Update on scalar singlet dark matter
Cline, J.M.; Scott, P.; Kainulainen, K.; Weniger, C.

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One of the simplest models of dark matter is where a scalar singlet field $S$ comprises some or all of the dark matter and interacts with the standard model through an $|H|^2S^2$ coupling to the Higgs boson. We update the present limits on the model from LHC searches for invisible Higgs decays, the thermal relic density of $S$, and dark matter searches via indirect and direct detection. We point out that the currently allowed parameter space is on the verge of being significantly reduced with the next generation of experiments. We discuss the impact of such constraints on possible applications of scalar singlet dark matter, including a strong electroweak phase transition, and the question of vacuum stability of the Higgs potential at high scales.

I. INTRODUCTION

Scalar singlet dark matter [1–3] is an attractive model due to its simplicity; the essential couplings are just its bare mass term and a cross-coupling to the standard model (SM) Higgs field,

$$V = \frac{1}{2} \mu_S^2 S^2 + \frac{1}{2} \lambda_{hs} S^2 |H|^2.$$  

(1)

After electroweak symmetry breaking, the $S$ boson mass receives contributions from both terms, giving

$$m_S = \sqrt{\mu_S^2 + \frac{1}{2} \lambda_{hs} v_0^2},$$

(2)

where $v_0 = 246.2$ GeV is the Higgs vacuum expectation value (VEV). Phenomenology of this model has been studied in Refs. [4–23].

The Higgs cross-term is generically expected to be present because it is a dimension-4 operator that is not forbidden by any symmetry. Apart from the $S$ kinetic term and its quartic self-coupling (which plays no observable role in phenomenology), the two terms in Eq. (1) are in fact the only renormalizable terms allowed by general symmetry arguments. Terms cubic or linear in $S$ are excluded if one demands that $S$ is absolutely stable, and therefore a viable dark matter (DM) candidate, by imposing the $Z_2$ symmetry, where we focus exclusively on the stable case.

The single $S^2|H|^2$ coupling is, however, enough to allow for a contribution to the invisible decay of the Higgs boson, scattering of $S$ on nucleons through Higgs exchange, and annihilation of $S$ into SM particles, leading to indirect detection signatures and an allowed thermal relic density. The scalar singlet model with $Z_2$ symmetry is, in essence, the simplest possible UV-complete theory containing a WIMP. It is intriguing that natural values of $\lambda_{hs} \lesssim 1$ and $m_S$ below a few TeV simultaneously reproduce the observed DM relic density and predict a cross section for scattering on nucleons that is not far from the current direct detection limit.

These aspects have of course been widely studied, with Refs. [25–28] providing the most recent comprehensive analyses. We believe it is worthwhile to update the results presented there, for several reasons:

1. Some [25,26] were done before the mass of the Higgs boson was measured by ATLAS and CMS, and the dependence of the results on $m_h$ was shown for only a limited number of Higgs masses.

2. With the exception of Ref. [28], these recent studies were performed prior to the release of updated direct detection constraints by the XENON100 Collaboration [29].

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*jecline@physics.mcgill.ca  †patscott@physics.mcgill.ca  ‡kimmo.kainulainen@jyu.fi  §c.weniger@uva.nl

1These upper limits based on perturbativity in the $\lambda_{hs}$ coupling are more stringent than the unitarity bounds on the annihilation cross section [24].
(3) The predicted direct detection cross section depends on the Higgs-nucleon coupling. Recent results from lattice studies \[30\text{–}40\] and chiral perturbation theory \[41\text{–}47\] have reduced the theoretical uncertainty in this quantity.

(4) Limits on the invisible width of the Higgs have improved \[48\] since all of the recent studies of this model, reducing the allowed parameter space in the region \(m_S < m_h/2\).

(5) The constraints on \(\lambda_{hS}\) from direct detection presented by Refs. \[25\text{,}27\text{,}28\] and from indirect detection in Ref. \[28\] were derived without taking into account the fact that larger values of \(\lambda_{hS}\) suppress the \(S\) relic density by increasing the annihilation cross section. This reduces the overall predicted signal for scattering on nucleons and annihilation into SM particles. Because of this effect, the dependence on \(\lambda_{hS}\) of the direct and indirect detection constraints is significantly different than one might have expected, as noted in Ref. \[26\]. We take the view here that singlet dark matter might provide only a fraction of the total dark matter density, which is a logical possibility.

(6) In some previous studies (e.g., Ref. \[26\]), the relic density has not been computed using the full thermal average of the annihilation cross section. It is necessary to do so when \(m_S\) is near \(m_h/2\) in order to obtain accurate results because the integral over DM velocities is sensitive to the degree of overlap with the resonance at center-of-mass energy \(E_{CM} = m_h\). This can change the result by orders of magnitude in comparison to using the threshold approximation.

(7) So far, Ref. \[28\] has been the only comprehensive study of scalar singlet DM to consider recent indirect detection constraints. The most important of these are gamma-ray constraints from Fermi observations of dwarf galaxies. Reference \[28\] implemented these limits in an approximate fashion, rescaling published 95\% limits on the cross sections for annihilation into an incomplete set of SM final states and ignoring the \(SS \rightarrow hh\) channel. Here, we calculate constraints self-consistently for the complete set of branching fractions to SM final states at every point of the parameter space, adding further constraints from the impact of \(SS\) annihilation on the cosmic microwave background (CMB) and providing projected constraints including the impact of the Čerenkov Telescope Array (CTA).

In the following, we outline updated constraints and projections from the Higgs invisible width (Sec. II), the \(S\) thermal relic density (Sec. III), indirect detection (Sec. IV), and direct detection (Sec. V). The relevance of these constraints to some applications of the model is discussed in Sec. VI. We give conclusions in Sec. VII.

II. HIGGS INVISIBLE WIDTH

For \(m_S < m_h/2\), the decay \(h \rightarrow SS\) is kinematically allowed and contributes to the invisible width \(\Gamma_{inv}\) of the Higgs boson. The LHC constraints on \(\Gamma_{inv}\) continue to improve as the properties of the Higgs boson are shown to be increasingly consistent with SM expectations. Reference \[48\] obtains a limit of 19\% for the invisible branching fraction at 2\(\sigma\), based on a combined fit to all Higgs production and decay channels probed by ATLAS, CMS, and the Tevatron.

The contribution to \(\Gamma_{inv}\) in the scalar singlet dark matter model is

\[
\Gamma_{inv} = \frac{\lambda_{hS}^2 v_0^2}{32 \pi m_h} (1 - 4 m_S^2/m_h^2)^{1/2},
\]

(this corrects a factor of 2 error in Eq. (3.2) of Ref. \[49\]). To compute the branching fraction \(\Gamma_{inv}/(\Gamma_{vis} + \Gamma_{inv})\), we take the visible contribution to the width to be \(\Gamma_{vis} = 4.07\) MeV for \(m_h = 125\) GeV.

In the left panel of Fig. 1, we show the limit imposed on the scalar singlet parameter space by the invisible width constraint. For \(m_S < m_h/2\), couplings larger than \(\lambda_{hS} \sim 0.02\text{–}0.03\) are ruled out. Here, we also show the region of parameter space that is projected to be in more than 1\(\sigma\) tension with data if no additional Higgs decays are detected at the 14 TeV LHC after 300 \(fb^{-1}\) of luminosity has been collected. This corresponds to a limit of 5\% on the invisible Higgs branching fraction \[50\].

