Okounkov's BC-Type Interpolation Macdonald Polynomials and Their q=1 Limit

Koornwinder, T.H.

Publication date
2015

Document Version
Final published version

Published in
Séminaire Lotharingien de combinatoire

Séminaire Lotharingien de Combinatoire **72** (2015), Article B72a

OKOUNKOV'S BC-TYPE INTERPOLATION MACDONALD POLYNOMIALS AND THEIR $q = 1$ LIMIT

TOM H. KOORNWINDER

Abstract. This paper surveys eight classes of polynomials associated with A-type and BC-type root systems: Jack, Jacobi, Macdonald and Koornwinder polynomials and interpolation (or shifted) Jack and Macdonald polynomials and their BC-type extensions. Among these the BC-type interpolation Jack polynomials were probably unobserved until now. Much emphasis is put on combinatorial formulas and binomial formulas for (most of) these polynomials. Possibly new results derived from these formulas are a limit from Koornwinder to Macdonald polynomials, an explicit formula for Koornwinder polynomials in two variables, and a combinatorial expression for the coefficients of the expansion of BC-type Jacobi polynomials in terms of Jack polynomials which is different from Macdonald's combinatorial expression. For these last coefficients in the two-variable case the explicit expression of Koornwinder and Sprinkhuizen [*SIAM J. Math. Anal.* **9** (1978), 457–483] is now obtained in a quite different way.

1. Introduction

In the past half century special functions associated with root systems became an active area of research with many interconnections and applications. The early results were strongly motivated by the notion of spherical functions on Riemannian symmetric spaces. An ambitious program, which still has not come to an end, started to do "zonal spherical analysis" without underlying group and for a wider parameter range than the discrete set of parameter values for which a group theoretic interpretation is possible. Another motivation came from applications in multivariate statistics. By the end of the eighties of the past century Heckman and Opdam consolidated the theory of Jacobi polynomials associated with root systems. In the same period Macdonald, in his *annus mirabilis* 1987, introduced the q-analogues of these Jacobi polynomials in several manuscripts which were circulated and eventually published: *Macdonald polynomials* $P_\lambda(x; q, t)$ (associated with A-type root systems) in [17] and [18, Ch. VI], *Macdonald polynomials associated with root systems* in [19], and scratch notes about hypergeometric functions (associated with BC-type root systems) in [21]. Again in the same period Dunkl introduced his *Dunkl operators*, which inspired Heckman, Opdam and in particular Cherednik to consider the Weyl group invariant (W-invariant) special functions as part of a more general theory of non-symmetric special functions which are eigenfunctions of operators having a reflection term. Special representations of graded

E-mail: `T.H.Koornwinder@uva.nl`.

and double affine Hecke algebras (DAHA's) were an important tool. This approach not only introduced new interesting special functions, but also greatly simplified the W-invariant theory.

The author [13] introduced a 5-parameter class of q-polynomials, on the one hand extending the 3-parameter class of Macdonald polynomials associated with root system BC_n [19] and on the other hand providing the n-variable analogue of the Askey-Wilson polynomials [1]. These polynomials became known in the literature as *Macdonald-Koornwinder* or *Koornwinder polynomials*. Cherednik's DAHA approach could also be used for these polynomials, see Sahi [34], [35] and Macdonald's monograph [20]. A different approach started by work of Sahi, Knop, Okounkov and Olshanski ([32], [12], [11], [27], [24], [25], [26]). It used the so-called *shifted* or *interpolation* versions of Jack and Macdonald polynomials. These could be characterized very briefly by their vanishing property at a finite part of a (q-)lattice, they could be represented by *combinatorial formulas* (tableau sums) generalizing those for Jack and Macdonald polynomials, and they occurred in generalized *binomial formulas*. In particular, Okounkov's [26] *BC_n type interpolation Macdonald polynomials* inspired Rains [29] to use these in the definition of Koornwinder polynomials, thus building the theory of these latter polynomials in a completely new way. An analogous approach then enabled Rains to develop a theory of *elliptic* analogues of Koornwinder polynomials, as surveyed in [30].

Jack and Macdonald polynomials in n variables play a double role, on the one hand as homogeneous orthogonal polynomials associated with root system A_{n-1}, on the other hand as generalized "monomials" (in the one-variable case ordinary monomials) in terms of which orthogonal polynomials associated with root system BC_n can be naturally expanded. This second role is emphasized in the approach using interpolation polynomials, in particular where it concerns binomial formulas.

The present paper surveys, mainly in Sections 4 and 5 and after some preliminaries in Section 3, the definition and properties of eight classes of polynomials: four associated with root system BC_n and four with root system A_{n-1}. Also four of these classes are for general q and four are for $q = 1$. Four of these classes can be considered as orthogonal polynomials while the other four (interpolation) classes only play a role as generalized monomials. There are many limit connections between these eight classes. For six of them (however, see [6] and Remark 6.1) combinatorial formulas are known, see such formulas mainly in Section 6. In a sense these combinatorial formulas are generalized hypergeometric series.

One of the eight classes, the BC_n-type interpolation Jack polynomials, seems to have been overlooked in the literature, although it occurs very naturally in the scheme formed by the limit connections. It will be defined in Section 7. All its properties will be obtained here as limit cases of properties of BC_n-type interpolation Macdonald polynomials, including the combinatorial formula for polynomials of this latter class.

Binomial formulas as they were already known for three classes of polynomials are surveyed in Section 8. The probably new binomial formula for BC_n-type interpolation Jack polynomials is given in Section 9. It gives a new approach to coefficients of the expansion of BC_n-type Jacobi polynomials in terms of Jack polynomials. As a consequence of the binomial formulas a new limit formula (8.4) and a new proof of an already known limit formula (9.2) will follow.

All classes of polynomials and formulas for them become much more elementary and explicit in the one-variable case. This is the subject of the Prelude in Section 2. The two-variable case is already more challenging, but explicit formulas are feasible. This is the topic of Sections 10 and 11. In particular, in Subsection 11.2 we arrive at an explicit expression for BC_2-type Jacobi polynomials which was earlier obtained in a very different way by the author together with Sprinkhuizen in [14].

Acknowledgement The material of this paper was first presented in lectures given at the 72nd Séminaire Lotharingien de Combinatoire, Lyon, France, 24–26 March 2014. I thank the organizers for the invitation. I thank Siddhartha Sahi, Ole Warnaar, Genkai Zhang and an anonymous referee for helpful remarks. Thanks also to Masatoshi Noumi for making available to me his unpublished slides on interpolation functions of type BC.

Notation See [8]. Throughout we assume $0 < q < 1$. (q)-shifted factorials are given by

$$(a)_k := a(a+1)\cdots(a+k-1), \qquad (a)_0 := 1, \qquad (a_1,\ldots,a_r)_k := (a_1)_k\cdots(a_r)_k\,;$$

$$(a;q)_k := (1-a)(1-aq)\cdots(1-aq^{k-1}), \quad (a;q)_0 := 1, \quad (a_1,\ldots,a_r;q)_k := (a_1;q)_k\cdots(a_r;q)_k\,.$$

For n a nonnegative integer we have terminating (q-)hypergeometric series

$${}_rF_s\left(\begin{matrix}-n,a_2,\ldots,a_r\\ b_1,\ldots,b_s\end{matrix};z\right) := \sum_{k=0}^n \frac{(-n)_k}{k!}\,\frac{(a_2,\ldots,a_r)_k}{(b_1,\ldots,b_s)_k}\,z^k,$$

$${}_r\phi_s\left(\begin{matrix}-n,a_2,\ldots,a_r\\ b_1,\ldots,b_s\end{matrix};q,z\right) := \sum_{k=0}^n \frac{(q^{-n};q)_k}{(q;q)_k}\,\frac{(a_2,\ldots,a_r;q)_k}{(b_1,\ldots,b_s);q_k}\,\big((-1)^kq^{\frac12 k(k-1)}\big)^{r-s+1}\,z^k.$$

2. Prelude: the one-variable case

Let us explicitly consider the most simple situation, for polynomials in one variable (in this section n will denote the degree rather than the number of variables). Then both Jack and Macdonald polynomials are simple monomials x^n. Put

$$P_n(x) := x^n, \quad P_n(x;q) := x^n, \quad P_k^{\rm ip}(x) := x(x-1)\cdots(x-k+1) = (-1)^k(-x)_k.$$

$P_k^{\rm ip}(x)$ is the unique monic polynomial of degree k which vanishes at $0,1,\ldots,k-1$. A binomial formula is given by

$$(x+1)^n = \sum_{k=0}^n \binom{n}{k} x^k, \quad \text{or} \quad P_n(x+1) = \sum_{k=0}^n \frac{P_k^{\rm ip}(n)}{P_k^{\rm ip}(k)}\,P_k(x). \tag{2.1}$$

In the q-case put

$$P_k^{\mathrm{ip}}(x;q) := (x-1)(x-q)\cdots(x-q^{k-1}) = x^k(x^{-1};q)_k.$$

$P_k^{\mathrm{ip}}(x;q)$ is the unique monic polynomial of degree k which vanishes at $1,q,\ldots,q^{k-1}$. A q-binomial formula (see [8, Exercise 1.6(iii)]) is given by

$$x^n = {}_2\phi_0\left(\begin{matrix} q^{-n}, x^{-1} \\ - \end{matrix}; q, q^n x\right) = \sum_{k=0}^n \frac{(q^{-n},x^{-1};q)_k}{(-1)^k q^{\frac12 k(k-1)}(q;q)_k}\,(q^n x)^k, \quad \text{or}$$

$$P_n(x;q) = \sum_{k=0}^n \frac{P_k^{\mathrm{ip}}(q^n;q)}{P_k^{\mathrm{ip}}(q^k;q)}\,P_k^{\mathrm{ip}}(x;q). \tag{2.2}$$

Identity (2.1) is the limit case for $q\uparrow 1$ of (2.2). The polynomials $P_k^{\mathrm{ip}}(x)$ and $P_k^{\mathrm{ip}}(x;q)$ are the one-variable cases of the interpolation Jack and the interpolation Macdonald polynomials, respectively.

In the one-variable case BC_n-type Jacobi polynomials become classical Jacobi polynomials and Koornwinder polynomials become Askey-Wilson polynomials. Their standard expressions as (q-)hypergeometric series are:

$$\frac{P_n^{(\alpha,\beta)}(1-2x)}{P_n^{(\alpha,\beta)}(1)} = \sum_{k=0}^n \frac{(-n)_k(n+\alpha+\beta+1)_k}{(\alpha+1)_k\,k!}\,x^k = {}_2F_1\left(\begin{matrix} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{matrix}; x\right) \tag{2.3}$$

and

$$\begin{aligned}\frac{p_n(\frac12(x+x^{-1});a_1,a_2,a_3,a_4\mid q)}{p_n(\frac12(a_1+a_1^{-1});a_1,a_2,a_3,a_4\mid q)} &= \sum_{k=0}^n \frac{(q^{-n},q^{n-1}a_1a_2a_3a_4,a_1x,a_1x^{-1};q)_k}{(a_1a_2,a_1a_3,a_1a_4,q;q)_k}\,q^k \\ &= {}_4\phi_3\left(\begin{matrix} q^{-n},q^{n-1}a_1a_2a_3a_4,a_1x,a_1x^{-1} \\ a_1a_2,a_1a_3,a_1a_4 \end{matrix}; q,q\right).\end{aligned} \tag{2.4}$$

Note that (2.3) gives an expansion in terms of monomials $P_k(x)=x^k$ (Jack polynomials in one variable), while (2.4) gives an expansion in terms of monic symmetric Laurent polynomials

$$P_k^{\mathrm{ip}}(x;q,a_1) := \prod_{j=0}^{k-1}(x+x^{-1}-a_1q^j-a_1^{-1}q^{-j}) = \frac{(a_1x,a_1x^{-1};q)_k}{(-1)^k\,q^{\frac12 k(k-1)}\,a_1^k}\,. \tag{2.5}$$

The monic symmetric Laurent polynomial (2.5) is characterized by its vanishing at a_1, a_1q, $\ldots$, a_1q^{k-1}. It is the one-variable case of Okounkov's BC-type interpolation Macdonald polynomial. If we consider (2.4) as an expansion of its left-hand side as a function of n, then we see that it is expanded in terms of functions $P_k^{\mathrm{ip}}(q^na_1';q,a_1')$ (using the definition in (2.5)), where $a_1' := (q^{-1}a_1a_2a_3a_4)^{\frac12}$. Furthermore, if we replace x by a_1x in (2.5), divide by a_1^k, and let $a_1\to\infty$, then we obtain the q-binomial formula (2.2). Therefore, Okounkov [26] calls (2.5), as well as its multi-variable analogue, also a binomial formula.

