"A terrible piece of bad metaphysics"? Towards a history of abstraction in nineteenth- and early twentieth-century probability theory, mathematics and logic
Verburgt, L.M.

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A. Ia. Khinchin

[184] We, mathematicians, are hardly concerned with the charges, which we regularly hear from technicians and natural scientists, that contemporary mathematics lacks a subject matter.¹ Since we are deeply convinced of the living reality of the subject which we study, we are justly convinced that these reproaches are caused by the naïve dissatisfaction of an individual who sees his current needs unmet by contemporary mathematics. Such an individual does not perceive any achievements of our science beyond his narrow horizon and does not anticipate the great resources he could reap from this science in the future.

However, since the last fifteen years,² there arose within the field of the mathematical sciences themselves a vigorous and almost fanatic campaign against...
mathematics’ present ambitions – a campaign that calls into question exactly those state-of-the-art achievements that mathematics has been proud of. Under the banner of this campaign, which is led by world-class scientists, there is an inscription that says, when slightly deciphered and corrected: struggle for a subject matter, struggle for the relentless expulsion from mathematics of anything that disguises the lack of a subject matter behind the appearance of a formally immaculate logical game! Those who want may fight against this campaign, but a mathematician who is genuinely interested in science cannot stay on the sidelines and ignore this challenging assault. This is not only and not so much because the assailants are led by scientists such as Brouwer and Weyl, but rather because the weapon of attack is an argument that is inescapable for every mathematician: the mathematician is told not about the uselessness of his science for technology and the natural sciences, but about its deep internal disease and trouble. This trouble threatens mathematics not only with formal contradictions but, what is even more terrifying, also with the degeneration into abstract combination, into a chess game.

What is most surprising about this revolutionary campaign is its unexpected and unprecedented success. This success is indeed indubitable, profound and, evidently, far-reaching, despite the resistance on the part of such experts as Hilbert and despite the fact that the new tendency, due to its consistent and radical revolutionary character, has earned the name of ‘mathematical bolshevism’.

One can only be bewildered by this success. It was just recently that mathematicians seemed to unanimously agree that everything that does not contain internal contradictions is legitimate and universally accepted in our science. ‘To exist’ in mathematics meant not to contradict oneself. No one raised the question about the reality of a subject matter. Such curiosity was considered a sign of backwardness and bad form. The definition of mathematics as

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3 In a paper of 1926, Ramsey spoke of the ‘Bolshevik menace’ of Brouwer and Weyl (Ramsey, 1926). It is interesting to read that Khinchin was surprised about the success of intuitionism despite its having been characterized as a ‘mathematical bolshevism’.

4 Hilbert defended this thesis his entire life. Where he, in his early work, considered the objects designated by axiomatic systems to be existent when the consistency of a system had been established, in his later work Hilbert declared that it was sufficient to show consistency in order to justify the mathematical existence of the ‘ideal elements’.
a science ‘where one never knows what one is talking about’ (Russell)\(^5\) was accepted not as an anecdote, but as a serious formulation of new trends in our science. Almost no one protested against this conception that naturally resulted from the brilliant successes of the mathematical mind in the direction of the formalization of science. An instrument of research was proclaimed to be an end in itself and nobody objected. But at the moment that Brouwer declared that it is wrong and that we should know what we study, that we should be sure of the reality of a subject matter, because otherwise science becomes a fruitless process of self-enjoyment, we witnessed an unexpected and significant spectacle. We observe how everything that is alive in our science comes gradually – sometimes immediately, with juvenile enthusiasm, and sometimes after long and persistent resistance – under the banner of this vigorous protest. This happens because we are confronted by a matter of life and death, because Russell’s formulation threatens mathematics with deadlock and scientific death amongst the stream of futile and pointless, albeit internally non-contradictory, logical combinations. Fashion is changing: in the West one can now hardly speak about fundamental questions in mathematics without taking into consideration the critique of the new trend, namely intuitionism as Brouwer called it in contrast to the usual formalism. In 1925\(^6\) Hilbert emphasized the distinction between contentual and formal types of reasoning in mathematics and thereby instigated, under the influence of Brouwer’s criticism,\(^7\) a radical revision of the foundations of mathematics. We suppose that the meaning of this new trend, as well as the pledge of its success, consists exactly in the slogan of a struggle for a subject matter. At first glance, such a statement may seem weird because Brouwer probably only criticized some logical inaccuracies in the substan-

