fixed, but rather as nascent, or in process of generation: and employs, as its primary conception, the thought of a flowing point [...] And in one of his own most important researches in pure Algebra [...] Lagrange employs the conception of continuous progression to show that a certain variable quantity may be made as small as can be desired.164

The second claim was that the fact that the ‘notion or intuition of Order in Time is not less but more deep-seated in the human mind than the notion or intuition of Order in Space’ can be justified with reference to the ‘intuitive’ or ‘unempirical’ truth that ‘a moment of time which we inquire, as compared with a moment which we know, must either coincide with or precede or follow it’. The fundamental third claim of Hamilton’s argument, namely that the ‘Mathematical Science of Time’ is possible, was ‘a conclusion to which the author has been led by all his attempts [to] analyse what is Scientific in Algebra’.165 And in the paper of 1837 this was done by resolving the ‘old difficulties’ of the negative and imaginaries on the basis of his new ‘contrapositives’ and ‘couples’ which are deduced from the Intuition or Original Mental Form of Time: the opposition of Negatives and Positives being referred [not] to the opposition of the operations of increasing and diminishing a magnitude, but to the simpler and more extensive contrast between the relations of Before and After, or between the directions of Forward and Backward; and Pairs of Movements being used to suggest [the] Conjugate Functions, which

gives reality and meaning to conceptions that were before Imaginary [or] Impossible [...] because Mathematicians had derived them from that bounded notion of Magnitude, instead of the [...] thought of Order in Progression.\textsuperscript{166}

Hamilton himself would later insist that ‘Kant’s \textit{Criticism of the Pure Reason} [sic]’ had convinced him that it was possible ‘to construct, a priori, a Science of Time’.\textsuperscript{167} But in his early paper the definitions of positive whole numbers in terms of successive moments and that of imaginaries as ordered pairs (or ‘couples’) of real numbers were put forward against the background of traditional geometrical considerations.

At the end of the 1837 paper, Hamilton spoke not only of extending his scientific time-algebra of number couples to arbitrary sets of moments in time so as to create higher order algebras, but also of considering whether ‘triplets [would] provide a new algebra of ‘real’ three-dimensional space’.\textsuperscript{168} Put differently, Hamilton proposed to determine whether ‘there were number \textit{triples} which would do for three-dimensional geometry what [the] couples and the standard \textit{imaginary} numbers could do for the two-dimensional case’.\textsuperscript{169} Because he believed that the ‘three-dimensional numbers’ could not but possess the same properties as the real numbers, Hamilton ‘wanted, at first, a system which would form an associative, commutative, division algebra over the reals’\textsuperscript{170} for these specific numbers. After years of repeated failure to provide an account of the triplets that would preserve the operations of ordinary algebra,\textsuperscript{171} Hamilton found an answer, not in triplets, but in new numbers with four components that lacked the commutative property of multiplication, namely ‘quaternions’. Where the search for triplets had resulted from the attempt to provide a new algebra of real three-dimensional space, the quaternions dawned on Hamilton when he realized ‘that we must admit, in some sense, a \textit{fourth dimension} of space

\begin{flushleft}
\textsuperscript{166} Hamilton, ‘Theory of conjugate functions’ (note 25), 7.  \\
\textsuperscript{167} Hamilton, Lectures (note 162), 2, f. 2.  \\
\textsuperscript{169} Rice, ‘Inexplicable?’ (note 27), 169.  \\
\textsuperscript{170} Koppelman, ‘The calculus of operations’ (note 29), 228.  \\
\textsuperscript{171} See, for example, Rice, ‘Inexplicable?’ (note 27), section 7.
\end{flushleft}
for the purpose of calculating with triplets'.\textsuperscript{172} Hamilton concluded that in quaternions he had discovered/constructed ‘not just the algebra of space, which he had hoped to find in triplets, but a natural algebra of space \textit{and time}, since the three-dimensions of space plus the one dimension of time required a quaternion of numbers for their expression’.\textsuperscript{173}

