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Diffusion of Behavior in Network Games with Threshold Dynamics

October 2016

Jia-Ping HUANG¹ Maurice KOSTER² Ines LINDNER³

Research Highlights:

- The main novelty of the paper is to assume that the thresholds are endogenously determined. Agents change their inclination by exposition to other inclinations in the social network.
- With our model we are able to explain a variety of adoption behavior. Of particular interest is the existence of non-monotonic behavior of the aggregate adoption rate which is not possible in the benchmark model without inclination. Our model is therefore able to explain “sudden” outbreaks of collective action.
- This suggests to reinvent the common static and exogenous concept of a *tipping point* by defining it endogenously generated by the network.

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Diffusion of Behavior in Network Games with Threshold Dynamics*

December 2015, revised October 2016

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Abstract

In this paper we propose a generalized model of network games to incorporate preferences as an endogenous driving force of innovation. Individuals can choose between two actions: either to adopt a new behavior or stay with the default one. A key element is an individual threshold, i.e. the number or proportion of others who must take action before a given actor does so. This threshold represents an individual's *inclination* to adopt the new behavior. The main novelty of the paper is to assume that the thresholds are endogenously determined. Agents change their inclination by exposition to other inclinations in the social network. This provides a coupled dynamical system of aggregate adoption rate and inclinations orchestrated by the network. With our model we are able to explain a variety of adoption behavior. Of particular interest is the existence of non-monotonic behavior of the aggregate adoption rate which is not possible in the benchmark model without inclination. Our model is therefore able to explain “sudden” outbreaks of collective action. This suggests to reinvent the common static and exogenous concept of a *tipping point* by defining it endogenously generated by the network.

Keywords: Collective Action, Threshold Dynamics, Diffusion, Social Networks, Tipping Point.

JEL classification: D70, D85.

***Audioslides** are available at <http://bit.ly/1YFuLbJ>.

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1 Introduction

The interest in understanding the driving forces of collective behavior has triggered an extensive literature that spans economics, sociology, marketing and epidemiology. In their seminal work, Schelling [1971] and Granovetter [1978] develop models of collective behavior for situations where individuals have two alternatives and the costs and/or benefits of each depend on how many individuals choose which option. Examples are numerous, including riot behavior, innovation and rumor diffusion, strikes, consumption network externalities, spread of fashions, etc. The key element of Schelling [1971] and Granovetter [1978] is the concept of a *threshold*, i.e., the number or proportion of others who must take action before a given individual does so, or equivalently, the point at which net benefit begins to exceed net cost for that particular individual. Characteristic of these models is that a particular action alternative will only be adopted on a large scale if it achieves some critical mass of support, often referred to as *tipping point*. This concept applies to various fields of collective behavior. In the context of revolutions, for example, it is the level of rioting behavior that triggers protests on a large scale. In marketing, it is usually referred to as a threshold that, once reached, will result in a cascade of additional sales (see e.g. Gladwell [2000]). Extensions of the threshold model with network structures have been studied by, e.g., Watts [2002] in a random network context and Morris [2000] with an infinite network structure. Jackson and Yariv [2005, 2007] studied Granovetter's model with social structure using mean-field approximation. A similar approach can be found in López-Pintado [2006, 2008]. In Jackson and Yariv [2005, 2007], individuals are connected by an undirected graph, the social network, and decide whether to switch to a new behavior, e.g. a new technology, or stay with the default alternative. Similar to the classical literature their model features the existence of a tipping point which is largely determined by the network.

A major drawback of threshold models is that they fail to provide an explanation where the critical mass comes from. In particular, these models cannot explain further changes once a cascade is completed. Volatility of collective behavior such as “sudden” outbreaks or change of collective action is usually introduced as an exogenous trigger. Our goal is to improve existing theories on the emergence and stability of collective behavior by introducing *inclination dynamics* (endogenous thresholds). This approach is based on the insight that agents who interact in a social network have similar values and are important references to each other. Not only the action is observed but, in addition, the experienced payoff or the intensity of feeling or conviction is communicated. Putting it differently, the static threshold approach is replaced by a dynamic approach of threshold contagion. In precisely those situations in which we mainly observe behavior and not inclinations (sentiments) as e.g. the hidden unpopularity of the standing regime or the enthusiasm about an innovation, the limitation to static thresholds might explain why we do not understand, and thus not foresee, sudden actions of groups. As noted in Valente [2005] in the context of diffusion of innovation, “Verbal accounts on how people make

decisions and adopt behavior usually reveal ... whims that are not independent of networks, but not easily captured in social influence models.” We believe that these whims (inclination dynamics) are the unobservable missing link between why some collective action events occur and others do not.

In his overview article about innovation diffusion, Young [2009] compares three broad classes of collective dynamics: contagion, social influence, and social learning. He identifies the dynamic characteristics of a fairly general class of learning models which are surprisingly simple. It boils down to Bayesian updating of beliefs about the quality of innovation relative to the status quo. Individuals start with different initial beliefs about the payoff of the innovation, based on their prior partial information, and update their beliefs through a random meeting process with adopters. In this paper, we mimic the belief updating approach and combine it with collective action on networks in order to model collective action with endogenous driving force. It is important to note here, however, that our main focus is on the arising group dynamics by influencing each others’ inclination to act rather than learning an exogenously given characteristic. We therefore rather speak of *inclination dynamics* modeled by individual thresholds that can change by exposition to other thresholds in the social network.

We follow Jackson and Yariv [2005, 2007] by working with a stylized model of network games. Each individual has to make a binary choice between action A (status quo) and B . The decision depends on the costs and benefits of switching, where the payoff is influenced by the number of neighbors of the individuals that have already switched. This can be seen as a generalized threshold model in the sense that every agent evaluates whether the threshold is reached based on observations in the network. Jackson and Yariv [2005, 2007] analyze the diffusion of behavior when in each period agents choose a best response to last periods behavior. We add another variable, the *inclination*, into the model. This invisible part orchestrates the visible collective action, adoption level respectively. Given the complexity of the system, we use a standard technique for estimating the solution (mean field approximation). With our model we are able to explain a variety of behavior of the adoption rates. In particular, we show how a dynamic process can solve the early mobilization problem of collective action (how to reach a typical mass). In this context we elaborate why the concept of a tipping point is too static in presence of inclination dynamics and introduce a dynamic variant. This dynamic version will be able to explain sparks that can let collective action seem to appear out of nowhere.

Our abstract model and findings can be taken as a metaphor for many applications. In sociology, for instance, it provides a step towards understanding the volatility of rioting behavior and the impact of the social network architecture. In marketing, our results could contribute to understanding why a new product becomes a success or failure. In financial markets, the results could advance the understanding of market sentiments.¹

¹Here, we think of “sentiment” of an investor as the attitude towards anticipated price developments in a market, and of “market sentiment” as the general prevailing attitude.

The structure of the paper is as follows. Section 2 discusses the basic model of Jackson and Yariv [2005, 2007] in which the strategic choice of adoption is a best response dynamics when the inclinations are given as parameter and vary exogenously. Section 3 introduces the endogenous process of inclination formation and shows how to couple the process to the best response dynamics of Section 2. We discuss existence and structure of equilibria for a selection of small networks. Other dynamic properties are illustrated by numerical simulation. Section 4 concludes. Proofs of propositions are shifted to the Appendix.