III. RELIC DENSITY

The relic density of singlet dark matter is mostly determined by Higgs-mediated \(s\)-channel annihilation into SM particles. A subdominant role is played by annihilation into \(hh\), via the direct 4-boson \(h^2S^2\) vertex and \(S\) exchange in the \(t\) channel. As discussed in Ref. \[49\], tree-level calculations for \(SS\) annihilation into two-body final states do not give a very accurate approximation close to the threshold for producing gauge boson pairs, as they miss the 3- and 4-body final states from virtual boson decays as well as QCD corrections for quarks in the final state. However, this can be overcome by using accurate computations of the full Higgs boson width as a function of invariant mass \(\Gamma(m_h)\) from Ref. \[51\] and factorizing the cross section for annihilation into all SM particles except \(h\) as

\[
\sigma v_{rel} = \frac{2 \lambda_{hS}^2 v_0^2 |D_h(s)|^2 \Gamma_h(\sqrt{s})}{\sqrt{s}},
\]

where

\[
|D_h(s)|^2 = \frac{1}{(s - m_h^2 + m_S^2 \Gamma_h^2/m_h^2)^3}.
\]

For \(m_S < m_h/2\), the width in the propagator \(D_h(s)\) (but not elsewhere) must be increased by the invisible contribution due to \(h \rightarrow SS\). For \(m_S > m_h\), Eq. (4) must be supplemented
by the extra contribution from $SS \rightarrow hh$. The perturbative tree-level result for the $SS \rightarrow hh$ cross section is given in Appendix A.

The tabulation of $\Gamma_h(m_h^*)$ in Ref. [51] assumes that $m_h^*$ is the true Higgs mass, associated with a self-coupling $\lambda = (m_h^*)^2/2v_0^2$. Here, $\lambda = 0.13$ is fixed by the true Higgs mass, however, and we find that, for $\sqrt{s} \approx 300$ GeV, we must revert to perturbative expressions for $\Gamma_h(\sqrt{s})$, or otherwise the Higgs 1-loop self-interactions included in the table of Ref. [51] begin to overestimate the width. Above $m_S = 150$ GeV, we revert to the tree-level expressions for the decay width, including all SM final states. The expressions we use can again be found in Appendix A.

To accurately determine the relic density for $m_S$ in the vicinity of the resonance at $4m_S^2 \sim m_h$ in Eq. (4), it is essential to carry out the actual thermal average [52]

$$\langle \sigma v_{\text{rel}} \rangle = \int_{4m_S^2}^{\infty} \frac{s^{3/2} - 4m_S^2 K_1(\sqrt{s}/T) \sigma v_{\text{rel}}}{16Tm_h^4 K_0^2(m_h/T)} \, ds,$$

where $K_1$, $K_2$ are modified Bessel functions of the second kind, and to solve the Boltzmann equation for the relic abundance [53].

The common approximation of setting the threshold value of $\sigma v_{\text{rel}}$ to the standard value of 1 pb · c fails badly close to the resonance. This is because the integral in Eq. (6) can be dominated by the resonance at $s = m_h^2$ even if $m_S$ is considerably below $m_h/2$, possibly increasing $\langle \sigma v_{\text{rel}} \rangle$ by orders of magnitude relative to the threshold value. If $m_S \approx m_h/2$, the thermal averaging pushes $\langle \sigma v_{\text{rel}} \rangle$ to lower values relative to the naive approximation. We compute $\langle \sigma v_{\text{rel}} \rangle$ as a function of temperature and solve the equation for the number density of thermal relic WIMPs numerically, using both a full numerical integration and a very accurate approximation described in Appendix B. The two methods agree to within less than 1%.

The resulting contours of constant relic density are shown in the plane of $m_S$ and the coupling $\lambda_{hS}$ in Fig. 1. We display them both over the entire likely range of dark matter mass values (45 GeV $\leq m_S \leq 5$ TeV) and in the region $m_S \sim m_h/2$ in which annihilation is resonantly enhanced. Constraints from the Higgs invisible width are also plotted in the low-mass region. Below $m_h/2$, the two constraints combine to rule out all but a small triangle in the $m_S$–$\lambda_{hS}$ plane, including masses in the range 52.5–62.5 GeV. In the region above $m_h/2$, the relic density constrains the coupling as a function of mass in a way that can be approximately fit by the dependence $\log_{10} \lambda_{hS} > -3.63 + 1.04 \log_{10}(m_S/\mbox{GeV})$. We plot up to $\lambda_{hS} \approx 8$, which is at the (generous) upper limit of where the theory can be expected to remain perturbative.

### IV. INDIRECT DETECTION

Annihilation of scalar singlet DM into SM particles offers similar opportunities for indirect detection as with other WIMP DM candidates [55–59]. The strongest current limits come from gamma-ray searches for annihilation in dwarf spheroidal galaxies [60–66] (for a recent general review, see Ref. [67]) and impacts of DM annihilation at $z \sim 600$ on the...
angular power spectrum of the CMB [68–76]. At large WIMP masses, it is expected [77,78] that CTA will provide strong constraints.

We calculate limits on the scalar singlet parameter space implied by indirect detection using a combined likelihood function:

$$\ln L_{\text{total}}(m_S, \lambda_{hS}) = \ln L_{\text{CMB}}(m_S, \lambda_{hS}) + \ln L_{\text{dwarfs}}(m_S, \lambda_{hS}) + \ln L_{\text{CTA}}(m_S, \lambda_{hS}).$$

(7)

In general, $L_{\text{total}}$ includes components from all three indirect searches, but we only include CTA when discussing projected limits. All three likelihood functions depend in a direct sense upon $m_S$, but only indirectly upon $\lambda_{hS}$, via the zero-velocity annihilation cross section $\langle \sigma v_{\text{rel}} \rangle_0$, the branching fractions $r_i$ to the $i$th SM annihilation channel, and the total relic density.

We scale all indirect signals for the appropriate relic density for each combination of $m_S$ and $\lambda_{hS}$ self-consistently, suppressing signals where $S$ constitutes only a fraction of the total dark matter. Where the thermal relic density of $S$ is actually larger than the observed dark matter relic density, we simply rescale signals in exactly the same way, increasing the expected signals. We choose to do this rather than fix the relic density to the observed value in this region for the sake of simplicity and illustration; this region is robustly excluded anyway by the relic density constraint, and the thermal abundance could only be reduced to the observed value if some additional nonthermal effects were added to the scalar singlet theory, which would not be in the spirit of our analysis here.

We calculate $\langle \sigma v_{\text{rel}} \rangle_0$ including all allowed 2-body SM final states as per Eqs. (4) and (A4) for $m_S \leq m_t$, or Eqs. (A4), (A1), and (A2) for $m_S > m_t$. To estimate $r_i$, we calculate $\langle \sigma v_{\text{rel}} \rangle_0, i$, for annihilation into a given channel $i$ using these cross sections$^3$ with the zero-velocity replacement $\sqrt{s} \rightarrow 2m_S$ and take $r_i = \langle \sigma v_{\text{rel}} \rangle_0/i/\langle \sigma v_{\text{rel}} \rangle_0$. For $m_S$ just below $m_W$ and $m_Z$, where $\langle \sigma v_{\text{rel}} \rangle_0$ comes from the factorization approximation, we assign any remaining branching fraction to 3- and 4-body final states arising from annihilation into virtual gauge bosons corresponding to the next most massive threshold, i.e., $(W^+W^-)^*$ for $m_S < m_W$ and $(ZZ)^*$ for $m_W \leq m_S < m_Z$.

The final yields of photons and electrons from annihilation into each SM final state that we use for CMB limits come from the PPIC4DMID [79]. The gamma-ray yields we use for Fermi and CTA calculations are from DARKSUSY [80], which we supplement with the photon yield for the $hh$ annihilation channel from PPIC4DMID. For channels in common, we find good agreement between the gamma yields of PPIC4DMID and Darksusy.