\(^5\) It was in his ‘Recent work on the principles of mathematics’ of 1901 that Bertrand Russell (1872-1970) wrote that ‘mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true’ (Russell, 1901, 84). If Russell first expressed his commitment to logicism in this specific article, it has been shown that Hilbert was also a logicist during the 1890s (Ferreirós, 2009).

\(^6\) Khinchin here referred to Hilbert’s ‘Über das Unendliche’ – a lecture delivered at the Weierstrass memorial meeting of the Westfälische Mathematische Gesellschaft in Münster on June 4, 1925. It was published in the Mathematische Annalen in 1926 (Hilbert, 1926).

\(^7\) It may be remarked that Hilbert, much to Brouwer’s despair (e.g. Brouwer, 1928 [1998]), never explicitly acknowledged the influence of Brouwer’s criticism.
tiation of contemporary mathematical analysis. But let us look closer at the matter and try to clarify, by a concrete example, the nature of intuitionism’s main goals.

In one of his latest papers, Hilbert rightly considered the contemporary crisis of mathematics to be rooted in the fact that the false notion (‘the imaginary idea’) of infinity had not yet been entirely banned from our science. This notion does not refer to any phenomenon in the real world. Neither does it exist in our mind. Everything is finite, both in the external world and in our psyche. Mathematics made a distinction between potential, emerging infinity and actual, accomplished infinity. Weierstrass was the one to put an end to the first type of infinity which, from that moment on, survived as a mere name for an historical remnant, whereas the phenomenon signified by this notion was entirely reduced to a finite phenomenon. We are perfectly clear on this matter.

However, the situation regarded actual infinity is much worse. Until now we have regarded the multiplicity of all whole numbers or all points of a segment as real infinite collections. This is exactly the source of all disasters because there is nothing infinite in the real world. We deal here with a false intuition, with a term that does not refer to any phenomenon. Discrediting this false idea, as

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8 Although it is true that Brouwer’s ‘De onbetrouwbareheid der logische principes’ (‘The unreliability of the logical principles’) of 1908 – which contained the published version (in Dutch) of his argument against the general validity of the (Aristotelian) PEM – was the first article that he published, Brouwer already put forward his intuitionistic reconstruction of mathematics in his doctoral dissertation Over de Grondslagen der Wiskunde (‘On the Foundations of Mathematics’) of 1907.

9 Khinchin again referred to Hilbert (1926).

10 Also here, Khinchin followed Hilbert (1926). Hilbert wrote that it was by means of a clarification of ‘all confused notions about the infinitesimal’ that Karl Weierstrass (1815-1897) was able to ‘provide a solid foundation for mathematical analysis’ (Hilbert, 1926, 161). He then reasoned as follows: ‘Weierstrass’s analysis did [...] eliminate the infinitely large and the infinity small by reducing statements about them to [statements about] relations between finite magnitudes. Nevertheless, the infinite still appears in the infinite numerical series which defines the real numbers and in the concept of the real number system which is thought of as a completed totality existing all at once [...] Hence the infinite can reappear in another guise in Weierstrass’s theory and thus escape the precision imposed by his critique [...] [Yet] [t]he recognition that many theorems which hold for the finite [...] cannot be immediately and unrestrictedly extended to the infinite [...] [has] been completely clarified, notably through Weierstrass’s acuity’ (Hilbert, 1926, 161, 163).
Weierstrass had once discredited potential infinity – i.e. to reveal that we, when speaking of ‘infinite multiplicity’, inevitably imply a certain real (and therefore necessarily finite) phenomenon, and to clarify the properties and laws inherent to this phenomenon – is the main and most honorable goal of contemporary mathematics.\footnote{Again, in the words of Hilbert ‘[s]omeone who wished to characterize briefly the new conception of the infinite which Cantor introduced might say that in analysis we deal with the infinitely large and the infinitely small only as limiting concepts, as something becoming, happening, i.e., with the potential infinite. But this is not the true infinite. We meet the true infinite when we regard the totality of numbers 1, 2, 3, 4, … itself as a completed unity, or when we regard the points of an interval as a totality of things which exists all at once. This kind of infinity is known as actual infinity’ (Hilbert, 1926, 164).}