If we replace in (2.4) the parameters a_1, a_2, a_3, a_4 by $q^{\alpha+1}, -q^{\beta+1}, 1, -1$ and let $q \uparrow 1$, then we arrive at (2.3), which therefore might also be called a binomial formula. If we consider (2.3) as an expansion of its left-hand side as a function of n, then we see that it is expanded in terms of functions $P_k^{\rm ip}(n+\alpha';\alpha')$, where $\alpha' := \frac{1}{2}(\alpha+\beta+1)$ and

$$P_k^{\rm ip}(x;\alpha) := \prod_{j=0}^{k-1} \big(x^2 - (\alpha+j)\big)^2 = (-1)^k\,(\alpha-x)_k\,(\alpha+x)_k\,, \tag{2.6}$$

a monic even polynomial of degree $2k$ in x which is characterized by its vanishing at α, $\alpha+1$, …, $\alpha+k-1$. This is the one-variable case of the BC-type interpolation Jack polynomial, which (possibly for the first time) will be defined in the present paper.

3. Preliminaries

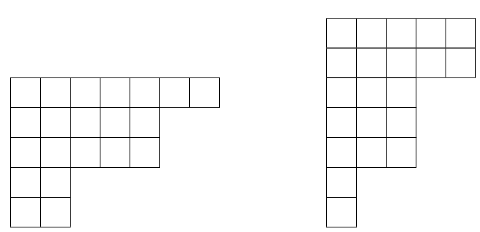
3.1. **Partitions.** We recapitulate some notions about partitions, diagrams and tableaux from Macdonald [18, § I.1]. However, in contrast to [18], we fix an integer $n \ge 1$ and always understand a *partition* λ to be of length $\le n$, i.e., $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ with $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0$. Write $\ell(\lambda) := |\{j \mid \lambda_j > 0\}|$ for the *length* of λ and $|\lambda| := \lambda_1 + \cdots + \lambda_n$ for its *weight*. Also put

$$n(\lambda) := \sum_{i=1}^{n} (i-1)\lambda_i. \tag{3.1}$$

We may abbreviate k parts of λ equal to m by m^k and we may omit 0^k at the end. For instance, $(2,2,1,0,0,0) = (2^2,1,0^3) = (2^2,1)$. There is the special partition

$$\delta := (n-1, n-2, \ldots, 1, 0). \tag{3.2}$$

A partition λ can be displayed by a *Young diagram*, also notated by λ, which consists of boxes (i,j) with $i = 1, \ldots, \ell(\lambda)$ and $j = 1, \ldots, \lambda_i$ for a given i. The *conjugate* partition λ' has diagram such that $(j,i) \in \lambda'$ if and only if $(i,j) \in \lambda$ The example below of the diagram of $\lambda = (7,5,5,2,2)$ and its conjugate $\lambda' = (5,5,3,3,3,1,1)$ will make clear how a diagram is drawn:



For (i,j) a box of a partition λ, the *arm-length* $a_\lambda(i,j)$ and *leg-length* $l_\lambda(i,j)$ are defined by

$$a_\lambda(i,j) := \lambda_i - j, \qquad l_\lambda(i,j) := |\{k > i \mid \lambda_k \ge j\}|.$$

Also the *arm-colength* $a'_\lambda(i,j)$ and *leg-colength* $l'_\lambda(i,j)$ are defined by

$$a'_\lambda(i,j) := j-1, \qquad l'_\lambda(i,j) := i-1.$$

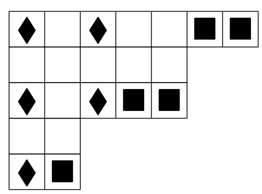
The *dominance partial ordering* $\le$ and the *inclusion partial ordering* $\subseteq$ are defined by

$$\begin{aligned} &\mu \le \lambda \quad \text{if and only if} \quad \mu_1 + \cdots + \mu_i \le \lambda_1 + \cdots + \lambda_i \quad (i = 1, \ldots, n);\\ &\mu \subseteq \lambda \quad \text{if and only if} \quad \mu_i \le \lambda_i \quad (i = 1, \ldots, n). \end{aligned}$$

Clearly, if $\mu \subseteq \lambda$ then $\mu \le \lambda$, while $\mu < \lambda$ implies that μ is less than λ in the lexicographic ordering. If $\mu \subseteq \lambda$ then we say that λ *contains* μ. Note that, for the dominance partial ordering, we do not make the usual requirement that $|\lambda| = |\mu|$.

For $\mu \subseteq \lambda$ define the *skew diagram* $\lambda - \mu$ as the set of boxes $\{s \in \lambda \mid s \notin \mu\}$. A *horizontal strip* is a skew diagram with at most one box in each column.

For a horizontal strip $\lambda - \mu$ define $(R\backslash C)_{\lambda/\mu}$ as the set of boxes which are in a row of λ intersecting with $\lambda - \mu$ but not in a column of λ intersecting with $\lambda - \mu$. Then clearly $(R\backslash C)_{\lambda/\mu}$ is completely contained in μ. For an example consider again $\lambda = (7,5,5,2,2)$ and take $\mu = (5,5,3,2,1)$. In the following diagram the cells of $\lambda - \mu$ have black squares and the cells of $(R\backslash C)_{\lambda/\mu}$ have black diamonds.



3.2. **Tableaux.** For λ a partition (of length $\le n$) we can fill the boxes s of λ by numbers $T(s) \in \{1, 2, \ldots, n\}$. Then T is called a *reverse tableau of shape λ with entries in* $\{1, \ldots, n\}$ if $T(i,j)$ is weakly decreasing in j and strongly decreasing in i. (Clearly, the number of different entries has to be $\ge \ell(\lambda)$. In [18, § I.1] tableaux rather than reverse tableaux are defined.) For an example consider again $\lambda = (7,5,5,2,2)$, so $\ell(\lambda) = 5$. Let us take $n = 6$. Then an example of a reverse tableau T of shape λ is given by

6	6	6	4	3	1	1
5	5	5	2	2		
4	4	2	1	1		
3	2					
2	1					

For T of shape λ and for $k = 0, 1, \ldots, n$ let $\lambda^{(k)}$ be the partition of which the Young tableau consists of all $s \in \lambda$ such that $T(s) > k$. Thus

$$0^n = \lambda^{(n)} \subseteq \lambda^{n-1} \subseteq \cdots \subseteq \lambda^{(1)} \subseteq \lambda^{(0)} = \lambda. \tag{3.3}$$

Then the skew diagram $\lambda^{(k-1)} - \lambda^{(k)}$ is actually a horizontal strip and it consists of all boxes s with $T(s) = k$. We call the sequence $(\mu_1, \ldots, \mu_n)$ with $\mu_k := |\lambda^{(k-1)} - \lambda^{(k}| = |T^{-1}(\{k\})$ the *weight* of T. For λ and T as in the example the inclusion sequence (3.3) becomes

$$() \subseteq (3) \subseteq (3,3) \subseteq (4,3,2) \subseteq (5,3,2,1) \subseteq (5,5,3,2,1) \subseteq (7,5,5,2,2).$$

and T has weight $(5,5,2,3,3,3)$.

For a skew diagram $\lambda - \mu$ a *standard tableau* T of shape $\lambda - \mu$ puts $T(s)$ in box s of $\lambda - \mu$ such that each number in $\{1, \ldots, |\lambda - \mu|\}$ occurs and $T(s)$ is strictly increasing in each row and in each column.

3.3. **Symmetrized monomials.** Write $x^\mu := x_1^{\mu_1} \cdots x_n^{\mu_n}$ for $\mu \in \mathbb{Z}^n$. We say that x^μ has degree $|\mu| := \mu_1 + \cdots + \mu_n$. By the degree of a Laurent polynomial $p(x)$ we mean the highest degree of a monomial occurring in the Laurent expansion of $p(x)$.

Let S_n be the symmetric group in n letters and $W_n := S_n \ltimes (\mathbb{Z}_2)^n$. For λ a partition and $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$ put

$$m_\lambda(x) := \sum_{\mu \in S_n \lambda} x^\mu, \qquad \widetilde{m}_\lambda(x) := \sum_{\mu \in W_n \lambda} x^\mu. \tag{3.4}$$

They form a basis of the space of S_n-invariant polynomials (respectively, W_n-invariant Laurent polynomials) in $x_1, \ldots, x_n$. Call an S_n-invariant polynomial (respectively, W_n-invariant Laurent polynomial) of degree $|\lambda|$ λ-*monic* if its coefficient of m_λ (respectively, of $\widetilde{m}_\lambda$) is equal to 1.

4. Macdonald and Koornwinder polynomials and $q = 1$ limits

From now on n will be the number of variables and we will assume $n > 1$.

4.1. **Macdonald polynomials.** See Eqs. (4.7), (9.3), (9.5) and the Remark on p. 372 in Ch. VI in Macdonald [18].

Let $0 < t < 1$. If $x = (x_1, \ldots, x_n)$ with $x_j \neq 0$ for all j then write $x^{-1} := (x_1^{-1}, \ldots, x_n^{-1})$. Put

$$\Delta_+(x) = \Delta_+(x; q, t) := \prod_{1 \le i < j \le n} \frac{(x_i x_j^{-1}; q)_\infty}{(t x_i x_j^{-1}; q)_\infty}, \qquad \Delta(x) := \Delta_+(x) \Delta_+(x^{-1}).$$

Macdonald polynomials (for root system A_{n-1}) are λ-monic S_n-invariant polynomials

$$P_\lambda(x; q, t) = P_\lambda(x) = \sum_{\mu \le \lambda} u_{\lambda,\mu} m_\mu(x) \tag{4.1}$$

such that (with q, t-dependence of P_λ and Δ understood)

$$\int_{\mathbb{T}^n} P_\lambda(x)\, m_\mu(x^{-1})\, \Delta(x)\, \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n} = 0 \quad \text{if } \mu < \lambda. \tag{4.2}$$

Here $\mathbb{T}^n$ is the n-torus in $\mathbb{C}^n$. It follows from (4.2) that

$$\int_{\mathbb{T}^n} P_\lambda(x)\, P_\mu(x^{-1})\, \Delta(x)\, \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n} = 0 \tag{4.3}$$

if $\mu < \lambda$, and that P_λ is homogeneous of degree $|\lambda|$. In fact, it can be shown that the orthogonality (4.3) holds for $\lambda \neq \mu$. This deeper and very important result will also be met for the three other orthogonal families discussed below.

Macdonald polynomials can be explicitly evaluated in a special point (see [18, Ch. VI, Eq. (6.11)]):

$$P_\lambda(t^\delta;q,t) = t^{n(\lambda)}\,\frac{\Delta_+(q^\lambda t^\delta;q,t)}{\Delta_+(t^\delta;q,t)} = t^{n(\lambda)} \prod_{1\le i<j\le n} \frac{(t^{j-i+1};q)_{\lambda_i-\lambda_j}}{(t^{j-i};q)_{\lambda_i-\lambda_j}}\,. \tag{4.4}$$

There is also the duality result (see [18, Ch. VI, Eq. (6.6)]):

$$\frac{P_\lambda(q^\nu t^\delta;q,t)}{P_\lambda(t^\delta;q,t)} = \frac{P_\nu(q^\lambda t^\delta;q,t)}{P_\nu(t^\delta;q,t)}\,. \tag{4.5}$$

4.2. **Jack polynomials.** See Eqs. (10.13), (10.14), (10.35) and (10.36) in Ch. VI in Macdonald [18], and see Stanley [36].

Let $\tau > 0$. Put

$$\Delta_+(x) = \Delta_+(x;\tau) := \prod_{1\le i<j\le n} (1 - x_i x_j^{-1})^\tau, \qquad \Delta(x) := \Delta_+(x)\Delta_+(x^{-1}). \tag{4.6}$$

Jack polynomials are λ-monic S_n-invariant polynomials

$$P_\lambda(x;\tau) = P_\lambda(x) = \sum_{\mu\le\lambda} u_{\lambda,\mu} m_\mu(x)$$

satisfying (4.2) with Δ given by (4.6). Hence they satisfy (4.3) if $\mu < \lambda$, and P_λ is homogeneous of degree $|\lambda|$. In fact, it can be shown that they satisfy (4.3) for $\lambda \neq \mu$.