Yields from the 3- and 4-body final states initiated by virtual gauge bosons are also required. As these are not already available, for Fermi and CTA, we estimate the photon yields by analytically extending those of the $WW$ and $ZZ$ channels below threshold. This is feasible because the integrated photon multiplicity per annihilation in the energy windows considered in each analysis is very close to linear with $m_S$. We therefore fit a straight line to this multiplicity over a few GeV above threshold in each case and use it to extrapolate a small way below threshold ($< 10$ GeV), in the region in which the emission of virtual gauge bosons is significant. This is an extremely good approximation for Fermi and reasonable for CTA also, although not as good as for Fermi due to the energy dependence of the CTA effective area in this region. If anything, the approximation is marginally optimistic for Fermi (in that the actual yield curve is ever so slightly concave down), whereas, for CTA, it is conservative (as the true yield curve is slightly concave up). We do not perform this exercise for CMB limits, as the actual limits near the $W$ and $Z$ thresholds are strongly dominated by Fermi anyway, and it would be more cumbersome to incorporate this into the CMB analysis; hence we assume that 3- and 4-body final states do not contribute anything to CMB limits, which gives a conservative limit in this region.

To show the relative importance of the various final states as a function of $m_S$, we plot their branching fractions in Fig. 2, along the line in $\{m_S, \lambda_{hS}\}$-space where $S$ constitutes the entire observed relic density. Here, we combine the branching fractions of on-shell and off-shell gauge bosons.

### A. CMB likelihood

We take the CMB likelihood function $L_{\text{CMB}}$ directly from the results presented for annihilation in Ref. [75] (which were partially based on earlier results in Refs. [73,74]), using tables of the effective fraction $f_{\text{eff}}$ of the DM rest mass injected as additional energy into the primordial gas. We interpolate $f_{\text{eff}}$ linearly in $m_s$, then use the calculated values of $r_i$ and $\langle \sigma v_{\text{rel}} \rangle_0$ for each combination of $m_S$ and $\lambda_{hS}$ to obtain the final likelihood. We extend the $f_{\text{eff}}$ tables of Ref. [75] in order to accommodate $S$ masses up to 5 TeV (see appendix C for high-mass $f_{\text{eff}}$ data). For calculating current constraints, we employ the WMAP 7-year likelihood function [81]. For projected constraints, we use the Planck predictions, which assume

$^3$For determining branching fractions, we simply use the tree-level versions; the QCD 1-loop correction has minimal impact above $\sim 70$ GeV, and below this, the exact partitioning into $b$, $c$, and $\tau$ has only a small effect on integrated gamma-ray yields and so modifies the overall limits from indirect detection only very slightly.

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For consistency with other channels, we use the $hh$ gamma-ray yields from PPIC4DMID uncorrected for electroweak bremsstrahlung, as none of the Darksusy yields takes this into account; for all values of $m_S$ we consider here, the impact of electroweak corrections on the yield of the $hh$ channel is less than 10%. 

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We calculate the corresponding Fermi constraints on the annihilation cross section of dark matter dwarf likelihood function integral. The dwarf spheroidals mainly differ in their R that is given by

\[ R = \frac{d}{C} h S \]

for a differential flux of dark matter signal photons that is included and existing WMAP limits. A factor of a few improvement in the signal-to-noise ratio; our projected sensitivities are therefore based on rescaling the official CTA sensitivities for searches for dark matter (8) works as follows. From the upper limits on \( \Phi_{pp} \) as a function of the confidence level \( \alpha \), we determine the inverse function \( \alpha = \alpha(\Phi_{pp}) \). Roughly speaking, this function returns the probability (in repeated experiments) of measuring less than the observed number of events, given some true value of \( \Phi_{pp} \). This can be mapped onto a likelihood function,

\[ -2 \ln L_{\text{dwarfs}}(\Phi_{pp}) = ISF[\alpha(\Phi_{pp})], \]  

where ISF\((x)\) is the inverse survival probability function of a \( \chi^2_{\nu-1} \) distribution. In this way, we obtain \(-2 \ln L(5.0 \times 10^{-30} \text{ cm}^3 \text{s}^{-1} \text{GeV}^{-2}) \approx 4.0\), as expected for a 95% C.L. limit.

When deriving projected limits, we assume that Fermi operates for a total of at least 10 years in the current survey mode and that it is able to add a further ten new southern dwarfs to its combined search. We assume conservatively that the limits on \( \langle \sigma v \rangle_{\text{rel}} \) will scale as \( \sqrt{N} \), following the improvement in the signal-to-noise ratio; our projected Fermi sensitivities are therefore based on rescaling the current limits by a factor of \( \sqrt{20/10 \times 10/3} = 2.68 \).

C. CTA likelihood

For the CTA likelihood function \( L_{\text{CTA}} \), we reconstruct the official CTA sensitivities for searches for dark matter annihilation toward the Galactic center [77], with a few reasonable alternative choices for different parameters. Specifically, we use the “ring method,” assume an Navarro-Frenk-White [85] DM profile, 500 h of observing time, and an effective area corresponding to an extended array including both European and proposed

\[ \text{FIG. 2 (color online). Branching fractions for SS to annihilation at threshold into various SM final states vs the DM mass. We have chosen } \lambda_{dS} \text{ at each dark matter mass such that the } 3 \text{ relic density exactly matches the observed value; these } \lambda_{dS} \text{ values can be seen along the } \Omega_S = \Omega_{DM} \text{ curve in Fig. 1.} \]
U.S. contributions [86]. We include a simple background model based on an $E^{-3}$ electron power law in the sensitivity calculation but neglect protons and do not consider possible systematic effects in the background determination. We caution that, although neglecting background systematics leads to good agreement with recent CTA projections [78], it may result in overly optimistic sensitivities. Full details are given in Appendix D.

D. Indirect detection results

In Fig. 3, we show the combined sensitivity of indirect detection to different parts of the scalar singlet parameter space. For current limits, incorporating existing data from the Fermi combined dwarf analysis and WMAP7, we give only a 1σ band. Almost no parameter space not already excluded by relic density considerations is excluded at much higher C.L. than this. The region $m_b/2 \leq m_S \leq 70$ GeV, where $S$ makes up all of the dark matter, can be seen to be in tension with existing indirect searches at slightly more than the 1σ level. The same is true for a small region at $m_S \leq 49$ GeV, but this is within the area already excluded by the invisible width constraint.

Future combined limits incorporating Planck polarization data, CTA, and extended Fermi dwarf observations will be able to probe the region where $S$ is all the dark matter for $m_b/2 \leq m_S \leq 74$ GeV at 90% C.L. The absence of a signal in any of these searches will place all scalar singlet masses from $m_b/2$ to over 5 TeV in tension with indirect detection at more than the 1σ level, if $S$ makes up all the DM. As mentioned earlier, however, CTA sensitivities should be taken with something of a grain of salt. In Fig. 4 we show the breakdown of the projected 90% C.L. limit into the three different searches. At low masses, Fermi dominates the limit, whereas above $m_S \sim m_b$, CTA takes over. The impact of neglecting 3- and

FIG. 3 (color online). Limits on scalar singlet dark matter from indirect searches for dark matter annihilation. The lowermost shaded region is ruled out because these models exceed the observed relic density. Regions below the other curves are in tension with indirect searches, or will be in the future: at more than 1σ according to current data from Fermi dwarf galaxy observations and WMAP 7-year CMB data (solid), at $\geq 90\%$ C.L. (dashes) and $\geq 1\sigma$ C.L. (dots) with CTA, Planck polarization data, and future Fermi observations. The area ruled out by the Higgs invisible width at $2\sigma$ C.L. is indicated by the shaded region in the upper left-hand corner of both plots. Note that all indirect detection signals are scaled for the thermal relic density of the scalar singlet, regardless of whether that density is greater than or less than the observed density of dark matter. Left: a closeup of the resonant annihilation region. Right: the full mass range.

