"A terrible piece of bad metaphysics"? Towards a history of abstraction in nineteenth- and early twentieth-century probability theory, mathematics and logic
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Chapter 12

On Aleksandr Iakovlevich Khinchin’s paper ‘Ideas of intuitionism and the struggle for a subject matter in contemporary mathematics’

0. Introduction

Aleksandr Iakovlevich Khinchin’s (1894-1959) paper ‘Ideas of intuitionism and the struggle for a subject matter in contemporary mathematics’\(^1\) appeared as the introduction to a special seminar on intuitionism organized, in the winter of 1925-1926, by the Department for Natural and Exact Sciences of the Communist Academy\(^2\) in Moscow. The seminar, which was supervised by Khinchin, was part of the mathematical section’s very popular\(^3\) series of meetings on general methodological issues ‘at which various philosophical and historical questions of mathematics were discussed’ (Yushkevich, 2007, 18). Among the participants were Igor Vladimirovich Arnol’d (1900-1948) and Grigorii Borisovich Gurevich (1898-1980) – who both delivered a lecture – and Andrei

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1 Originally published as ‘Idei intuitsionizma i bor’ba za predmet v sovremennoy matematike’ in Vestnik Kommunisticheskoi akademii 16, (1926), 184-192.

2 The Socialist Academy was founded by the Bolshevik authorities in 1918 with the task of ‘making Marxism a unified and coherent system of philosophical propositions’ (Vucinich, 1999, 108, see also Seneta, 2004, 342). It was renamed the Communist Academy in 1923 and abolished in 1936. The Section for Natural and Exact Sciences was headed by Otto Yul’evich Schmidt (1891-1956) and its members were Veniamin F. Kagan (1869-1953), H.P. Kasterin (?-?), V.A. Kostitsin (1883-1963), Vasily G. Fesenkov (1889-1972) and Khinchin (see Ermolaeva, 1999, 262). Yushkevich also mentioned Lazar A. Liusternik (1899-1981) and Liutsian M. Likhtenbaum (?-1969) as its ‘scientific workers’ (Yushkevich, 2007, 18).

3 Yushkevich noted that ‘[m]any hundreds of listeners attended the seminar sessions’ (Yushkevich, 2007, 18).
Nikolaevich Kolmogorov (1903-1987), Valerii Ivanovich Glivenko (1896-1940), Sof’ia Alexandrova Ianovskaia (1896-1966) and ‘some others’ (ibid.) – who commented, in the form of presentations, on Khinchin’s lecture which was subsequently published in the Bulletin of the Communist Academy (Vestnik Kommunisticheskoii akademii) in 1926.

The English translation of the paper is presented here for the first time. What follows shall provide a brief introduction to Khinchin’s somewhat loosely formulated, but philosophically and historically revealing endeavor to caution Marxist philosophers of mathematics ‘against the sweeping rejection of intuitionism as an idealistic orientation in the foundations of mathematics’ (Vucinich, 1999, 111).

1. Aleksandr Iakovlevich Khinchin (1894-1959)

Khinchin was born on July 19, 1894 in the village of Kondrovo (in the Kaluga Oblast) and died in Moscow on November 18, 1959, at the age of sixty-five. He entered the Faculty of Physics and Mathematics (Fizmat) of Moscow University in the year 1911. Already before graduating in 1916, with Nikolai Nikolaevich Luzin (1883-1950) as his supervisor, Khinchin presented an original contribution to the theory of functions at a 1914 meeting of Pavel Aleksandrovich Florenskii’s (1882-1937) and Luzin’s Student Mathematical

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4 Although Khinchin did not mention Kolmogorov as one of the seminar’s lecturers, the then twenty-two-year-old student might have presented his now famous ‘On the tertium non datur principle’ (Kolmogorov, 1925 [1999]) there; he finished this paper on September 30, 1925 (Uspensky, 1992, 386). Yushkevich, who ‘first heard Kolmogorov in Khinchin’s seminar’, wrote that ‘Kolmogorov’s short and oratorically unsophisticated presentation was especially striking. His interpretation of intuitionistic logic was highly original’ (Yushkevich, 2007, 18).


6 The topic was the same as that of the 1916 Comtes Rendus paper, namely the generalization of certain results found in the work of Arnaud Denjoy (1884-1974) on the asymptotic behavior of integral functions. The 1914 presentation introduced a generalization of the concept of what is known today as an ‘asymptotic derivative’ (see Gnedenko, 1961, 2).

7 Florenskii was one of the founders, in 1902, of the Student Circle of the Moscow Mathematical Society in which he was not only a leading figure but for which he also served as the first secretary – a position which he would later offer to his friend Luzin. At the meetings of the Circle, Florenskii himself, ‘other students, and sometimes even the professors, gave lectures on topics including set theory and the theory of functions of a real variable’ (Demidov & Ford, 2005, 599).

8 Following several trips to France and Germany, Mlodzeevskii, ‘in the academic year 1900-1901, gave the first course in the theory of functions of a real variable at Moscow University [...] [His] younger colleague Egorov attended these lectures, as did a first year student, [...] Luzin’ (Phillips, 1978, 280).

9 Given that he was the ‘most prominent and active member’ of the seminar (Vucinich, 1970, 355) it seems probable that the name ‘Lusitania’ was derived from Luzin. However, there are several other explanations of the origin of the name (Graham & Kantor, 2009, 101-102).

10 ‘Kruzhok’, the literal translation of which is ‘circle’, is the Russian equivalent of a ‘working group’.

Khinchin’s professional career started around 1918 – the year in which he taught his first class at the Moscow Women’s Polytechnical Institute. He was elected dean of the Faculty of Mathematics and Physics of the then newly-founded

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11 Vucinich has noted that the ‘Moscow school […] received its initial impetus not from ‘practical’ demands, but from a need to improve the formal tools of mathematical operations. It dwelt in the world of pure abstractions, and sought to generalize its theoretical formulations as far as possible by elucidating the inner logic of their mathematical expression’ (Vucinich, 1970, 356).

12 For example, Aleksandrov wrote that ‘[t]he richness of […] Luzin’s creative ideas was so great that almost every meeting with him was, for his students, a source of new problems and unexpected ideas […] Luzin possessed the extraordinary art of presenting a mathematical result so that the listener was forced to participate in the process of obtaining it, thus transforming the lecture into a kind of self made laboratory of thought’ (Aleksandrov, 1955, 19). See Phillips (1978), Dolenc (2012, chapter 1) and Tikhomirov (2007) for other reminiscences of Luzin’s ‘completely new methods of working with youth’ (Tikhomirov, 2007, 276).

13 The descriptive theory of sets and functions is the theory of the structural properties of sets which are not related to the concept of measure – for example the convergence of a function everywhere. Luzin himself gave the name ‘descriptive theory of sets’ to the study of the (semi-intuitionist) theory of effective sets – i.e. those sets which can be constructed without the axiom of choice (e.g. ‘Borel-sets’) (see section 2.2.1).

14 The metric theory of functions is the name given to that part [of the Lebesgue measure of a (Cantorian) linear point set] which is concerned with the idea of measure [of the length of an interval]; related to it are the theories of integration, convergence almost everywhere and other ‘almost everywhere’ properties, mean convergence and so on (Liusternik, 1967a, 144).
Pedagogical Institute in Ivanovo-Voznesensk\textsuperscript{15} (now Ivanovo) in 1919/1920 and in 1922/1923 he became affiliated, as a part-time researcher, with the Scientific Research Institute for Mathematics and Mechanics (NIIMM) at Moscow State University.\textsuperscript{16} Khinchin was appointed as a Professor at the Mathematics Institute of Moscow State University in 1927 where he later also occupied the chairs of Probability Theory and Mathematical Analysis. The following year, Khinchin spent several summer weeks at David Hilbert’s world-famous mathematics faculty at the University of Göttingen (see Khinchin, 1952, 11) and participated in the International Congress of Mathematicians in Bologna (Italy) in September as a member of the Soviet delegation. Khinchin’s foreign travels reflect the international orientation of his fellow Moscow mathematicians. For example, where Egorov and Luzin had once traveled to Berlin, Paris and Göttingen, Aleksandrov and Pavel S. Urysohn (1898-1924) were allowed to visit Klein, Hilbert and Noether (Göttingen), Hausdorff (Bonn) and Brouwer (Amsterdam) in 1923 and 1924 as the first two Soviet mathematicians to cross the border (e.g. Freudenthal & Heyting, 1966, Lorentz, 1999, 173, Van Dalen, 1999 [2013], 397).\textsuperscript{17} Furthermore, in 1930-1931 Kolmogorov and Aleksandrov went on a ten-month trip to Germany and France where they met, among

\textsuperscript{15} Demidov wrote that ‘[t]he years of revolution, Civil War and profound economic devastation badly affected the activity of [‘Luzin’s’] school. Harsh living conditions made many researchers move to other towns in order to survive. [Luzin] himself and a group of his pupils, including Khinchin, [V.S.] Fedorov, Menshov, Suslin, found themselves in Ivanovo-Voznesenski’ (Demidov, 2007, 37).