Jack polynomials are limits of Macdonald polynomials:

$$\lim_{q\uparrow 1} P_\lambda(x;q,q^\tau) = P_\lambda(x;\tau). \tag{4.7}$$

Our notation of Jack polynomials relates to Macdonald's notation by $P_\lambda(x;\tau) = P_\lambda^{(\tau^{-1})}(x)$. Alternatively, [18, Ch. VI, Eq. (10.22)] and [36, Theorem 1.1] work with $J_\lambda^{(\alpha)}(x) = J_\lambda(x;\alpha)$, respectively. Then (Eqs. (10.22) and (10.21) in [18, Ch. VI])

$$J_\lambda^{(\alpha)} = c_\lambda(\alpha)\, P_\lambda^{(\alpha)}, \quad c_\lambda(\alpha) = \prod_{s\in\lambda} (\alpha a(s) + l(s) + 1).$$

We have the evaluation (see [36, Theorem 5.4])

$$P_\lambda(1^n;\tau) = \prod_{1\le i<j\le n} \frac{((j-i+1)\tau)_{\lambda_i-\lambda_j}}{((j-i)\tau)_{\lambda_i-\lambda_j}}\,. \tag{4.8}$$

The following limit is formally suggested by (4.7):

$$\lim_{q\uparrow 1} P_\lambda(q^{\tau\delta};q,q^\tau) = P_\lambda(1^n;\tau). \tag{4.9}$$

It follows rigorously by comparing (4.4) and (4.8).

4.3. **Koornwinder polynomials.** See Koornwinder [13].

Let $|a_1|,|a_2|,|a_3|,|a_4|\le 1$ such that $a_ia_j\ne 1$ if $i\ne j$, and such that non-real a_j occur in complex conjugate pairs. Let $0<t<1$. Put

$$\Delta_+(x)=\Delta_+(x;q,t;a_1,a_2,a_3,a_4) \\ :=\prod_{j=1}^n\frac{(x_j^2;q)_\infty}{(a_1x_j,a_2x_j,a_3x_j,a_4x_j;q)_\infty}\prod_{1\le i<j\le n}\frac{(x_ix_j,x_ix_j^{-1};q)_\infty}{(tx_ix_j,tx_ix_j^{-1};q)_\infty}.$$

Put $\Delta(x):=\Delta_+(x)\Delta_+(x^{-1})$. *Koornwinder polynomials* are λ-monic W_n-invariant Laurent polynomials

$$P_\lambda(x;q,t;a_1,a_2,a_3,a_4)=P_\lambda(x)=\sum_{\mu\le\lambda}u_{\lambda,\mu}\widetilde{m}_\mu(x) \tag{4.10}$$

such that

$$\int_{\mathbb{T}^n}P_\lambda(x)\,\widetilde{m}_\mu(x)\,\Delta(x)\,\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n}=0\quad\text{if }\mu<\lambda. \tag{4.11}$$

It follows from (4.11) that

$$\int_{\mathbb{T}^n}P_\lambda(x)\,P_\mu(x)\,\Delta(x)\,\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n}=0 \tag{4.12}$$

if $\mu<\lambda$, and that P_λ is symmetric in a_1,a_2,a_3,a_4. In fact, it can be shown that the orthogonality (4.12) holds for $\lambda\ne\mu$. Koornwinder polynomials are a 5-parameter generalization of Macdonald's [19] 3-parameter q-polynomials associated with root system BC_n.

Van Diejen [3, § 5.2] showed that the Macdonald polynomial $P_\lambda(x;q,t)$ is the term of highest degree $|\lambda|$ of $P_\lambda(x;q,t;a_1,a_2,a_3,a_4)$:

$$\lim_{r\to\infty}r^{-|\lambda|}P_\lambda(rx;q,t;a_1,a_2,a_3,a_4)=P_\lambda(x;q,t). \tag{4.13}$$

For the following result we will need dual parameters a_1',a_2',a_3',a_4':

$$a_1':=(q^{-1}a_1a_2a_3a_4)^{\frac12},\quad a_1'a_2'=a_1a_2,\quad a_1'a_3'=a_1a_3,\quad a_1'a_4'=a_1a_4. \tag{4.14}$$

Below the ambiguity in taking a square root will cause no harm because Δ_+ is invariant under the transformation $(x,a_1,a_2,a_3,a_4)\to(-x_1,-a_1,-a_2,-a_3,-a_4)$, by which P_λ will also have this invariance, up to a factor $(-1)^{|\lambda|}$.

An evaluation formula for Koornwinder polynomials was conjectured by Macdonald (1991, unpublished; see [4, Eq. (5.5)]). It reads

$$\begin{aligned} P_\lambda(t^\delta a_1;q,t;a_1,a_2,a_3,a_4) &= t^{-\langle\lambda,\delta\rangle}a_1^{-|\lambda|}\,\frac{\Delta_+(q^\lambda t^\delta a_1';q,t;a_1',a_2',a_3',a_4')}{\Delta_+(t^\delta a_1';q,t;a_1',a_2',a_3',a_4')} \\ &= t^{-\langle\lambda,\delta\rangle}a_1^{-|\lambda|}\prod_{j=1}^n \frac{(t^{n-j}a_1'^2;q)_{\lambda_j}}{(t^{2n-2j}a_1'^2;q)_{2\lambda_j}}\,(t^{n-j}a_1a_2,t^{n-j}a_1a_3,t^{n-j}a_1a_4;q)_{\lambda_j} \\ &\qquad \prod_{1\le i<j\le n}\frac{(t^{2n-i-j+1}a_1'^2;q)_{\lambda_i+\lambda_j}}{(t^{2n-i-j}a_1'^2;q)_{\lambda_i+\lambda_j}}\,\frac{(t^{j-i+1};q)_{\lambda_i-\lambda_j}}{(t^{j-i};q)_{\lambda_i-\lambda_j}}\,. \end{aligned} \tag{4.15}$$

It was proved by van Diejen [4, Eq. (5.5)] in the self-dual case $a_1 = a_1'$. In that case he also proved [4, Eq. (5.4)] Macdonald's duality conjecture (1991):

$$\frac{P_\lambda(q^\nu t^\delta a_1;q,t;a_1,a_2,a_3,a_4)}{P_\lambda(t^\delta a_1;q,t;a_1,a_2,a_3,a_4)} = \frac{P_\nu(q^\lambda t^\delta a_1';q,t;a_1',a_2',a_3',a_4')}{P_\nu(t^\delta a_1';q,t;a_1',a_2',a_3',a_4')}\,. \tag{4.16}$$

Sahi [34] proved (4.16) in the general case. As pointed out in [4, Section 7.2], this also implies (4.15) in the general case. Macdonald independently proved his conjectures in his book [20], see Eqs. (5.3.12) and (5.3.5), respectively, there.

4.4. BC_n**-type Jacobi polynomials.** See [10] and [37, Definition 3.5 and (3.18)].
Let $\alpha,\beta > -1$ and $\tau > 0$. Put

$$\Delta(x) = \Delta(x;\tau;\alpha,\beta) := \prod_{j=1}^n x_j^\alpha(1-x_j)^\beta \prod_{1\le i<j\le n}|x_i-x_j|^{2\tau}.$$

BC_n-type Jacobi polynomials are λ-monic S_n-invariant polynomials

$$P_\lambda(x;\tau;\alpha,\beta) = P_\lambda(x) = \sum_{\mu\le\lambda} a_{\lambda,\mu} m_\mu(x)$$

such that

$$\int_{[0,1]^n} P_\lambda(x)\,m_\mu(x)\,\Delta(x)\,dx_1\cdots dx_n = 0 \quad \text{if } \mu<\lambda. \tag{4.17}$$

It follows from (4.17) that

$$\int_{[0,1]^n} P_\lambda(x)\,P_\mu(x)\,\Delta(x)\,dx_1\cdots dx_n = 0 \tag{4.18}$$

if $\mu < \lambda$. It can be shown, see [10, Corollary 3.12], that (4.18) holds more generally if $\lambda \ne \mu$.

The case $c = 1$, $d = -1$ of [37, Eq. (5.5)] says that

$$\lim_{q\uparrow 1} P_\lambda(x;q,q^\tau;q^{\alpha+1},-q^{\beta+1},1,-1) = (-4)^{|\lambda|}\,P_\lambda(\tfrac14(2-x-x^{-1});\tau;\alpha,\beta). \tag{4.19}$$

Furthermore, it was pointed out in [31, Eq. (4.8)] that the Jack polynomial $P_\lambda(x;\tau)$ is the term of highest degree $|\lambda|$ of the BC_n-type Jacobi polynomial $P_\lambda(x;\tau;\alpha,\beta)$:

$$\lim_{r\to\infty} r^{-|\lambda|} P_\lambda(rx;\tau;\alpha,\beta) = P_\lambda(x;\tau). \tag{4.20}$$

This is the $q=1$ analogue of the limit (4.13).

An evaluation formula for Jacobi polynomials associated with root systems, including BC_n, was given by Opdam [28, Corollary 5.2]. See reformulations of this result in the BC_n case by van Diejen [5, Eq. (6.43d)] and by Hallnäs [9, p. 1594]. The formula can be given very explicitly as follows:

$$\begin{aligned}P_\lambda(0;\tau;\alpha,\beta) = (-1)^{|\lambda|} &\prod_{j=1}^n \frac{((n-j)\tau+2\alpha')_{\lambda_j}\,((n-j)\tau+\alpha+1)_{\lambda_j}}{((2n-2j)\tau+2\alpha')_{2\lambda_j}}\\ &\times \prod_{1\le i<j\le n} \frac{((2n-i-j+1)\tau+2\alpha')_{\lambda_i+\lambda_j}}{((2n-i-j)\tau+2\alpha')_{\lambda_i+\lambda_j}}\,\frac{((j-i+1)\tau)_{\lambda_i-\lambda_j}}{((j-i)\tau)_{\lambda_i-\lambda_j}}. \end{aligned}\tag{4.21}$$

Here

$$\alpha' := \tfrac12(\alpha+\beta+1). \tag{4.22}$$

The following limit is formally suggested by (4.19):

$$\lim_{q\uparrow 1} P_\lambda(q^{\tau\delta+\alpha+1};q,q^\tau;q^{\alpha+1},-q^{\beta+1},1,-1) = (-4)^{|\lambda|}\,P_\lambda(0;\tau;\alpha,\beta). \tag{4.23}$$

It follows rigorously by comparing (4.15) and (4.21).

5. Interpolation polynomials

5.1. **Interpolation Macdonald polynomials.** See Sahi [33, Theorem 1.1], Knop [11, Theorem 2.4(b)], and Okounkov [25, Eqs. (4.2), (4.3)].

Let $0<t<1$. The *interpolation Macdonald polynomial* (or *shifted Macdonald polynomial*) $P^{\rm ip}_\lambda(x;q,t)$ is the unique λ-monic S_n-invariant polynomial of degree $|\lambda|$ such that $P^{\rm ip}_\lambda(q^\mu t^\delta;q,t)=0$ for each partition $\mu\ne\lambda$ with $|\mu|\le|\lambda|$. Here $q^\mu t^\delta = (q^{\mu_1}t^{n-1},q^{\mu_2}t^{n-2},\ldots,q^{\mu_n})$.

Our $P^{\rm ip}_\lambda$ is related to Sahi's R_λ, Knop's P_λ (use [11, Theorem 3.11]) and Okounkov's P^*_λ (use [25, Eq. (4.11)]), respectively, as follows:

$$P^{\rm ip}_\lambda(x;q,t) = R_\lambda(x;q^{-1},t^{-1}) = t^{(n-1)|\lambda|}\,P_\lambda(t^{-(n-1)}x) = t^{(n-1)|\lambda|}\,P^*_\lambda(xt^{-\delta}).$$

Okounkov [25] speaks about *shifted* polynomials because in his notation the polynomials are only symmetric after a (multiplicative) shift.