Let us consider the number $\pi$ and express it in decimal form – which, as is well known, contains an infinite number of digits. Let us ask the following question: will we ever see one hundred successive zeros among the digits? This task may be very complicated and even unsolvable. However, according to Brouwer, it has a very specific meaning to the eyes of an ordinary mathematician, or a formalist, as we would call him henceforth: whether the hundred successive zeros exist or not, tertium non datur.\footnote{This ‘honorable goal’ was, of course, opposed to the goal of Hilbert which was to save the notion of actual infinity, but it also conflicted with the goals of Brouwer’s set theory in which the (denumerably) infinite is given in the form of a ‘becoming (choice) sequence’. If Khinchin might have been indebted to Luzin’s (semi-intuitionist) idea of the ‘effective’ – or, for that matter, the rejection of ‘choice’ –, it seems that the Russian mathematicians were not familiar with Brouwer’s publications on intuitionist set theory.}

It is exactly against this interpretation of the problem, against applying the law of excluded middle here, that an intuitionist protests. The intuitionist presumes rather that the whole infinite representation of decimal digits of the number $\pi$ does not exist as a real phenomenon. We can construct as many digits as we

\footnote{Khinchin, like, for example, Paul Finsler (1894-1970) in his 1926 article ‘Gibt es Widersprüche in der Mathematik?’, set out to explain Brouwer’s rejection of the PEM by means of the possibility of unsolvable problems – something which, obviously, goes against Hilbert’s famous claim that there is ‘no Ignorabimus for the mathematician’ (it may here be recalled that Brouwer, reflecting on Hilbert’s work, identified the PEM with the PSMP). This argumentative strategy resembled that of the so-called ‘Brouwerian counter-examples’ to well-known classical theorems which rest upon the inability to prove something (e.g. Posy, 1980, Van Dalen, 1999).}
would like and, as a matter of fact, we possess an instrument\textsuperscript{14} for unlimited construction, but the constructed series of these digits thus accomplished is a false intuition. This series does not exist at all, in any world whatsoever, and consequently cannot be conceived because only that which is real can be conceived. Hence, we have in front of us merely an instrument for the infinite construction of decimal digits. That is why we must, and can, understand the assertion ‘the series somewhere contains one hundred successive zeros’ only in such a way that the generating instrument has the following property: ‘The essence of the instrument that generates the factorization of a number π into a decimal fraction is such that the instrumental inevitably generates (or, in contrast, cannot generate) one hundred successive zeros’.

To start with, let us consider a second (negative) answer. The claim that ‘the object $R$ contradicts in its essence the hypothesis $N$’ is always understood in mathematics in such a way that the hypothesis $N$ considered together with the definition of the object $R$ leads to a formal-logical contradiction. Hence, in view of the above, we should understand the assertion ‘there will not be one hundred successive zeros among the decimal digits of the number π’ as follows:

\begin{itemize}
  \item A) When considered together, the definition of the instrument that generates the decimal representation of the number $\pi$ and the assumption that there will be one hundred successive zeros lead to a formal-logical contradiction.
  \item B) There will be one hundred successive zeros among the decimal digits of the number $\pi$. Suppose (an empirically possible fact) that someone succeeded to prove thesis $A$ wrong. This could have happened, for instance, if this someone would have deduced from thesis $A$, by means of a finite chain of syllogisms, the correlation $1 = 0$.
\end{itemize}