\textsuperscript{16} There were no mathematical activities in Moscow during the years of ‘War Communism’, that is, the years of the Bolshevik Revolution and the Russian Civil War (November 1917-October 1921). For Menshov, Suslin and Khinchin, Luzin ‘found refuge in Ivanovo to the northeast of Moscow where a new Polytechnical Institute was founded […] After the […] introduction of Lenin’s New Economic Policy (NEP), conditions of life improved [and] [m]athematicians could return to Moscow’ (Lorentz, 1999, 172).

\textsuperscript{17} Here, it is important to refer to the Rapallo Treaty of April 16, 1922 with which the two isolated countries Soviet Russia and Weimar Germany sealed their by then already friendly relations in terms of an economic, military and political partnership. At least from around 1920 until the end of Lenin’s NEP in 1928, the Soviet-planners sought for a modus vivendi with the national and international community of ‘bourgeois’ specialists – allowing Russian scientists and intellectuals to travel abroad, hosting visiting delegations, supporting joint research and managing purchases and translations of foreign books (e.g. David-Fox, 1997, Joravsky, 1961 [2013], Josephson, 1991). The contacts between Soviet and ‘Western’ mathematicians (e.g. Maltsev, 1971, 68, Hollings, 2014, 5) disappeared by the end of the 1930s due to the increasing partiinost (‘Party-orientation’) of mathematics (e.g. Vucinich, 2000).
others, Hilbert, Landau and Weyl (Göttingen), Carathéodory (Munich) and Borel, Lebesgue, Fréchet and Lévy (Paris) (e.g. Kolmogorov, 1986 [2000], 153-157, Shiryaev, 2000, 27) and until the end of the 1930s Khinchin,\textsuperscript{18} Kolmogorov and Glivenko corresponded with Fréchet, Lévy and Brouwer’s pupil Heyting (e.g. Troelstra, 1990).

The Moscow school of mathematics was very much an outgrowth of the increasing withdrawal of several of Egorov’s and Luzin’s students from the original Luzitian agenda. Where Khinchin’s early contributions to the metrical theory of functions of the late-1910s can be considered as an off-shoot of Luzin’s endeavors, in the years in which he combined his lecturing at Ivanovo-Voznesensk with research work in Moscow, he came to devote himself to the application of the new set- and function-theoretical ideas from Moscow to the more classical St. Petersburgian topics of probability- and number theory (see Gnedenko, 1961, 3). The decay of Luzitania and its transformation into a large complex of movements associated with Khinchin and Aleksandrov (topology) was very much ‘a consequence of its rapid rise – it became too narrow’ (Liusternik, 1967c, 66).\textsuperscript{19} Khinchin delivered his first lectures on probability theory (‘A proposition in the theory of probability’) and number theory (‘Approximation of algebraic numbers by rational fractions’, ‘Some questions in the theory of Diophantine approximations’ and ‘The theory of Diophantine approximations’) at the Moscow Mathematical Society in 1923 (see Liusternik, 1967c, 63, see also Khinchin, 1923). His first (‘Borelian’)\textsuperscript{20} papers on probabilistic topics related to number theory (the law of iterated logarithm) and the theory of (measurable) functions (convergence of series of independent random variables) were published in 1924 (see Khinchin, 1924a, 1924b). Khinchin’s early ‘systematic application of concepts and of tools of

\textsuperscript{18} Jan von Plato (e-mail correspondence) has drawn attention to the fact that Troelstra (1990) lists two letters from Khinchin to Heyting of 1936. These letters are to be found in the Heyting Nachlass (Rijksarchief Noord-Holland, Haarlem).

\textsuperscript{19} Liusternik recalled that Khinchin’s withdrawal ‘was not so dramatic in the emotional sense as that of the topological school [of Aleksandrov and Urysohn] […] because of [his] great isolation in Luzitania’ (Liusternik, 1967c, 75). See also footnote 27.

\textsuperscript{20} For example, Khinchin’s famous law of the iterated logarithm of 1924 arose out of the number-theoretical attempt, of 1923, to sharpen Hardy and Littlewood’s estimation of the asymptotic behavior of the so-called oscillation of frequency of zeros and ones in a binary representation of real numbers as posited by Borel’s strong law of large numbers of 1909 (e.g. Gnedenko, 1961, Von Plato, 1994, section 2.3).
the theory of functions and of the theory of sets to the theory of probability’ (Gnedenko, 1961, 3, see also Liusternik, 1967a, 147) soon attracted the attention of Kolmogorov with whom Khinchin would briefly collaborate to obtain several important results in the theory of summation of independent random variables (see Khinchin & Kolmogorov, 1925). Their work formed the beginning of the Moscow school of probability theory (see Gnedenko, 1990, Vere-Jones, 2008) and, together with that of Sergei N. Bernshtein (1880-1968) and Evgeny Slutsky (1880-1948), of the ‘Soviet school’ in the foundations of mathematical probability (Barbut, Locker & Mazliak, 2014, 11).

Despite Khinchin’s wide scientific interests and his contributions to number theory (e.g. the study of metric properties of different classes of irrationals), his ‘fundamental role in the progress of mathematics [was] connected with the theory of probability’ (Gnedenko, 1961, 6). During the 1920s-1930s, Khinchin published some fifty papers on applications of the theory (statistical physics, queuing problems, information theory etc.) and on classical probabilistic problems (summation of independent random variables) and new probabilistic problems related to number- and function theory (iterated logarithm and convergence of series of independent random variables, respectively).22 His work was ‘highly evaluated and regarded’ (Rogosin & Mainardi, 2010, xii) by many of his famous contemporaries, such as Maurice Fréchet (1878-1973), Paul Lévy (1886-1971) and Borel, and honored with, for example, an election to the USSR Academy of Sciences (1939), the State Prize (1940) and the prestigious Stalin Prize (1941).

21 Khinchin and Kolmogorov’s short paper of 1925 contained the so-called ‘Kolmogorov-Khinchin two series theorem’, the ‘Kolmogorov three series theorem’ and the ‘Kolmogorov-Khinchin criterion’ (see Shiryaev, 2000). About their brief collaboration, Von Plato remarks that ‘Kolmogorov’s first work on probability was a joint paper with [Khinchin] in 1925. From then on he appeared as the ingenious proof-maker, many times strengthening the limit theorems obtained by [Khinchin] [...] Probability theory became a field of very active research. The publications of [Khinchin] and Kolmogorov show that a close collaboration often stood behind the new developments’ (Von Plato, 1994, 199, see also Shiryaev, 1989, 874).

22 His complete bibliography contains some 150 contributions to mathematical probability theory (see Cramér, 1962, Rogosin & Mainardi, 2010, Gnedenko, 1961).
2. Khinchin and the foundations of mathematics in the late-1920s

The late-1920s was a time of great turmoil for the Soviet mathematical community, and perhaps especially for the one in Moscow (see Lorentz, 1999, Vucinich, 1999) – and this both in terms of politics, with Stalin’s steady rise to absolute power after Lenin’s death in 1924, and mathematics, with the paradoxes of set theory giving rise to a crisis in the whole of classical infinitary mathematics. It was only several years after the Bolshevik Revolution (1917) that Marxist ideologues began to turn their attention to the establishment of a Soviet philosophy of (modern) mathematics (Hollings, 2013, 1448, Vucinich, 1999, 107). But the ‘Marxist classics’ of Engels and Lenin ‘left no readily accessible and sustained comments’ on mathematics and the contemporary developments (in set theory, logic, foundations etc.) ‘were so new that they were not included in school curricula at the time when most [writers] received their secondary and higher education’ (Vucinich, 1999, 107) (Engels, 1927 [1940], Lenin, 1909, see also Aleksandrov, 1970, Kennedy, 1997, Struik, 1948, Van Heijenoort, 1985). This situation had several consequences. Firstly, by the end of the 1930s the Communist Academy’s task of turning Marxism into a unified and consistent theory that was able to express everything (science and mathematics included) in dialectical materialist terms was still in its initial phase (e.g. Bogolyubov & Rozhenko, 1991, Kolmogorov, 1938). Secondly, the Soviet-Marxist philosophy of mathematics of the 1920s was ‘fragmented’, ‘generally vague’ and ‘superficial’ and the ideologues occupied themselves with the criticism of particular manifestation of ‘mathematical idealism’ in the West on the basis of a handful of ‘rather disconnected suppositions’ (Vucinich, 1999, 108, see ibid., 108-110). The two major sources for the fact that ‘the promises that were made for this philosophy, and all that it might do for mathematics, were never realized’ (Hollings, 2013, 1449) were the following; ‘an ideological compulsion to exaggerate the ‘idealistic’ leaning of many modern mathematicians and a doctrinaire rigidity in identifying the branches of mathematics characterized as impractical and as targets of direct attack’ (Vucinich, 1999, 110). Thirdly, because of their many and thoroughgoing disagreements, the Marxist writers were unable to have much impact on mathematics and at least until the