$P^{\rm ip}_\lambda$ has the extra vanishing property (see [11, p. 93] or [25, Eq. (4.12)])

$$P^{\rm ip}_\lambda(q^\mu t^\delta;q,t)=0 \quad \text{if } \mu \text{ is a partition not containing } \lambda.$$

By [25, Eq. (4.11)], $P^{\rm ip}_\lambda$ can be expanded in terms of Macdonald polynomials as follows:

$$P^{\rm ip}_\lambda(x;q,t) = \sum_{\mu\subseteq\lambda} b_{\lambda,\mu}\,P_\mu(x;q,t) \tag{5.1}$$

for certain coefficients $b_{\lambda,\mu}$, where $b_{\lambda,\lambda} = 1$ by λ-monicity. This has several consequences. By combination with (4.1) we see that

$$P_\lambda^{\rm ip}(x;q,t) = \sum_{\mu\le\lambda} c_{\lambda,\mu}\, m_\mu(x) \tag{5.2}$$

for certain coefficients $c_{\lambda,\mu}$, and $c_{\lambda,\lambda} = 1$. Furthermore, by (5.1), $P_\lambda(x;q,t)$ is the term of highest degree $|\lambda|$ of the polynomial $P_\lambda^{\rm ip}(x;q,t)$:

$$\lim_{r\to\infty} r^{-|\lambda|} P_\lambda^{\rm ip}(rx;q,t)) = P_\lambda(x;q,t). \tag{5.3}$$

Although Knop [11] did not give (5.1), he did give (5.2) and (5.3), proved differently (see Theorem 3.11 and Theorem 3.9 in [11]). The result (5.3) is also proved by Sahi [33, Theorem 1.1].

5.2. **Interpolation Jack polynomials.** See Sahi [32, Theorem 1], Knop and Sahi [12, pp. 475, 478], Okounkov and Olshanski [27, p. 70], and Okounkov [25, Section 7].

Let $\tau > 0$. The *interpolation Jack polynomial* (or *shifted Jack polynomial*) $P_\lambda^{\rm ip}(x;\tau)$ is the unique λ-monic S_n-invariant polynomial of degree $|\lambda|$ such that $P_\lambda^{\rm ip}(\mu+\tau\delta;\tau) = 0$ for each partition $\mu \ne \lambda$ with $|\mu| \le |\lambda|$. It can be expressed in terms of $P_\lambda^{\tau\delta}$ from [12] and in terms of $P_\lambda^*(\,.\,;\tau)$ from [27], [25] as follows:

$$P_\lambda^{\rm ip}(x;\tau) = P_\lambda^{\tau\delta}(x) = P_\lambda^*(x-\tau\delta;\tau).$$

It has the extra vanishing property (see [12, Theorem 5.2])

$$P_\lambda^{\rm ip}(\mu+\tau\delta;\tau) = 0 \quad \text{if } \mu \text{ is a partition not containing } \lambda.$$

It has an expansion of the form

$$P_\lambda^{\rm ip}(x;\tau) = \sum_{\mu\le\lambda} c_{\lambda,\mu}\, m_\mu(x)$$

for certain coefficients $c_{\lambda,\mu}$, and $c_{\lambda,\lambda} = 1$ (see [12, Corollary 4.6]). The term of highest degree $|\lambda|$ of the polynomial $P_\lambda^{\rm ip}(x;\tau)$ is the Jack polynomial $P_\lambda(x;\tau)$ (see [12, Corollary 4.7]):

$$\lim_{r\to\infty} r^{-|\lambda|} P_\lambda^{\rm ip}(rx;\tau) = P_\lambda(x;\tau). \tag{5.4}$$

Interpolation Jack polynomials are a limit case of interpolation Macdonald polynomials (see [25, Eq. (7.1)]):

$$\lim_{q\uparrow 1} (q-1)^{-|\lambda|} P_\lambda^{\rm ip}(q^x;q,q^\tau) = P_\lambda^{\rm ip}(x;\tau). \tag{5.5}$$

5.3. BC_n-type interpolation Macdonald polynomials.

Definition 5.1 (*BC_n-type interpolation* (or *BC_n-type shifted*) *Macdonald polynomials*)**.** Let $0 < t < 1$ and let $a \in \mathbb{C}$ be generic. $P_\lambda^{\mathrm{ip}}(x; q, t, a)$ is the unique W_n-invariant λ-monic Laurent polynomial of degree $|\lambda|$ such that

$$P_\lambda^{\mathrm{ip}}(q^\mu t^\delta a; q, t, a) = 0 \quad \text{if } \mu \text{ does not contain } \lambda. \tag{5.6}$$

(In particular, $P_\lambda^{\mathrm{ip}}(q^\mu t^\delta a; q, t, a) = 0$ if $|\mu| \le |\lambda|$, $\mu \ne \lambda$.)

These polynomials were first introduced by Okounkov [26] in a different notation and normalization. Okounkov [26, p. 185, Section 1] specifies the genericity of the parameter a (in his notation s) by the condition $q^i t^j a^k \ne 1$ for $i, j, k \in \mathbb{Z}_{>0}$. However, this may be too strong on the one hand and too weak on the other hand. Certainly the right-hand side of (5.10) (equivalently the normalization constant in [26, Definition 1.2]) should be nonzero. A requirement for this is that $q^i t^j a^2 \ne 1$ for $i \in \mathbb{Z}_{>0}$, $j \in \mathbb{Z}_{\ge 0}$.

Different approaches were given by Rains [29], and later by Noumi in unpublished slides of a lecture given in 2013 at a conference at Kyushu University. Our normalization follows Rains [29]. In terms of Rains' $\bar{P}_\lambda^{*(n)}$ and Okounkov's P_λ^*, we have (cf. [29, p. 76, Remark 1])

$$P_\lambda^{\mathrm{ip}}(x; q, t, a) = \bar{P}_\lambda^{*(n)}(x; q, t, a) = (t^{n-1}a)^{|\lambda|} P_\lambda^*(x t^{-\delta} a^{-1}; q, t, a). \tag{5.7}$$

Note that Okounkov's $P_\lambda^*(x; q, t, s)$ is W_n-symmetric in the variables $x_i t^{n-i} s$ ($i = 1, 2, \ldots, n$).

Just as for $P_\lambda^{\mathrm{ip}}(x; q, t)$, the top homogeneous term of $P_\lambda^{\mathrm{ip}}(x; q, t, a)$ equals the Macdonald polynomial $P_\lambda(x; q, t)$ (see [26, Section 4]):

$$\lim_{r \to \infty} r^{-|\lambda|} P_\lambda^{\mathrm{ip}}(rx; q, t, a)) = P_\lambda(x; q, t). \tag{5.8}$$

There is also a limit from $P_\lambda^{\mathrm{ip}}(x; q, t, a)$ to $P_\lambda^{\mathrm{ip}}(x; q, t)$ (see [29, p. 75]):

$$\lim_{a \to \infty} a^{-|\lambda|} P_\lambda^{\mathrm{ip}}(ax; q, t, a) = P_\lambda^{\mathrm{ip}}(x; q, t). \tag{5.9}$$

By combination of (5.7) with [26, Definitions 1.1 and 1.2], we get the following evaluation formula (with $\langle\, .\, ,\, .\, \rangle$ denoting the standard inner product on $\mathbb{R}^n$):

$$P_\lambda^{\mathrm{ip}}(q^\lambda t^\delta a; q, t, a) = q^{-\langle \lambda, \lambda \rangle} t^{-\langle \lambda, \delta \rangle}\, a^{-|\lambda|} \prod_{(i,j) \in \lambda} (1 - q^{\lambda_i - j + 1} t^{\lambda'_j - i})(1 - a^2 q^{\lambda_i + j - 1} t^{\lambda'_j - i + 2(n - \lambda'_j)}). \tag{5.10}$$

By [29, Lemma 2.1] (see also [29, Corollary 3.7]) this can be rewritten as

$$\begin{aligned} P_\lambda^{\mathrm{ip}}(q^\lambda t^\delta a; q, t, a) = q^{-\langle \lambda, \lambda \rangle} t^{-\langle \lambda, \delta \rangle}\, a^{-|\lambda|} \prod_{j=1}^n \frac{(q t^{n-j}; q)_{\lambda_j} (t^{2n-2j} a^2; q)_{2\lambda_j}}{(t^{n-j} a^2; q)_{\lambda_j}} \\ \times \prod_{1 \le i < j \le n} \frac{(t^{2n-i-j} a^2; q)_{\lambda_i + \lambda_j}}{(t^{2n-i-j+1} a^2; q)_{\lambda_i + \lambda_j}} \, \frac{(q t^{j-i-1}; q)_{\lambda_i - \lambda_j}}{(q t^{j-i}; q)_{\lambda_i - \lambda_j}}\,. \end{aligned} \tag{5.11}$$

An elementary consequence of the definition of $P_\lambda^{\mathrm{ip}}(x;q,t,a)$ is a reduction formula (see [26, Proposition 2.1]):

$$P_\mu^{\mathrm{ip}}(x;q,t,a)=(-a)^{n\mu_n}q^{-\frac12 n\mu_n(\mu_n-1)}\prod_{j=1}^n\big((x_ja;q)_{\mu_n}(x_j^{-1}a;q)_{\mu_n}\big)P^*_{\mu-\mu_n1^n}(x;q,t,q^{\mu_n}a). \tag{5.12}$$

6. Combinatorial formulas

6.1. **Combinatorial formula for Macdonald polynomials.** The combinatorial formula for Macdonald polynomials which we will use is a special case of [18, Ch. VI, Eq. (7.13′)]:

$$P_\lambda(x;q,t)=\sum_T\psi_T(q,t)\prod_{s\in\lambda}x_{T(s)}, \tag{6.1}$$

where the sum is over all tableaux T of shape λ with entries in $\{1,\ldots,n\}$ and with $\psi_T(q,t)$ defined in [18, Ch. VI] by formula (7.11′), by formula (ii) on p. 341 with $C_{\lambda/\mu}$ and $S_{\lambda/\mu}$ given after (6.22), and by formula (6.20). See [15, Section 1] for a summary of these results. Since the Macdonald polynomial is symmetric, we may as well sum over reverse tableaux instead of tableaux, with the definition of $\psi_T(q,t)$ accordingly adapted. We will now give $\psi_T(q,t)$ explicitly. See Subsections 3.1 and 3.2 for notation.

Recall that for a reverse tableaux T of shape λ with entries in $\{1,\ldots,n\}$ we write $0^n=\lambda^{(n)}\subseteq\lambda^{(n-1)}\subseteq\cdots\subseteq\lambda^{(0)}=\lambda$ with $T(s)=i$ for s in the horizontal strip $\lambda^{(i-1)}-\lambda^{(i)}$. Now

$$\psi_T(q,t):=\prod_{i=1}^n\psi_{\lambda^{(i-1)}/\lambda^{(i)}}(q,t),\qquad \psi_{\mu/\nu}(q,t)=\prod_{s\in(R\setminus C)_{\mu/\nu}}\frac{b_\nu(s;q,t)}{b_\mu(s;q,t)}\,, \tag{6.2}$$

where

$$b_\mu(s;q,t):=\frac{1-q^{a_\mu(s)}t^{l_\mu(s)+1}}{1-q^{a_\mu(s)+1}t^{l_\mu(s)}}\,. \tag{6.3}$$

By (3.4) and (4.1) we obtain that

$$P_\lambda(x;q,t)=\sum_{\mu\le\lambda}u_{\lambda,\mu}(q,t)m_\mu(x)\quad\text{with}\quad u_{\lambda,\mu}(q,t)=\sum_T\psi_T(q,t), \tag{6.4}$$

where the T-sum is over all reverse tableaux T of shape λ and weight μ, see [18, p. 378].

Let T_λ be the tableau of shape λ for which $T(i,j)=n+1-i$. This is the unique tableau of shape λ which has weight $(\lambda_n,\lambda_{n-1},\ldots,\lambda_1)$. Since $P_\lambda(x;q,t)$ is λ-monic, it follows from (6.1) that

$$\psi_{T_\lambda}(q,t)=1. \tag{6.5}$$

6.2. **Combinatorial formulas for (BC_n) interpolation Macdonald polynomials.** For interpolation Macdonald polynomials $P_\lambda^{\rm ip}(x;q,t)$ and BC_n-type interpolation Macdonald polynomials $P_\lambda^{\rm ip}(x;q,t,a)$ there are combinatorial formulas similar to (6.1) and also involving $\psi_T(q,t)$ given by (6.2):

$$P_\lambda^{\rm ip}(x;q,t)=\sum_T \psi_T(q,t)\prod_{s\in\lambda}\big(x_{T(s)}-q^{a'_\lambda(s)}t^{n-T(s)-l'_\lambda(s)}\big), \tag{6.6}$$

$$\begin{aligned}P_\lambda^{\rm ip}(x;q,t,a)=\sum_T \psi_T(q,t)\prod_{s\in\lambda}&\Big(x_{T(s)}-q^{a'_\lambda(s)}t^{n-T(s)-l'_\lambda(s)}a\Big)\\ &\cdot\Big(1-\big(q^{a'_\lambda(s)}t^{n-T(s)-l'_\lambda(s)}a\big)^{-1}x_{T(s)}^{-1}\Big).\end{aligned} \tag{6.7}$$

See [25, Eq. (1.4)] for (6.6) and [26, Eq. (5.3)] for (6.7). The sums are over all reverse tableaux T of shape λ with entries in $\{1,\ldots,n\}$. Note that the limits (5.3), (5.8) and (5.9) also follow by comparing (6.1), (6.6) and (6.7).