2 Diffusion Dynamics on Networks

2.1 The Basic Model

We consider a society of individuals, each of them chooses an action between two alternatives A and B . Assume A is the default behavior (the status quo). Choosing B can be interpreted as adopting a new technology, learning another language, etc.

The underlying social network structure is given and characterized by the degree distribution $P(d)$ for $d \in \mathcal{D}$ where $\mathcal{D}$ is the set of all degrees and $\sum_{d \in \mathcal{D}} P(d) = 1$. We assume that $0 \notin \mathcal{D}$, i.e., there are no isolated individuals in the network. Individual i 's degree is denoted by d_i . Put

$$\tilde{P}(d) = \frac{P(d)d}{\bar{d}}, \quad \bar{d} = \sum_{k \in \mathcal{D}} P(k)k, \quad (1)$$

where $\bar{d}$ denotes the average degree of the network. Here, $\tilde{P}(d)$ represents the probability of some agent having degree d conditional on that agent being at the end of a randomly chosen link.²

Agents have a choice between taking either action A or B . Without loss of generality, we define action A to be the default behavior (for example, the status-quo technology). We normalize the payoff from taking action A to be 0. Each individual i has an idiosyncratic cost $c_i > 0$ of choosing B instead of the default behavior A . Agent i 's payoff from adopting behavior B when she has d_i neighbors and expects them each to independently choose B with probability x_i is given by $v(d_i, x_i) - c_i$. She hence prefers to choose B instead of A if

$$v(d_i, x_i) > c_i. \quad (2)$$

Following Jackson and Yariv [2005, 2007] we concentrate on the special case in which $v(d, x) = g(d)x$ such that v is linear in x . With this functional form the adoption rule (2) reads

$$g(d_i)x_i > c_i. \quad (3)$$

²For a derivation of (1) see Newman [2010, Chapter 13.3].

We assume the costs c_i are randomly and independently assigned to each individual i according to a probability distribution. For computational convenience, we follow Jackson and Yariv [2005] in considering the distribution function of the cost reciprocal, i.e., for each individual we consider the distribution of $1/c_i$ instead of c_i . More precisely, let $C \geq 0$ denote the random variable of individual cost, and let $F : \mathbb{R}_+ \rightarrow [0, 1]$ be the cumulative distribution function of $1/C$. Rearranging (3) provides

$$\frac{1}{c_i} > \frac{1}{g(d_i)x_i}. \quad (4)$$

Individual i 's probability of choosing action B is therefore

$$\begin{aligned} \Pr[v(d_i, x_i) > c_i] &= \Pr\left[\frac{1}{c_i} > \frac{1}{g(d_i)x_i}\right] = 1 - \Pr\left[\frac{1}{c_i} \leq \frac{1}{g(d_i)x_i}\right] \\ &= 1 - F\left(\frac{1}{g(d_i)x_i}\right). \end{aligned} \quad (5)$$

2.2 Dynamics and Equilibria

This section introduces a dynamic model in which agents at each time period reconsider whether (2) holds and choose A otherwise (with payoff 0). At each $t > 0$, agents best respond to the distribution of agents choosing B in period $t - 1$, presuming that their neighbors will be a random draw from the population.

Let $x_d^{(t)}$ denote the fraction of degree d agents who have adopted B by time t . Put

$$x^{(t)} = \sum_{d \in \mathcal{D}} \tilde{P}(d) x_d^{(t)} \quad (6)$$

for each $t > 0$. Recall that $\tilde{P}(d)$ represents the probability that any given neighbor of some agent is of degree d . This implies that $x^{(t)}$ represents the link-weighted average of adoption of neighbors in the society.

At time $t = 0$, a fraction $x_d^{(0)}$ of each degree d is exogenously and randomly assigned to action B . Subsequently, each agent best responds which leads to a new $x_d^{(1)}$ for each d . Iterating on this process provides the diffusion process which will converge to an equilibrium, i.e. a steady state in which no agent wishes to change her adoption decision.

Due to the complexity of the system Jackson and Yariv [2005, 2007] propose mean-field analysis to estimate the adoption rates which is a standard technique for estimating solutions in complex stochastic models. In a nutshell, this means that each agent bases her decision on the neighbor average of the population given by (6). At time $t = 0$, we start with the assumption that each agent faces the same fraction of neighbors having adopted, $x^{(0)}$. Subsequently, the fraction of degree d types best responding to $x^{(0)}$ by adopting is

$$x_d^{(1)} = 1 - F\left(\frac{1}{g(d)x^{(0)}}\right).$$

The new link-weighted average of adoption follows as $x^{(1)} = \sum_{d \in \mathcal{D}} \tilde{P}(d)x_d^{(1)}$. Iterating on this provides

$$x_d^{(t)} = 1 - F\left(\frac{1}{g(d)x^{(t-1)}}\right). \quad (7)$$

By combining (6) and (7) and using the property $\sum_{d \in \mathcal{D}} \tilde{P}(d) = 1$, we get

$$x^{(t)} = 1 - \sum_{d \in \mathcal{D}} \tilde{P}(d)F\left(\frac{1}{g(d)x^{(t-1)}}\right). \quad (8)$$

Put

$$h(x) := 1 - \sum_{d \in \mathcal{D}} \tilde{P}(d)F\left(\frac{1}{g(d)x}\right), \quad (9)$$

such that the steady states adoption rates of (8) are given as the solutions of the fixed point equation

$$x = h(x). \quad (10)$$

2.3 Diffusion Process with Time Varying Inclination

Jackson and Yariv [2005] discuss the model for the setting³

$$g(d) = \alpha \cdot d^\beta, \quad \alpha, \beta \in \mathbb{R}.$$

The role of the parameter α is to amplify or decrease the effect of individual degrees on the overall adoption rate as indicated in (3). In particular, a higher α implies a lower individual threshold. This in turn implies a higher individual *inclination* to adopt action B , or, in the terms of equation (5), a higher probability of adoption. Jackson and Yariv [2005] treat α as an exogenously given parameter. In Section 3 we will reintroduce α as a dynamic variable capturing the inclination dynamics. In order to focus on endogenous inclination and simplify our analysis we assume $\beta = 0$ throughout the paper. Note that in this case (8) simplifies to $x^{(t)} = 1 - F(1/(\alpha x^{(t-1)}))$. This implies that for $\beta = 0$ the degree distribution is irrelevant in the benchmark model of Jackson and Yariv [2007]. In Section 3 we will show that with inclination dynamics the network becomes relevant again.

To get some feeling for the influence of inclination $\alpha^{(t)}$ on the adoption rate $x^{(t)}$, we first discuss a simple case where $\alpha^{(t)}$ is time varying but exogenously determined. Jackson and Yariv [2007] show for constant α that the adoption rate over time exhibits an S-shape (see their Proposition 7). In other words, starting from small initial levels, the increase in adoption will gain speed up to a certain level, and will then start to slow down until eventually reaching the steady state. The following example illustrates that this property is not necessarily preserved under the model with time varying inclinations $\alpha^{(t)}$.

³As an extension, Jackson and Yariv [2007] treat $v(d, x)$ as a general function without explicitly considering its shape.