FIG. 4 (color online). Contributions of different searches for dark matter annihilation to the combined future 90% C.L. exclusion curve. The limit from future Fermi searches for annihilation in dwarf galaxies alone are shown by the dotted line, assuming 10 years of exposure and the discovery of a further ten southern dwarfs. The impact of Planck alone, including polarization data, can be seen from the solid line, and the projected impact of CTA is shown as a dashed line. The parameter space excluded by the relic density appears once more as a dark-shaded area in the lower part of the plot.
V. DIRECT DETECTION

We begin our discussion of the limits from direct searches with a fresh analysis of the complementary determinations of the Higgs-nucleon coupling, which enters in the cross section for singlet dark matter scattering on nuclei. Thanks to vigorous activity within the lattice and the theoretical communities, this coupling seems to be better determined now than it was just a few years ago. For further historical details and impacts of nuclear uncertainties on dark matter direct detection, see Refs. [87–89].

A. Higgs-nucleon coupling

In the past, one of the largest uncertainties in the analysis of singlet DM couplings to nucleons has been the Higgs-nucleon coupling, \( f_N m_N / v_0 \), which depends upon the quark content of the nucleon for each quark flavor. Here, \( m_N = 0.946 \text{ GeV} \) is the nucleon mass (we ignore the small differences between neutrons and protons here). In general, \( f_N \) can be expressed in the form
\[
 f_N = \sum_q f_q = \sum_q m_q \langle N | \bar{q} q | N \rangle, \tag{10}
\]
where the sum is over all quark flavors. The contributions from heavy quarks \( q = c, b, t \) can be expressed in terms of the light ones,
\[
 \sum_{q = c, b, t} f_q = \frac{2}{9} \left( 1 - \sum_{q = u, d, s} f_q \right). \tag{11}
\]
by the following argument [90]. First, by equating the trace of the stress-energy tensor at low and high scales,
\[
 m_N \Delta N = \sum_q m_q \langle \bar{q} q \rangle - (7 \alpha_s / 8 \pi) G_{\mu \nu} G^{\mu \nu}, \tag{12}
\]
and taking the nucleon matrix element, one gets the relation
\[
 m_N = m_N \sum_q f_q + \frac{21}{2} A, \tag{13}
\]
with
\[
 A = -\frac{1}{12 \pi} \langle N | G_{\mu \nu} G^{\mu \nu} | N \rangle. \tag{14}
\]
Second, \( \langle N | \bar{q} q | N \rangle \) for the heavy quarks comes from the triangle diagram that generates the \( h G_{\mu \nu} G^{\mu \nu} \) coupling. Therefore, the heavy-quark \( f_q \) values are related to \( A \) through \( f_q = A / m_N \) for \( q = c, b, t \). Eliminating \( A \) from these equations leads to the claimed relation between the heavy and light quark \( f_q \) values. From the above argument, the overall coupling is
\[
 f_N = \frac{2}{9} + \frac{7}{9} \sum_{q = u, d, s} f_q. \tag{15}
\]
The contributions from \( u, d, \) and \( s \) are related to the light quark matrix element \( \sigma_i \) (which is related to the pion-nucleon isoscalar amplitude \( \Sigma_{\pi N} \); see, e.g., Ref. [37]),
\[
 \sigma_i = m_i \langle N | \bar{u} u + \bar{d} d | N \rangle, \tag{16}
\]
where \( m_i \equiv \frac{1}{2} (m_u + m_d) \), and the nonsinglet combination
\[
 \sigma_0 = m_0 \langle N | \bar{u} u + \bar{d} d - 2 s \bar{s} | N \rangle \tag{17}
\]
and the fairly well known isospin breaking ratio 6
\[
 z = \frac{\langle N | \bar{u} u - s \bar{s} | N \rangle}{\langle N | \bar{d} d - s \bar{s} | N \rangle} = 1.49. \tag{18}
\]
In principle, these relations suffice to determine all light quark \( f_q \) values. Indeed, if we further define the strangeness content through the ratio
\[
 y = -\frac{2 \langle N | s \bar{s} | N \rangle}{\langle N | \bar{u} u + \bar{d} d | N \rangle} = 1 - \frac{\sigma_0}{\sigma_i}, \tag{19}
\]
we can solve
\[
 f_u = \frac{m_u}{m_u + m_d} \frac{\sigma_i}{m_N} \frac{2 z + y (1 - z)}{1 + z}, \tag{20}
\]
\[
 f_d = \frac{m_d}{m_u + m_d} \frac{\sigma_i}{m_N} \frac{2 y (1 - z)}{1 + z}, \tag{20}
\]
\[
 f_s = \frac{m_S}{m_u + m_d} \frac{\sigma_i}{m_N} y. \tag{20}
\]
The quantities \( \sigma_i \) and \( \sigma_0 \) have been evaluated by chiral perturbation theory (ChPT), pion-nucleon scattering, and lattice simulations, with some scatter in the results. For a long time, the canonical ChPT value of \( \sigma_0 \) was \( \sigma_0 \approx 55 \pm 7 \text{ MeV} \) [41–43], but a recent computation found \( \sigma_0 = 58 \pm 9 \text{ MeV} \) [44]. Similarly, for \( \sigma_i \), the older perturbation theory result was \( \sigma_i = 45 \text{ MeV} \), whereas Ref. [45] found \( \sigma_i = 59 \pm 7 \text{ MeV} \). The new result is in good agreement with partial wave analysis of pion-nucleon scattering \( (\sigma_i = 64 \pm 8 \text{ MeV} \) [46]) and in particular with a recent lattice evaluation \( (\sigma_i = 58 \pm 9 \text{ MeV} \) [40]). Depending on which of these sets one accepts, there is a wide range of possible strangeness contents of the nucleon. Fortunately, there also exist many recent, direct lattice evaluations of the strangeness matrix element,
\[
 \sigma_s = m_s \langle N | s \bar{s} | N \rangle, \tag{21}
\]
using \( 2 + 1 \) dynamical quark flavors [30–40]. For a recent review, see Ref. [37]. Although there still is some scatter also in these results, all evaluations agree that \( \sigma_s \) is quite small. Based on a subset of more constraining studies, Refs. [36,47] reported world averages of \( \sigma_s = 43 \pm 8 \text{ MeV} \)

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6This corrects a typo in the definition of \( z \) given in Ref. [25].
and \( \sigma_s = 40 \pm 10 \text{ MeV} \), respectively. However, Ref. [37] arrived at a looser result, \( \sigma_s = 40 \pm 30 \text{ MeV} \), by including also less constraining results in the analysis. (The difference between different sets may be associated with taking the correct continuum limit.)

We have made a statistical analysis of what \( f_N \) might be in light of these constraints on the nucleon matrix elements. We choose to use the isospin breaking ratio \( z \) (Eq. (18)) and the lattice determinations for \( \sigma_l \) and \( \sigma_s \) as inputs. We choose \( \sigma_s \) because there is a consensus on its value when evaluated three different ways and \( \sigma_l \) because lattice simulations agree in the prediction that it is small. To be precise, we shall use a fixed value for isospin breaking \( z = 1.49 \) and \( \sigma_l = 58 \pm 9 \text{ MeV} \) with a Gaussian distribution. For \( \sigma_s \), we explore two possibilities: either \( \sigma_s = 43 \pm 8 \text{ MeV} \) with a Gaussian distribution or \( \sigma_s < 70 \text{ MeV} \) with a top-hat distribution. In addition, we allow the light quark masses to be Gaussian distributed with \( m_q = m_{q,0} + \delta m_q \) with \( \delta m_q = 0.5 \delta m_{q+} + 0.5 \delta m_{q-} \), where [47]

\[
\begin{align*}
    m_{u,0} &= 2.5 \quad \delta m_{u,+} = 0.6 \quad \delta m_{u,-} = 0.8 \\
    m_{d,0} &= 5 \quad \delta m_{d,+} = 0.7 \quad \delta m_{d,-} = 0.9 \\
    m_{s,0} &= 100 \quad \delta m_{s,+} = 30 \quad \delta m_{s,-} = 20.
\end{align*}
\]

Here, all units are in MeV. Finally, the nucleon mass is \( m_N = (m_u + m_d)/2 = 938.95 \text{ MeV} \).