6.3. **Combinatorial formulas for (interpolation) Jack polynomials.** The combinatorial formula for Jack polynomials can be obtained as a limit case of the combinatorial formula (6.1) for Macdonald polynomials by using the limit (4.7) (see [18, p. 379]), but it can also be obtained independently, as was first done by Stanley [36, Theorem 6.3]:

$$P_\lambda(x;\tau)=\sum_T \psi_T(\tau)\prod_{s\in\lambda}x_{T(s)}, \tag{6.8}$$

where the sum is over all reverse tableaux T of shape λ with entries in $\{1,\ldots,n\}$. For the definition of $\psi_T(\tau)$ take $0^n=\lambda^{(n)}\subseteq\lambda^{(n-1)}\subseteq\cdots\subseteq\lambda^{(0)}=\lambda$ as before and put

$$\psi_T(\tau):=\prod_{i=1}^n\prod_{s\in(R\backslash C)_{\lambda^{(i-1)}/\lambda^{(i)}}}\frac{b_{\lambda^{(i)}}(s;\tau)}{b_{\lambda^{(i-1)}}(s;\tau)} \tag{6.9}$$

with

$$b_\mu(s;\tau):=\frac{a_\mu(s)+\tau(l_\mu(s)+1)}{a_\mu(s)+\tau l_\mu(s)+1}\,.$$

Note that

$$\lim_{q\uparrow 1}b_\mu(s;q,q^\tau)=b_\mu(s,\tau)\quad\text{and}\quad\lim_{q\uparrow 1}\psi_T(q,q^\tau)=\psi_T(\tau).$$

Hence, by (6.5) we have

$$\psi_{T_\lambda}(\tau)=1. \tag{6.10}$$

Similarly as for (6.4) we derive immediately that

$$P_\lambda(x;\tau)=\sum_{\mu\le\lambda}u_{\lambda,\mu}(\tau)m_\mu(x)\quad\text{with}\quad u_{\lambda,\mu}(\tau)=\sum_T\psi_T(\tau),$$

where the T-sum is over all reverse tableaux T of shape λ and weight μ.

The combinatorial formula for interpolation Jack polynomials (see [27, Eq. (2.4)]) was obtained in [25, Section 7] as a limit case of the combinatorial formula (6.6) for interpolation Macdonald polynomials by using the limit (5.5):

$$P_\lambda^{\rm ip}(x;\tau)=\sum_T \psi_T(\tau)\prod_{s\in\lambda}\big(x_{T(s)}-a'_\lambda(s)-\tau(n-T(s)-l'_\lambda(s))\big), \tag{6.11}$$

with the sum over all tableaux T of shape λ with entries in $\{1,\ldots,n\}$ and $\psi_T(\tau)$ given by (6.9). The limit (5.4) also follows by comparing (6.11) and (6.8). Furthermore, by comparing (6.7) and (6.8) we obtain the limit

$$\lim_{q\uparrow 1} P_\lambda^{\rm ip}(x;q,q^\tau,q^\alpha)=P_\lambda(x+x^{-1}-2;\tau), \tag{6.12}$$

and from (6.6) and (6.8) we obtain

$$\lim_{q\uparrow 1} P_\lambda^{\rm ip}(x;q,q^\tau)=P_\lambda(x-1^n;\tau). \tag{6.13}$$

Remark 6.1. When we compare combinatorial formulas in the case of n and of $n-1$ variables, we see that, in general, a combinatorial formula is equivalent to a *branching formula*, which expands a polynomial P_λ in $x_1,\ldots,x_n$ in terms of polynomials P_μ in $x_1,\ldots,x_{n-1}$ with the expansion coefficients depending on x_n. In particular, the branching formula for Macdonald polynomials is (see [15, Eqs. (1.9), (1.8)])

$$P_\lambda(x_1,\ldots,x_{n-1},x_n;q,t))=\sum_\mu P_{\lambda/\mu}(x_n;q,t)\,P_\mu(x_1,\ldots,x_{n-1};q,t), \tag{6.14}$$

where the sum runs over all partitions $\mu\subseteq\lambda$ of length $<n$ such that $\lambda-\mu$ is a horizontal strip, and where, in notation (6.2),

$$P_{\lambda/\mu}(z;q,t)=\psi_{\lambda/\mu}(q,t)\,z^{|\lambda|-|\mu|}. \tag{6.15}$$

The coefficients $\psi_{\lambda/\mu}(q,t)$ can be expressed in terms of Pieri coefficients for Macdonald polynomials by interchanging q and t and by passing to conjugate partitions λ',μ': $\psi_{\lambda/\mu}(q,t)=\psi'_{\lambda'/\mu'}(t,q)$, see [18, Ch. VI, Eq. (6.24)].

Van Diejen and Emsiz [6] recently obtained a branching formula for Koornwinder polynomials. It has the same structure as (6.14), but the analogue of (6.15) becomes a sum of terms in the right-hand side. Each term is similar to the right-hand side of (6.15), with the monomial being replaced by a quadratic q-factorial, also depending on the parameter a_1. The analogues of the coefficients $\psi_{\lambda/\mu}$ can be expressed in terms of (earlier known) Pieri-type coefficients for Koornwinder polynomials. By taking highest degree parts in both sides of the new branching formula and by using (4.13), we are reduced to (6.15). However, the combinatorial formula (6.7) and its corresponding branching formula for BC_n-type interpolation Macdonald polynomials are quite different from the results in [6].

7. BC_n-type interpolation Jack polynomials

In view of the results surveyed until now, the following definition is quite natural.

Definition 7.1. Let $\tau > 0$ and let $\alpha \in \mathbb{C}$ be generic. The *BC_n-type interpolation Jack polynomial* $P_\lambda^{\mathrm{ip}}(x;\tau,\alpha)$ is given as a limit of BC_n-type interpolation Macdonald polynomials,

$$P_\lambda^{\mathrm{ip}}(x;\tau,\alpha) := \lim_{q\uparrow 1}(1-q)^{-2|\lambda|}P_\lambda^{\mathrm{ip}}(q^x;q,q^\tau,q^\alpha). \tag{7.1}$$

Concerning the genericity of $\alpha \in \mathbb{C}$ we should have at least that the evaluation (7.5) is nonzero, i.e., $i + j\tau + 2\alpha \neq 0$ for $i \in \mathbb{Z}_{>0}$ and $j \in \mathbb{Z}_{\geq 0}$. That the limit (7.1) exists can be seen by substituting (6.7) in the right-hand side of (7.1). We obtain

$$P_\lambda^{\mathrm{ip}}(x;\tau,\alpha) = \sum_T \psi_T(\tau)\prod_{s\in\lambda}\Big(x_{T(s)}^2 - \big(a'_\lambda(s) + \tau(n - T(s) - l'_\lambda(s)) + \alpha\big)^2\Big) \tag{7.2}$$

with the sum over all reverse tableaux T of shape λ with entries in $\{1,\ldots,n\}$ and $\psi_T(\tau)$ given by (6.9). From (7.1), (7.2) and the properties of $P_\lambda^{\mathrm{ip}}(x;q,t)$ we see that $P_\lambda^{\mathrm{ip}}(x;\tau,\alpha)$ is a W_n-invariant polynomial of degree $2|\lambda|$ in x, where $(\mathbb{Z}_2)^n$ now acts on the polynomial by sending some of the variables x_i to $-x_i$ rather than to x_i^{-1}. By (6.10) it follows from (7.2) that $P_\lambda^{\mathrm{ip}}(x;\tau,\alpha)$ is (2λ)-monic.

It follows from (5.6) and (7.1) that

$$P_\lambda^{\mathrm{ip}}(\mu + \tau\delta + \alpha;\tau,\alpha) = 0 \quad \text{if } \mu \text{ does not contain } \lambda.$$

By comparing (7.2), (6.11) and (6.8) we obtain the limits

$$\lim_{r\to\infty} r^{-2|\lambda|}P_\lambda^{\mathrm{ip}}(rx;\tau,\alpha) = P_\lambda(x^2;\tau), \tag{7.3}$$

$$\lim_{\alpha\to\infty}(2\alpha)^{-|\lambda|}P_\lambda^{\mathrm{ip}}(x+\alpha;\tau,\alpha) = P_\lambda^{\mathrm{ip}}(x;\tau). \tag{7.4}$$

From (5.10) and (5.11) together with (7.1), we obtain the evaluation formula

$$\begin{aligned}
&P_\lambda^{\mathrm{ip}}(\lambda + \tau\delta + \alpha;\tau,\alpha)\\
&\quad= \prod_{(i,j)\in\lambda}\big(\lambda_i - j + 1 + \tau(\lambda'_j - i)\big)\big(2\alpha + \lambda_i + j - 1 + \tau(\lambda'_j - i + 2(n - \lambda'_j))\big)\\
&\quad= \prod_{j=1}^n \frac{((n-j)\tau + 1)_{\lambda_j}(2(n-j)\tau + 2\alpha)_{2\lambda_j}}{((n-j)\tau + 2\alpha)_{\lambda_j}}\\
&\qquad\times \prod_{1\leq i<j\leq n}\frac{((2n-i-j)\tau + 2\alpha)_{\lambda_i+\lambda_j}}{((2n-i-j+1)\tau + 2\alpha)_{\lambda_i+\lambda_j}}\,\frac{((j-i-1)\tau + 1)_{\lambda_i-\lambda_j}}{((j-i)\tau + 1)_{\lambda_i-\lambda_j}}.
\end{aligned} \tag{7.5}$$

By (5.12) and (7.1) we get a reduction formula

$$P_\lambda^{\mathrm{ip}}(x;\tau,\alpha) = (-1)^{n\lambda_n}\prod_{j=1}^n\big((\alpha + x_j)_{\lambda_n}(\alpha - x_j)_{\lambda_n}\big)\,P_{\lambda-\lambda_n 1^n}^{\mathrm{ip}}(x;\tau,\lambda_n + \alpha). \tag{7.6}$$

8. Binomial formulas

8.1. **Binomial formula for Koornwinder polynomials.** Okounkov [26, Theorem 7.1] obtained the *binomial formula* for Koornwinder polynomials:

$$\frac{P_\lambda(x;q,t;a_1,a_2,a_3,a_4)}{P_\lambda(t^\delta a_1;q,t;a_1,a_2,a_3,a_4)}=\sum_{\mu\subseteq\lambda}\frac{P_\mu^{\rm ip}(q^\lambda t^\delta a_1';q,t,a_1')}{P_\mu^{\rm ip}(q^\mu t^\delta a_1';q,t,a_1')}\,\frac{P_\mu^{\rm ip}(x;q,t,a_1)}{P_\mu(t^\delta a_1;q,t;a_1,a_2,a_3,a_4)}\,. \tag{8.1}$$

As pointed out in [26], the duality (4.16) immediately follows from (8.1) in the self-dual case $a_1=a_1'$. In the general case (4.16) will follow from (8.1) together with the identity

$$\frac{P_\mu(t^\delta a_1;q,t;a_1,a_2,a_3,a_4)}{P_\mu^{\rm ip}(q^\mu t^\delta a_1;q,t,a_1)}=\frac{P_\mu(t^\delta a_1';q,t;a_1',a_2',a_3',a_4')}{P_\mu^{\rm ip}(q^\mu t^\delta a_1';q,t,a_1')}\,,$$

which is a consequence of the evaluation formulas (4.15) and (5.11).

Rains [29, Section 5] gives an alternative definition of Koornwinder polynomials involving triangularity and evaluation symmetry by which a version of (8.1) is an immediate consequence.