Figure 1: Dynamics of $\{x^{(t)} : t = 0, \dots, 10\}$ with $x^{(0)} = 0.54$, where $\alpha^{(t)} = \alpha^{(0)} + \sum_{s=1}^t (1/2)^s$ for $t = 1, \dots, 10$, $\alpha^{(0)} = 1/2$, and $F \sim \text{Uniform}[0, 5]$. The dynamics follows $x^{(t+1)} = 1 - F\left(\frac{1}{\alpha^{(t)} x^{(t)}}\right)$.

Example 1. Assume $\alpha^{(t)}$ is monotonically increasing in t defined by $\alpha^{(t)} = \alpha^{(0)} + \sum_{s=1}^t (1/2)^s$ with $\alpha^{(0)} = 1/2$. Let F be uniform on $[0, 5]$. Although $\alpha^{(t)}$ grows in a monotonic way, the effect on the adoption rate $x^{(t)}$ can be non-monotonic as illustrated by Figure 1. The dynamics start at low levels such that $x^{(t)}$ moves towards zero. However, since $\alpha^{(t)}$ “heats up” in the background, the graph of $h(x | \alpha^{(t)})$ shifts to the top left quickly enough to “catch” the current level of $x^{(t)}$. In other words, individual thresholds of adopting B decrease quickly enough such that the tipping point falls below $x^{(t)}$ before it drops to zero. After $x(t)$ starts growing it first accelerates until time $t = 4$ and then decelerates.

As we will discuss in the upcoming sections, this “erratic” behavior of x is a systemic phenomenon in the study of endogenous inclination dynamics. For the moment we conclude that the common concept of the “tipping point” or “moment of critical mass” (see e.g. Granovetter [1978] and Jackson and Yariv [2005]) is too short-sighted as it assumes static inclinations.

3 Evolution of Inclination by Social Dynamics

In this section we consider a class of social dynamic models in which individuals observe the inclination of prior adopters. As Young [2009] points out the literature on social dynamics in general, and social learning in particular, is extensive. By means of some simplifying assumptions, however, he introduces a fairly general class of social learning models that still allows for heterogeneous characteristics. We will reintroduce and slightly modify his general framework. First, rather than learning a certain truth (quality of the product) we interpret changes of inclinations (thresholds) as a result of being “charged” by the environment. Second, we add to

the model the feature that social dynamics is governed by the social network.⁴ Finally, we will couple the mechanism of inclination updating with the dynamics of the adoption rate (8).

3.1 The Model

Let $\alpha_d^{(t)}$ denote the inclination of degree d agents such that

$$g(d) = \alpha_d^{(t)}. \quad (11)$$

The time dependent inclination captures the idea that the agent's inclination to adopt B will depend on the inclinations of the neighborhood.⁵ As time proceeds each agent updates her inclination (threshold) due to exposition to the inclination of the social network. For each individual of degree d let $\alpha_d^{(0)} \in (0, \infty)$ denote the prior (initial) inclination towards switching behavior.

Similar to (6), put

$$\alpha^{(t)} = \sum_{d \in \mathcal{D}} \tilde{P}(d) \alpha_d^{(t)}. \quad (12)$$

such that $\alpha^{(t)}$ represents the link-weighted average of neighbor inclination. At $t > 0$, each individual will have met a random draw of neighbors at $t - 1$. If an individual has d neighbors and the overall fraction of adopters is $x^{(t-1)}$, the expected number of independent observations of adopters is $dx^{(t-1)}$ with an average inclination $\alpha^{(t-1)}$.⁶ Following Young [2009], the updating mechanism is such that the posterior inclination is just a weighted average of the own prior inclination and the observed mean, where the weight on the mean is the number of independent observations that produced it⁷

$$\alpha_d^{(t)} = \frac{\tau_d \alpha_d^{(0)} + dx^{(t-1)} \alpha^{(t-1)}}{\tau_d + dx^{(t-1)}} \quad \text{for } d \in \mathcal{D}. \quad (13)$$

⁴Young [2009] models flow of information by a Poisson arrival process including a parameter measuring the extent to which an individual “gets around”, which could be interpreted as a degree in a social network. He notes, however, that this parameter will not be sufficient to describe the impact of the network topology on the dynamics. (See his Footnote 16.)

⁵In a social learning setup, Young [2009] discusses the case $g(d) = Ad^\beta$, with $\beta > 0$ and where A is a normally distributed random variable with mean $\mu > 0$ and variance σ^2 , independent and identically distributed among individuals and time periods. Here, μd^β is the mean individual payoff gain per period of switching to B . Ex ante, however, individuals are not informed about the true value of μ and start with different inclinations at $t = 0$. In order to demonstrate dynamic effects of initial expectations we leave the initial levels as a degree of freedom.

⁶Similar to the method of Section 2 this is a mean field approach.

⁷See equation (12) in Young [2009].

Here, $\tau_d \in (0, \infty)$ reflects *flexibility* in changing inclinations (see Groot [1970], Young [2009]).⁸ Low values of τ_d indicate that relatively little network exposure is necessary to change the agent's inclination.⁹ Note that there is no inclination updating in absence of adoption, $x^{(t-1)} = 0$. For notational convenience we drop the index in the overall inclination profile $\{\alpha_d\}_{d \in \mathcal{D}}$ and represent it by $\{\alpha_d\}$.

It is now an easy task to combine the social dynamics (13) with the dynamics of adoption as discussed in the previous section. Substituting $g(d) = \alpha_d^{(t-1)}$ into Equation (8) gives the update formula for aggregate adoption rate

$$x^{(t)} = 1 - \sum_{d \in \mathcal{D}} \tilde{P}(d) F\left(\frac{1}{\alpha_d^{(t-1)} x^{(t-1)}}\right). \quad (14)$$

We conclude that (13) and (14) represents a $D + 1$ dimensional system pinpointing the evolution of $(\{\alpha_d^{(t)}\}, x^{(t)})$. Note that prior inclinations $\{\alpha_d^{(0)}\}$ explicitly enter the update process (13) which complicates the discussion of solutions.

We summarize (13) and (14) as

$$\begin{bmatrix} \{\alpha_d^{(t)}\} \\ x^{(t)} \end{bmatrix} = H \left(\begin{bmatrix} \{\alpha_d^{(t-1)}\} \\ x^{(t-1)} \end{bmatrix} \middle| \{\alpha_d^{(0)}\} \right), \quad (15)$$

where $H : \mathbb{R}_+^D \times [0, 1] \rightarrow \mathbb{R}_+^D \times [0, 1]$ represents the right hand side of (13) and (14), i.e.,

$$H \left(\begin{bmatrix} \{\alpha_d\} \\ x \end{bmatrix} \middle| \{\alpha_d^{(0)}\} \right) := \begin{bmatrix} \left\{ \frac{\tau_d \alpha_d^{(0)} + dx(\sum_{d \in \mathcal{D}} \tilde{P}(d) \alpha_d)}{\tau_d + dx} \right\} \\ 1 - \sum_{d \in \mathcal{D}} \tilde{P}(d) F\left(\frac{1}{\alpha_d x}\right) \end{bmatrix}. \quad (16)$$

3.2 Equilibrium Structure

A state $(\{\alpha_d\}, x)$ is an equilibrium of (15) if

$$(\{\alpha_d^{(t)}\}, x^{(t)}) = (\{\alpha_d\}, x) \quad \text{for all } t \geq 0. \quad (17)$$

We will refer to (17) as an *individual* equilibrium since no agent changes inclination or behavior.

