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Fatal Heyting Algebras and Forcing Persistent Sentences

Abstract. Hamkins and Löwe proved that the modal logic of forcing is **S4.2**. In this paper, we consider its modal companion, the intermediate logic **KC** and relate it to the fatal Heyting algebra $\mathbf{H}_{\mathsf{ZFC}}$ of forcing persistent sentences. This Heyting algebra is equationally generic for the class of fatal Heyting algebras. Motivated by these results, we further analyse the class of fatal Heyting algebras.

Keywords: forcing, intermediate logics, Heyting algebra.

0. Personal remarks of the second author

Over many years, there have been strong research ties between the *Institute for Logic, Language and Computation* in Amsterdam and the logic group built by Leo Esakia in Tbilisi. When I was visiting Tbilisi in late May 2007, Leo's group asked me to give two talks in their logic seminar and I presented the material in the papers [6] and [9]. After the presentation on the joint work with Joel Hamkins on the *Modal Logic of Forcing*, Leo suggested to use the ideas from [6] in the setting of intermediate logics: in the ensuing lively discussion, Leo developed his ideas in Russian and David Gabelaia translated for me.

I had almost forgotten about this bilingual discussion when I returned to Tbilisi later that year (October 2007) and Leo informed me then that our "joint paper was progressing well". In early 2009, Leo sent me notes containing the results on intermediate logics we had talked about, but also additional material on fatal Heyting algebras that he had worked on in the meantime. In the month of March 2009, Leo and I collaborated via e-mail to transform his notes into a submission for the conference *Topology, Algebra, and Categories in Logic* (TACL 2009). Since I felt that I had made no contribution to the additional material on fatal Heyting algebras, we eventually decided to remove it from the presentation I gave on 7 July 2009. §§ 1 and 2 of the present paper contain the material presented at TACL 2009. The editors of this special issue encouraged me to submit the paper as it had been intended by Leo when he wrote the notes in 2009; Guram Bezhanishvili,

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David Gabelaia and Mamuka Jibladze volunteered to provide the proofs of the material on fatal Heyting algebras in an appendix. § 3 contains the material on fatal Heyting algebras essentially as it was in Leo's notes that he had written in 2009, and all of the results in this section are due to Leo alone. The proofs for the statements in § 3 have been provided by Bezhanishvili, Gabelaia and Jibladze in the appendix on the basis of Leo's handwritten notes. I would like to thank them for their crucial assistance in this project.

1. Introduction

In this paper, we shall deal with three different languages, the language $\mathcal{L}$ of propositional logic, the language $\mathcal{L}_\Box$ of modal propositional logic, and the language $\mathcal{L}_{\mathsf{ZFC}}$ of set theory (i.e., first-order logic with a binary relation symbol $\in$). We identify the languages with their sets of formulæ or sentences.

The Lindenbaum algebra of ZFC is the Boolean algebra $\mathbf{B}_{\mathsf{ZFC}}$ of classes of provably equivalent $\mathcal{L}_{\mathsf{ZFC}}$-sentences (i.e., formulæ with no free variables). More precisely the elements of $\mathbf{B}_{\mathsf{ZFC}}$ are the classes $[\varphi] = \{\psi\,;\, \mathsf{ZFC} \vdash \varphi \leftrightarrow \psi\}$ and the Boolean operations are the induced ones, e.g., $[\varphi] \wedge [\psi] := [\varphi \wedge \psi]$, $\bot := [\varphi \wedge \neg\varphi]$, $\top = [\varphi \vee \neg\varphi]$. If $\mathbf{B} = \langle B, \wedge, \vee, \neg, 0, 1\rangle$ is a Boolean algebra and $\Box$ is a unary operation on B, we call $\langle \mathbf{B}, \Box\rangle$ an **interior algebra** if $\Box p \leq p$, $\Box\Box p = \Box p$, $\Box(p \wedge q) = \Box p \wedge \Box q$, and $\Box 1 = 1$. Note that if for an $\mathcal{L}_{\mathsf{ZFC}}$-sentence φ, we define $\Box\varphi$ to be the $\mathcal{L}_{\mathsf{ZFC}}$-formalization of "in every forcing extension, φ holds", then $\langle \mathbf{B}_{\mathsf{ZFC}}, \Box\rangle$ is an interior algebra.