Hamilton’s creation of a new algebra was, thus, inspired by two geometrical considerations that can also be found in the work of Gregory and Walton: firstly, the ‘old difficulties’ related to the geometrical representation of negative and imaginary numbers and, secondly, the notion of the generation of lines by the movement of points of geometrical fluxional analysis. It was on the basis of his idea that the principles of science are laws ‘of the mind which are correlative with the laws of nature but are not derived from nature’ that Hamilton was able to transform these considerations into constructions with an abstractness unreachable within the abstractionist framework of Gregory and Walton. In other words, abstract algebra begun at the moment that the intuitive meaningfulness of time, rather than abstraction from the empirical concept of motion in time, was said to ground algebra as an independent science.

5.3 \textit{Cayley’s criticism}

For Hamilton ‘the question of three-dimensional geometrical representation had been answered by quaternions’\textsuperscript{174} and since he believed that, in so far as geometry is the a priori science of space, no one ‘can doubt the truth of the chief properties’ put forward ‘by Euclid in his \textit{Elements}’\textsuperscript{175} he continued to insist on the importance of a geometrical interpretation, in ‘geometrical algebra’, of the four-dimensional quaternions that connected them back to the three-dimensional world described in ‘algebraical geometry’.\textsuperscript{176} Another Kantian critic of Gregory and Walton, Cayley, was one of the first to realize that Hamilton’s quaternions suggested that other number systems in even higher dimensions were possible.

\begin{itemize}
\item \textsuperscript{172} Hamilton in Rice, ‘Inexplicable?’ (note 27), 23.
\item \textsuperscript{173} Hankins, ‘Triplets and triads’ (note 168), 176, emphasis in original.
\item \textsuperscript{174} Rice, ‘Inexplicable?’ (note 27), 173.
\item \textsuperscript{175} Hamilton, ‘Theory of conjugate functions’ (note 25), 4.
\item \textsuperscript{176} The three-dimensional properties of quaternions were, indeed, soon to be adopted for use in physics by another mathematician associated with the CMJ, James Clerk Maxwell (1831-1879).
\end{itemize}
And it was in 1847 that Cayley created an algebra with numbers with one real and seven imaginary components – so-called ‘octaves’ or ‘octonions’ that were neither commutative nor associative. His search for these numbers took place within the science of, what he himself came to call, ‘abstract geometry’:

[This] science presents itself in two ways; – as a legitimate extension of the ordinary two- and three-dimensional geometries; and as a need in these geometries and in analysis generally. In fact whenever we are concerned with quantities connected together in any manner [...] then the nature of the relation between the quantities is frequently rendered more intelligible by regarding them [...] as the co-ordinates of a point in a plane or in space; for more than three quantities there is, from the greater complexity of the case, the greater need of such a representation; [and] this can only be obtained by means of the notion of a space of the proper dimensionality; and to use such representation we require the geometry of such a space.¹⁷⁷

Although Cayley did not refer explicitly to the distinction, his criticism of Gregory and Walton was that their work on ‘algebraical geometry’ and ‘geometrical algebra’ implied a change of the delicate balance between the essential subject matter of geometry, which was Euclidean space, and the analytical forms used to describe it.¹⁷⁸ Cayley’s ally George Salmon (1819-1904) wrote that ‘we know what a circle is before we know anything about the equation $x^2 + y^2 = a^2$ and any interpretation of this equation differing [...] from our previous geometrical conception must be rejected’.¹⁷⁹ Because Gregory and Walton’s contributions to ‘geometrical algebra’ seemed to undermine the geometrical ideas symbolized in ‘algebraical geometry’ these contributions were to be considered an ‘intrusion on the territory of geometry’.¹⁸⁰ The reason for this was that reality was claimed for notions, such as ‘imaginary curves, which were, in fact, nothing but figures of speech – or, for that matter, analytical constructions. Cayley himself sometimes referred to real space of higher dimensionality, but from their work it is clear that ‘neither Salmon or