⁸Note that this is more general than assuming a common flexibility τ for all agents. The most general case would be to assume idiosyncratic τ_i as in the original setting of Young [2009]. The model approach of the present paper, however, is based on representative agents for every degree d . A natural assumption is that τ_d decreases in d such that those "getting around" are more prone to incorporating external experiences. Details of this correlation, however, depend on the context of the application.

⁹This definition corresponds to an *infinite memory property*, which means that every individual remembers her prior inclination and updates it forever. There are other possible definitions, e.g., replacing $\tau_d \alpha_d^{(0)}$ by $\tau_d \alpha_d^{(t-1)}$. In the latter case, the memory of previous periods wears off gradually, resulting in a faster convergence of inclination. Hence the effect caused by heterogenous inclination lasts only for a few periods, which makes unique movement of adoption rate, such as non-monotonicity as we will see, difficult to appear.

The aggregate state (α^*, x^*) is an *aggregate* equilibrium of (15) if

$$(\alpha^{(t)}, x^{(t)}) = (\alpha^*, x^*) \quad \text{for } t \geq 0.$$

An individual equilibrium is always an aggregate equilibrium, while the converse is not true. This easily follows from the fact that α is a weighted average which can be constant although its components might be changing.

Note that the right-hand side of (14) tends to 0 for $x \rightarrow 0$ which implies that $x = 0$ is a fixed point solution. We shall refer to equilibria with zero adoption rate as *trivial* equilibria and conclude that any profile $(\{\alpha_d\}, 0)$ constitutes a trivial equilibrium. We shall refer to individual equilibria with positive adoption level $x > 0$ as non-trivial. From (13) we conclude that this is the case if and only if $\alpha_d^0 = \alpha^0$ for all $d \in \mathcal{D}$. The intuition is straightforward. If inclinations are identical there is no change in dynamics. If there is no change in dynamics then inclinations must be identical as the agents of different degrees sooner or later “bump” into each other and communicate inclinations. We shall refer to this equilibrium simply as an *identical* equilibrium.

The following Lemma summarizes our findings as well as the insight that an aggregate equilibrium induces constant individual inclinations after one time period (statement (III)).

Lemma 1.

- (I) *An individual equilibrium always induces an aggregate equilibrium.*
- (II) *Any non-trivial individual equilibrium is identical.*
- (III) *Non-trivial aggregate equilibria are of two kinds. The corresponding individual inclinations*
 - (a) *change once from period 0 to period 1 and stay unchanged thereafter.*
 - (b) *stay with their initial values. In this case the individual inclinations are identical and thus also constitutes an individual equilibrium. Here, in absence of changing actions, this kind of equilibrium is represented by any (α^*, x^*) that solves (10).*

Proof. Statement (III) remains to be proven.

Assume an aggregate equilibrium such that aggregate inclination $\alpha^{(t)}$ and adoption rate $x^{(t)}$ are constant for all $t \geq 0$. By definition, the initial values $\{\alpha_d^{(0)}\}$ are constant for all $d \in \mathcal{D}$ such that it follows from (13) that $\alpha_d^{(t)} = \alpha_d^{(t-1)}$ for all $t \geq 2$. We conclude that starting from an aggregate equilibrium changing individual inclinations can occur once, if at all, from period 0 to period 1. $\square$

Remark 1. *If we let $\tau_d = 0$ for $d \in \mathcal{D}$ in (13), individual inclinations become memoryless which are simply the link-average inclination of the previous period, i.e.,*

$$\alpha_d^{(t)} = \alpha^{(t-1)} = \sum_{d \in \mathcal{D}} \tilde{P}(d) \alpha_d^{(t-1)}.$$

This leads to an immediate coincidence of individual inclinations to $\alpha^{(0)}$ in the very first period after the process started. On the other hand, if we let $\tau_d \rightarrow \infty$, every individual will keep her initial inclination forever and not update at all. Interestingly, the aggregate inclination will be $\alpha^{(t)} = \alpha^{(0)}$ for $t > 0$, which coincides with the case of $\tau_d = 0$. In both cases, the dynamical process is reduced to the one-dimensional system discussed in Section 2.1.

Due to finding (II) of Lemma 1 we will restrict our equilibrium discussion to the aggregate process $(\alpha^{(t)}, x^{(t)})$. The following insight gives a hint about the set of possible equilibria. Note that $\alpha^{(1)} = \alpha^{(0)}$ and constant $x^{(t)}$ implies that individual inclinations given by (13) are constant after one time period. In case of an aggregate equilibrium, however, the individual inclinations change once from $t = 0$ to $t = 1$ which affects the adoption rate (14). We conclude that the equilibrium equations for the adoption rate read $x^{(2)} = x^{(1)} = x^{(0)}$.

Corollary 2. *An aggregate equilibrium is determined by the three independent equations*

$$x^{(2)} = x^{(1)} = x^{(0)} \quad \text{and} \quad \alpha^{(1)} = \alpha^{(0)}. \quad (18)$$

This implies that if there are more than two different degrees the system (18) is usually underdetermined such that the set of solutions is infinite. The complexity of the dynamical system does not allow closed form solutions under general settings. However, we will discuss some special cases of networks and costs when the number of possible degrees is small.

Example 2. *Consider a network with degrees $\mathcal{D} = \{1, 2, 3\}$ represented by $\tilde{P}(1) = \frac{1}{4}$, $\tilde{P}(2) = \frac{1}{2}$, $\tilde{P}(3) = \frac{1}{4}$. Let F be uniform on $[0, 10]$, $\tau_1 = 1$, $\tau_2 = 2$, and $\tau_3 = 3$. Put $\alpha_3^{(0)} = 10$ and numerically solve Equation (18), one has*

$$(\alpha_1^{(0)}, \alpha_2^{(0)}, x^{(0)}) = (10.9998, 11.7354, 0.9909),$$

resulting in an aggregate equilibrium $(\alpha^, x^*) = (11.1176, 0.9909)$.*

We close this section by insights on the existence of non-trivial equilibria for small numbers of possible degrees.

Proposition 3. *Suppose the set of possible degrees is given by $\mathcal{D} = \{d_1, d_2\}$, $d_1 \neq d_2$ and $\tau_{d_1}, \tau_{d_2} \in (0, \infty)$. The existence of a non-trivial aggregate equilibrium requires $\tau_{d_1}/\tau_{d_2} = d_1/d_2$.*

Proposition 4. *Suppose the set of possible degrees is given by $\mathcal{D} = \{d_1, d_2, d_3\}$. Without loss of generality, let $d_1 < d_2 < d_3$. Put $\tau_d = \tau > 0$ for all $d \in \mathcal{D}$. The set of equilibria is empty if the values of $\{\alpha_d^{(0)}\}$, $d \in \mathcal{D}$, are weakly monotonic in d .¹⁰*

¹⁰Here weakly monotonic means either $\alpha_{d_1}^{(0)} \leq \alpha_{d_2}^{(0)} \leq \alpha_{d_3}^{(0)}$ or $\alpha_{d_1}^{(0)} \geq \alpha_{d_2}^{(0)} \geq \alpha_{d_3}^{(0)}$.