A **Heyting algebra H** $= \langle H, \wedge, \vee, \to, 0, 1\rangle$ is a structure such that $\langle H, \wedge, \vee, 0, 1\rangle$ is a lattice with smallest and largest element and the equations $p \to p = 1$, $p \wedge (p \to q) = p \wedge q$, $q \wedge (p \to q) = q$, and $p \to (q \wedge r) = (p \to q) \wedge (p \to r)$ hold. We write $\neg p := p \to 0$. As usual, we shall not distinguish between $\mathbf{H}$ and its underlying set H, and write "$p \in \mathbf{H}$" when we mean "$p \in H$". Recall that a Heyting algebra $\langle H, \wedge, \vee, \to, 0\rangle$ is a Boolean algebra if and only if for every $p \in \mathbf{H}$, we have $p \vee \neg p = 1$. Algebraic terms in Boolean and Heyting algebras can be naturally identified with $\mathcal{L}$-formulas, whereas algebraic terms in interior algebras can be identified with $\mathcal{L}_\Box$-formulas. So, if Λ is any modal logic extending **S4**, we can say "an interior algebra $\langle B, \Box\rangle$ satisfies Λ" and mean that we identify each theorem φ of Λ with a term t_φ in the interior algebra, and $\langle B, \Box\rangle \models t_\varphi = 1$. From now on, we shall write "$\varphi = 1$" for this.

We denote the class of Heyting algebras by HA. Note that one of the two de Morgan laws is satisfied in every Heyting algebra, namely $\neg(p \vee q) = \neg p \wedge \neg q$ for every p and q in $\mathbf{H}$.

Definition 1. We call a Heyting algebra **fatal** if every prime filter is contained in only one maximal filter. The class of fatal Heyting algebras will be denoted by fHA.

For an interior algebra $\langle \mathbf{B}, \Box \rangle$, we define its **Heyting core** to be $\mathrm{H}(\mathbf{B}, \Box) := \{\Box p\, ;\, p \in \mathbf{B}\}$. On $\mathrm{H}(\mathbf{B}, \Box)$, we define $p \to q := \Box(\neg p \vee q)$, and see that $\langle \mathrm{H}(\mathbf{B}, \Box), \wedge, \vee, \to, 0, 1\rangle$ is a Heyting algebra.

Fact 2. *For a Heyting algebra* $\mathbf{H}$ *the following are equivalent:*

1. $\mathbf{H}$ *is fatal,*
2. *Stone's Principle holds, i.e.,* $\neg p \vee \neg\neg p = 1$,
3. $\neg(p \wedge q) = \neg p \vee \neg q$,
4. $\neg\neg p \to p = p \vee \neg p$.

Proof. (1) $\Rightarrow$ (2): If there is $p \in \mathbf{H}$ such that $\neg p \vee \neg\neg p \neq 1$, then there is a prime filter x of $\mathbf{H}$ with $\neg p \vee \neg\neg p \notin x$. Therefore, $\neg p \notin x$ and $\neg\neg p \notin x$. Let F be the filter generated by x and $\neg p$ and G be the filter generated by x and $\neg\neg p$. If there is $a \in x$ such that $a \wedge \neg p = 0$, then $a \leq \neg\neg p$, so $\neg\neg p \in x$, a contradiction. Therefore, F is proper. Similarly if there is $b \in x$ with $b \wedge \neg\neg p = 0$, then $b \leq \neg\neg\neg p = \neg p$, so $\neg p \in x$, a contradiction. Thus, G is also proper. Clearly F and G are incomparable as $\neg p \in F \backslash G$ and $\neg\neg p \in G \backslash F$. Consequently, they extend to two different maximal filters y and z. So there are two different maximal filters containing x, and so $\mathbf{H}$ is not fatal.