... Does it thus follow that the validity of thesis $B$ is established? Look at both theses closely and you will see that it does not follow at all; and this could also...\textsuperscript{187}

\textsuperscript{14} It is important to observe that where Khinchin spoke of a ‘(generating) instrument’, Brouwer’s intuitionism was based on a ‘subject’ constructing (mathematical) objects from a ‘primordial consciousness’. This, of course, fits Khinchin’s ‘materialist’ interpretation of Brouwer.
be confirmed by the most ardent opponent of intuitionism. If so, then theses $A$ and $B$ are not contradictory and, accordingly, applying the law of excluded middle to this disjunction is not correct.

Here follows the conclusion: 1) There will be one hundred successive zeros among the decimal digits of the number $\pi$, 2) there will not be one hundred successive zeros among the decimal digits of the number $\pi$ – these two assertions, if understood correctly and in the only way possible, are not contradictory and, therefore, applying the law of excluded middle to this disjunction is ungrounded.

In this respect we can imagine the following dialogue between an intuitionist and a formalist:

**F.** Fine. We accept your triple disjunction: 1) one hundred zeros exist, 2) according to your interpretation, it does not exist, 3) tertium. If we assume that tertium occurs, then the one hundred zeros we are looking for cannot be indicated in any case (because it would lead to the 1st and not to the 3rd case). To me, this merely means that one hundred zeros do not exist at all and that your disjunction acquires a different meaning in my language, namely: 1) one hundred zeros exist, 2) one hundred zeros do not exist, and this can be proved, 3) one hundred zeros do not exist and this cannot be proved. I combine the second and third case, thereby I simply say that one hundred zeros do not exist and I do not bother about the provability of this fact. You see now that we are just arguing about terminology.

**I.** This is incorrect. Thesis 1 of my disjunction can only mean that one hundred zeros are actually found (similar to the second thesis, which can only mean that the proof of the nonexistence of one hundred zeros is actually found). Therefore, my tertium does not exclude the possibility of finding one hundred zeros. Moreover, your formulation of this tertium does not make any sense to me because ‘the absence of one hundred zeros’ is a property of the generating instrument. If this property does not follow from its definition, which means that it cannot be proved, then it does not exist because once it has been defined a mathematical phenomenon possesses only those properties which follow from its definition and are thus provable.
F. But can a mathematician ever accept such a view? Can an answer to a mathematical problem depend on what one of us can or cannot do?

I. I wash my hands hereof. The question about the existence of one hundred zeros has not been raised by me. It is of no concern to me, as there is no point in this question from a scientific point of view. Since you have raised this issue, you should clarify what this issue means. I honestly declare that for me the problem of existence of one hundred zeros either has a meaning, which I have indicated, i.e. it is reduced to a historical note deprived of any scientific meaning, or has no meaning at all. We, intuitionists, do not pose and do not consider such problems of existence. When someone gives me a basket of apples and asks me to find out whether there could be an orange among the apples, I understand what it means because the filled basket stands in front of me. But when someone raises a question concerning the existence of one hundred successive zeros, which do not exist as a finished whole, I do not understand what it means. It is clear for me that the very definition of the problem is grounded on a false intuition. It seems to you that the whole finished series of decimal digits of the number π is right in front of you and that you can sort it out like a basket of apples in order to solve the question, without reference to any tertium, whether one hundred digits can be found in the basket. However, such a finished series does not exist as a matter of fact; assertions about the existence or non-existence of one hundred zeros can only be understood in the way that I have indicated. Furthermore, these assertions are, certainly, not contradictory and hence leave room for tertium, and, consequently, the problem of existence, as such, loses any meaning.

F. But almost all contemporary mathematicians use general assertions and negations of existence. Your position threatens mathematics with devastation and impoverishment.