2.1 Khinchin, Moscow mathematics and Soviet ideology

The older, deeply-religious, representatives of the ‘otherworldly’ Moscow school of mathematics Egorov and Luzin were among the first and most famous victims of the official war on the scientific intelligentsia in the Stalin era (e.g. Demidov, 2007, 44-51, Joravsky, 1961, 242-244, Seneta, 2004, 343-344). After some of his own ‘proletarian’ (post-graduate) students had turned against him on 21 December 1929, the ‘reactionary churchman’ and ‘counter-revolutionary’ Egorov was replaced by Schmidt and Kolman as the Director of the Faculty of Physics and Mathematics at Moscow State University and Head of the Moscow Mathematical Society, respectively. With Egorov arrested and exiled to Kazan, where he would die on 10 September 1931, Kolman, the zealous Stalinist mathematical ideologue who ‘occupied a great number of significant functions in the Party [and] in many scientific and social organizations’ (Kovaly, 1972, 337) organized an ‘Initiative Group for the Reorganization of the Mathematical Society’, consisting of Luzin’s former students Liusternik, Shnirelman, Gelfond and Pontryagin, that, in 1932, would elect Aleksandrov as President and, among others, Khinchin and Ivanovskaya as members of the new presidium. In 1936, Kolman would initiate, through an anonymous Pravda article (‘On enemies hiding behind a Soviet mask’), and then orchestrate the famous ‘Luzin affair’ in which Luzin was accused of a great number

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23 As is well known, under Stalin dialectical materialism was used to terrorize intellectual life to such a degree that upholding a scientific or mathematical theory branded as ‘idealistic’ or ‘bourgeois’ immediately led to ‘political disloyalty [...] humiliating criticism [and/or] imprisonment’ (Graham, 1993, 121).


25 Egorov was arrested by the secret police and exiled to Kazan in 1930, where he died on 10 September 1931.

of ‘anti-Soviet’ activities (e.g. Demidov & Ford, 1997, Kutateladze, 2007, 2012, 2013, Levin, 1990).27 An extraordinary investigative commission of the USSR Academy of Sciences dismissed Luzin from all his official positions and threatened to have him arrested, but due to Stalin’s intervention this punishment was never executed. The commission consisted of eleven of Luzin’s academic colleagues among which were three of his former students who, for various reasons,28 ‘had been on rather cool personal terms with [him] for quite some time’ (Siegmund-Schultze, 2004, 382, see also Luzin, 2002, 202) – Aleksandrov, playing the ‘key role’ with his ‘aggressive ideological accusations’ against Luzin (Demidov, 2007, 49), and Shnirel’man and Khinchin, being ‘rather active’ at the meetings, but demonstrating more ‘reserve’ (see ibid., 49, Kutateladze, 2012, 2).

2.2 Khinchin, the Communist Academy and Marxist philosophy of mathematics

Khinchin’s personal and theoretical position within the public controversies on politically correct mathematics of the late-1920s and early-1930s ‘made him acceptable to the Marxists, but it did not entitle him to call himself a Marxist’ (Vucinich, 2000, 61). Although he was never a member of the Communist Party and did not publish in the Party journal Pod Znamenem Marksizma (Under the Banner of Marxism), Khinchin closely collaborated with the Marxist writers and mathematicians of the Communist Academy, of which he became a member in 1926 (Ermolaeva, 1999, 262); some of his articles appeared in the Vestnik Kommunisticheskoi akademii (Herald (or Bulletin) of the

27 For example, Kolman accused Luzin of ‘moral unscrupulousness and scientific dishonesty with deeply concealed enmity and hatred to every bit of Soviet life’, publishing ‘would-be scientific papers and of standing ‘close to the ideology of the [far-right] ‘black hundred’’ (Kolman quoted in Kutateladze, 2012, 85).

28 The personal relations between, on the one hand, Aleksandrov and Khinchin and, on the other hand, Luzin seems to have deteriorated when Aleksandrov and Khinchin gave up the (‘Luzitanian’) study of function-theoretic issues and devoted themselves to topology and probability and number theory, respectively. Yushkevich recalled that at one of the meetings of the abovementioned commission Aleksandrov said that when he went over to topology Luzin told him that ‘[a]s long as you study topology we can have no contact’ (Yushkevich, 2007, 13).
Communist Academy) (Khinchin, 1926 [2015], 1929a, 1929b)\textsuperscript{29} and he actively contributed to its newly-erected Department for Natural and Exact Sciences headed by the Party-member and new Director (1930-1931) of Egorov’s Fizmat Schmidt. The Communist Academy, which had been formally established, in 1918, ‘from above’ by the Bolshevik authorities had the official task of the ‘rigid advocacy of the standpoint of dialectical materialism [in] the natural sciences, and [the] repudiation of the survivals of idealism’ (Decree of the All-Union Congress of Soviets, 26 November 1926 quoted in Price, 1977 [2012], 255).\textsuperscript{30} Its informal function was to answer the call ‘from below’ from the, largely non-Party and (former) Menshevik,\textsuperscript{31} intelligentsia which, in opposition to the group of graduates of the Institute of Red Professors\textsuperscript{32} at Moscow State University,\textsuperscript{33} worked on ‘the assumption that the very survival of Marxist philosophy depended on its […] adjustment to basic changes in scientific knowledge’ (Vucinich, 1999, 119) (Joravsky, 1961 [2013], chapter 5, see also Graham, 1967, Shapiro, 1976). Where the editors of Under the Banner of Marxism\textsuperscript{34} requested its readers, in 1925, to step up against the ‘idealism nourished by nondialectical interpretations of the substance and the methods of mathematics’ (Bammel

\textsuperscript{29} For example, the mathematical statistician Evgenii Evgenievich Slutskii (1880-1948) ‘was never tempted to contribute to such debates in the more overtly political journals’ (Barnett, 2011, 100).

\textsuperscript{30} This statement of the All-Union Congress of Soviets of 1926 was connected to the development of ‘a small section in the natural sciences’, headed by the ‘expert in algebra, old communist [and] important official’ (Demidov, 2007, 42) Schmidt, that was to compete with the ‘older academy in the natural sciences [of] the bourgeois Academy of Sciences’ (Graham, 1993, 86).

\textsuperscript{31} Joravsky has noted that ‘as late as 1928 twenty-six of the Academy’s fifty-nine ‘senior scholarly colleagues’ were not Party members, and a good many of the remaining thirty-three were very likely former Mensheviks only recently accepted into the Bolshevik Party’ (Joravsky, 1961 [2013], 84). It may here be remarked that Schmidt himself was a former Left Menshevik.

\textsuperscript{32} The Institute of Red Professors was ‘the Party’s only graduate-level institution of higher learning […] [Its] special mission and identity [was] of all the party institutions most closely bound up with the great revolutionary theme of the red expert […] The graduates expected from [it] would be both revolutionaries and scholars, Bolsheviks and intellectuals, reds and experts’ (David-Fox, 1997, 133-134).

\textsuperscript{33} About this specific group, Vucinich writes that that, ‘loyally echoing the ideas [of] Bolshevik leaders, [it made] Marxist theory a closed system of […] principles [that] were accepted dogmatically and no theoretical improvement was required’ (Vucinich, 1999, 119).

\textsuperscript{34} It was the second group of dogmatic Marxists that ‘had easy access to Under the Banner of Marxism’ (Vucinich, 1999, 119).
quoted in Vucinich, 1999, 117), the *Herald of the Communist Academy* showed a lenient view toward the Western orientations in mathematics.