(2) $\Rightarrow$ (1): Suppose there is a prime filter x contained in two different maximal filters y and z. Then there is $p \in \mathbf{H}$ contained in y but not in z. Since z is maximal, $\neg p \in z$. Therefore, $\neg p \notin y$ and $\neg\neg p \notin z$. Thus, $\neg p, \neg\neg p \notin x$, and since x is prime, $\neg p \vee \neg\neg p \notin x$, so $\neg p \vee \neg\neg p \neq 1$.

That (2) is equivalent to (3) is well known; cf., e.g., [7, p. 10].

(2) $\Rightarrow$ (4). As $\neg\neg p \wedge (p \vee \neg p) \leq p$, it follows that $p \vee \neg p \leq \neg\neg p \to p$ in any Heyting algebra $\mathbf{H}$. Let $\mathbf{H}$ satisfy (2). We show that if $q \leq \neg\neg p \to p$, then $q \leq p \vee \neg p$. Let $q \leq \neg\neg p \to p$. Then $q \wedge \neg\neg p \leq p$. Therefore, $(q \wedge \neg\neg p) \vee \neg p \leq p \vee \neg p$. Thus, $(q \vee \neg p) \wedge (\neg\neg p \vee \neg p) \leq p \vee \neg p$, and so by (2), $q \vee \neg p \leq p \vee \neg p$, which implies $q \leq p \vee \neg p$. Consequently, $\neg\neg p \to p \leq p \vee \neg p$, and so $\neg\neg p \to p = p \vee \neg p$.

(4) $\Rightarrow$ (2): Substituting $\neg p$ for p in (4) gives $\neg\neg\neg p \to \neg p = \neg p \vee \neg\neg p$. The left hand side is obviously 1, so (2) follows. ∎

2. Forcing persistent sentences

We say that an $\mathcal{L}_{\mathsf{ZFC}}$-sentence φ is **forcing persistent** if whenever φ is true in a model M then it is true in every forcing extension of M. As before, we use $\Box$ for the "true in all forcing extensions" operator. We say that a function $H : \mathcal{L}_{\Box} \to \mathcal{L}_{\mathsf{ZFC}}$ is a **forcing translation** if H commutes with propositional connectives and $H(\Box\varphi)$ is the formalization of "$H(\varphi)$ is true in all forcing extensions". We call a formula $\varphi \in \mathcal{L}_{\Box}$ a **valid principle of forcing** if for all forcing translations H, we have that $\mathsf{ZFC} \vdash H(\varphi)$. By [6, Theorem 3], all theorems of **S4.2** are valid principles of forcing ("the soundness of **S4.2**").

THEOREM 3 (Hamkins-Löwe). *If* ZFC *is consistent, then for every $\varphi \in \mathcal{L}_{\Box}$, we have* **S4.2** $\vdash \varphi$ *if and only if for all forcing translations H the sentence $H(\varphi)$ is provable in* ZFC *[6, Main Theorem 6].*

We follow [6, p. 1798] and call an $\mathcal{L}_{\mathsf{ZFC}}$-sentence φ a **button** if there is a forcing extension such that $\Box\varphi$ is true and a **switch** if in all forcing extensions, both $\Diamond\varphi$ and $\Diamond\neg\varphi$ are true. Every $\mathcal{L}_{\mathsf{ZFC}}$-sentence is either a button or the negation of a button or a switch.

PROPOSITION 4. *Let $\varphi \in \mathcal{L}_{\mathsf{ZFC}}$. Then the following are equivalent:*

1. *the sentence φ is forcing persistent, and*
2. *the sentence φ is equivalent to a statement of the form $\Box\psi$.*

PROOF. If φ is forcing persistent, then $\varphi \leftrightarrow \Box\varphi$, so (1.)$\Rightarrow$(2.) is obvious.

To see (2.)$\Rightarrow$(1.) check the three possible cases. If ψ is a button, then $\Box\psi$ is a button (and forcing persistent). If ψ is the negation of a button, then $\Box\psi$ is provably false (since buttons are necessarily buttons by **S4.2**; cf. [6, p. 1798]), and hence forcing persistent. If ψ is a switch, then $\Box\psi$ is provably false as well. ■

We denote the set of equivalence classes of forcing persistent sentences by $\mathbf{H}_{\mathsf{ZFC}}$. It is not hard to see that $\mathbf{H}_{\mathsf{ZFC}}$ is a sublattice of $\mathbf{B}_{\mathsf{ZFC}}$. By Proposition 4, we get that $\mathbf{H}_{\mathsf{ZFC}} = \mathrm{H}(\mathbf{B}_{\mathsf{ZFC}}, \Box)$, and thus $\mathbf{H}_{\mathsf{ZFC}}$ is a Heyting algebra.