3.3 Long Run Behavior and Convergence: Non-monotonicity

This section illustrates the dynamics of (15) by numerical simulation. Naturally, it is possible that the trajectory of $(\{\alpha_d^{(t)}\}, x^{(t)})$ converges to a limit $(\{\alpha_d^*\}, x^*)$ as t increases. We refer to the corresponding aggregate level (α^*, x^*) as an *attractor*.¹¹ To be precise, a pair $(\alpha^*, x^*) \in \mathbb{R}_+ \times [0, 1]$ is an attractor of (15) if there exists a trajectory $(\{\alpha_d^{(t)}\}, x^{(t)})$ such that

$$\lim_{t \rightarrow \infty} (\alpha^{(t)}, x^{(t)}) = (\alpha^*, x^*)$$

where $\alpha^{(t)} = \sum_{d \in \mathcal{D}} \alpha_d^{(t)}$.

The main focus of our discussion is the existence of non-trivial attractors and non-monotonic behavior of aggregate adoption rate $x^{(t)}$. We assume the cost distribution F to be uniform on $[0, b]$. In this special setting (14) reads

$$x^{(t)} = 1 - \sum_{d \in \mathcal{D}} \tilde{P}(d) \min \left\{ 1, \frac{1}{b\alpha_d^{(t-1)}x^{(t-1)}} \right\} = \sum_{d \in \mathcal{D}} \tilde{P}(d) \left[1 - \min \left\{ 1, \frac{1}{b\alpha_d^{(t-1)}x^{(t-1)}} \right\} \right].$$

We choose a scale free network $P(d) \propto d^{-2.5}$, where the symbol $\propto$ means “proportional to”, with $\mathcal{D} = \{1, \dots, 100\}$. The flexibility parameters $\{\tau_d\}$ are taken as identical, i.e. $\tau_d = \tau$ for all $d \in \mathcal{D}$. The dynamics are qualitatively similar under different values of τ and b .¹² The initial levels of individual inclinations have a subtle impact on the dynamics and demonstrates the effect of the network given by P . We consider a decreasing series of $\{\alpha_d^{(0)}\}$ in d for demonstrating purpose. However, similar observations can also be obtained for $\{\alpha_d^{(0)}\}$ increasing in d .

Figure 2 depicts some points of $(\alpha^{(t)}, x^{(t)})$, represented by black dots, and their moving directions, represented by colored arrows, on the x - α plane for $t = 0, 1, 2, 3, 5$ and 10 . The subfigure on the upper-left illustrates the starting scenario with a selection of initial points located on a grid in the x - α plane. Their moving directions from $t = 0$ to $t = 1$ are represented as arrows. For illustrative convenience the lengths of the arrows are fixed and hence do not represent velocities. Analogously, the remaining subfigures illustrate the location of these points at time t , where arrows indicate the moving direction from t to $t + 1$.

The initial points are taken as a two dimensional 21×21 grid. The (m, n) -th point has initial adoption rate $x^{(0)} = 0.05(m - 1)$, $m = 1, \dots, 21$, such that initial adoption rates are equally distributed over $[0, 1]$. Moreover, we put for the initial inclinations

$$\alpha^{(0)} = 0.1(n - 1) + 0.2,$$

¹¹In general dynamical systems, an attractor is usually also an equilibrium state. In our model (15), however, a state that the system converges to is not necessarily an equilibrium. The name attractor only describes the convergence, i.e., it has a corresponding basin of attraction.

¹²For Figure 2 and 3, we put $\tau = 0.2$ and $b = 10$, i.e. a (weak) stickiness to initial inclinations and a broad range of uniform inverse cost distribution F on $[0, 10]$.

Figure 2: Moving directions of $(\alpha^{(t)}, x^{(t)})$ on the x - α plane for $t = 0, 1, 2, 3, 5, 10$. Different directions are illustrated by different colors: $\nearrow$, $\nearrow$, $\nearrow$, $\downarrow$, $\nwarrow$, $\nwarrow$, $\nwarrow$, some do not appear in the plots.

$n = 1, \dots, 21$, such that $\alpha^{(0)}$ is equally distributed over $[0.2, 2.2]$.¹³ We still have to specify how to allocate the initial individual inclination $\alpha_d^{(0)}$ which are distributed with mean $\alpha^{(0)}$ according to (19). Here we choose a series such that $\{\alpha_d^{(0)}\}$ is decreasing in d .¹⁴

Note that the system (15) cannot be described by a standard vector field plot since the initial inclinations $\alpha_d^{(0)}$ keep influencing the dynamics. This implies for Figure 2 that the same state (x, α) may have different moving directions at any point in time t , depending on initial inclinations.

The subfigure on the upper-left of Figure 2 shows an upwards movement of aggregate inclinations at $t = 0$, except the starting points with zero adoption rate. The latter observation is explained by the fact that updating inclinations can only take place in presence of agents adopting B .

All plots of Figure 2 illustrate a U -shaped function $f(x)$ which divides the x - α plane such that the points above $f(x)$ are increasing in x and decreasing below, as indicated by red (increasing) and violet (decreasing). The complexity of (15) prevents a closed form solution of $f(x)$. The plots illustrate that aggregate states with zero adoption levels $x = 0$ belong to the set of attractors. If the inclination levels are sufficiently high, however, the trajectories can start with decreasing adoption rate only to cross $f(x)$ after some time and start increasing again.

The existence of attractors with non-zero adoption rate is illustrated in Figure 3. Four sample paths with different initial states are illustrated in 3-(a) through 3-(c). Of particular interest is sample 1 which exhibits non-monotonic evolution of adoption rate and converges to a non-trivial attractor while aggregate inclination keeps increasing. This behavior is similar to what has been observed in Figure 1, where parameter $\alpha^{(t)}$ was considered to be exogenously given and increasing in time. Although growing inclination levels are necessary for reaching a non-trivial attractor, it is not sufficient. This is illustrated by sample 2 which starts at a slightly lower aggregate inclination comparing to sample 1, but eventually converges to the trivial attractor. In this case the increase of inclination is not quick enough to catch the dropping adoption rate and hence does not succeed in lifting it back on a growing track. Note that in absence of adoption, $x = 0$, the inclination levels determined by (13) jump back to initial levels which explains the sudden drop to initial levels of sample path 2 and 4. Figure 3-(d) depicts the basins of attraction¹⁵ of trivial and non-trivial attractors.

Our example illustrates paths of non-monotonic behavior of x , in particular abrupt changes in levels. The reason is the underlying endogenous inclination dynamics that orchestrates adoption by means of the network. This subtle interaction of adoption and inclinations could be an explanation of “sparks” of collective behavior. In particular, this demonstrates how inclination

¹³Due to restricted space we restrict the values of $\alpha^{(0)}$ to this interval. However, the graph can be easily extended.

¹⁴In particular, we set $\alpha_d^{(0)} = \text{constant} - \sum_{k=1}^d P(k) + \alpha^{(0)}$, where the constant ensures $\alpha^{(0)} = \sum_{d \in \mathcal{D}} \tilde{P}(d) \alpha_d^{(0)}$.

¹⁵This is the set of initial states which lead the long run behavior of a dynamical system to an attractor.

(a) Sample paths starting with different initial values

(b) Dynamical behavior of aggregate inclination

(c) Dynamical behavior of aggregate adoption rate

(d) Basins of attraction: blue/red points indicate the initial states that converge to attractors with zero/non-zero adoption rate.