With these inputs, we generate \( 10^7 \) random realizations, from which we construct the distributions for the strangeness content \( y \), the matrix element \( \sigma_0 \), and finally \( f_N \). Results are displayed in Fig. 5. Note that \( \sigma_0 \) distribution is a prediction here. It is satisfying to see that it does not depend much on the strangeness input, and that the distribution \( (\sigma_0 = 55 \pm 9 \text{ MeV}) \) agrees very well with the recent ChPT calculation [44]. This lends support to the self-consistency of our analysis. The strangeness content \( y \) mostly reflects the input choices; the top-hat choice assumes only an upper bound for the strangeness matrix element, so \( y \) is only restricted from above. This upper bound is almost the same as the upper bound in the Gaussian case, which is not consistent with \( y = 0 \).

However, what interests us is that both strangeness input choices give comparable, almost Gaussian, distributions for the Higgs-nucleon coupling. In the top-hat case, we find \( f_N = 0.341 \pm 0.021 \), and in the Gaussian case, \( f_N = 0.348 \pm 0.015 \). All uncertainties quoted are formal 1-\( \sigma \) (68.3\% C.L.) limits.

Thus, the error in the determination of \( f_N \) is quite a lot smaller than one might believe—less than 10\% according to our analysis. Taking the mean value of the two different strangeness input choices, we will use \( f_N = 0.345 \) in our analysis. This is also exactly the central value that we would have obtained (with an uncertainty of 0.016) had we instead used \( \sigma_s = 40 \pm 10 \text{ MeV} \), as advocated in Ref. [36].

### B. Direct detection limits

The cross section for spin-independent scattering of singlet DM on nucleons is given by

\[
\sigma_{SI} = \frac{\lambda_{hs}^2 f_N^2}{4\pi} \frac{\mu^2 m_n^2}{m_S^2},
\]

where \( \mu = m_n/(m_n + m_p) \) is the DM-nucleon reduced mass. The current best limit on \( \sigma_{SI} \) comes from the XENON100 experiment [29]. In our analysis, we allow for the singlet to provide a fraction of the total dark matter, as indicated by the contours in Fig. 1. We thus apply the 90\% C.L. limits of Ref. [29] (which assume a local DM density of 0.3 GeV cm\(^{-3}\)), appropriately weighted by the fraction of dark matter in the singlet component.

In the standard analysis in which only a single component of DM with the full relic density is assumed, the differential rate of detection \( dR/dE \) is proportional to \( (\rho_S/\rho_{DM})\sigma_{SI} \), where \( \rho_S \) is the local DM mass density. Thus, the appropriate rescaling of the limiting value of \( \sigma_{SI} \) is by the fraction \( f_{rel} = \Omega_S/\Omega_{DM} \) of energy density contributed by \( S \) to the total DM density. We assume that there is no difference in the clustering properties of the singlet component and the dominant component so that the local energy density of \( S \) is \( f_{rel}\rho_S \). We therefore demand for every value of \( \{\lambda_{hs}, m_S\} \) that

FIG. 5 (color online). Predicted distributions (in arbitrary units) of the strangeness content \( y \) of the nucleon (left), the nucleon matrix element \( \sigma_0 \) (center), and the Higgs-nucleon coupling factor \( f_N \) (right). These are drawn from a random sample generated using experimental and theoretical constraints, as explained in the text.
where \( \sigma_{Xe} \) is the 90\% C.L. limit from XENON100. As with indirect signals, for simplicity, we perform the same rescaling even if the thermal relic density exceeds the observed value.

The resulting constraints in the \( m_S - \lambda hS \) plane are shown in Fig. 6 as well as projections for how these limits will improve in future xenon-based experiments, assuming that the sensitivity as a function of mass scales relative to that of XENON100 simply by the exposure. The contours showing improvements in the current sensitivity by a factor of 5 or 20 will be relevant in the coming year as LUX expects to achieve such values [91,92], while XENON1T projects a factor of 100 improvement [93,94] within two years. The left panel of Fig. 6 focuses on the resonant annihilation region

\[
\sigma_{eff} \equiv f_{rel} \sigma_{SI} \leq \sigma_{Xe},
\]

where \( \sigma_{Xe} \) is the 90\% C.L. limit from XENON100. As with indirect signals, for simplicity, we perform the same rescaling even if the thermal relic density exceeds the observed value.

The shape of the exclusion contours in the \( m_S - \lambda hS \) plane of course carries over into a similar shape in the \( m_S - \sigma_{SI} \) plane, which is the more customary one for direct detection constraints. We nevertheless replot them in this form in Fig. 7 to emphasize that they look very different.
from the usual ones, being mostly vertical rather than horizontal. Normally, the DM relic density is assumed to take the standard value because the annihilation cross section \( h/C_{27v} \) \( \rho_{\text{DM}} \) is distinct from that for detection, \( C_{27} \). Only because they are so closely related in the present model do we get limits that are modified by the changing relic density as one scans the parameter space.

VI. APPLICATIONS

The singlet model we have considered, or modest elaborations of it, has implications for a number of purposes other than just explaining the dark matter or one of its components. These include strengthening the electroweak phase transition, explaining tentative evidence for 130 GeV and continuum gamma rays from the Galactic center, hints of an extra component of dark radiation from analysis of the cosmic microwave background, a candidate for the curvaton mechanism, and impacting the stability of the Higgs potential near the Planck scale. We briefly discuss these issues in the present section.

A. Strong electroweak phase transition

Recently, it was pointed out that a strong electroweak phase transition (EWPT), with \( \langle \sigma v_{\text{rel}} \rangle \geq 1 \) at the critical temperature, can be obtained in the scalar singlet dark matter model if \( \lambda_{h5} \geq 0.1 \) \( [49] \), thus requiring the singlet to comprise a subdominant component of the total dark matter density. The criterion \( \langle \sigma v \rangle > 1 \) is needed for a successful model of electroweak baryogenesis (also considered in Ref. [49]). The effect of the singlet on the EWPT depends upon an additional operator \( C_{21} S S \), which was not relevant for the preceding analysis. By scanning over \( C_{21} S \), Ref. [49] produced many random realizations of models giving a strong enough EWPT. Here, we have repeated this procedure in order to display the range of viable models in the space of \( \mu_S \) for comparison with Figs. 1–7.

In these models, the \( Z_2 \) symmetry \( S \rightarrow -S \) is temporarily broken by a VEV, \( S_c \), at the critical temperature. It is convenient to parametrize the \( S^4 \) coupling as \( \lambda_S = (\lambda_{h5}^4/4)((\mu_S/S_c)^4) \), where \( \lambda_{h5} = 0.13 \) is the Higgs quartic coupling. We consider \( (\mu_S/S_c)^4 \) in the range 0.1–10, corresponding to \( \lambda_S \in [3 \times 10^{-3}, 3] \). The results are shown in Fig. 8. In the left panel, shaded bands of models correspond to intervals of \( \mu_S/S_c \) as shown in the key of the figure; larger \( \mu_S/S_c \) corresponds to larger \( \mu_S \) at a given mass \( m_S \). There is an island of small \( \mu_S \) near \( m_S \sim m_h/2 \) where SS annihilations are resonantly enhanced. These correspond to \( \mu_S/S_c < 1 \). In the right panel, we take several discrete values of \( \mu_S/S_c \) to better illustrate the dependence of \( \mu_S/S_c \) on the parameters \( m_S, \lambda_{h5} \). For a given value of \( \mu_S/S_c \), there is always a maximum mass \( m_S \) beyond which there is no longer a strong phase transition. For large \( \mu_S/S_c \), this occurs at strong couplings \( \lambda_{h5} > 5 \) that we do not consider.