PROPOSITION 5. *For every forcing persistent φ, $\Box\neg\varphi \vee \Box\Diamond\varphi$ is a* ZFC-*theorem. Therefore, $\mathbf{H}_{\mathsf{ZFC}}$ satisfies Stone's Principle, and thus (by Fact 2) is a fatal Heyting algebra.*

PROOF. Suppose $\Box\neg\varphi$ is not the case. Then there is a forcing extension in which φ holds. Since φ is forcing persistent, $\Box\varphi$ holds in this extension, so in the original model, we have $\Diamond\Box\varphi$. But by the soundness of **S4.2**, we get $\Box\Diamond\varphi$. ■

We furthermore have the **Gödel translation** $T : \mathcal{L} \to \mathcal{L}_\Box$ of the intuitionistic propositional calculus **HC** into the modal system **S4** defined by $T(p) = \Box p$, $T(\varphi \wedge \psi) = T(\varphi) \wedge T(\psi)$, $T(\varphi \vee \psi) = T(\varphi) \vee T(\psi)$, and $T(\neg\varphi) = \Box\neg T(\varphi)$.

THEOREM 6 (Gödel). *For every $\varphi \in \mathcal{L}$, we have that* $\mathbf{HC} \vdash \varphi$ *if and only if* $\mathbf{S4} \vdash T(\varphi)$ *[5, 11].*

Via the identification of formulæ in $\mathcal{L}$ and terms in Heyting algebras (as well as formulæ in $\mathcal{L}_\Box$ and terms in interior algebras), we can consider T as a map between the Heyting core of an interior algebra and the surrounding interior algebra.

PROPOSITION 7. *If $\langle \mathbf{B}, \Box \rangle$ is an interior algebra and $\varphi \in \mathcal{L}$, then $\langle \mathbf{B}, \Box \rangle \models T(\varphi) = 1$ if and only if* $\mathrm{H}(\mathbf{B}, \Box) \models \varphi = 1$.

PROOF. Cf., e.g., [3, § 8.3]. ■

An **intermediate logic** is an $\mathcal{L}$-logic extending **HC** and contained in classical propositional logic. One particular intermediate logic is the logic **KC**, also known as "the logic of the weak law of excluded middle", "Jankov logic", "testability logic", or "De Morgan logic" which has been originally introduced and studied by Dummett and Lemmon in [4]. The logic **KC** is axiomatized by adding to **HC** the following weak law of excluded middle: $\neg\varphi \vee \neg\neg\varphi$. By Fact 2, it is obvious that the class fHA provides an adequate algebraic semantics of the intermediate logic **KC**.

Dummett and Lemmon showed, among other things, that the modal system **S4.2** (which was also originally introduced in [4]) interprets **KC** by the Gödel translation T. We call **KC** the **modal companion** of **S4.2** via the Gödel translation.

THEOREM 8 (Dummett/Lemmon). *For every $\varphi \in \mathcal{L}$, we have that* $\mathbf{KC} \vdash \varphi$ *if and only if* $\mathbf{S4.2} \vdash T(\varphi)$ *[4].*

We are now considering the compositions $H \circ T : \mathcal{L} \to \mathcal{L}_{\mathsf{ZFC}}$ of forcing translations with the Gödel translation.

PROPOSITION 9. *Let $\varphi \in \mathcal{L}_{\mathsf{ZFC}}$. Then the following are equivalent:*

1. *the sentence φ is forcing persistent, and*

2. there is a $\psi \in \mathcal{L}$ and a forcing translation H such that $\varphi = H(T(\psi))$.

PROOF. By Proposition 4, we know that the forcing persistent sentences are of the form $\Box\psi$ for a button ψ or provably false.