Figure 3: Sample paths and basins of attraction

dynamics can solve the early mobilization problem of collective action. This suggests that the concept of a tipping point is too static in presence of inclination dynamics. In fact, the curve dividing the basins of attraction in Figure 3-(d) takes over that role which implies an infinite set of tipping points.

4 Concluding Remarks

The aim of this paper was to improve existing theories on the emergence and stability of collective behavior by explicitly modeling the development of hidden forces of action (sentiments). Our main focus was to explain volatility of collective action. The major challenge was to find a model approach

- with a high level of generality with respect to heterogeneous characteristics (costs, benefits, etc.)
- with the capacity to model network structures
- which is analytically tractable
- that allows to implement and to be tested against empirical data.

Our model approach extends the network model of Jackson and Yariv [2005, 2007] by a dimension of inclination (preference) dynamics such that individual actors influence the others preferential attachment through the social network. Our network model can be interpreted as a generalization of the classical threshold models of Schelling [1971] and Granovetter [1978]. The static threshold approach is replaced by thresholds that are dynamic and endogenously orchestrated by the network.

The main finding is that the aggregate adoption rate can display non-monotonic or abrupt changes in its evolution. This could help to understand “sparks” of collective action, i.e., sudden change in behavior orchestrated by the “invisible hand” of the inclination dynamics. In particular, our model shows that the traditional definition of tipping points is too shortsighted as its level depends on inclinations whose levels change in time.

Due to the complexity of the model our approach is only partly analytically tractable. The fact that the initial inclinations explicitly enter the update process at every time step makes it particularly difficult to get a grip on the interplay of the dynamics of adoption rate and inclination. However, our approach is analytically tractable enough to allow theoretical analysis for networks with low dimensionality, in particular equilibria discussions. For networks with a larger number of possible degrees we have seen by simulation that the properties of the network architecture leave their characteristic footprints on the collective dynamics. Naturally, further discussions are conceivable. A systematic analysis on which kind of networks tend to

display high volatility and the impact of the network structure on the set of (generalized) tipping points is a natural direction to proceed. Another interesting variant could be to think about applications with a different inclination dynamics. For example, social influence suggests that thresholds are driven by a conformity motive. Contagion mechanisms would suggest that inclinations are infectious much like epidemics. Examples for recent work related to collective inclination dynamics include Dauhoo et al. [2016] on stochastic evolution of rumor spreading and Eger [2016] on opinion dynamics and wisdom of crowds. Last but not least the network itself may be subject to dynamics, say, following laws of homophily such that the likelihood of a link increases with similarity of preferences.

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Appendix: Proofs of Propositions

Proof of Proposition 3

We first assume $\tau_{d_1} = \tau_{d_2} = \tau$ and relax it later. Suppose (α, x) is an aggregate equilibrium of (15) with $x > 0$. If $\{d_1, d_2\}$ is the set of possible degrees the equilibrium condition reads

$$\alpha = \tilde{P}(d_1) \frac{\tau \alpha_{d_1}^{(0)} + d_1 x \alpha}{\tau + d_1 x} + \tilde{P}(d_2) \frac{\tau \alpha_{d_2}^{(0)} + d_2 x \alpha}{\tau + d_2 x}. \quad (19)$$

It follows that

$$\begin{aligned} \alpha &= \frac{\tilde{P}(d_1)(\tau \alpha_{d_1}^{(0)} + d_1 x \alpha)(\tau + d_2 x)}{(\tau + d_1 x)(\tau + d_2 x)} + \frac{\tilde{P}(d_2)(\tau \alpha_{d_2}^{(0)} + d_2 x \alpha)(\tau + d_1 x)}{(\tau + d_1 x)(\tau + d_2 x)} \\ &= \frac{\tilde{P}(d_1)[\tau^2 \alpha_{d_1}^{(0)} + \tau x(d_1 \alpha + d_2 \alpha_{d_1}^{(0)}) + x^2 \alpha d_1 d_2]}{\tau^2 + \tau x(d_1 + d_2) + x^2 d_1 d_2} \\ &\quad + \frac{\tilde{P}(d_2)[\tau^2 \alpha_{d_2}^{(0)} + \tau x(d_2 \alpha + d_1 \alpha_{d_2}^{(0)}) + x^2 \alpha d_1 d_2]}{\tau^2 + \tau x(d_1 + d_2) + x^2 d_1 d_2} \\ &= \frac{\tau^2 \alpha^{(0)} + \tau x[\tilde{P}(d_1)(d_1 \alpha + d_2 \alpha_{d_1}^{(0)}) + \tilde{P}(d_2)(d_2 \alpha + d_1 \alpha_{d_2}^{(0)})] + x^2 \alpha d_1 d_2}{\tau^2 + \tau x(d_1 + d_2) + x^2 d_1 d_2} \\ &= \frac{\tau^2 \alpha^{(0)} + \tau x[\{\tilde{P}(d_1)d_1 + \tilde{P}(d_2)d_2\}\alpha + \{\tilde{P}(d_1)d_2 \alpha_{d_1}^{(0)} + \tilde{P}(d_2)d_1 \alpha_{d_2}^{(0)}\}] + x^2 \alpha d_1 d_2}{\tau^2 + \tau x(d_1 + d_2) + x^2 d_1 d_2} \\ &= \alpha + E_1, \end{aligned}$$

where

$$\begin{aligned} E_1 &= \frac{\tau^2(\alpha^{(0)} - \alpha)}{\tau^2 + \tau x(d_1 + d_2) + x^2 d_1 d_2} \\ &\quad + \frac{\tau x[\{(\tilde{P}(d_1) - 1)d_1 + (\tilde{P}(d_2) - 1)d_2\}\alpha + \{\tilde{P}(d_1)d_2 \alpha_{d_1}^{(0)} + \tilde{P}(d_2)d_1 \alpha_{d_2}^{(0)}\}]}{\tau^2 + \tau x(d_1 + d_2) + x^2 d_1 d_2} \\ &= \frac{\tau^2(\alpha^{(0)} - \alpha) + \tau x[\{-\tilde{P}(d_2)d_1 - \tilde{P}(d_1)d_2\}\alpha + \{\tilde{P}(d_1)d_2 \alpha_{d_1}^{(0)} + \tilde{P}(d_2)d_1 \alpha_{d_2}^{(0)}\}]}{\tau^2 + \tau x(d_1 + d_2) + x^2 d_1 d_2} \\ &= \frac{\tau^2(\alpha^{(0)} - \alpha) + \tau x[\tilde{P}(d_1)d_2(\alpha_{d_1}^{(0)} - \alpha) + \tilde{P}(d_2)d_1(\alpha_{d_2}^{(0)} - \alpha)]}{\tau^2 + \tau x(d_1 + d_2) + x^2 d_1 d_2}. \end{aligned}$$