¹⁷⁸ Richards, Mathematical Visions (note 38), 52.
¹⁷⁹ Salmon, Treatise on Higher Plane Curves (note 144), 303.
¹⁸⁰ Richards, Mathematical Visions (note 38), 52.
Cayley would have considered the possibility that spaces of higher dimension might be real in the same sense that Euclid’s was. If their criticism ‘eclipsed the kind of algebraic perspective [of] Gregory [and Walton], in which algebraic forms would define geometrical concepts’, that is, ‘algebraical geometry’, and reinforced Euclid’s *Elements* as the final arbiter of what was valid in ‘geometrical algebra’, Cayley, Salmon and many others engendered a new debate on the epistemological status of higher dimensional spaces needed to interpret algebraical problems ‘which did not have straightforward, three-dimensional interpretations’. Cayley’s solution, namely to accept higher-dimensional numbers as free mental constructions, once again demonstrated that the limitedness of Gregory and Walton’s symbolical algebra had to do with the empirical origins of the abstractions with which algebra was concerned.

6. Conclusion

This paper has had a specific and a more general aim. The specific aim has been to provide a detailed analysis of the renovation of Peacock’s symbolical algebra as it was carried out in Gregory and Walton’s contributions to ‘algebraical geometry’ and ‘geometrical algebra’. What these contributions, first and foremost, made manifest was the importance of the negotiation of the connection between algebra and geometry or, more specifically, of the geometrical interpretation of imaginaries. Their contributions are notable not only for what they reveal about the increasing abstractness of symbolical algebra but also for their importance to the eventual establishment of abstract algebra. With regard to that larger process, the general aim of the paper has been to show that a picture of the transition of algebra in Britain from a ‘pre-modern’ to a ‘modern’ state is too simplistic. There was no clear rupture of that sort. Nor was there a straightforward gradual transition. The present paper has suggested instead that there is a ‘middle way’ between these views in which, firstly, symbolical and abstract algebraists are recognized as allies vis-à-vis the idea of algebra’s status as a science and, secondly, the growth of abstractness is attributed to a fundamental change in views about mathematical abstraction.

The symbolical algebraists advanced abstraction but only as abstraction from empirically grounded quantities of number and magnitude. The abstract algebraists were able to exercise the freedom of algebra suggested, but never exercised by, Peacock and Gregory in so far as they adopted a conception of abstraction in which mathematical objects exist as generalized constructions from certain pure a priori intuitions. If it is, thus, the case that both the symbolical algebraists and the abstract algebraists sought to establish algebra on a scientific or demonstrative basis rather than on a purely formal one, the latter’s presentation of algebra as a system of meaningful truths went hand-in-hand with a dismissal of the former’s approach as formal. An appreciation of such conceptual and philosophical complexities is important not only for a more faithful account of the development of British algebra, but also for an acknowledgment of the need for a more nuanced understanding of what it means to explain the modernist transformation of mathematics in terms of notions such as ‘abstraction’ and ‘formalism’.\(^{183}\)

\(^{183}\) See, for example, Gray, *Plato’s Ghost* (note 20). Following Mehrtens, *Moderne* (note 20), Gray distinguishes between ‘moderns’ and ‘countermoderns’. Where the former held that mathematics needs no independent referent to justify the existence of the objects about which it speaks, the latter argued that the existence of mathematical objects cannot be derived solely from their function within a formal system. Gray writes that ‘British algebraists of the first half of the nineteenth century can be seen as quite formal in their study of systems of meaningless symbols’, and then attributes the destruction of this ‘pre-modern’ British view to Hamilton’s ‘modernist’ formalism. Gray, *Plato’s Ghost* (note 20), 28. But this leaves unexplained not only the ground of the possibility of referring to different meanings of ‘formal’, but also that and in what sense there is something ‘modern’ about the work of the ‘pre-moderns’ Peacock and Gregory, and ‘countermodern’ about the ‘modern’ work of Hamilton.
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