For (2.)⇒(1.), we only need to check that for every $\psi \in \mathcal{L}$, $T(\psi)$ is **S4.2**-equivalent to a boxed formula. However, this is clear by induction (note that $\Box p \vee \Box q \leftrightarrow \Box(\Box p \vee \Box q)$).

For (1.)⇒(2.), let $\varphi = \Box\psi$, and use the formula $p \in \mathcal{L}$. Any forcing translation H with $H(p) = \psi$ witnesses $\varphi = H(T(p))$. ■

Combining Theorems 3 and 8 with Proposition 9, we immediately get:

COROLLARY 10. *If $\varphi \in \mathcal{L}$, then* **KC** $\vdash \varphi$ *if and only if for every forcing translation H, we have* $\mathsf{ZFC} \vdash H(T(\varphi))$. *Thus the intermediate logic* **KC** *is precisely the logic of* ZFC*-provable forcing persistent sentences.*

THEOREM 11. *The fatal Heyting algebra* $\mathbf{H}_{\mathsf{ZFC}}$ *is equationally generic for the class* fHA*, i.e., each element of* fHA *is a homomorphic image of a subalgebra of a product of copies of* $\mathbf{H}_{\mathsf{ZFC}}$.

PROOF. In general, if $\mathfrak{K}$ is a class of algebras, then we call $A \in \mathfrak{K}$ **functionally free** if for any two terms s and t, we have that $A \models s = t$ if and only if for all $K \in \mathfrak{K}$, we have that $K \models s = t$. Tarski proved that being functionally free is equivalent to being equationally generic in the sense of the theorem [13, p. 164].

Let $\mathrm{BAO}_{\mathbf{S4.2}}$ be the class of Boolean algebras with an operator that satisfy **S4.2** (i.e., they are all interior algebras). If $\langle \mathbf{B}, \Box \rangle$ is functionally free for $\mathrm{BAO}_{\mathbf{S4.2}}$, then its Heyting core $\mathrm{H}(\mathbf{B}, \Box)$ is functionally free in fHA, and thus by Tarski's theorem equationally generic. [We identify formulæ with their canonical terms in the algebras. By Theorem 8, $\mathbf{KC} \vdash \psi$ is equivalent to $\mathbf{S4.2} \vdash T(\psi)$. By assumption, this is equivalent to $\langle \mathbf{B}, \Box \rangle \models T(\psi) = 1$, and this in turn to $\mathrm{H}(\mathbf{B}, \Box) \models \psi = 1$ by Proposition 7.]

But Theorem 3 implies that $\langle \mathbf{B}_{\mathsf{ZFC}}, \Box \rangle$ is functionally free for $\mathrm{BAO}_{\mathbf{S4.2}}$, thus completing the proof. ■

3. Observations about fatal Heyting algebras

In the course of analysing fatal Heyting algebras in the setting of intermediate logics, we also established some specific properties of fatal Heyting algebras which may be of independent interest. The following subsets of a Heyting algebra $\langle H, \wedge, \vee, \rightarrow, 0 \rangle$ are of special importance: the *centre* of $\mathbf{H}$,

$c(\mathbf{H}) = \{p \in \mathbf{H}\,;\, p \vee \neg p = 1\}$, the *dense part* $d(\mathbf{H}) = \{p \in \mathbf{H}\,;\, \neg p = 0\}$, the *fatal part* of $\mathbf{H}$, $f(\mathbf{H}) = \{p \in \mathbf{H}\,;\, \neg p \vee \neg\neg p = 1\}$ and the *set of regular elements* of $\mathbf{H}$, $\mathrm{rg}(\mathbf{H}) = \{p \in \mathbf{H}\,;\, p = \neg\neg p\}$.

PROPOSITION 12. *Let $\langle H, \wedge, \vee, \rightarrow, 0\rangle$ be a Heyting algebra. Then* $f(\mathbf{H})$ *is a Heyting subalgebra of* $\mathbf{H}$.