For an equilibrium it must hold that $E_1 = 0$. Since $\tau > 0, d_1 > 0, d_2 > 0$, and $x \neq 0$, we get

$$\begin{aligned}
 & \tau^2(\alpha^{(0)} - \alpha) + \tau x [\tilde{P}(d_1)d_2(\alpha_{d_1}^{(0)} - \alpha) + \tilde{P}(d_2)d_1(\alpha_{d_2}^{(0)} - \alpha)] = 0 \\
 \Leftrightarrow & \tau^2\alpha^{(0)} + \tau x [\tilde{P}(d_1)d_2\alpha_{d_1}^{(0)} + \tilde{P}(d_2)d_1\alpha_{d_2}^{(0)}] = \tau^2\alpha + \tau x [\tilde{P}(d_1)d_2 + \tilde{P}(d_2)d_1]\alpha \\
 \Leftrightarrow & \alpha = \frac{\tau^2\alpha^{(0)} + \tau x [\tilde{P}(d_1)d_2\alpha_{d_1}^{(0)} + \tilde{P}(d_2)d_1\alpha_{d_2}^{(0)}]}{\tau^2 + \tau x [\tilde{P}(d_1)d_2 + \tilde{P}(d_2)d_1]} \\
 & = \alpha^{(0)} + \frac{\tau x [\tilde{P}(d_1)d_2(\alpha_{d_1}^{(0)} - \alpha^{(0)}) + \tilde{P}(d_2)d_1(\alpha_{d_2}^{(0)} - \alpha^{(0)})]}{\tau^2 + \tau x [\tilde{P}(d_1)d_2 + \tilde{P}(d_2)d_1]}. \tag{20}
 \end{aligned}$$

Here we use a small trick. The term $\alpha_{d_1}^{(0)} - \alpha^{(0)}$ can be rewritten as

$$\begin{aligned}
 \alpha_{d_1}^{(0)} - \alpha^{(0)} &= \alpha_{d_1}^{(0)} - \tilde{P}(d_1)\alpha_{d_1}^{(0)} - \tilde{P}(d_2)\alpha_{d_2}^{(0)} = \{1 - \tilde{P}(d_1)\}\alpha_{d_1}^{(0)} - \tilde{P}(d_2)\alpha_{d_2}^{(0)} \\
 &= \tilde{P}(d_2)(\alpha_{d_1}^{(0)} - \alpha_{d_2}^{(0)}).
 \end{aligned}$$

Similarly,

$$\alpha_{d_2}^{(0)} - \alpha^{(0)} = \tilde{P}(d_1)(\alpha_{d_2}^{(0)} - \alpha_{d_1}^{(0)}).$$

This provides for (20)

$$\begin{aligned}
 \alpha &= \alpha^{(0)} + \frac{\tau x [\tilde{P}(d_1)d_2\tilde{P}(d_2)(\alpha_{d_1}^{(0)} - \alpha_{d_2}^{(0)}) + \tilde{P}(d_2)d_1\tilde{P}(d_1)(\alpha_{d_2}^{(0)} - \alpha_{d_1}^{(0)})]}{\tau^2 + \tau x [\tilde{P}(d_1)d_2 + \tilde{P}(d_2)d_1]} \\
 &= \alpha^{(0)} + \frac{\tau x (d_2 - d_1)\tilde{P}(d_1)\tilde{P}(d_2)(\alpha_{d_1}^{(0)} - \alpha_{d_2}^{(0)})}{\tau^2 + \tau x [\tilde{P}(d_1)d_2 + \tilde{P}(d_2)d_1]}. \tag{21}
 \end{aligned}$$

From the equilibrium condition $\alpha^{(0)} = \alpha$ we conclude that the second term on the right-hand side of (21) has to be 0. For $d_1 \neq d_2$ it must hold that $\alpha_{d_1}^{(0)} = \alpha_{d_2}^{(0)}$, implying that for $\tau_{d_1} = \tau_{d_2}$ non-trivial equilibria do not exist.

For the case of $\tau_{d_1} \neq \tau_{d_2}$, the same procedure can be used. After some algebra we obtain the following equation

$$\alpha = \alpha^{(0)} + \frac{x(d_2\tau_{d_1} - d_1\tau_{d_2})\tilde{P}(d_1)\tilde{P}(d_2)(\alpha_{d_1}^{(0)} - \alpha_{d_2}^{(0)})}{\tau_{d_1}\tau_{d_2} + x[\tilde{P}(d_1)d_2\tau_{d_1} + \tilde{P}(d_2)d_1\tau_{d_2}]}. \tag{22}$$

Since $\alpha = \alpha^{(0)}$, the second term on the right-hand side of (22) must equal 0 in equilibrium. This implies that either $\alpha_{d_1}^{(0)} = \alpha_{d_2}^{(0)}$ or $d_2\tau_{d_1} = d_1\tau_{d_2}$. Therefore $\tau_{d_1}/\tau_{d_2} = d_1/d_2$ is a necessary condition for existence of non-identical and non-trivial aggregate equilibrium under $\tau_{d_1} \neq \tau_{d_2}$.

□

Proof of Proposition 4

Since we apply a similar argument as in the proof of Proposition 3 we restrict ourselves to outline of analysis and ignore the details of algebra. Suppose (α, x) is an aggregate equilibrium of (15) with $x > 0$. Then α satisfies the following equation.

$$\begin{aligned}\alpha &= \sum_{d \in \{d_1, d_2, d_3\}} \tilde{P}(d) \frac{\tau \alpha_d^{(0)} + dx\alpha}{\tau + dx} \\ &= \tilde{P}(d_1) \frac{\tau \alpha_{d_1}^{(0)} + d_1 x \alpha}{\tau + d_1 x} + \tilde{P}(d_2) \frac{\tau \alpha_{d_2}^{(0)} + d_2 x \alpha}{\tau + d_2 x} + \tilde{P}(d_3) \frac{\tau \alpha_{d_3}^{(0)} + d_3 x \alpha}{\tau + d_3 x} .\end{aligned}\quad (23)$$

Reducing the right hand side of (23) to a common denominator provides

$$\alpha = \alpha + \frac{E_2^N}{E_2^D},$$

with

$$\begin{aligned}E_2^N &= \tau^3(\alpha^{(0)} - \alpha) \\ &\quad + \tau^2 x \left[d_1 \{ \tilde{P}(d_2)(\alpha_{d_2}^{(0)} - \alpha) + \tilde{P}(d_3)(\alpha_{d_3}^{(0)} - \alpha) \} \right. \\ &\quad \quad + d_2 \{ \tilde{P}(d_3)(\alpha_{d_3}^{(0)} - \alpha) + \tilde{P}(d_1)(\alpha_{d_1}^{(0)} - \alpha) \} \\ &\quad \quad \left. + d_3 \{ \tilde{P}(d_1)(\alpha_{d_1}^{(0)} - \alpha) + \tilde{P}(d_2)(\alpha_{d_2}^{(0)} - \alpha) \} \right] \\ &\quad + \tau x^2 \left[d_1 d_2 \tilde{P}(d_3)(\alpha_{d_3}^{(0)} - \alpha) + d_2 d_3 \tilde{P}(d_1)(\alpha_{d_1}^{(0)} - \alpha) + d_3 d_1 \tilde{P}(d_2)(\alpha_{d_2}^{(0)} - \alpha) \right]\end{aligned}$$

and

$$E_2^D = (\tau + d_1 x)(\tau + d_2 x)(\tau + d_3 x) > 0 .$$

Putting $E_2^N = 0$ and solving for α yields $\alpha = \alpha^{(0)} + \frac{E_3^N}{E_3^D}$ where