15 These three sentences (‘I wash … point of view’) are remarkable for two reasons. Firstly, Brouwer himself explicitly raised the issue of ‘mathematical existence’; if he focused on the existence of mathematical systems, this was directly linked to his views on the ontological status of mathematical objects. Secondly, Khinchin seems to reduce the issue of ‘mathematical existence’ – whether in the infinite or finite domain – to existence as such which he, in turn, equated with ‘empirical reality’. In other words, even though he followed Brouwer in writing that only ‘individual constructions’ can bring forward actual mathematical objects, Brouwer himself did not define these constructions in terms of ‘concrete matters of fact’.
I. What you call devastation and impoverishment means to me cleansing and recovery. Incautiously applying the law of excluded middle and using false intuition; contemporary mathematics is stuck in futile imaginary problems and damaged by ungrounded assertions. What you call devastation indicates rather a humble wish to admit all of it honestly and to try to discard everything that scientific conscience cannot accept after having resisted the temptation of the attractive, yet internally pointless and ungrounded, achievements of our science. One should recognize that general arguments regarding so-called infinite totalities lack any scientific content and do not create any added value to our science. Only the separate and the concrete, only an individual construction, can provide mathematics with valuable achievements. Mathematics is a specific practice, rather than a doctrine (*mehr ein Tun als eine Lehre*, Brouwer).  

As has become clear in the above, intuitionism first of all rejects problems concerning the existence of an individual with some properties in a certain totality as soon as this totality is considered infinite – because in this case totality is no longer something finished and certain, but rather a process in progress with respect to which the disjunction ‘exists – does not exist’ loses any fundamental meaning. As a result, intuitionism logically established its unique view on the value of mathematical achievements – a view that conflicts with all main trends within contemporary mathematics.

Weierstrass had succeeded in constructing a continuous function having no derivatives.  

Contemporary mathematics takes this distinguished achievement to be the solution to the following crucial problem, namely: does there exist a non-differentiable function in the set of all continuous functions? For an intuitionist, this problem makes no sense. Does this imply that the intuitionist sees no value in Weierstrass’ work? Not at all. In the first place, he would evaluate the very construction of Weierstrass’ function as an exemplar of a mathematical construction. Accordingly, Weierstrass’ construction would...
reveal that it is hopeless to seek for another construction which proves that the continuity of a function entails differentiability. The intuitionist considers this very valuable but cannot agree that Weierstrass’ construction can solve any problem of existence at all.\textsuperscript{18}

This difference of opinion, even though quite deep and fundamental, should not per se have led to a conflict. Everyone is free to perceive and appreciate the achievements of our science as one pleases. But here the formalist asserts as part of his argument: ‘Let us suppose, first of all, that there are one hundred successive zeros among the decimal digits of the number π, this implies the following, etc. Let us suppose, in the second place, that one hundred zeros do not exist, and this implies the same. Given tertium non datur, our claim is proved’.

It is this assertion that the intuitionist vehemently opposes. In his opinion, tertium non datur is not grounded on anything and that is why proof based on this law is invalid. These situations are not as uncommon in mathematics as one might think. It suffices to indicate that the theorem of the existence of a supremum\textsuperscript{19} – a theorem without which one cannot do anything in the modern theory of functions – is decisively rejected by intuitionists because one is forced to ground the proof of it on the illegitimate application of the law of excluded middle.

\textsuperscript{18} If this whole paragraph (‘Weierstrass had succeeded […] existence at all’) may seem surprising in light of the fact that the ‘pre-intuitionist’ Leopold Kronecker (1823-1891) identified Weierstrass’ analysis as being an exemplar of the ‘horrible dream’ of formalism (Van Dalen, 1999 [2013], 625-627), Khinchin’s goal seems to have been to express his dedication to the ‘Russian aim’ of promoting a kind of intuitionism that did not contradict classical analysis. This aim was not only the inspiration for Kolmogorov’s 1925 paper, but also reflects Luzin’s emphasis, in the introduction to his Integral and Trigonometric Series, of the ‘historical continuity of mathematics by asserting that [for example] the new theory of functions was not to be viewed as a repudiation of classical analysis, but rather as its natural extension’ (Phillips, 1978, 286).