COROLLARY 13. *The fatal part* $f(\mathbf{H})$ *of any Heyting algebra* $\mathbf{H}$ *is the largest subalgebra of* $\mathbf{H}$ *which is a fatal Heyting algebra.*

It is not hard to see that the assignment of $f(\mathbf{H})$ to each Heyting algebra $\mathbf{H}$ can be expanded to yield a functor from the category HA into the category fHA of fatal Heyting algebras.

Consider the following typical example of a fatal Heyting algebra: Let X be an arbitrary topological space. We say that $U \subseteq X$ is a *steadily open set* if both U and the closure of U are open. The set of steadily open subsets of X constitutes a fatal Heyting algebra. Since a space is extremally disconnected if and only if every open set is steadily open, we obtain: X is extremally disconnected if and only if the Heyting algebra of open sets of X is fatal.

PROPOSITION 14. *Let* $\mathbf{H}$ *be a Heyting algebra; then* $f(\mathbf{H}) = \{b \wedge d\,;\, b \in c(\mathbf{H}), d \in d(\mathbf{H})\}$.

FACT 15. *A Heyting algebra* $\mathbf{H}$ *is fatal iff* $\mathrm{rg}(\mathbf{H}) = c(\mathbf{H})$, *i.e., every regular element is complemented.*

PROOF. Cf. [7, Chapter I.1.13]. ■

COROLLARY 16. *The algebra* $f(\mathbf{H})$ *is the largest subalgebra* $\mathbf{H}_0$ *of* $\mathbf{H}$ *which satisfies the condition* $\mathrm{rg}(\mathbf{H}_0) = c(\mathbf{H}_0)$.

Combining Fact 2 with Corollary 16, we immediately obtain:

PROPOSITION 17. *The center* $c(\mathbf{H})$ *of an arbitrary fatal Heyting algebra* $\mathbf{H}$ *is a retract of* $\mathbf{H}$.

The following observation expresses an interesting connection between fHA and HA:

THEOREM 18. *The inclusion functor* $I : \mathrm{fHA} \rightarrow \mathrm{HA}$ *has a left adjoint F and a right adjoint G, i.e., the subcategory* fHA *is a reflective and coreflective subcategory of the category of all Heyting algebras* HA.

Furthermore, the functor G preserves varieties, i.e., if $\mathcal{V}$ is a subvariety of HA *then* $G(\mathcal{V}) = \{G(\mathbf{H})\,;\, \mathbf{H} \in \mathcal{V}\}$ *also constitutes a variety.*

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A. Proofs for § 3

The proofs in this appendix were provided by Guram Bezhanishvili, David Gabelaia, and Mamuka Jibladze (see § 0).

Proof of Proposition 12. Clearly $0, 1 \in \mathrm{f}(\mathbf{H})$. Let $p, q \in \mathrm{f}(\mathbf{H})$. Then:

$$\begin{aligned}\neg(p \vee q) \vee \neg\neg(p \vee q) &= (\neg p \wedge \neg q) \vee \neg(\neg p \wedge \neg q)\\ &= (\neg p \vee \neg(\neg p \wedge \neg q)) \wedge (\neg q \vee \neg(\neg p \wedge \neg q))\\ &\geq (\neg p \vee \neg\neg p) \wedge (\neg q \vee \neg\neg q)\\ &= 1\end{aligned}$$

and

$$\begin{aligned}\neg(p \wedge q) \vee \neg\neg(p \wedge q) &\geq (\neg p \vee \neg q) \vee (\neg\neg p \wedge \neg\neg q)\\ &= (\neg p \vee \neg q \vee \neg\neg p) \wedge (\neg p \vee \neg q \vee \neg\neg q)\\ &\geq (\neg p \vee \neg\neg p) \wedge (\neg q \vee \neg\neg q)\\ &= 1.\end{aligned}$$

Therefore, $p \vee q, p \wedge q \in \mathrm{f}(\mathbf{H})$. Moreover,

$$\begin{aligned}\neg(p \to q) \vee \neg\neg(p \to q) &\geq \neg(p \to \neg\neg q) \vee \neg\neg(\neg p \vee q)\\ &= \neg(\neg q \to \neg p) \vee \neg(\neg\neg p \wedge \neg q).\end{aligned}$$