Contours showing the current direct detection limit [29] and projected ones for experiments with 5 and 20 times greater sensitivity are also shown in the right panel of Fig. 8. A large region of the remaining parameter space
will be excluded by the LUX experiment [91,92], which plans to achieve a factor of better than 10 times improvement relative to Ref. [29] by the end of 2013. Within two years, XENON1T expects to reach 100 times the sensitivity of the XENON100 (2012) [93,94].

The island of models near $m_S/m_h^2$ is squeezed on the left by the requirement $m_S > m_h^2$ due to the constraint on the invisible width of the Higgs and on the right by the direct detection bound. This region will become increasingly narrow as the XENON bounds improve, as shown close up in Fig. 9.

B. 130 GeV gamma-ray line

There has been significant interest in tentative evidence for a 130 GeV gamma-ray line from the Galactic center found in Fermi-LAT data [97–104], which might be interpreted as coming from annihilation of dark matter. In Ref. [105], it was suggested that the scalar singlet dark matter model could provide an explanation, if one added an additional interaction $\lambda_{S,\sigma} S^2|\sigma|^2$ with a charge-two singlet $\sigma$, transforming in the fundamental representation of a new SU(N) gauge interaction. Then, $SS$ can annihilate into $\gamma\gamma$ through a virtual loop of $\sigma$, producing gamma rays of the observed energy if $m_S = 130$ GeV.

To get a large enough cross section into photons, $S$ should be the dominant dark matter particle; hence, $\lambda_{S,\sigma}$ should be close to 0.05. From the right panel of Fig. 6 and the previous discussion, it is clear that these values will be probed in the coming year by LUX. This conclusion could be evaded if glueballs of the new SU(N) are lighter than 130 GeV, however; in that case, $\lambda_{S,\sigma}$ could be much less than 0.05 to evade the direct detection limit, while the $S$ relic density could be achieved by annihilation of $SS$ into glueballs, via the $\sigma$ loop.

C. Continuum gamma rays from the Galactic center

An excess of continuum gamma rays has also been claimed in Fermi-LAT data toward the Galactic center [106–109]. This has been interpreted as consistent with the annihilation of dark matter with a mass of 30–50 GeV and a cross section of $\langle s|\sigma v_{rel}|i\rangle \sim 6–8 \times 10^{-27}$ cm$^3$ s$^{-1}$.
into b quarks \cite{107,109}. Considering that the Fermi-LAT dwarf limit on annihilation into $b\bar{b}$ is $(\sigma v_{\text{rel}})_{b\bar{b}} \lesssim 4 \times 10^{-26}$ cm$^3$s$^{-1}$ at a mass of 50 GeV \cite{64,65}, and remembering that $\sigma v_{\text{rel}}$ scales roughly as $\lambda_{bS}^2$ for fixed $m_S$, we see that all models that could approximately fit this signal (i.e., with appropriate cross sections and masses below $\sim 60$ GeV) lie less than 1 order of magnitude above the indirect limit shown in Fig. 3. At low masses, all these models are therefore excluded by the Higgs invisible width, and above 53 GeV, their thermal relic densities all grossly exceed the observed cosmological abundance of dark matter. Scalar singlet dark matter therefore cannot be responsible for the observed continuum gamma rays at the Galactic centre unless the theory is supplemented by some additional physics that would suppress the thermal relic density.

D. Complex singlet dark matter

Another natural generalization of scalar singlet dark matter is the case in which $S$ is a complex scalar. With no additional interactions, this would be equivalent to two real singlets, and the potential is most naturally written in the form

$$V = \mu^2 |S|^2 + \lambda_{bS}|S|^2|H|^2,$$

(26)

with $S = (S_1 + i S_2)/\sqrt{2}$ giving the relation to the canonically normalized real singlets $S_{1,2}$. The relic density $n$ would thus be doubled relative to the real singlet model with the same values of $m_S$ and $\lambda_{bS}$, and since $n$ scales as $1/\langle \sigma v_{\text{rel}} \rangle \sim \lambda_{bS}^{-2}$, our relic density contours would thus move upward by $\delta \log_{10} \lambda_{bS} \approx 0.15$. The direct detection signal scales roughly as $N/m_S$ for $N$ components of degenerate dark matter, so the contours for direct detection would move to the right by $\delta \log_{10} m_S \approx 0.3$.

It was recently suggested that hints from the CMB of an extra component of dark radiation could be explained in the context of fermionic singlet dark matter if the U(1) symmetry $\psi \rightarrow e^{i\alpha} \psi$ for dark matter number conservation is spontaneously broken near the weak scale. This leads to Nambu–Goldstone bosons comprising the dark radiation and a small mass splitting between the two dark matter components \cite{110}. Scalar singlet dark matter as we consider here offers an alternative implementation of this idea; by adding an extra scalar $X$ that carries dark matter charge 1 or 2 and for which the potential gives it a VEV, we can achieve a similar result. We leave the details for future investigation.

E. Curvaton model

The same model as we are studying as a dark matter candidate was recently considered as a curvaton candidate in Ref. \cite{111}. The curvaton is a massive field for which the fluctuations during inflation later come to dominate the universe, before they decay and produce the primordial density fluctuations. This is an interesting alternative to inflaton fluctuations in the case in which the latter are subdominant. In the present model, $S$ cannot decay, but its annihilations through resonant preheating can convert its fluctuations into Higgs particles which then decay into other standard model particles.

The region of interest in the parameter space $\{m_S, \lambda_{bS}\}$ considered by Ref. \cite{111} is $m_S \in [10^2, 10^{11}]$ GeV, $\lambda_{bS} \in 2 \times [10^{-2}, 10^{-30}]$, which, according to our analysis, should be entirely ruled out. However, we have assumed that the dark matter thermalizes at high temperatures and freezes out in the standard way, whereas the curvaton decay process is a nonthermal one, which can only be reliably calculated until the not-too-late stages of preheating. If the universe thermalizes in this scenario to a maximum temperature below the standard freeze-out value for the dark matter, then it is possible that $S$ could be the curvaton and evade our constraints, while possibly even attaining the right relic density through this nonthermal mechanism. However, it would be numerically very challenging to test the scenario given the current limitations of lattice codes for preheating.

F. Higgs potential stability

A curious feature of the recently determined value of the Higgs boson mass is that it is slightly below what would be needed to maintain positivity of the quartic Higgs coupling $\lambda_h$ under renormalization group group running up to the Planck scale assuming only the standard model \cite{112}. The top quark gives a large negative contribution to the running of $\lambda_h$, which is not quite offset by the positive contribution from $\lambda_h$ itself. However, the coupling $\lambda_{bS}$ gives an additional positive contribution that has the potential to bring about stability of $\lambda_h$. This effect was previously studied in Refs. \cite{13,113–115}.