Because the inclusion partial ordering is compatible with the lexicographic ordering, we can use induction in λ with respect to the lexicographic ordering in order to show from (8.1) that

$$P_\lambda^{\rm ip}(x;q,t,a_1)=\sum_{\mu\subseteq\lambda}b_{\lambda,\mu}\,P_\mu(x;q,t;a_1,a_2,a_3,a_4)$$

for certain coefficients $b_{\lambda,\mu}$ (more explicitly given in [29, Theorem 5.12]). Together with (4.10) this implies that

$$P_\lambda^{\rm ip}(x;q,t,a_1)=\sum_{\mu\le\lambda}c_{\lambda,\mu}\,\widetilde m_\mu(x) \tag{8.2}$$

for certain coefficients $c_{\lambda,\mu}$.

8.2. **Binomial formula for Macdonald polynomials.** Okounkov [24, Eq. (1.11)] gave the binomial formula for Macdonald polynomials:

$$\frac{P_\lambda(x;q,t)}{P_\lambda(t^\delta;q,t)}=\sum_{\mu\subseteq\lambda}\frac{P_\mu^{\rm ip}(q^\lambda t^\delta;q,t)}{P_\mu^{\rm ip}(q^\mu t^\delta;q,t}\,\frac{P_\mu^{\rm ip}(x;q,t)}{P_\mu(t^\delta;q,t)}\,. \tag{8.3}$$

The binomial formula (8.3) immediately implies the duality formula (4.5) for Macdonald polynomials.

By comparing the binomial formulas (8.1) and (8.3) we can obtain a new limit formula.

Theorem 8.1. *There is the limit*

$$\lim_{a_1\to\infty}a_1^{-|\lambda|}\,P_\lambda(a_1x;q,t;a_1,a_2,a_3,a_4)=P_\lambda(x;q,t). \tag{8.4}$$

Proof. Rewrite (8.1) as

$$\frac{a_1^{-|\lambda|}\,P_\lambda(a_1x;q,t;a_1,a_2,a_3,a_4)}{a_1^{-|\lambda|}\,P_\lambda(t^\delta a_1;q,t;a_1,a_2,a_3,a_4)} = \sum_{\mu\subseteq\lambda} \frac{a_1^{-|\mu|}P_\mu^{\rm ip}(q^\lambda t^\delta a_1';q,t,a_1')}{a_1^{-|\mu|}P_\mu^{\rm ip}(q^\mu t^\delta a_1';q,t,a_1')}\,\frac{a_1^{-|\mu|}\,P_\mu^{\rm ip}(a_1x;q,t,a_1)}{a_1^{-|\mu|}\,P_\mu(t^\delta a_1;q,t;a_1,a_2,a_3,a_4)}\,.$$

Now let $a_1\to\infty$. The result follows by (5.9) and (8.3) if we can show that (8.4) holds for $x=t^\delta$. This, in its turn, follows by comparing (4.15) and (4.4). □

As pointed out to me by Ole Warnaar, the limit (8.4) also follows from the limit (4.13) (yielding Macdonald polynomials as highest degree part of Koornwinder polynomials) together with the known fact (although not in the literature) that the coefficients $u_{\lambda,\mu}$ ($|\mu|<|\lambda|$) in (4.10) are bounded as $a_1\to\infty$.

8.3. **Binomial formula for Jack polynomials.** The binomial formula for Jack polynomials was given by Okounkov and Olshanski [27, p. 72]:

$$\frac{P_\lambda(1+x,\tau)}{P_\lambda(1;\tau)} = \sum_{\mu\subseteq\lambda} \frac{P_\mu^{\rm ip}(\lambda+\tau\delta;\tau)}{P_\mu^{\rm ip}(\mu+\tau\delta;\tau)}\,\frac{P_\mu(x;\tau)}{P_\mu(1;\tau)}\,. \tag{8.5}$$

It is a limit case of (8.3) because of the limits (4.7), (4.9) (twice), (5.5) (twice) and (6.13). If we compare (8.5) with Macdonald [21, Eqs. (6.15), (6.24)], Lassalle [16, § 3] or Yan [38, Eq. (10)], we see that $P_\mu^{\rm ip}(\lambda+\tau\delta;\tau)/P_\mu^{\rm ip}(\mu+\tau\delta;\tau)$ equals the generalized binomial coefficient $\binom{\lambda}{\mu}$ defined in these references.

9. Binomial formula for BC_n-type Jacobi polynomials

Let α' be given by (4.22).

Theorem 9.1. *For BC_n-type Jacobi polynomials we have the binomial formula*

$$\frac{P_\lambda(x;\tau;\alpha,\beta)}{P_\lambda(0;\tau;\alpha,\beta)} = \sum_{\mu\subseteq\lambda} \frac{P_\mu^{\rm ip}(\lambda+\tau\delta+\alpha';\tau,\alpha')}{P_\mu^{\rm ip}(\mu+\tau\delta+\alpha';\tau,\alpha')}\,\frac{P_\mu(x;\tau)}{P_\mu(0;\tau;\alpha,\beta)}\,. \tag{9.1}$$

Proof. From (8.1) we obtain

$$\frac{P_\lambda(x;q,q^\tau;q^{\alpha+1},-q^{\beta+1},1,-1)}{P_\lambda(q^{\tau\delta+\alpha+1};q,q^\tau;q^{\alpha+1},-q^{\beta+1},1,-1)} = \sum_{\mu\subseteq\lambda} \frac{P_\mu^{\rm ip}(q^{\lambda+\tau\delta+\alpha'};q,q^\tau,q^{\alpha'})}{P_\mu^{\rm ip}(q^{\mu+\tau\delta+\alpha'};q,q^\tau,q^{\alpha'})}\,\frac{P_\mu^{\rm ip}(x;q,q^\tau,q^{\alpha+1})}{P_\mu(q^{\tau\delta+\alpha+1};q,q^\tau;q^{\alpha+1},-q^{\beta+1},1,-1)}\,.$$

Now let $q\uparrow 1$ and apply (4.19), (4.23) (twice), (7.1) (twice) and (6.12). □

By comparing the binomial formulas (9.1) and (8.5) we arrive at the limit

$$\lim_{\alpha\to\infty} P_\lambda(x;\tau;\alpha,\beta) = P_\lambda(x-1^n;\tau), \tag{9.2}$$

where 1^n is the n-vector with all coordinates equal to 1. The limit (9.2) was given slightly more generally in [31, Theorem 4.2] and goes back to an unpublished result by Beerends and the author. For the proof of (9.2) let $\alpha\to\infty$ in the right-hand side of (9.1) and use (7.4). Then we will obtain the right-hand side of (8.5) if we can prove that (9.2) is valid for $x=0$. But in that case (9.2) follows by comparing (4.21) and (4.8).

Remark 9.2. In (9.1) we have an expansion

$$P_\lambda(x;\tau;\alpha,\beta) = \sum_{\mu\subseteq\lambda} b_{\lambda,\mu}\, P_\mu(x;\tau) \tag{9.3}$$

with

$$b_{\lambda,\mu} = \frac{P_\lambda(0;\tau;\alpha,\beta)}{P_\mu(0;\tau;\alpha,\beta)}\,\frac{P_\mu^{\rm ip}(\lambda+\tau\delta+\alpha';\tau,\alpha')}{P_\mu^{\rm ip}(\mu+\tau\delta+\alpha';\tau,\alpha')}\,, \tag{9.4}$$

where, by (7.2),

$$P_\mu^{\rm ip}(\lambda+\tau\delta+\alpha';\tau,\alpha') = \sum_T \psi_T(\tau) \prod_{s\in\mu}\big((\lambda_{T(s)}+\tau\delta_{T(s)}+\alpha')^2-(\tau\delta_{T(s)}+\alpha'+a'_\lambda(s)-\tau l'_\lambda(s))^2\big). \tag{9.5}$$

On the other hand, Macdonald [21, p. 58] (also in Beerends and Opdam [2, Eq. (5.12)]) gives (9.3) with

$$b_{\lambda,\mu} = (-1)^{|\lambda|-|\mu|}\frac{P_\lambda(1;\tau)}{P_\mu(1;\tau)}\, c_{\lambda/\mu}(2\tau(n-1)+\alpha'+1) \prod_{j=1}^n (\mu_j+\tau(n+j-2)+\alpha+1)_{\lambda_j-\mu_j}\,. \tag{9.6}$$

Here

$$c_{\lambda/\mu}(C) = \sum_T f_T(C) \tag{9.7}$$

for a certain function f_T, and the sum is over all standard tableaux T of shape λ/μ. Macdonald derives (9.7) by solving a recurrence relation [21, Eq. (9.16)] for $c_{\lambda/\mu}(C)$ (C and λ fixed). Thus both in Macdonald's formula (9.6) and in formula (9.4) the expansion coefficients $b_{\lambda,\mu}$ are essentially given combinatorially by tableau sums ((9.7) and (9.5), respectively). However the tableau sums are quite different: over standard tableaux of shape λ/μ in Macdonald's case, and over reverse tableaux T of shape μ with entries in $\{1,\ldots,n\}$ in the case of (9.5). I do not see how to match these tableau sums with each other.

10. Explicit expressions for $n = 2$

The combinatorial formulas (6.1), (6.6), (6.7) for Macdonald polynomials and their (BC_n-type) interpolation versions all have the form

$$\sum_T \psi_T(q,t) \prod_{s\in\lambda} f(x_{T(s)}). \tag{10.1}$$

Here $\psi_T(q,t)$ is given by (6.2), while f is an elementary (Laurent) polynomial which may also depend on $s, T(s), t$ and a. The sum is over all reverse tableaux T of shape λ with entries in $\{1,\ldots,n\}$.

Now let $n=2$ and $\lambda := (m,0)$. Then the possible reverse tableaux T of shape $(m,0)$ with entries in $\{1,2\}$ are the tableaux T_k ($k=0,1,\ldots,m$) given by

$$T_k(1,j) = 2 \quad \text{if } j=1,\ldots,k \quad \text{and} \quad =1 \quad \text{if } j=k+1,\ldots,m. \tag{10.2}$$

So we have to compute $\psi_{T_k}(q,t)$, which is given by (6.2) as a double product involving $(R\backslash C)_{\lambda^{(0)}/\lambda^{(1)}} = \{(1,1),\ldots,(1,k)\}$ and $(R\backslash C)_{\lambda^{(1)}/\lambda^{(2)}} = \emptyset$. Hence

$$\psi_{T_k}(q,t) = \prod_{s\in(R\backslash C)_{\lambda^{(0)}/\lambda^{(1)}}} \frac{b_{(k)}(s;q,t)}{b_{(m)}(s;q,t)}\,,$$

where $b_\mu(s;q,t)$ is defined by (6.3). This yields for $j=1,\ldots,k$ that

$$b_{(k)}((1,j);q,t) = \frac{1-q^{k-j}t}{1-q^{k-j+1}}\,, \qquad b_{(m)}((1,j);q,t) = \frac{1-q^{m-j}t}{1-q^{m-j+1}}\,.$$

Hence

$$\psi_{T_k}(q,t) = \frac{(q^{k-1}t;q^{-1})_k\,(q^m;q^{-1})_k}{(q^k;q^{-1})_k\,(q^{m-1}t;q^{-1})_k} = \frac{(t,q^{-m};q)_k}{(q,q^{1-m}t^{-1};q)_k}\,(qt^{-1})^k. \tag{10.3}$$

10.1. **Explicit expression for BC_2-type interpolation Macdonald polynomials.** Consider in (10.1) $\prod_{j=1}^m f(x_{T_k(1,j)})$ for the case of (6.1). By taking into account (10.2) this product becomes

$$\prod_{j=1}^k \big((x_2-q^{j-1}a)(1-(q^{j-1}a)^{-1}x_2^{-1})\big) \prod_{j=k+1}^m \big((x_1-q^{j-1}ta)(1-(q^{j-1}ta)^{-1}x_1^{-1})\big)$$
$$= (-1)^m q^{-\frac12 m(m-1)}(ta)^{-m}(tax_1,tax_1^{-1};q)_m\,\frac{(ax_2,ax_2^{-1};q)_k}{(tax_1,tax_1^{-1};q)_k}\,t^k. \tag{10.4}$$