### 2.2.1 Khinchin's 1926 paper at the colloquium on intuitionism

It was in the winter of 1925-1926 that Khinchin organized a special colloquium on intuitionism as part of his regular seminar on the foundations of analysis\(^{35}\) held at the mathematics section of Schmidt’s Department for Natural and Exact Sciences at the Communist Academy.\(^{36}\) The seminar itself was established with the aim of creating working relations with prominent mathematicians such as Kolmogorov, Liusternik, Shnirel’man and Gelfond – all of whom would present papers there in 1928-1929 with ‘direct mathematical information without Marxist elaboration’ to familiarize Marxist theorists with trends in modern mathematics (Vucinich, 1999, 121, see also Ermolaeva, 1999, 263). Khinchin’s paper for the colloquium, entitled ‘Ideas of intuitionism and the struggle for a subject matter in contemporary mathematics’, took up the more explicit task of cautioning the other members of the Academy against the dogmatic rejection of Western orientations in the foundations of mathematics with avowed ‘idealist’ or ‘subjective’ leanings, especially that of L.E.J. Brouwer (1881-1966).

Following the dismissal either of dogmatic Marxist theorists upholding mathematical utilitarianism or ‘practicalism’ or of outdated technicians and scientists misunderstanding the practical promises of mathematics (cf. Vucinich, 1999, 117, Ermolaeva, 1999, 262),\(^{37}\) Khinchin wrote that ‘we, mathematicians, are […] [d]eeply convinced of the living reality of the subject which we study’. Khinchin then put forward an interpretation of Brouwer’s intuitionism in which it was presented as an orientation amenable to dialectical materialism.

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\(^{35}\) This seminar was run by Khinchin (see Liusternik, 1967c, 75, Yushkevich, 2007, 18).

\(^{36}\) Two points may be mentioned here. Firstly, Khinchin’s seminar at the mathematics section apparently took place before the official establishment of Schmidt’s Department in 1926. Secondly, some authors have referred to the mathematics section in terms of a ‘Circle of Mathematicians and Physicists-Materialists’ which existed within the framework of the Communist Academy (see Ermolaeva, 1999, 262).

\(^{37}\) Vucinich writes that the fact that Khinchin found technicians and natural scientists, rather than Marxist theorists, ‘guilty of misunderstanding the practical promises of new mathematics was merely a tactical move to preserve the state of relative tranquility in the relations between mathematicians and interpreters of Marxist thought’ (Vucinich, 1999, 117). This statement may be put in doubt with reference to Ermolaeva emphasis on the mid-1920s characterization of university professors, funda-mental scientists and technicians as class enemies (Ermolaeva, 1999, 262).
in so far as it maintained that there is always a mathematical content standing behind the mathematical formalism (e.g. Khinchin, 1946 [2000], 1). On the one hand, intuitionism only accepted axioms that are ‘both “intuitive” in origin and “real” in reference to nature as the subject matter of scientific inquiry’ (Vucinich, 1999, 113). On the other hand, this enabled the intuitionist to protest against the ‘degeneration [of mathematics] into abstract combination, into a chess game’ – as carried out in the work of the formalist David Hilbert (1862-1943). Because Khinchin attributed to Hilbert the statement that the infinite exists neither in nature nor in thought and there should only be finite mathematics, in the sense of Brouwer’s mathematics of ‘real phenomena’, he was able to conclude that Hilbert agreed with Brouwer who could thus consider himself the winner of the *Grundlagenstreit*.

Given Khinchin’s standing as an internationally acclaimed mathematician and his acquaintance with a fair amount of the relevant literature,38 his simplification of the positions of Hilbert and Brouwer was instrumental in his attempt to ascribe the victory of intuitionism to its materialistic outlook and, thus, to present its victory as a proof not only of its promises for a Marxist philosophy of mathematics but also of the correctness of this very philosophy.39

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38 Khinchin’s 1926 paper on intuitionism contained only one explicit reference, namely to Hilbert’s 1925 lecture on the infinite as it was published in 1926 (Hilbert, 1926), and one implicit reference to a 1921 paper by Weyl (Weyl, 1921). Given that his close colleague and friend Kolmogorov referred to Brouwer (1918), Brouwer (1919), Brouwer (1921) and Brouwer (1925) in his 1925 paper on intuitionistic logic (Kolmogorov, 1925 [1991], 68) it seems highly unlikely that Khinchin wrote his 1926 paper in ignorance of Brouwer’s work. At the same time, their colleague Glivenko apparently came to know Kolmogorov’s 1925 paper as late as ‘somewhere between July 4 and October 13, 1928’ (Van Atten, 2004, 423). (But in light of footnote 4 it also seems possible that Glivenko simply did not refer to Kolmogorov in his own papers of 1928 and 1929 because the paper was in Russian (Coquand, 2007, 25)).

39 The issue of what was tactical and what was real scientific interest in Khinchin’s 1926 paper will be taken up in section 3.3.
Khinchin was not the only person from Soviet Russia to publish a positive treatment of intuitionism.\textsuperscript{40} Around 1928/1929 Kolmogorov would present a paper at the mathematics section of Schmidt’s Department at the Communist Academy entitled ‘Intuitionism and the Moscow School (of N.N. Luzin)’). This suggests that Khinchin, Kolmogorov and Glivenko became interested in intuitionism via their teacher Luzin who, together with Aleksandrov and Suslin, worked on the theory of Borel sets and analytic sets which was initiated by the French (semi-)intuitionists (e.g. Lorentz, 2001). Where Khinchin was interested in intuitionism as a foundational orientation, the participants of the 1925/1926 colloquium and the discussants of Khinchin’s 1926 paper Glivenko (1928, 1929) and Kolmogorov (1925 [1991], 1932)\textsuperscript{41} made actual contributions to its further development. The two Luzitanians accepted Brouwer’s criticism of classical mathematics – or, more specifically, of its illegitimate application of the (Aristotelian) principle of excluded middle (PEM) to infinite rather than finite domains (see section 3.1) –, and devoted their attention to proving that classical mathematics is translatable into intuitionistic mathematics on the basis of an intuitionistically correct system of logic (e.g. Coquand, 2007, Hyland, 1990, Troelstra, 1990, Uspensky, 1992, Mancosu & Van Stigt, 1998). Here, it may be worthwhile to emphasize, firstly, that the results obtained from their presentation of intuitionistic mathematics in the language of an axiomatized logical calculus conflicted with some of Brouwer’s deepest commitments and, secondly, that Ianovskaia would later point to Kolmogorov and Glivenko as the founders of the Soviet constructivist school whose dismissal of idealistic philosophy distinguished their work from that of Brouwer (see Ianovskaia, 1948).

\textsuperscript{40} The Bolshevik ideologue Yanovskaya wrote a survey of these papers, entitled ‘Mathematics in the USSR during the thirty-year period 1917-1947’, that was published in 1948 (Yanovskaya, 1948).

\textsuperscript{41} As is well known, where Kolmogorov’s formalization of what became known as minimal propositional logic pre-dated the one given by Heyting in 1930 (Heyting, 1930), his meta-logical translation of classical into intuitionistic logic predated the translations of Kurt Gödel (1907-1978) (Gödel, 1933a, 1933b) and Gerhard Gentzen (1909-1945) (Gentzen, 1933). It may also be noted that Kolmogorov published a general article on the debate between Brouwer and Hilbert in 1929 (Kolmogorov, 1929 [2006]).
2.3.1 Luzin, the semi-intuitionist

Luzin has been characterized as a ‘semi-intuitionist’ (‘Halbintuitionistisch’) (Becker, 1927, Bockstaele, 1949, 40-41, Graham & Kantor, 2009, 147, Hesling, 1999, 212) – a position which he developed under the header of ‘effectivism’ during the course of his career (Bazhanov, 2001a, 213, Bazhanov, 2009, 131, Phillips, 1978, 291, Suzuki, 2009, 368). Following the French tradition of (half-axiomatic, half-constructivist) semi- or pre-intuitionism (as Brouwer came to call it)

French semi- or pre-intuitionists (as Brouwer came to call them) such as Henri Poincaré (1854-1912), René-Louis Baire (1874-1932), Borel and Lebesgue, Luzin rejected Ernst Zermelo’s (1871-1953) so-called axiom of choice (‘Axiom der Auswahl’)\(^\text{42}\) for its reliance on George Cantor’s (1845-1918) purely non-constructive existence proofs,\(^\text{43}\) but accepted the other basic axioms of set theory.\(^\text{44}\)

The objection against the axiom of choice was twofold: firstly, that it postulated the existence of a choice-set without showing how it is to be constructed or ‘effectively defined’ and, secondly, that it extended the applicability of the axiom from finite to infinite domains.