Since $\neg q \vee \neg\neg q = 1$, we have $\neg q \in \mathrm{c}(\mathbf{H})$, and so $\neg q \to r = \neg\neg q \vee r$ for any r (cf., e.g., [2, Lemma 16]). Thus, $\neg(\neg q \to \neg p) = \neg(\neg\neg q \vee \neg p)$, and so

$$\begin{aligned}\neg(\neg q \to \neg p) \vee \neg(\neg\neg p \wedge \neg q) &= \neg(\neg\neg q \vee \neg p) \vee \neg(\neg\neg p \wedge \neg q)\\ &= (\neg q \wedge \neg\neg p) \vee \neg(\neg\neg p \wedge \neg q)\\ &\geq (\neg q \wedge \neg\neg p) \vee (\neg p \vee \neg\neg q)\\ &= (\neg q \vee \neg p \vee \neg\neg q) \wedge (\neg\neg p \vee \neg p \vee \neg\neg q)\\ &= 1.\end{aligned}$$

Consequently, $p \to q \in \mathrm{f}(\mathbf{H})$, and hence $\mathrm{f}(\mathbf{H})$ is a Heyting subalgebra of $\mathbf{H}$. ■

Proof of Corollary 13. By Proposition 12, $\mathrm{f}(\mathbf{H})$ is a Heyting subalgebra of $\mathbf{H}$. Moreover, if $\mathbf{H}'$ is a Heyting subalgebra of $\mathbf{H}$ and $\mathbf{H}'$ is fatal, then $\neg p \vee \neg\neg p = 1$ for each $p \in \mathbf{H}'$, and so $\mathbf{H}' \subseteq \mathrm{f}(\mathbf{H})$. Thus, $\mathrm{f}(\mathbf{H})$ is the largest Heyting subalgebra of $\mathbf{H}$ that is a fatal Heyting algebra. ■

Proof of Proposition 14. Let $p \in \mathbf{H}$. Then

$$p = (\neg\neg p \wedge p) \vee (\neg\neg p \wedge \neg p) = \neg\neg p \wedge (p \vee \neg p).$$

Clearly $p \vee \neg p \in \mathrm{d}(\mathbf{H})$, and if $p \in \mathrm{f}(\mathbf{H})$, then $\neg\neg p \in \mathrm{c}(\mathbf{H})$. Therefore, $p = b \wedge d$ with $b \in \mathrm{c}(\mathbf{H})$ and $d \in \mathrm{d}(\mathbf{H})$. Conversely, let $b \in \mathrm{c}(\mathbf{H})$ and $d \in \mathrm{d}(\mathbf{H})$. Then $b = \neg\neg b$ (cf., e.g., [2, Lemma 16]) and $\neg(b \wedge d) = b \to \neg d = b \to 0 = \neg b$. Thus, $\neg(b \wedge d) \vee \neg\neg(b \wedge d) = \neg b \vee (\neg\neg b \wedge \neg\neg d) = \neg b \vee \neg\neg b = b \vee \neg b = 1$, and so $b \wedge d \in \mathrm{f}(\mathbf{H})$. ■

Corollary 16 is deduced immediately from Proposition 14 the same way Corollary 13 is deduced from Proposition 12.

Proof of Proposition 17. Let $\mathbf{H}$ be a fatal Heyting algebra. By Fact 15, $\mathrm{c}(\mathbf{H}) = \mathrm{rg}(\mathbf{H})$. Clearly $\mathrm{c}(\mathbf{H})$ is a Heyting subalgebra of $\mathbf{H}$ (see, e.g., [2, Lemma 16]). On the other hand, $\neg\neg : \mathbf{H} \to \mathrm{rg}(\mathbf{H})$ is a Heyting epimorphism, and so $\mathrm{rg}(\mathbf{H})$ is a homomorphic image of $\mathbf{H}$ (see, e.g., [12, Chapter IV.6]). Thus, $\mathrm{c}(\mathbf{H})$ is a retract of $\mathbf{H}$. ■