Although higher-order corrections are needed to make an accurate prediction, one can reasonably approximate the size of the effect using the 1-loop contributions to the beta function $\beta_{\lambda_h}$ in order to make a rough estimate of the magnitude of $\lambda_{bS}$ needed in order to have an impact on the vacuum stability question. It was shown in Ref. \cite{112} that a shift in the top quark mass $\delta m_t = -2$ GeV would be sufficient to yield positivity of $\lambda_h$ up to the Planck scale for $m_h = 125$ GeV. This corresponds to a shift in $\beta_{\lambda_h}$ of \cite{116}

$$\delta \beta_{\lambda_h} = -24 \frac{\delta m_t}{m_t} \frac{y_t^2}{16\pi^2} \approx 0.28 \frac{16\pi^2}{16\pi^2},$$

(27)

where $y_t$ is the top quark Yukawa coupling. On the other hand, the scalar singlet contributes an amount

$$\delta \beta_{\lambda_{bS}} = \frac{1/2}{16\pi^2} \lambda_{bS}^2.$$

(28)

According to this estimate, values near $\lambda_{bS} \sim 0.75$ could be sufficient to achieve stability of the Higgs potential, which would correspond to DM masses $m_S \sim 3$ TeV.
The previous argument ignores the effect of the $\lambda_S S^4$ coupling on the running of $\lambda_h$, which was shown in Ref. [113] to reduce the effectiveness of $\lambda_{hs}$ for improving vacuum stability. Inspection of their results (see Fig. 1 of Ref. [113]) confirms the above estimate for the needed size of $\lambda_{hs} \sim 0.75$.

VII. CONCLUSIONS

The model of scalar singlet dark matter $S$ was proposed at least 28 years ago. We have reconsidered the prospects for its discovery by direct or indirect signals and found that the next 2 years are likely to be crucial. In particular, the XENON1T experiment should discover or rule out the singlet scalar DM cannot be ruled out by any of the forthcoming observations. However, in this region, our momentum-independent relic density calculation, which solves only for the abundance rather than the DM distribution function, should be verified by use of a full momentum-dependent Boltzmann code. We argued that the theoretical uncertainty in the Higgs-nucleon coupling, which has long affected predictions, is now significantly smaller than it was until rather recently.

If the model is excluded by direct searches, then constraints from indirect detection will no longer be competitive, but the situation will be more interesting if there is a direct detection. In that case, complementary information will be required to see whether the singlet model is preferred over other possible models. We have shown (Fig. 4) that there is a region of parameter space in which $S$ provides a not-too-small fraction of the total dark matter while still giving an observable signal in gamma rays that might be detected by the Čerenkov Telescope Array. Interestingly, this includes a theoretically motivated region where the singlet’s effect on the running of the Higgs self-coupling $\lambda_h$ could push it back to a positive value at the Planck scale.

Unfortunately, for most values of the mass $m_S$, there is typically a rather large range of values of its coupling $\lambda_{hs}$ to the Higgs for which direct detection, but not indirect detection, would be possible. These include the regions in which $S$ could help to induce a strong electroweak phase transition. The prospects for indirect detection would be dramatically improved if $S$ couples to some new charged particles, which has been suggested as a scenario for explaining hints of 130 GeV dark matter annihilating into gamma rays at the Galactic center. This intriguing possibility, too, will be settled in the near future, both by improvements in direct detection sensitivity and imminent observations by the HESS-II experiment [117].

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APPENDIX A: $s$-DEPENDENT CROSS SECTIONS

As explained in the main text, we cannot use Eq. (4) with the tabulated values of Ref. [51] for Higgs boson widths $\sqrt{s} \approx 300$ GeV. Instead, we have to use the perturbative cross sections for annihilation into kinematically open channels, which are dominated by the gauge bosons and the top quark. The cross section into gauge bosons is

$$v_{rel} \sigma_{VV} = \frac{\lambda_h^2}{8\pi} \delta_{\nu\nu} |D_h(s)|^2 (1 - 4x + 12x^2), \quad (A1)$$

where $x = M_V^2/s$, $\nu_V = \sqrt{1 - 4x}$, $\delta_{\nu\nu} = 1$, $\delta_{\nu\nu} = 1$, and $[D_h(s)]^2$ is defined in Eq. (5). Annihilation into fermion final states is given by

$$v_{rel} \sigma_{fi} = \frac{\lambda_h^2 m_f^2}{4\pi} X_i |D_h(s)|^2, \quad (A2)$$

where $v_f = \sqrt{1 - 4m_f^2/s}$ and $X_i = 1$ for leptons, while, for quarks, it incorporates a color factor of 3 and an important 1-loop QCD correction [118]:

$$X_q = 3 \left[ 1 + \left( \frac{3}{2} \log \frac{m_0^2}{s} + \frac{9}{4} \right) \frac{4\alpha_s}{\pi} \right], \quad (A3)$$

where $\alpha_s$ is the strong coupling for which we take the value $\alpha_s = 0.12$. Using QCD-corrected annihilation rates for light quarks is an excellent approximation below the lower limit $\sqrt{s} = 90$ GeV to which Ref. [51] gives tabulated results. Neglecting QCD corrections, there would lead to an error of order $O(1)$. Of course, this region turns out to be ruled out. In the large mass region, the QCD correction on the top-quark final state is quite small.

Finally, the annihilation cross section to the Higgs boson pairs is given by

$$v_{rel} \sigma_{hh} = \frac{\lambda_h^2}{16\pi s^3 v_S} \left[ (a_R^2 + a_T^2) s v_S v_h \right. \right.$$

$$+ 4\lambda_{sh} v_0^2 \left( a_R - \frac{\lambda_{sh} v_0^2}{s - 2m_h^2} \right) \log \left( \frac{m_S^2 - t_+}{m_S^2 - t_-} \right) \left. \right.$$
\[
a_R = 1 + 3m_R^2(s - m_R^2)|D_h(s)|^2
\]

\[
a_I = 3m_R^2\sqrt{\Gamma_h(m_h)|D_h(s)|^2}.
\]  

(A5)

In the zero-velocity limit \( \sqrt{s} = 2m_s \), this cross section immediately reduces to the expression given in Eq. (4.1) of Ref. [49].

**APPENDIX B: SOLUTION OF THE BOLTZMANN EQUATION**

The Lee–Weinberg equation for the number density can be written as

\[
\frac{dY}{dx} = Z(x)[Y_{eq}^2(x) - Y^2(x)],
\]

(B1)

where \( Y = n/s \) is the ratio of the WIMP number density \( n \) to entropy \( s \), \( x \equiv m/T \), and

\[
Z(x) = \sqrt{\frac{\pi}{45}} \frac{m_SM_p}{x^2} \sqrt{g_*(\nu_{rel}\sigma)}(x),
\]

(B2)

where the average cross section \( \langle \nu_{rel}\sigma \rangle \) is given in Eq. (6) and

\[
\sqrt{g_*} = \frac{h_{eff}}{\sqrt{g_{eff}}} \left(1 + \frac{T}{3h_{eff}} \frac{dh_{eff}}{dT}\right),
\]

(B3)

where \( h_{eff} \) and \( g_{eff} \) are the effective entropy and energy degrees of freedom, which we compute assuming standard model particle content. Finally,

\[
Y_{eq}(x) = \frac{45}{4\pi^4} \frac{x^2}{h_{eff}(T)} K_2(x)
\]

(B4)

in the Maxwell–Boltzmann approximation. We solve Eq. (B1) both numerically and in a semianalytic freeze-out approximation, which differs slightly from the one usually presented in the literature [52,119]. For a similar treatment, see, however, Ref. [120]. We begin by defining \( Y = (1 + \delta)Y_{eq} \) and rewriting the Lee–Weinberg equation as an equation for \( \delta \):

\[
\frac{d\delta}{dx} + (1 + \delta) \frac{d\log Y_{eq}}{dx} = -Z(x)Y_{eq}(x)\delta(\delta + 2).
\]

(B5)