Luzin agreed with the semi-intuitionists’ criticism of set theory’s non-constructive existence proofs of infinite mathematical objects and shared their insistence on finite laws of construction for, and the effective definability of,

\(^\text{42}\) The aim of Zermelo’s axiom, which was first formulated in 1904, intended to prove that every set can be well-ordered. In brief, it asserts that whenever there is a set such that, on the one hand, each of its members is in turn itself a non-empty set and, on the other hand, each pair of its members have no elements in common, there exists a ‘choice-set’ that contains one element from each member of the original set. For an account of the origins, development and influence of Zermelo’s axiom of choice see, for example, Moore (1982).

\(^\text{43}\) A Cantorian non-constructive existence proof accepts the existence of a mathematical object (e.g. the infinite sequence of rational numbers thought of as a completed totality) when its introduction does not lead to a contradiction, that is, without a means for defining it and/or finding an object for which the theorem describing it is true.

\(^\text{44}\) Graham and Kantor have suggested that the esoteric Christian ideas of Pavel A. Florensky (1882-1937) on ‘name worshipping’ were instrumental for Luzin’s particular approach to set theory (Graham & Kantor, 2009, see also Graham, 2011, Ford, 1998, Kantor, 2011)
mathematical objects. At the same time, it was under the influence of the Cantor-inspired work of the French mathematicians that Luzin, for example in his *The Integral and Trigonometric Series* of 1915, showed the ‘success with which set theory can be applied in function theory and measure theory’ (Hesseling, 1999, 15). His view was thus that, on the one hand, the *foundations* of set theory itself were not well-established and, on the other hand, that intuitionism could answer the ‘cardinal questions *beyond the reach* of set-theoretical considerations’ (Vucinich, 1999, 113). If the descriptive theory of sets – in which only those (*effective*) sets are considered that can be constructed without the axiom of choice – was Luzin’s answer to this situation, it is clear that this theory was developed, ‘after his return from Paris in 1914’, in response to ‘Les polémiques sur le transfini et sur une démonstration de M. Zermelo’ (Borel, 1914 [1950]) which included the famous ‘Cinq letters’ of 1905 exchanged between [Jacques] Hadamard [1865-1963], [René] Baire [1871-1932], Lebesgue and Borel’ (Phillips, 1978, 291) (Bockstaele, 1949, Kuznetsov, 1974, Michel, 2008).

In his paper ‘On the views of set theory’, which was read at the International Congress of Mathematicians in 1928, Luzin explained the ‘effective’ middle position between the ‘realist’ Brouwer and the ‘idealist’ Hilbert in terms of a ‘*fatique* with Cantor’s Paradise’ (Luzin quoted in Graham & Kantor, 2009, 201, see also Luzin, 1928). It is of some interest to note that Luzin expressed his commitment to intuitionism in Bologna in that specific year – for one year earlier, in the spring of 1927, he, supposedly, ‘abandoned his earlier neutrality by attacking Brouwer’s theories as having a ‘destructive character’ (Keldysh, 1974, 188, my emphasis) at a mathematical congress in Moscow.

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45 The following passage is insightful in this context: ‘In 1912, Luzin left Göttingen for Paris, where he remained until 1914 [...] In Paris, Luzin became acquainted with the leading mathematicians who were working on problems in the theory of functions: Lebesgue, Borel, Denjoy and Picard [...] In retrospect, Borel’s lectures – which contained a generalization of the idea of an analytic function – seemed the most interesting. For after completing *The Integral and Trigonometric Series*, Luzin became increasingly occupied with the foundations of analysis, with such questions as when is a function or set defined?’ (Phillips, 1978, 284).

46 It was in an attempt to warn his audience about Brouwer’s intuitionist reconstruction of mathematics (see section 3.2) that Hilbert, in 1926, made the famous remark that ‘no one shall expel us from the Paradise which Cantor created for us’.
2.3.2 Luzin and non-Aristotelian logic

Around the year 1912/1913 Brouwer distanced himself from his French predecessors on the basis of the observation that ‘[n]o matter how much inventiveness was invested by the French school, it would never produce a coherent constructive mathematics while sticking to the principle of the excluded middle [and] similar sins’ (Van Dalen, 1999 [2013], 7). Given Brouwer’s claim that Aristotelian logic and intuitionism are in conflict, another influence on Luzin, namely the logical work of Samuil O. Shatunovsky (1859-1929) and Nicolai A. Vasiliev (1880-1940), must be mentioned.

Shatunovsky had criticized the unrestricted use of the PEM in the context of the ‘thorough study of the specifics of mathematical proof as applied to infinite sets’ (Anellis, 1994b, 231) carried out in his 1917 textbook on algebra. He produced an axiomatic method – which he employed to establish the foundations of, for example, geometry and algebraic fields – independent of Hilbert, but is best known for his formulation of ‘a constructive, i.e. without the law of excluded middle, presentation of the Galois theory’ (Levin, 1994, 320). Although it is not known whether Luzin had first-hand knowledge of these particular achievements, it is possible that Yanovskaya – who attended, in 1923 and lead, from 1925 on, the seminar on methodological issues at the Communist Academy (O’Connor & Robertson, 2006) – provided him with information that she gathered during the years in which she studied under Shatunovsky at the Novorossiisk University in Odessa (Bazhanov, 2001b, Trakhtenbrot, 1997).

As is well-known, the work of the orthodox Marxist and Party functionary Yanovskaya was instrumental in the attacks against the Moscow Mathematical Society in 1929-1931 and the so-called ‘Luzin affair’ of 1936 – in which her co-author Ernst Kol’man (1892-1979) prosecuted the ‘anti-Soviet’ scholar Luzin with the goal of ‘destroy[ing] the remnants of the reactionary Moscow philosophical-mathematical School abhorrent to him’ (Demidov & Esakov quoted in Seneta, 2004, 339, see also Demidov & Ford, 1997, Kutateladze. 2007, 2013, Levin, 1990, Lorentz, 2002).

This early ‘campaign’, which treated formal logic ‘as metaphysical heritage, alien to the revolutionary proletariat’ (Bazhanov, 2005, 45), was led by Gr. Bammel (1907) and A. Varjas (1923) (e.g. Bammel, 1925, Varjas, 1923).
logic which included the rejection of the law of contradiction and excluded middle on the ground of its reliance on ‘dialectical logic’.\(^{50}\)

Vasiliev, the son of the ‘bourgeois professor’ and ‘subjective idealist’ Aleksandr V. Vasiliev (1853-1929) (Vucinich, 1999, 113-116), was the first to develop a non-Aristotelian, or non-classical, logic in a series of papers published between 1910-1913 (Anellis, 1994, Bazhanov, 1990, Bazhanov, 1994, Bazhanov, 1998). Influenced by N.I. Lobachevski’s discovery of an ‘imaginary’ non-Euclidean geometry in which Euclid’s fifth (‘parallel’) postulate is not valid, Vasiliev constructed a new ‘imaginary’ non-Aristotelian logic in which two of Aristotle’s fundamental principles are discarded. The core of his logic was the distinction between a variable ‘ontological’ or ‘material’ level standing in relation to observable objects and an absolute ‘epistemological’ or ‘formal’ meta-level consisting of laws that hold in any logical system (Puga, Newton & Da Costa, 1988, Smirnov, 1989). Vasiliev acknowledged that the Aristotelian principles make sense as empirical generalizations from so-called ‘positive sensations’, but reasoned that

‘[i]f one imagines a world in which […] positive [and] negative sensations are possible, then such a world will […] require a different logic and the introduction of supplementary judgments. Just as Euclidean geometry possesses an empirical foundation, which makes the fifth postulate sensibly evident, this logic possesses its own empirical foundation through the law of contradiction. If one rejects [this] law, [then] alongside positive and negative […] it is possible to introduce another type [of] ‘indifferent judgments’. For a logic which operates with three forms of judgments, the law of excluded middle is not required’ (Bazhanov, 1990, 338).