\textsuperscript{19} The supremum of a set is its least upper bound (defined as the smallest real number that is greater than or equal to every number in this specific set) and the infimum is its greatest lower bound (defined as the biggest real number that is smaller than or equal to every number in that specific set). In light of Khinchin’s discussion it is merely important to recognize that the supremum and infimum of a set need not exist.
Hence, intuitionism’s main criticism is directed against the logical groundlessness of certain methods of argumentation that are accepted in contemporary mathematics. The question concerning a subject matter has not been raised and the struggle for a subject matter has not yet begun. Let us see, however, what happens next.

Intuitionism’s first attack is refuted in a distinctive and sophisticated way – as suggested by Hilbert several years ago. He states: ‘If you are afraid that the ungrounded application of the law of excluded middle and arguments about existence can plunge you into the depths of logical contradictions, I can prove to you that your apprehension is groundless’. Presumably, Hilbert has, indeed, succeeded, or will succeed in the near future, to prove that conjunctions ensuing from the finite can never lead to formal-logical contradictions when applied to ‘infinite’ totalities. If so, then we, following the main trend of modern mathematics, can legitimately accept the possibility of applying the law of excluded middle as an axiom because in that way a formally non-contradictory logical system would become the field of our research, and this is all that modern mathematics requires nowadays. What more would you need? Is not the struggle against the construction of mathematics identical to the struggle against non-Euclidean geometry, or against introducing ideal elements, which is so popular in contemporary science? Moreover, does it not resemble the obscurantism involved in the resistance to the introduction of complex numbers into mathematics?

However, the intuitionist remains implacable. Perhaps Hilbert has, indeed, proved that that the unconditional application of the law of excluded middle cannot lead to contradictions. But this is not enough. This proof alone does not render the law acceptable; a lie does not become truth because no one notices the lie. We deal here with a phenomenon with which we are familiar, namely with totalities of numbers, points, functions, etc. Yet, someone now requests us to assume that the rules of the problem of existence apply to these phenomena as well, and attempts to substantiate this assumption on the ground that it would never lead to a contradiction. Can scientific conscience admit such an assumption? Can I consider a problem meaningful if I, in my humble opinion, do not see any sense in it? Is it not the same as telling a lie when I have been assured that my lie would remain unpunished?
Undoubtedly, there is a way out of this situation, namely to accept Russell’s formula: mathematics never knows what it deals with? We speak not about a real, naturally given phenomenon, but rather about ‘something’ that is subject to certain axioms. From this perspective, any non-contradictory logical system is a mathematical discipline.

At what price does this way-out come? It comes at the price of rejecting real phenomena. We would thus have to forget what we study, we have to reject the very thought about the subject of our research. The logical structure of the phenomenon, being deprived of its real substance, becomes the only aim of our investigation. Even stronger, we no longer study the logical form of the phenomenon, for this form has already been distorted by the introduction of new axioms, but rather a self-sufficient formal-logical system separated from any reality at all. This system precludes any attempt at going back to the phenomena. It is said to be spending and harmonious; this makes it even worse for those who yield to the temptation. However, we stay loyal to the phenomena because we want to do research, to study the real world and not to engage in formal-logical self-indulgence.

The analogy with non-Euclidean geometry and the introduction of ideal elements is not very effective and does not stand up to criticism. Non-Euclidean geometry is a sub-division of general geometry and as such has a right to existence. Her raison d’être is the legitimate right of mathematics to doubt the compulsory significance of the Euclidean assumptions. This postulate applies to some spaces, but it does not to others. If the question would concern a mere assumption of the possibility of applying the law of excluded middle to certain totalities, no one would object to it. But we are requested to admit that it is applicable to all cases. This is the same as if someone would offer to admit that the same geometry would be applicable to all spaces, with the only difference that the last thesis can be logically rejected.