Then take (10.1), a single sum over $T=T_k$ ($k=0,\ldots,m$), with (10.3) and (10.4) substituted. The result is

$$P^{\mathrm{ip}}_{m,0}(x_1,x_2;q,t,a) = \frac{(tax_1,tax_1^{-1};q)_m}{q^{\frac12 m(m-1)}(-ta)^m}\ {}_4\phi_3\!\left(\begin{matrix} q^{-m},t,ax_2,ax_2^{-1} \\ q^{1-m}t^{-1},tax_1,tax_1^{-1}\end{matrix};q,q\right). \tag{10.5}$$

Next we want to obtain an explicit expression for $P^{\rm ip}_{m_1,m_2}$ by using the reduction formula (5.12). In the present case, this takes the form

$$\begin{aligned}&P^{\rm ip}_{m_1,m_2}(x_1,x_2;q,t,a)\\&\qquad\qquad=a^{-2m_2}q^{-m_2(m_2-1)}(ax_1,ax_1^{-1},ax_2,ax_2^{-1};q)_{m_2}\,P^{\rm ip}_{m_1-m_2,0}(x_1,x_2;q,t,q^{m_2}a).\end{aligned}$$

In combination with (10.5) this gives

$$\begin{aligned}P^{\rm ip}_{m_1,m_2}(x_1,x_2;q,t,a)&=\frac{q^{-\frac12m_1(m_1-1)-\frac12m_2(m_2-1)}}{(-t)^{m_1-m_2}a^{m_1+m_2}}\,(ax_1,ax_1^{-1},ax_2,ax_2^{-1};q)_{m_2}\\&\times(q^{m_2}tax_1,q^{m_2}tax_1^{-1};q)_{m_1-m_2}\;{}_4\phi_3\left(\begin{matrix}q^{-m_1+m_2},t,q^{m_2}ax_2,q^{m_2}ax_2^{-1}\\q^{1-m_1+m_2}t^{-1},q^{m_2}tax_1,q^{m_2}tax_1^{-1}\end{matrix};q,q\right).\end{aligned}\tag{10.6}$$

In order to make the symmetry in x_1,x_2 in (10.6) obvious, we may expand the second line of this formula as

$$\sum_{k=0}^{m_1-m_2}\frac{(q^{-m_1+m_2},t;q)_k\,q^k}{(q^{1-m_1+m_2}t^{-1},q;q)_k}\,(q^{m_2+k}tax_1,q^{m_2+k}tax_1^{-1};q)_{m_1-m_2-k}\,(q^{m_2}ax_2,q^{m_2}ax_2^{-1};q)_k.$$

Now substitute there a version of the q-Pfaff-Saalschütz formula [8, Eq. (II.12)]:

$$\frac{(q^{m_2+k}tax_1,q^{m_2+k}tax_1^{-1};q)_{m_1-m_2-k}}{(q^kt,q^{k+2m_2}ta^2;q)_{m_1-m_2-k}}={}_3\phi_2\left(\begin{matrix}q^{-m_1+m_2+k},q^{m_2}ax_1,q^{m_2}ax_1^{-1}\\q^{1-m_1+m_2}t^{-1},q^{2m_2+k}ta^2\end{matrix};q,q\right)$$

and expand the ${}_3\phi_2$. Then (10.6) takes the form

$$\begin{aligned}P^{\rm ip}_{m_1,m_2}(x_1,x_2;q,t,a)&=\frac{q^{-\frac12m_1(m_1-1)-\frac12m_2(m_2-1)}}{(-t)^{m_1-m_2}a^{m_1+m_2}}\,(t,q^{2m_2}ta^2;q)_{m_1-m_2}\\&\times(ax_1,ax_1^{-1},ax_2,ax_2^{-1};q)_{m_2}\sum_{\substack{j,k\ge0\\j+k\le m_1+m_2}}\frac{(q^{-m_1+m_2};q)_j}{(q^{1-m_1+m_2}t^{-1},q;q)_j}\,(q^{m_2}ax_1,q^{m_2}ax_1^{-1};q)_j\\&\qquad\times\frac{(q^{-m_1+m_2};q)_k}{(q^{1-m_1+m_2}t^{-1},q;q)_k}\,(q^{m_2}ax_1,q^{m_2}ax_1^{-1};q)_k\,\frac{q^{j+k}}{(q^{2m_2}ta^2;q)_{j+k}}\,.\end{aligned}\tag{10.7}$$

10.2. **(Interpolation) Macdonald polynomials for $n=2$.** For explicit formulas of $P^{\rm ip}_{m_1,m_2}(x_1,x_2;q,t)$ and $P_{m_1,m_2}(x_1,x_2;q,t)$ we may either give a derivation analogous to the one for (10.6), now starting from (6.6) or (6.1), or apply, much quicker, the limits (5.9) or (5.8) to (10.6). For $P^{\rm ip}_{m_1,m_2}(x_1,x_2;q,t)$ we obtain

$$\begin{aligned}P^{\rm ip}_{m_1,m_2}(x_1,x_2;q,t)&=x_1^{m_1}x_2^{m_2}\,(x_1^{-1},x_2^{-1};q)_{m_2}(q^{m_2}tx_1^{-1};q)_{m_1-m_2}\\&\qquad\qquad\times\,{}_3\phi_2\left(\begin{matrix}q^{-m_1+m_2},t,q^{m_2}x_2^{-1}\\q^{1-m_1+m_2}t^{-1},q^{m_2}tx_1^{-1}\end{matrix};q,\frac{qx_2}{tx_1}\right).\end{aligned}$$

By series inversion, this yields

$$P^{\rm ip}_{m_1,m_2}(x_1,x_2;q,t)=x_1^{m_2}x_2^{m_1}\,(x_1^{-1};q)_{m_2}\,(x_2^{-1};q)_{m_1}\;{}_3\phi_2\left(\begin{matrix}q^{-m_1+m_2},t,q^{-m_1+1}t^{-1}x_1\\ q^{1-m_1+m_2}t^{-1},q^{-m_1+1}x_2\end{matrix};q,q\right). \tag{10.8}$$

Different explicit expansions for interpolation Macdonald polynomials in the case where $n=2$ are given by Morse [22, Theorem 1.1], [23, Theorem 1].

For $P_{m_1,m_2}(x_1,x_2;q,t)$ we obtain

$$P_{m_1,m_2}(x_1,x_2;q,t)=x_1^{m_1}x_2^{m_2}\;{}_2\phi_1\left(\begin{matrix}q^{-m_1+m_2},t\\ q^{1-m_1+m_2}t^{-1}\end{matrix};q,\frac{qx_2}{tx_1}\right) \tag{10.9}$$

$$=\frac{(q;q)_{m_1-m_2}}{(t;q)_{m_1-m_2}}\sum_{j=0}^{m_1-m_2}\frac{(t;q)_j(t;q)_{m_1-m_2-j}}{(q;q)_j(q;q)_{m_1-m_2-j}}\,x_1^{m_1-j}x_2^{m_2+j} \tag{10.10}$$

$$=\frac{(q;q)_{m_1-m_2}}{(t;q)_{m_1-m_2}}\,(x_1x_2)^{\frac12(m_1+m_2)}\,C_{m_1-m_2}\left(\frac{x_1+x_2}{2(x_1x_2)^{\frac12}};t\mid q\right). \tag{10.11}$$

Here we used the *q-ultraspherical polynomial* (see [1, Eq. (4.4)]):

$$C_m(\cos\theta;t\mid q):=\sum_{j=0}^{m}\frac{(t;q)_j(t;q)_{m-j}}{(q;q)_j(q;q)_{m-j}}\,e^{i(m-2j)\theta}. \tag{10.12}$$

Formula (10.11) was earlier given by Morse [23, Theorem 2].

10.3. **Jack polynomials and (BC_n) interpolation Jack polynomials for $n=2$.** We can get explicit formulas for the $n=2$ cases of BC_n-type interpolation Jack polynomials, interpolation Jack polynomials and Jack polynomials by applying the limit formulas (7.1), (5.5), (4.7) to (10.6), (10.8), (10.9), respectively. We obtain

$$\begin{aligned}P^{\rm ip}_{m_1,m_2}(x_1,x_2;\tau,\alpha)&=(-1)^{m_1+m_2}\,(\alpha+x_1,\alpha-x_1,\alpha+x_2,\alpha-x_2)_{m_2}\\ &\quad\times(m_2+\tau+\alpha+x_1,m_2+\tau+\alpha-x_1)_{m_1-m_2}\\ &\quad\times\;{}_4F_3\left(\begin{matrix}-m_1+m_2,\tau,m_2+\alpha+x_2,m_2+\alpha-x_2\\ 1-m_1+m_2-\tau,m_2+\tau+\alpha+x_1,m_2+\tau+\alpha-x_1\end{matrix};1\right),\end{aligned} \tag{10.13}$$

$$\begin{aligned}&P^{\rm ip}_{m_1,m_2}(x_1,x_2;\tau)\\ &\quad=(-1)^{m_1+m_2}(-x_1)_{m_2}(-x_2)_{m_1}\;{}_3F_2\left(\begin{matrix}-m_1+m_2,t,-m_1+1-\tau+x_1\\ 1-m_1+m_2-\tau,-m_1+1+x_2\end{matrix};1\right).\end{aligned} \tag{10.14}$$

$$
\begin{aligned}
P_{m_1,m_2}(x_1,x_2;\tau) &= x_1^{m_1}x_2^{m_2}\;{}_2F_1\left(\begin{matrix}-m_1+m_2,\tau\\ 1-m_1+m_2-\tau\end{matrix};\frac{x_2}{x_1}\right)\\
&= \frac{(m_1-m_2)!}{(\tau)_{m_1-m_2}}\sum_{j=0}^{m_1-m_2}\frac{(\tau)_j(\tau)_{m_1-m_2-j}}{j!\,(m_1-m_2-j)!}\,x_1^{m_1-j}x_2^{m_2+j}\\
&= \frac{(m_1-m_2)!}{(\tau)_{m_1-m_2}}\,(x_1x_2)^{\frac12(m_1+m_2)}\,C^\tau_{m_1-m_2}\left(\frac{x_1+x_2}{2(x_1x_2)^{\frac12}}\right). \qquad (10.15)
\end{aligned}
$$

Here we used the *ultraspherical polynomial* (see [7, Section 10.9]):

$$
C^\tau_m(\cos\theta) := \sum_{j=0}^{m}\frac{(\tau)_j(\tau)_{m-j}}{j!\,(m-j)!}\,e^{i(m-2j)\theta}. \qquad (10.16)
$$

11. Binomial formulas for $n=2$

In this section we will only discuss the $n=2$ case of the binomial formulas for BC-type polynomials (Koornwinder and Jacobi), because they lead to explicit expressions of these polynomials. We will not discuss here the $n=2$ cases of the binomial formulas for the Macdonald and Jack polynomials, although they have interesting aspects from the point of view of special functions.