\(^{50}\) Vucinich explains that, during the 1920s, Marxist theorists reasoned that ‘mathematics cannot depend on formal logic alone. The dialectic – the synthesis of contradictions – is a particularly valuable method of mathematical advancement. By recognizing and unifying contradictions in the existing mathematical conceptualization – and by rising above the syllogistic limitations of formal logic – dialectical logic opens new vistas for the advancement […] of the mathematical mode of thinking […] by means of ‘dialectical synthesis’ – by unifying contradictions that defy formal, logical procedures’ (Vucinich, 1999, 109-110).
Vasiliev was unable to further develop his new ideas on logic due to ill-health and, despite the fact that it initially attracted positive commentary, his logical system was soon forgotten. It was Luzin who would revive Vasiliev’s ‘imaginary logic’ in a critical review of 1927. Here, he wrote that

‘Vasiliev’s works on logic are of great importance in connection with investigations of the principles of thought as a whole, but [on] account of the new tendencies in mathematics [i.e. intuitionism and effectivism] Vasiliev’s ideas coincide remarkably with the latest efforts to which mathematicians must resort by force of facts’ (Luzin quoted in Bazhanov, 1990, 340).

It is definitely true that Vasiliev’s rejection of the PEM – ‘at the same time that Brouwer did, but independently of him’ (Bazhanov, 1990, 340) – added ‘a tributary to both intuitionist logic and intuitionism as a major orientation in the foundations of mathematics’ (Vucinich, 1999, 116). But Luzin first and foremost wished to emphasize that Vasiliev and, albeit implicitly, Shatunovski, had anticipated that the foundational crisis in mathematics in the 1920s would eventually turn into a debate on the validity of Aristotelian logic in the finite and infinite domain (Bazhanov, 1987).

3. The debate on the foundations of mathematics in the 1920s

It was in 1920 that Hilbert’s former student and one of Brouwer’s earliest followers, Hermann Weyl (1885-1955) claimed that the publication of ‘Foundations of set theory independent of the Principle of Excluded Middle’ (Brouwer, 1918, 1919) had caused a ‘new crisis in the foundations of mathematics’ (Weyl, 1921) – one in which Brouwer and his new theory of sets and the continuum represented the revolution. After writing, in 1922, that ‘Brouwer is not, as Weyl thinks, the revolution […] but [a] coup (Putsch)’ (Hilbert, 1922), Hilbert set out to answer their criticism in the form of the presentation of the re-foundation of the whole of (modern) mathematics from the ‘finitary viewpoint’ (i.e. ‘Hilbert’s program’) (e.g. Hilbert, 1922 [1998], 1923, 1926, 1928, 1929, 1931 [1998], see also Detlefsen, 1986, Sieg, 1999, Zach, 2007). The Lusitani-
ans were familiar\textsuperscript{51} with several fundamental papers of Brouwer, Hilbert and Weyl and in light of Luzin’s knowledge of the doubtful status of transfinite reasonings (section 2.2.1 and 2.2.2) it may come as no surprise that his students showed interest in the foundational debate.

### 3.1 Brouwer’s intuitionism

Brouwer’s aim was to reconstruct mathematics from the viewpoint of a form of ‘subjectivist’ metaphysical idealism (Placek, 1999, chapter 2). He defined mathematics as a free thought-creation that has no existence outside the human mind; in so far as the individual consciousness is the source of all knowledge, ‘mathematics is identified with the whole of the constructive [process] on and with the elements of the Primordial Intuition [of Time]’ (Van Stigt, 1998, 7).\textsuperscript{52} If mathematical reality and truth are found ‘in the present and past experiences of consciousness’ (Brouwer, 1948, 1243) and mathematical entities are nothing but past and completed constructions consisting of finite sequences of constructive steps, Brouwer defined mathematical existence as ‘having been constructed in time’. It was on the basis of this ‘intuitive mathematics’ that Brouwer not only argued that mathematics is independent of language and logic – such that ‘[m]athematical language follows upon mathematical activity, and logic consists of looking at that language in a mathematical way’ (Hesseling, 1999, 35). But he also dismissed, for example, Hilbert’s foundational view of mathematics as the theory of formal systems in which mathematical existence is said to result from the proof of the consistency of the axioms for mistaking mathematical language for mathematics proper.

‘It does not follow from the consistency of the axioms that the supposed corresponding mathematical system exists. Neither does it follow from the existence of such a system of mathematical reasoning that the linguis-

\textsuperscript{51} Where Kolmogorov’s 1925 paper contained references to Hilbert (1923) and Brouwer (1925 [1967], 1921 [1998], 1918, 1919), Khinchin referred directly only to Hilbert (1926).

\textsuperscript{52} As the abstraction of pure time awareness, this ‘Primordial Intuition’, or ‘Primordial Happening’, is ‘nothing but the fundamental intellectual phenomenon of the falling apart of a moment of life into two qualitatively different things of which one is experienced as giving way to the other and yet is retained by an act of memory’ (Brouwer, 1929 [1998], 45).
tic system is alive, i.e. that it accompanies a chain of thought, and even less that this chain of thought is a mathematical construction’ (Brouwer quoted in Van Stigt, 1998, 10).

Where Hilbert assumed the PEM – which he identified with the principle of the solvability of every mathematical problem (PSMP) – as part of his means of proving the consistency of formal systems, Brouwer held that ‘the justification of formalistic mathematics by means of its consistency contains a vicious circle, since this justification rests upon [...] the [contentual] correctness of the [PME]’ (Brouwer, 1928 [1998], 41, see also Brouwer, 1923 [1967]). Because mathematical statements express the completion of a (finite) thought-construction and there is no guarantee that such a construction can be completed in the case of infinite systems (Van Stigt, 1998, 9-10), Brouwer argued that the PEM and the PSMP are

‘dogmas that have their origin in the practice of first abstracting the system of classical logic from the mathematics of subsets of a definite finite set and then attributing to this system an a priori existence independent of mathematics, and finally applying it wrongly – on the basis of its reputed a priori nature – to the mathematics of the infinite sets’ (Brouwer, 1921 [1998], 27, f. 4).

Brouwer’s publication ‘The foundations of set theory independent of the logical principle of the excluded middle’ (Brouwer, 1918, 1919) was the culmination of his search for an alternative for the classical set-theoretical treatment of the continuum as a closed totality of all numbers. His early attempt at making constructive sense out of Cantorian set theory had consisted of the insight that in so far as only finite or denumerably infinite sets could be constructed, the nondenumerability of the continuum led to the need for the ‘superimposition’ of the ‘measurable continuum’ on the intuitive continuum in the form of the notion of a ‘denumerably unfinished set’.\footnote{If this meant that he settled for a continuum that is poorer than the intuitive one – namely in so far as the constructions of the measurable continuum are denumerable (or ‘denumerably unfinished’), while Cantor had shown that the points on the (‘intuitive’) continuum are not – Brouwer rejected part of generally accepted mathematics (e.g. Cantor and Dedekind’s actually infinite, non-denumerable sets) as a consequence of his intuitionist views. See, for example, Brouwer (1930 [1998]).} This ad hoc solution relied on the
intuitionist definition of a denumerably infinite sequence as a completed construction of real number understood as ‘the algorithm or ‘law’ by which each element of the sequence is uniquely determined’ (Van Stigt, 1998, 8). It was in 1918/1919 that Brouwer, for reasons internal to intuitionism, developed these (lawlike) infinite sequences into choice sequences that allowed for ‘as-yet-uncompleted’ elements determined, with complete freedom, by the ‘creative subject’ (Niekus, 2005, Troelstra, 1968, 1982). The admission of unfinished, lawless constructions – with an initial, finite and completed, ‘segment’ and a ‘tail’ to be constructed in the open future – as legitimate mathematical objects enabled Brouwer to formulate not only an intuitionistic set theory, but also an intuitionistic account of the (nondenumerable and measurable) continuum as a ‘medium of free becoming’ (Weyl, 1925-1927, 133) (Brouwer, 1918, 1919, 1921 [1998], 1930 [1998]).

The implications of Brouwer’s intuitionist reconstruction of the foundations of mathematics were twofold. Firstly, it rejected large parts of classical mathematics, namely those which were not based on ‘the intuitive (contentual) theory of laws of [the] construction of […] mathematical formulae [for which] the intuitionist mathematics of the set of natural numbers is indispensable’ (Brouwer, 1928 [1998], 41). And, secondly, it contained features – related to its introduction of choice sequences leading to a new theory of the continuum – that were not found in classical mathematics.

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54 Brouwer discovered that his theory of the measurable continuum ‘depended on two suppositions whose intuitionist acceptability had yet to be shown (the suppositions of the individualized constructability of a point set and of the internal dissectability of every individualized point set)’ (Van Atten, 2007, 35). If this is indicative of Brouwer’s growing criticism of his French intuitionist predecessors, his admission of ‘choice’ within the intuitionist framework placed him directly at odds with figures such as Borel and Lebesgue (Michel, 2008).