If we want to stay loyal to the phenomena [192], we can obviously accept only those axioms that are intuitively justified – hence the name (‘intuitionism’). The requirement of the non-contradictoriness of axioms certainly remains valid, but does not suffice. It is inappropriate to impose those properties on the phenomenon that are not inherent to it and to attach meaning to the problems that do not really have meaning. This would imply the end of the phenome-
non, a deliberate rejection of it. The abhorrent nature of this approach remains unnoticed only because these tendencies have become usual and universally accepted in contemporary mathematics. It is the main disease of mathematics that should be cured at any price.

Now, we have considered the way in which intuitionism arrives at the slogan of the struggle for a subject matter, thereby touching a nerve of modern mathematics. This is evident from intuitionism's success as well as from the growing number of mathematicians involved in this struggle. It is no longer possible to ignore the assault of this powerful trend within academic thought. No matter how we would evaluate this trend, we have to recognize that intuitionism has already refuted the intellectual perspective that prevailed in mathematics fifteen years ago. It has brought into the resources of the philosophy and the methodology of our discipline something that, perhaps, has not yet assumed its complete form and has not yet been fully comprehended, but that is clearly perceived by us as an inspiring and tangible achievement – one which promises to put us back onto the ground that we have almost lost from under our feet. That is the Zeitgeist today. Even Hilbert, in his recent paper ‘Über das Unendliche’ published just some months ago, seriously takes into account this Zeitgeist. In one of his opening theses he asserts that we can conceive only the finite in substantial terms and that there is no infinity either in nature or in our perception (we have already referred to this thesis). Hilbert emphasizes the need to rescue all the distinguished achievements of the last decennia and declares that such a rescue is possible, or even stronger, has already been accomplished by him.

He reasons as follows; alongside the finite subjects of mathematical reflections that are familiar to us, we take into consideration new ideal elements (purely fictitious constructions devoid of objective reality) and subjugate the whole system to a certain set of axioms that is surely non-contradictory. Meanwhile, the law of excluded middle is assumed with regard to infinite totalities (ideal constructions). This constructed logical system is liable to evolve naturally. In the course of this evolution, one can achieve certain results that exclusively concern finite (real) elements of the system (so that the ideal elements play the role of temporary auxiliary constructions similar to, for instance, complex numbers in mechanics or in the theory of numbers, that are eventually to disappear). Undoubtedly, such a result is quite substantial and firmly grounded; since it concerns only finite objects, its validity or falsity can be verified. [192]
It cannot be false, as it has been an outcome of an initially non-contradictory system which comprises a set of axioms applied to finite subjects of mathematical reflection. Hence, the mathematics of the infinite is essentially transformed into a certain heuristic method that is at the service of finite mathematics.

We do not yet know how intuitionism will react to this new situation.\(^{20}\) Apparently, it can principally consider itself a winner; further debate is already moving from the fundamental to the practical ground in so far as the advantage of the heuristic method is measured, obviously, by its usefulness.

The Department of Natural and Exact Sciences of the Communist Academy organized a special seminar on intuitionism under the supervision of the author of this article. The present article comprises introductory information; the seminar intends to publish its proceedings in the near future. The author takes the opportunity to thank the participants of the seminar he supervised, namely I.V. Arnol’d and G.B. Gurevich, whose lectures (given in the winter of 1925-26) at the seminar significantly helped him to write the present article.

\(^{20}\) In this passage (‘Hence, mathematics of [...] this new situation’) Khinchin does not seem to be aware of the fact – or, perhaps, does not want to realize – that where Hilbert identified classical mathematics with finite mathematics plus ideal elements, Brouwer came to argue that classical and intuitionistic mathematics are incompatible. Put differently, the question of how intuitionism would react to the new situation is premised on two problematic assumptions. Firstly, that Hilbert, from the publication of ‘On the infinite’ onwards, committed himself to an ‘intuitionist finitism’. And secondly, that the ground of Hilbert’s ‘finite mathematics’ was similar to that of Brouwer’s ‘finite mathematics’, namely empirical reality.
References