11.1. **Binomial formula for Koornwinder polynomials for $n=2$.** The binomial formula (8.1) takes for $n=2$ the form

$$
\begin{aligned}
\frac{P_{m_1,m_2}(x_1,x_2;q,t;a_1,a_2,a_3,a_4)}{P_{m_1,m_2}(ta_1,a_1;q,t;a_1,a_2,a_3,a_4)} &= \sum_{k_2=0}^{m_2}\sum_{k_1=k_2}^{m_1}\frac{P^{\rm ip}_{k_1,k_2}(q^{m_1}ta_1',q^{m_2}a_1';q,t,a_1')}{P^{\rm ip}_{k_1,k_2}(q^{k_1}ta_1',q^{k_2}a_1';q,t,a_1')}\\
&\qquad\times\frac{P^{\rm ip}_{k_1,k_2}(x_1,x_2;q,t,a_1)}{P_{k_1,k_2}(ta_1,a_1;q,t;a_1,a_2,a_3,a_4)}. \qquad (11.1)
\end{aligned}
$$

This gives an explicit expression for Koornwinder polynomials for $n=2$, since we have explicit expressions for everything on the right-hand side. For the first quotient on the right-hand side of (11.1) we get by (10.6):

$$
\begin{aligned}
\frac{P^{\rm ip}_{k_1,k_2}(q^{m_1}ta_1',q^{m_2}a_1';q,t,a_1')}{P^{\rm ip}_{k_1,k_2}(q^{k_1}ta_1',q^{k_2}a_1';q,t,a_1')} &= \frac{(q^{m_1}ta_1'^2,q^{-m_1}t^{-1},q^{m_2}a_1'^2,q^{-m_2};q)_{k_2}}{(q^{k_1}ta_1'^2,q^{-k_1}t^{-1},q^{k_2}a_1'^2,q^{-k_2};q)_{k_2}}\\
\times\frac{(q^{m_1+k_2}t^2a_1'^2,q^{m_2-k_1};q)_{k_1-k_2}}{(q^{k_1+k_2}t^2a_1'^2,q^{k_2-k_1};q)_{k_1-k_2}}&\;{}_4\phi_3\left(\begin{matrix}q^{-k_1+k_2},t,q^{m_2+k_2}a_1'^2,q^{m_2-k_2}\\ q^{1-k_1+k_2}t^{-1},q^{m_2+k_2}t^2a_1'^2,q^{-m_1+k_2}\end{matrix};q,q\right). \qquad (11.2)
\end{aligned}
$$

As for the second quotient on the right-hand side the numerator is obtained from (10.6), i.e.,

$$P^{\mathrm{ip}}_{k_1,k_2}(x_1,x_2;q,t,a_1)=\frac{q^{-\frac12 k_1(k_1-1)-\frac12 k_2(k_2-1)}}{(-t)^{k_1-k_2}a_1^{k_1+k_2}}\,(a_1x_1,a_1x_1^{-1},a_1x_2,a_2x_2^{-1};q)_{k_2}$$
$$\times\,(q^{k_2}ta_1x_1,q^{k_2}ta_1x_1^{-1};q)_{k_1-k_2}\ {}_4\phi_3\!\left(\begin{matrix}q^{-k_1+k_2},t,q^{k_2}a_1x_2,q^{k_2}a_1x_2^{-1}\\ q^{1-k_1+k_2}t^{-1},q^{k_2}ta_1x_1,q^{k_2}ta_1x_1^{-1}\end{matrix};q,q\right), \quad (11.3)$$

and the denominator is obtained from (4.15), i.e.,

$$P_{k_1,k_2}(ta_1,a_1;q,t;a_1,a_2,a_3,a_4)=t^{-k_1}a_1^{-k_1+k_2}\,\frac{(ta_1'^2;q)_{k_1}\,(a_1'^2;q)_{k_2}}{(t^2a_1'^2;q)_{2k_1}\,(a_1'^2;q)_{2k_2}}$$
$$\times\,(ta_1a_2,ta_1a_3,ta_1a_4;q)_{k_1}\,(a_1a_2,a_1a_3,a_1a_4;q)_{k_2}\,\frac{(t^2a_1'^2;q)_{k_1+k_2}}{(ta_1'^2;q)_{k_1+k_2}}\,\frac{(t^2;q)_{k_1-k_2}}{(t;q)_{k_1-k_2}}. \quad (11.4)$$

11.2. **Binomial formula for BC_2-type Jacobi polynomials.** We can specialize the binomial formula (9.1) for BC_n-type Jacobi polynomials to $n=2$ and get everything on the right-hand side explicitly:

$$\frac{P_{m_1,m_2}(x_1,x_2;\tau;\alpha,\beta)}{P_{m_1,m_2}(0,0;\tau;\alpha,\beta)}$$
$$=\sum_{k_2=0}^{m_2}\sum_{k_1=k_2}^{m_1}\frac{P^{\mathrm{ip}}_{k_1,k_2}(m_1+\tau+\alpha',m_2+\alpha';\tau,\alpha')}{P^{\mathrm{ip}}_{k_1,k_2}(k_1+\tau+\alpha',k_2+\alpha';\tau,\alpha')}\,\frac{P_{k_1,k_2}(x_1,x_2;\tau)}{P_{k_1,k_2}(0,0;\tau;\alpha,\beta)}. \quad (11.5)$$

The first quotient on the right-hand side can be explicitly evaluated as a limit case of (11.2) by using (7.1):

$$\frac{P^{\mathrm{ip}}_{k_1,k_2}(m_1+\tau+\alpha',m_2+\alpha';\tau,\alpha')}{P^{\mathrm{ip}}_{k_1,k_2}(k_1+\tau+\alpha',k_2+\alpha';\tau,\alpha')}=\frac{(m_1+\tau+2\alpha',-m_1-\tau,m_2+2\alpha',-m_2)_{k_2}}{(k_1+\tau+2\alpha',-k_1-\tau,k_2+2\alpha',-k_2)_{k_2}}$$
$$\times\,\frac{(m_2+k_2+2\tau+2\alpha',m_2-k_2)_{k_1-k_2}}{(k_1+k_2+2\tau+2\alpha',k_2-k_1)_{k_1-k_2}}$$
$$\times\ {}_4F_3\!\left(\begin{matrix}-k_1+k_2,\tau,m_2+k_2+2\alpha',m_2-k_2\\ 1-k_1+k_2-\tau,m_2+k_2+2\tau+2\alpha',-m_1+k_2\end{matrix};1\right). \quad (11.6)$$

For the numerator of the second quotient we have (10.15):

$$P_{k_1,k_2}(x_1,x_2;\tau)=\frac{(q;q)_{k_1-k_2}}{(t;q)_{k_1-k_2}}\,(x_1x_2)^{\frac12(k_1+k_2)}\,C^{\tau}_{k_1-k_2}\left(\frac{x_1+x_2}{2(x_1x_2)^{\frac12}}\right). \quad (11.7)$$

The denominator of the second quotient can be obtained by specializing (7.5):

$$P_{k_1,k_2}(0,0;\tau;\alpha,\beta) = (-1)^{k_1+k_2} \frac{(\tau+2\alpha',\tau+\alpha+1)_{k_1}\,(2\alpha',\alpha+1)_{k_2}}{(2\tau+2\alpha')_{2k_1}\,(2\alpha')_{2k_2}} \times \frac{(2\tau+2\alpha')_{k_1+k_2}}{(\tau+2\alpha')_{k_1+k_2}}\,\frac{(2\tau)_{k_1-k_2}}{(\tau)_{k_1-k_2}}. \quad (11.8)$$

The explicit formula for BC_2-type Jacobi polynomials which is given by combination of (11.5), (11.6), (11.7) and (11.8) was obtained in a very different way by Koornwinder and Sprinkhuizen [14, Corollary 6.6].

References

[1] R. Askey and J. Wilson, *Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials*, Memoirs Amer. Math. Soc. 54 (1985), no. 319.

[2] R. J. Beerends and E. M. Opdam, *Certain hypergeometric series related to the root system* BC, Trans. Amer. Math. Soc. 339 (1993), 581–609.

[3] J. F. van Diejen, *Commuting difference operators with polynomial eigenfunctions*, Compos. Math. 95 (1995), 183–233.

[4] J. F. van Diejen, *Self-dual Koornwinder-Macdonald polynomials*, Invent. Math. 126 (1996), 319–339.

[5] J. F. van Diejen, *Properties of some families of hypergeometric orthogonal polynomials in several variables*, Trans. Amer. Math. Soc. 351 (1999), 233–270.

[6] J. F. van Diejen and E. Emsiz, *Branching formula for Macdonald-Koornwinder polynomials*, arχiv:1408.2280v1, 2014.

[7] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher transcendental functions, Vol. II*, McGraw-Hill, 1953.

[8] G. Gasper and M. Rahman, *Basic hypergeometric series*, Cambridge University Press, second ed., 2004.

[9] M. Hallnäs, *Multivariable Bessel polynomials related to the hyperbolic Sutherland model with external Morse potential*, Internat. Math. Res. Notices (2009), 1573–1611.

[10] G. J. Heckman, *An elementary approach to the hypergeometric shift operators of Opdam*, Invent. Math. 103 (1991), 341–350.

[11] F. Knop, *Symmetric and non-symmetric quantum Capelli polynomials*, Comment. Math. Helv. 72 (1997), 84–100.

[12] F. Knop and S. Sahi, *Difference equations and symmetric polynomials defined by their zeros*, Internat. Math. Res. Notices (1996), no. 10, 473–486.

[13] T. H. Koornwinder, *Askey-Wilson polynomials for root systems of type* BC, in: *Hypergeometric functions on domains of positivity, Jack polynomials, and applications*, Contemp. Math. 138, Amer. Math. Soc., 1992, pp. 189–204.

[14] T. H. Koornwinder and I. G. Sprinkhuizen, *Generalized power series expansions for a class of orthogonal polynomials in two variables*, SIAM J. Math. Anal. 9 (1978), 457–483.

[15] A. Lascoux and S. O. Warnaar, *Branching rules for symmetric Macdonald polynomials and* $\mathfrak{sl}_n$ *basic hypergeometric series*, Adv. in Appl. Math. 46 (2011), 424–456.

[16] M. Lassalle, *Une formule du binôme généralisée pour les polynômes de Jack*, C. R. Acad. Sci. Paris Sér. I Math. 310 (1990), 253–256.

[17] I. G. Macdonald, *A new class of symmetric functions*, Sém. Lothar. Combin. 20 (1988), B20a, 41 pp.

[18] I. G. Macdonald, *Symmetric functions and Hall polynomials*, Oxford University Press, second edition, 1994.
[19] I. G. Macdonald, *Orthogonal polynomials associated with root systems*, unpublished manuscript, 1987; Sém. Lothar. Combin. 45 (2000), B45a, 40 pp.
[20] I. G. Macdonald, *Affine Hecke algebras and orthogonal polynomials*, Cambridge University Press, 2003.
[21] I. G. Macdonald, *Hypergeometric functions I*, unpublished manuscript, 1987; `arχiv:1309.4568`, 2013.
[22] J. Morse, *Some classical expansions for Knop-Sahi and Macdonald polynomials*, Sém. Lothar. Combin. 41 (1998), B41a, 29 pp.
[23] J. Morse, *Bivariate Knop-Sahi and Macdonald polynomials related to q-ultraspherical functions*, Discrete Math. 217 (2000), 293–299.
[24] A. Okounkov, *Binomial formula for Macdonald polynomials and applications*, Math. Res. Lett. 4 (1997), 533–553.
[25] A. Okounkov, *(Shifted) Macdonald polynomials: q-integral representation and combinatorial formula*, Compos. Math. 112 (1998), 147–182.
[26] A. Okounkov, *BC-type interpolation Macdonald polynomials and binomial formula for Koornwinder polynomials*, Transform. Groups 3 (1998), 181–207.
[27] A. Okounkov and G. Olshanski, *Shifted Jack polynomials, binomial formula, and applications*, Math. Res. Lett. 4 (1997), 69–78.
[28] E. M. Opdam, *Some applications of hypergeometric shift operators*, Invent. Math. 98 (1989), 1–18.
[29] E. M. Rains, *BC_n-symmetric polynomials*, Transform. Groups 10 (2005), 63–132.
[30] E. M. Rains, *Elliptic analogues of the Macdonald and Koornwinder polynomials*, in *Proceedings of the International Congress of Mathematicians. Volume IV*, Hindustan Book Agency, New Delhi, 2010, pp. 2530–2554.
[31] M. Rösler, T. Koornwinder and M. Voit, *Limit transition between hypergeometric functions of type BC and type A*, Compos. Math. 149 (2013), 1381–1400.
[32] S. Sahi, *The spectrum of certain invariant differential operators associated to a Hermitian symmetric space*, in: *Lie theory and geometry*, Progr. Math. 123, Birkhäuser, 1994, pp. 569–576.
[33] S. Sahi, *Interpolation, integrality, and a generalization of Macdonald's polynomials*, Internat. Math. Res. Notices (1996), no. 10, 457–471.
[34] S. Sahi, *Nonsymmetric Koornwinder polynomials and duality*, Ann. of Math. (2) 150 (1999), 267–282.
[35] S. Sahi, *Some properties of Koornwinder polynomials*, in: *q-Series from a contemporary perspective*, Contemporary Math. 254, Amer. Math. Soc., 2000, pp. 395–411.
[36] R. P. Stanley, *Some combinatorial properties of Jack symmetric functions*, Adv. Math. 77 (1989), 76–115.
[37] J. V. Stokman and T. H. Koornwinder, *Limit transitions for BC type multivariable orthogonal polynomials*, Canad. J. Math. 49 (1997), 373–404.
[38] Z. Yan, *A class of generalized hypergeometric functions in several variables*, Canad. J. Math. 44 (1992), 1317–1338.

T. H. Koornwinder, Korteweg-de Vries Institute, University of Amsterdam,, P.O. Box 94248, 1090 GE Amsterdam, The Netherlands; *E-mail*: `T.H.Koornwinder@uva.nl`