$$\begin{aligned}E_3^N &= \tau x \left[(\tau + d_3 x)(d_2 - d_1) \tilde{P}(d_1) \tilde{P}(d_2)(\alpha_{d_1}^{(0)} - \alpha_{d_2}^{(0)}) \right. \\ &\quad + (\tau + d_1 x)(d_3 - d_2) \tilde{P}(d_2) \tilde{P}(d_3)(\alpha_{d_2}^{(0)} - \alpha_{d_3}^{(0)}) \\ &\quad \left. + (\tau + d_2 x)(d_3 - d_1) \tilde{P}(d_1) \tilde{P}(d_3)(\alpha_{d_1}^{(0)} - \alpha_{d_3}^{(0)}) \right]\end{aligned}$$

and

$$\begin{aligned}E_3^D &= \tau^3 + \tau^2 x \left[d_1 \{ \tilde{P}(d_2) + \tilde{P}(d_3) \} + d_2 \{ \tilde{P}(d_3) + \tilde{P}(d_1) \} + d_3 \{ \tilde{P}(d_1) + \tilde{P}(d_2) \} \right] \\ &\quad + \tau x^2 \left[d_1 d_2 \tilde{P}(d_3) + d_2 d_3 \tilde{P}(d_1) + d_3 d_1 \tilde{P}(d_2) \right] > 0 .\end{aligned}$$

Since α is the aggregate inclination in equilibrium, it must hold that $\alpha = \alpha^{(0)}$ which in turn implies $E_3^N = 0$. Note that E_3^N contains a sum of three terms which cannot be all positive or negative. The next argument is a case differentiation for different values of $\{\alpha_{d_1}^{(0)}, \alpha_{d_2}^{(0)}, \alpha_{d_3}^{(0)}\}$. Non-identical initial inclinations can be categorized as follows.

- 1) $\alpha_{d_1}^{(0)} = \alpha_{d_2}^{(0)} < \alpha_{d_3}^{(0)} \Rightarrow E_3^N < 0$
 $\alpha_{d_1}^{(0)} = \alpha_{d_2}^{(0)} > \alpha_{d_3}^{(0)} \Rightarrow E_3^N > 0$
 $\alpha_{d_1}^{(0)} = \alpha_{d_3}^{(0)} \neq \alpha_{d_2}^{(0)} \Rightarrow E_3^N = 0$ if $\frac{\tau + d_3x}{\tau + d_1x} = \frac{(d_3 - d_2)\tilde{P}(d_3)}{(d_2 - d_1)\tilde{P}(d_1)}$
 $\alpha_{d_2}^{(0)} = \alpha_{d_3}^{(0)} < \alpha_{d_1}^{(0)} \Rightarrow E_3^N > 0$
 $\alpha_{d_2}^{(0)} = \alpha_{d_3}^{(0)} > \alpha_{d_1}^{(0)} \Rightarrow E_3^N < 0$
- 2) $\alpha_{d_1}^{(0)} < \alpha_{d_2}^{(0)} < \alpha_{d_3}^{(0)} \Rightarrow E_3^N < 0$
 $\alpha_{d_1}^{(0)} < \alpha_{d_3}^{(0)} < \alpha_{d_2}^{(0)} \rightarrow$ see the discussion below
 $\alpha_{d_2}^{(0)} < \alpha_{d_1}^{(0)} < \alpha_{d_3}^{(0)} \rightarrow$ see the discussion below
 $\alpha_{d_2}^{(0)} < \alpha_{d_3}^{(0)} < \alpha_{d_1}^{(0)} \rightarrow$ see the discussion below
 $\alpha_{d_3}^{(0)} < \alpha_{d_1}^{(0)} < \alpha_{d_2}^{(0)} \rightarrow$ see the discussion below
 $\alpha_{d_3}^{(0)} < \alpha_{d_2}^{(0)} < \alpha_{d_1}^{(0)} \Rightarrow E_3^N > 0$

The proof of Proposition 4 is completed with the first row ow 2). To see how necessary equilibrium conditions can be derived when $\alpha_d^{(0)}$'s are all different and non-monotonic consider for example the special case $\alpha_{d_2}^{(0)} < \alpha_{d_1}^{(0)} < \alpha_{d_3}^{(0)}$. Other cases follow analogously.

Putting $E_3^N = 0$ provides

$$\begin{aligned} & (\tau + d_3x)(d_2 - d_1)\tilde{P}(d_1)\tilde{P}(d_2)(\alpha_{d_1}^{(0)} - \alpha_{d_2}^{(0)}) \\ & = (\tau + d_1x)(d_3 - d_2)\tilde{P}(d_2)\tilde{P}(d_3)(\alpha_{d_3}^{(0)} - \alpha_{d_2}^{(0)}) \\ & \quad + (\tau + d_2x)(d_3 - d_1)\tilde{P}(d_1)\tilde{P}(d_3)(\alpha_{d_3}^{(0)} - \alpha_{d_1}^{(0)}). \end{aligned} \quad (24)$$

Both sides of (24) are larger than 0. Solving this equation for $\alpha_{d_3}^{(0)}$ provides

$$\alpha_{d_3}^{(0)} = C_1\alpha_{d_1}^{(0)} + C_2\alpha_{d_2}^{(0)},$$

with

$$\begin{aligned} C_1 &= \frac{(\tau + d_3x)(d_2 - d_1)\tilde{P}(d_1)\tilde{P}(d_2) + (\tau + d_2x)(d_3 - d_1)\tilde{P}(d_1)\tilde{P}(d_3)}{(\tau + d_1x)(d_3 - d_2)\tilde{P}(d_2)\tilde{P}(d_3) + (\tau + d_2x)(d_3 - d_1)\tilde{P}(d_1)\tilde{P}(d_3)}, \text{ and} \\ C_2 &= \frac{(\tau + d_1x)(d_3 - d_2)\tilde{P}(d_2)\tilde{P}(d_3) - (\tau + d_3x)(d_2 - d_1)\tilde{P}(d_1)\tilde{P}(d_2)}{(\tau + d_1x)(d_3 - d_2)\tilde{P}(d_2)\tilde{P}(d_3) + (\tau + d_2x)(d_3 - d_1)\tilde{P}(d_1)\tilde{P}(d_3)}. \end{aligned}$$

It is easy to see that $C_1 + C_2 = 1$, $C_1 > 0$ and $C_2 < 1$. This means that $\alpha_{d_3}^{(0)}$ is an affine combination of $\alpha_{d_1}^{(0)}$ and $\alpha_{d_2}^{(0)}$. Since $\alpha_{d_2}^{(0)} < \alpha_{d_1}^{(0)} < \alpha_{d_3}^{(0)}$ we conclude $C_1 > 1$. Equivalently,

$$\begin{aligned} & (\tau + d_3x)(d_2 - d_1)\tilde{P}(d_1)\tilde{P}(d_2) > (\tau + d_1x)(d_3 - d_2)\tilde{P}(d_2)\tilde{P}(d_3) \\ \Leftrightarrow & \frac{\tau + d_3x}{\tau + d_1x} > \frac{(d_3 - d_2)\tilde{P}(d_3)}{(d_2 - d_1)\tilde{P}(d_1)}. \end{aligned} \quad (25)$$

We conclude that (25) is a necessary condition for existence of equilibrium when $\alpha_{d_2}^{(0)} < \alpha_{d_1}^{(0)} < \alpha_{d_3}^{(0)}$. $\square$