55 Brouwer rejected Cantor’s Principle of Comprehension, according to which each property determines a set (or a subset of a given set), and accepted, instead, ‘sets of choice sequences’ [i.e. ‘spreads’], infinitely proceeding sequences of mathematical entities previously constructed, where each successive element was allowed to be chosen more or less freely, restricted only by the finitely many choices already made and by the law determining the set’ (Moschovakis, 2009, 109-110).

56 Brouwer now identified points on the continuum (real numbers) with choice sequences satisfying certain conditions. It was the identification of real number with the whole of the so-called ‘Indefinitely Proceeding Sequence’ that ensures ‘the nondenumerability and measurability of the continuum’ (Van Stigt, 1998, 13).
3.2 Hilbert’s program

Hilbert first responded to ‘The new foundational crisis in mathematics’ (Weyl, 1921) in his ‘New foundations of mathematics’ of 1922.\(^{57}\) Although he did not agree with the intuitionist claim that (constructive) evidence is the only guarantee in mathematics, Hilbert’s ‘new’\(^ {58}\) proposal was to secure the formal, meaningless parts of (classical) mathematics by means of proving their consistency in contentual meta-mathematical terms –,\(^ {59}\) thereby preserving ‘the formally simple rules of ordinary Aristotelian logic’ (Hilbert, 1926, 174). More in specific, the core of ‘Hilbert’s program’ was

‘to show the admissibility of all of mathematics by establishing that the axiomatic systems for the various branches of mathematics cannot lead [to] a contradiction. The proof of consistency will only make use of contentual reasoning, characterized by its evidence, but the mathematics expressed in those systems will not have the same evidential status of the metamathematical considerations. [Hilbert] agreed with Brouwer and Weyl that mathematics goes well beyond what can be founded by purely contentual reasonings, but this is not a reason to jettison those areas of mathematics that go beyond such contentual reasoning’ (Mancosu, 1998, 156).

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\(^{57}\) Hilbert set out from the statement that Brouwer and Weyl ‘seek to ground mathematics by throwing overboard all phenomena that make them uneasy and by establishing a dictatorship of prohibitions. This means to dismember and mutilate our science, and if we follow such reformers, we run the danger of losing a large number of our most valuable treasures’ (Hilbert, 1922 [1998], 200).

\(^{58}\) It is well-known that roots of this proposal can already be found in Hilbert’s ‘On the foundations of logic and arithmetic’ which was first presented in 1904 and published in 1905 (Hilbert, 1905 [1967]).

\(^{59}\) In his ‘Intuitionist reflections on formalism’ of 1928, Brouwer emphasized that Hilbert’s distinction between these two parts of formalist mathematics already appeared in Brouwer’s doctoral thesis ‘On the foundations of mathematics’ of 1907 in the form of the distinction between ‘mathematical language’ and ‘second-order mathematics’ (Brouwer, 1928 [1998], see also Van Dalen, 1999 [2013], section 14.1).
The program had two parts; a descriptive part that contained, in the form of ‘axioms and provable theorems’ (Hilbert, 1931 [1998], 269),\(^{60}\) the formalization of (the ‘ideal parts’ and ‘transfinite statements’ of) classical mathematics and a justificatory part of (‘real’) inferential procedures that provided the proof of the consistency of the axioms and theorems in terms of (‘finite’)\(^{61}\) arithmetic and combinatorics (‘meta-mathematics’).\(^{62}\) Hilbert’s aim was, thus, ‘to use the trustworthy parts of mathematics indirectly to [secure] the more problematic, ideal parts’ (Posy, 1998, 295). The first (meta-mathematical) part, which consists of contentual reasonings operating on ‘extra-logical discrete objects which exist intuitively as immediate experience before all thought’ (Hilbert, 1922 [1998], 202),\(^{63}\) shows that

> ‘[i]f logical inference is to be certain, then these objects must be capable of being completely surveyed in all their parts, and their presentation, their difference, their succession (like the objects themselves) must exist for us immediately’ (ibid.).

Hilbert’s view that in meta-mathematics it is only allowed to make inferences about finite collections is reflective of the fact that he, in his ‘The logical foundations of mathematics’ and ‘On the infinite’ (Hilbert, 1923, 1926), implicitly accepted the intuitionistic statement that ‘whereas the application of the classical laws of logic is perfectly safe in a finite context, the extension to infinite sets

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\(^{60}\) ‘The axioms and provable theorems, i.e. the formulae that arise in this alternating game (Wechselspiel) are the images of thoughts that make up the usual procedure of traditional mathematics’ (Hilbert, 1931 [1998], 269).

\(^{61}\) The contentual arguments that are supposed to ground mathematics are derived from ‘finite logic’ and ‘purely intuitive thought (which includes recursion and intuitive induction for finite existing totalities’. In these cases ‘it is not necessary to apply any dubious or problematic mode of inference’ (Hilbert, 1923, 1139).

\(^{62}\) In Hilbert’s own words, ‘[e]verything that makes up mathematics in the traditional sense is rigorously formalized, so that mathematics proper […] becomes a stock of formulae […] This proper, formalized mathematics is accompanied by a mathematics that is to a certain extent new – a metamathematics that is necessary to secure formalized mathematics. In this metamathematics – in contrast to the purely formal modes of inference of mathematics proper – contentual inference is applied, but only to prove the consistency of the axioms’ (Hilbert, 1998 [1931], 269).

\(^{63}\) Hilbert wrote that ‘[t]he elementary number theory of numbers can be obtained […] by means of ‘finite’ logic and purely intuitive thought (which includes recursion and intuitive induction for finite existing totalities); here it is not necessary to apply any dubious or problematic mode of inference’ (Hilbert, 1923 [1998], 164).
Hilbert, thus, acknowledged that the second, ideal, part of mathematics (analysis, set theory etc.) is meaningless and that theories resulting from the application of trans-finite modes of inference are mere ‘games of symbols’. But in so far as their consistency was to be proven through finitary inferences, Hilbert could claim that the infinite is an ideal notion and still aim at the foundation of classical mathematics.

The foundational debate (1921-1928) between Brouwer and Hilbert, thus, centered around the decision for either a radical reconstruction of the whole of mathematics on the basis of intuitionist doctrines or the refoundation of classical mathematics by means of the meta-mathematical proof that the abstract concepts of the formal mathematical theories are free from contradictions. This sharp conflict between two parties that would ‘divide the mathematical world’ (Brouwer, 1913, 93) in the 1920s came to an abrupt end in 1928 after Hilbert’s dismissal of Brouwer from the editorial board of the *Mathematische Annalen* (Posy, 1998, Van Dalen, 1990). The personal ‘coup d’état’ of Hilbert was the motivation for the ‘Putschist’64 Brouwer’s withdrawal from the foundational debate.

3.3 *The role of ‘mathematical intuition’ in the foundational debate*

Given the importance of the interpretation of the notion of ‘intuition’ for the disagree-ment between Khinchin and the Marxist philosophers of mathemat-ics, it is worthwhile to end this introduction to his 1926 paper on intuitionism with a description of the complex role of ‘mathematical intuitionism’ within the foundational debate itself. For if the ‘inner circle’ (Brouwer and Weyl, Hilbert and Bernays) tended to polemically exaggerate the mathematical differences and downplay the philosophical similarities between ‘intuitionism’ and ‘for-

64 After Hilbert’s characterization of the work of Brouwer as representing a ‘Putsch’, the name ‘Putschist’ survived as a nickname for intuitionist. It was in a paper of 1926 that Frank Ramsey (1903-1930) spoke of the ‘Bolshevik menace of Brouwer and Weyl’ (Ramsey, 1926, 380). For a detailed description of the role of political metaphors in the ‘Grundlagenstreit’ see Hesseling (1999, chapter 6).
It may here be remarked that Hilbert nowhere addressed in detail Brouwer’s intuitionistic philosophy of mathematics or acknowledged the influence of its insights on his proof theoretical program. Brouwer, for his part, was ‘responsible for one of the worst misconceptions of Hilbert’s formalism; namely, that according to which it says that mathematics is a “game” played with symbols’ (Detlefsen, 1993a, 299).

Obviously, this introduction is not the place to further develop this Kantian connection between Brouwer and Hilbert – for which Posy (1998) provides a valuable starting-point.