The freeze-out approximation is based on the observation that \( \delta \) starts to grow slowly, such that \( d\delta/dx \ll \delta \) until \( \delta \sim \mathcal{O}(1) \). When this holds, one can neglect the \( \delta \) derivative and reduce Eq. (B5) into an algebraic equation for \( \delta = \delta(x) \). We turn this argument around by assuming that the approximation holds until some freeze-out value \( \delta_I \) and solve the corresponding freeze-out \( x_I = x(\delta_I) \) from the ensuing condition:

\[\text{Note that, due to the leading exponential behavior at large } x, \frac{d\log Y_{eq}}{dx} \approx -1.\]

8Note that, due to the leading exponential behavior at large \( x \), \( \frac{d\log Y_{eq}}{dx} \approx -1. \)

\[ \lambda_s = 1, \delta_I = 1 \]

FIG. 10 (color online). Shown is the relative difference \( \Delta = (y^\text{full} - y^\text{today})/y^\text{today} \), where \( y^\text{today} \) is obtained from Eq. (B7) and \( y^\text{full} \) from a direct numerical integration of the Lee–Weinberg equation.

\[
x_I = \log \left( \frac{\delta_f(2 + \delta_f)}{1 + \delta_f} \frac{\tilde{Y}_{eq}^2}{y_{eq} - \frac{dy_{eq}}{dx}} \right),
\]

(B6)

where \( \tilde{Y}_{eq} = e^x y_{eq} \). Equation (B6) is simple to solve by iteration. At \( x = x_f \), one then has \( Y_I = (1 + \delta_f)Y_{eq}(x_f) \). For \( x > x_I \), one may safely neglect the \( Y_{eq}^2 \) term (backward reaction), which allows us to integrate the equation exactly to the final result,

\[
Y_{today} = \frac{Y_I}{1 + Y_I A_f},
\]

(B7)

where

\[
A_f = \int_{x_f}^\infty dx Z(x).
\]

(B8)

The \( A_f \)-integral is easy to do numerically. We show the comparison of the numerical and the freeze-out solution of the Lee–Weinberg equation (B1) in Fig. 10 for \( \lambda_{hs} = 1 \) and \( \delta_I = 1 \). Overall, the freeze-out approximation (B7) is found to be accurate to 0.3%–0.7% over most of the parameter space in our model, the exception being close to the Higgs resonance where the error can reach 1.7%. The dependence of the freeze-out solution on \( \delta_I \) is at subpercent level for \( \delta_I = 0.5–1.5 \). Let us point out that, if the quantity \( \sqrt{g_*(\nu_{rel}\sigma)} \) is weakly dependent on \( x \), one can approximate \( A_f \approx x_I Z_f \). This approximation is typically accurate to a few percent at large masses and away from resonances, but it becomes much less accurate near resonances or places where \( \sqrt{g_*(\nu_{rel}\sigma)} \) has abrupt features as a function of \( x \).

**APPENDIX C: CMB \( f_{\text{eff}} \) AT WIMP MASSES ABOVE 1 TeV**

As a supplement to the results of Ref. [75], in Table I, we give values of \( f_{\text{eff}} \) for WMAP7 and Planck at WIMP masses \( m_X \) of 3 and 10 TeV.
The expected number of events in the observable energy window (approximately 30 GeV–8 TeV for this array) is then

\[
\theta_k = \theta_{k,\text{BG}} + \theta_{k,\text{DM}}
\]

\[
= \theta_{k,\text{BG}} + t_{\text{obs}} \frac{J}{E \pi m_{\chi}^2} \int_0^\infty \sum_i dN_i \frac{dE}{dE} A_{\text{eff}}(E) dE.
\]  

(D1)

Here, \(k \in \{\text{ON, OFF}\}\) is a label indicating the region on the sky (signal/on or background/off), whereas \(\theta_{k,\text{BG}}\) and \(\theta_{k,\text{DM}}\) are the expected number of events in region \(k\) from background processes and DM annihilation, respectively. These events are photons in the case of DM annihilation but will be mostly cosmic rays in the case of the background. The term \(dN_i/dE\) is the differential photon yield from the \(i\)th annihilation channel. We assume an integration time \(t_{\text{obs}}\) of 500 h, as in Ref. [78].

The ring method, and on-off analyses generally, are designed to consider the difference between the observed rates in the signal and background regions. If the background rate is expected to be uniform across the entire ring, then after correction for the ratio of sky areas covered by the signal and background regions \(\alpha = \Delta \Omega_{\text{ON}} / \Delta \Omega_{\text{OFF}}\), the expected difference in the observed counts reflects only signal processes

\[
\theta_{\text{diff}} \equiv \theta_{\text{ON}} - \alpha \theta_{\text{OFF}}
\]

\[
= \theta_{\text{ON,BG}} + \theta_{\text{ON,DM}} - \alpha \theta_{\text{OFF,BG}} - \alpha \theta_{\text{OFF,DM}}
\]

\[
= \theta_{\text{ON,DM}} - \alpha \theta_{\text{OFF,DM}}.
\]  

(D2)

In the case of the ring geometry that we adopted for array B from Ref. [77], \(\Delta \Omega_{\text{ON}} = 9.97 \times 10^{-3} \text{ sr}\), \(\Delta \Omega_{\text{OFF}} = 4.05 \times 10^{-3} \text{ sr}\) \(\Rightarrow \alpha = 0.246\). Our value of \(\Delta \Omega_{\text{ON}}\) is \(~4\%\) smaller than stated in Ref. [77], but this can likely be explained by the number of significant figures with which Ref. [77] gave its optimized ring method parameters.

We model the likelihood of observing a given difference \(N_{\text{diff}} \equiv N_{\text{ON}} - \alpha N_{\text{OFF}}\) between the on-region and scaled off-region counts, as the difference of two Poisson processes. This is known as a Skellam distribution [123]:

\[
L_S(N_{\text{diff}}|\theta_{\text{ON}}, \alpha \theta_{\text{OFF}}) = e^{-\theta_{\text{ON}} + \alpha \theta_{\text{OFF}}} \left(\frac{\theta_{\text{ON}}}{\alpha \theta_{\text{OFF}}}\right)^{N_{\text{diff}}} I_{N_{\text{diff}}}(2\sqrt{\alpha \theta_{\text{ON}} \theta_{\text{OFF}}}). \]

(D3)

where \(I_n\) is the \(n\)th modified Bessel function of the first kind. To determine the expected limit as we do here, one simply calculates this likelihood assuming \(N_{\text{diff}} = 0\). Because the dominant background for CTA comes from misidentified electron events, to obtain \(\theta_{k,\text{BG}}\), we model the expected background flux \(\Phi_{\text{BG}}\) with an electron spectrum \(E^2 \Phi_{\text{BG}} = 1.5 \times 10^{-2} \text{ GeV}^2 \text{ cm}^{-2} \text{ s}^{-1} \text{ sr}^{-1}\), as seen by Fermi [124]. Our final effective likelihood function is the ratio of the signal + background likelihood function [Eq. (D3)] to the background-only version.

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\[ \mathcal{L}_{\text{CTA}}(m_s, \lambda_{bg}) = \frac{\mathcal{L}_S[0] \theta_{ON}(m_s, \lambda_{bg}), \alpha \theta_{OFF}(m_s, \lambda_{bg})]}{\mathcal{L}_S[0] \theta_{ON,BG}, \alpha \theta_{OFF,BG}}. \]  

(D4)

In deriving expected limits, we know the best-fit likelihood to occur where the signal contribution is zero, so Eq. (D4) has a maximum \( \mathcal{L}_{\text{CTA}} = 1 \) at \( \langle \sigma v_{rel} \rangle = 0 \). The Skellam distribution is already almost a Gaussian, so by the central limit theorem, the ratio, Eq. (D4), is very close to Gaussian. We can therefore safely consider this likelihood ratio to be \( \chi^2 \) distributed with 1 degree of freedom and derive confidence limits accordingly.