Brouwer provided an extensive discussion of Kant’s views in his thesis of 1907 and in inaugural address of 1909, entitled ‘The nature of geometry’. Here, he argued that ‘Euclidean and non-Euclidean geometry have equal rights, but they contradict each other, and it can no longer be maintained that the former is a priori in mathematics’ (Brouwer, 1909 [1919], 114).

More in specific, Brouwer wrote that ‘[t]he only a priori element in science is time [...] Of course we mean here intuitive time which must be distinguished from scientific time’ (Brouwer, 1909 [1919], 61).
between Kant’s ‘transcendental ego’ and the ‘psychological subject’, the Brou-\newerian ‘subject’ is capable of constructing ‘full blooded’ objects on the basis of a mathematical intuition that is ‘fully and directly referential’ (Massimi, 2008, 187). Because he also wrote that for the properties of objects deduced from the PEM in finite systems ‘it is always certain that we can arrive at their empirical corroboration if we have a sufficient amount of time at our disposal’ (Brouwer, 1923 [1967], 336), Brouwer’s intuitionism has been said to be ‘very close to radical empiricism’ (Von Mises, 1951 [1968], 129). But Brouwer was a ‘subjective idealist’ committed to the view that

‘the Subject[’s] mathematical power to generate sequences enables man to create in his individual thought-world an interpretation of ‘Nature’, the outside world, which is manmade and mathematical. ‘Things’ […] are no more than repeated sequences [and] [b]ecause of the individual nature of human thought, [the] universe of ‘things’ is wholly private’ (Van Stigt, 1998, 6).

This means that logic, which itself has its origin in the finite mathematics constructed from the pure intuition of time, is applied to a physical world which,

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69 In other words, Brouwer had it that ‘[w]thin a specific finite ‘main system’ we can always test (i.e. either prove or reduce to absurdity) properties of systems […] On the basis of [this] testability, there hold, for properties conceived within a specific finite main system, the principle of excluded middle, that is, the principle that for every system every property is either correct or impossible [etc.]’ (Brouwer, 1923 [1967], 335). Importantly, Brouwer added that complete empirical confirmations are mostly ‘a priori materially impossible’.

70 The logical-empiricist Von Mises wrote the following about Brouwer’s intuitionism: ‘Disregarding certain rather mystic formulations that Brouwer gave to his doctrine, one recognizes his point of view as very close to a radical empiricism. The thesis that the fundamental assumptions [are] subject to continued examination and possible supplementation by intuition […] corresponds exactly to our conception. The opposite view is that of Kantian a-priorism, according to which the basic mathematical concepts [arise from] the properties of [the human] reasoning power’ (Von Mises, 1951 [1968], 129, my emphasis).
being a result of the mental activity of mathematical abstraction, is identified with ‘the Exterior World of the Subject’.\textsuperscript{71}

Hilbert, for his part, appealed to Kant’s ‘sensuous pure intuition’ in two contexts: firstly, elementary geometry, in so far as ‘it is given […] through the senses’ (Hilbert quoted in Hallet & Majer, 2004, 22), arises from ‘spatial intuition’ (Corry, 2006, Majer, 1995) and, secondly, elementary arithmetic (in the sense of the theory of real numbers) describes ‘concrete objects […] which exist \textit{intuitively} as immediate experience before all thought […] [These] objects […] can be generally and certainly recognized by us – independently of space and time’ (Hilbert, 1922 [1998], 202, my emphasis). Hilbert’s emphasis on the relation between perceptual experience and (spatial) intuition in geometry has given rise to a revision of the widespread image – first introduced by Brouwer in his ‘Intuitionism and formalism’ (1912, 1913) – of Hilbert as a formalist.\textsuperscript{72}

In the case of arithmetic, it is possible to observe a shift in Hilbert’s oeuvre between an \textit{empirical} conception of intuition and the Kantian notion of pure, a priori, intuition (Mancosu, 2010, 125-158).\textsuperscript{73} Where Hilbert, for example in ‘On the infinite’ (1926), spoke of the concrete objects of the finite mode of thought in terms of ‘perceptual intuition’, in his ‘The grounding of elementary number theory’ (1931), exactly these objects are referred to as arising from ‘the

\textsuperscript{71} Van Stigt explains that, according to Brouwer, ‘[t]he scientific observation of regularity in Nature, linking things and events in time as sequences, is a creative, mathematical process of the individual Mind […] Brouwer rejects any universal objectivity of things as well as their ‘causal coherence’, basing his argument on the essential individuality of thought and mind. [H]e emphatically denies the existence of a collective or ‘plural’ mind’ (Van Stigt, 1998, 6).

\textsuperscript{72} Corry characterizes Hilbert’s conception of geometry as ‘strongly empiricistic’ (Corry, 2006, 157). For an account of the connection between Hilbert and (logical-) empiricism see Stöltzner (2001).

\textsuperscript{73} Mancosu admits that it is ‘not easy to ascertain whether ‘primitive intuitive cognition’ refers [to] a Kantian pure intuition or whether […] the intuition mentioned here is an empirical one. I tend to read the [early] characterization of intuition as an empiricist one; that is, we are not dealing at this point with a Kantian pure intuition but rather with an empirical one’ (Mancosu, 1998, 169). It may here be remarked that in so far as, one the one hand, Kant claimed that space and time are pure intuitions and, on the other hand, Hilbert wrote that ‘[t]here are propositions that Kant regarded as a priori, and that we ascribe to experience [namely] the elementary properties of space and matter’ (Hilbert, 1931 [1998], 267), Hilbert’s later definition of intuition did not coincide with Kant’s pure intuition.
intuitive *a priori* mode of thought’ (Hilbert, 1931 [1998], 266, my emphasis). Because mathematics is conditioned on the fact that ‘something must already be given in representation’, Hilbert, who himself never used the term ‘formalism’ to characterize his foundational work, was able to write that it is *not* like a [meaningless] game determined by arbitrarily stipulated rules’ (Hilbert, 1919-1920 [1992], 14).

**An afterword**

Brouwer’s and Hilbert’s use of a somehow mathematical ‘intuition’ is demonstrative of the Kantian or, in more general terms, idealist framework within which the debate on the foundations of mathematics of the 1930s took place. Khinchin, for his part, attempted to present Brouwerian intuitionism as an orientation that found an empirical subject matter behind the empty mental constructions of Hilbertian formalism – such that Hilbert’s alleged acceptance of this criticism in his ‘On the infinite’ (the famous article in which he introduced his finitistic metamathematics) could be interpreted as a sign not only of Brouwer’s victory, but, thereby, also of the importance of the materialistic idea of mathematics being ‘real’ in reference for the foundational debate. Given the superficiality of some of Khinchin’s arguments this, of course, raises the question of what was tactical, what was real scientific interest and what

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74 It may here be noted that Hilbert repeated the passage, in his (1922 [1998]) on the ‘extralogical discrete objects, which exist intuitively as immediate experience before all thought’ *ad verbatim* in his (1931 [1998]).

75 Hilbert himself used the expression ‘axiomatics’ or ‘proof theory’ to describe his views on the foundations of mathematics. If Brouwer coined the terms ‘intuitionism’ and ‘formalism’ in his 1912 address ‘Intuitionism and formalism’, Richard Baldus (1885-1945) was the first to characterize Hilbert as the arch-formalist in his rector’s lecture of 1923 (Hesseling, 1999, 136-138).

76 Khinchin’s seems to have ‘mistaken’ Hilbert’s ‘finitism’ for a position which rejects the ‘infinite’ as such; where he ascribed to Hilbert the claim that infinity exists neither in nature nor in thought, Hilbert wrote his article (‘On the infinite’) exactly to justify the place of infinity within mathematics by finitary means.

77 It is somewhat remarkable to note that Khinchin was not alone in his (ideology-laden) misunderstanding of the debate in terms of ‘empiricism’ versus ‘idealism; for example, the Geneva professor Rolin Wavre (1896-1949), following Weyl’s private discussion in his Grundlagenkrise paper (see Hesseling, 1999, 144), contrasted Hilbert’s ‘logique formelle’ with Brouwer’s ‘logique empiriciste’ (see Wavre, 1926). See also Von Mises’s remark in footnote 72.
was due to ignorance in his 1926 paper. Although this is, of course, far from a
decisive answer, it seems to have been the case that it was all of these things
at once; Khinchin wanted to caution his audience against a sweeping rejection
both of intuitionism, as an orientation in the foundations of mathematics, and
of more specialized intuitionistic studies without, perhaps, being fully aware
of the sense in which these could conflict with his own set- and measure-theo-
retical studies.
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