Proof of Theorem 18. That fHA is a reflective subcategory of HA is clear; in fact, any subvariety of any variety is an (epi)reflective subcategory; the reflector F assigns to an algebra its quotient by the identities of the subvariety (see, e.g., [1, Corollary 10.21]). Define $G : \mathrm{HA} \to \mathrm{fHA}$ by assigning $\mathrm{f}(\mathbf{H})$ to each Heyting algebra $\mathbf{H}$. This clearly defines a functor since the restriction of any Heyting algebra homomorphism $h : \mathbf{H} \to \mathbf{H}'$ to $\mathrm{f}(\mathbf{H})$ has its image in $\mathrm{f}(\mathbf{H}')$. Moreover, this functor is a coreflector since any homomorphism $\mathbf{K} \to \mathbf{H}$ from a fatal Heyting algebra $\mathbf{K}$ to any Heyting algebra $\mathbf{H}$ factors uniquely through $\mathrm{f}(\mathbf{H})$.

In order to see the "furthermore" part of the theorem, let $\mathcal{V}$ be a subvariety of HA and let $\mathbf{H} \in \mathcal{V}$. Then $G(\mathbf{H})$ is in $\mathcal{V}$ as a subalgebra of $\mathbf{H}$, so we obtain:

1. any homomorphic image $\mathbf{H}'$ of $G(\mathbf{H})$ is both in $\mathcal{V}$ and in fHA, so we get that $\mathbf{H}' = \mathrm{f}(\mathbf{H}') = G(\mathbf{H}')$, and thus $\mathbf{H}' \in G(\mathcal{V})$;
2. the same argument shows that any subalgebra $\mathbf{H}'$ of $G(\mathbf{H})$ must be in $G(\mathcal{V})$;
3. any product of algebras from $G(\mathcal{V})$ is again in $G(\mathcal{V})$, since G, being a right adjoint, preserves products. ■

In order to strengthen Theorem 18 to a result that was not included in Leo's notes (Theorem 20 below), we shall need the following lemma:

Lemma 19. *Let $\mathcal{V}'$ be a coreflective subvariety of a variety $\mathcal{V}$. If $\mathcal{V}$ has the amalgamation property, then so does $\mathcal{V}'$.*

Proof. It is well known (and easy to see) that a variety $\mathcal{V}$ has the amalgamation property iff for any $A_1, A_2 \in \mathcal{V}$ and any common subalgebra $A \subseteq A_i$, $i = 1, 2$, the canonical homomorphisms $A_i \to A_1 \amalg_A A_2$, $i = 1, 2$, into the pushout of A_1 and A_2 over A in $\mathcal{V}$ are injective maps (cf., e.g., [8]). Now suppose A, A_1 and A_2 belong to $\mathcal{V}'$. Then since $\mathcal{V}'$ is coreflective, i.e., the inclusion functor $I : \mathcal{V}' \to \mathcal{V}$ has a right adjoint, it follows that I preserves all colimits; in particular, $A_1 \amalg_A A_2$ belongs to $\mathcal{V}'$ too. This proves the amalgamation property for $\mathcal{V}'$. ■

Theorem 20. *The only nontrivial proper coreflective subvarieties of* HA *are the variety* BA *of Boolean algebras and the variety* fHA *of fatal Heyting algebras.*

Proof. That BA is a nontrivial proper coreflective subvariety of HA is well known. For fHA, this was proved in Theorem 18. To prove the converse, suppose $\mathcal{V}$ is a nontrivial proper coreflective subvariety of HA. Then by Lemma 19, $\mathcal{V}$ has the amalgamation property. Therefore, by [10], $\mathcal{V}$ is one of the following six varieties:

1. BA.
2. The variety generated by the three element chain $\mathbf{3}$.
3. The variety generated by all finite chains.
4. The variety generated by the Heyting algebra $\mathbf{4} \oplus 1$, where $\mathbf{4}$ is the four element Boolean algebra and $- \oplus 1$ is the operation of adjoining a new top.

5. The variety generated by $0 \oplus B \oplus 1$, where B is an arbitrary finite Boolean algebra and $0 \oplus -$ is the operation of adjoining a new bottom.
6. fHA.

If $\mathcal{V} \neq \mathrm{BA}$, then the three element Heyting algebra $\mathbf{3}$ belongs to $\mathcal{V}$. Since $\mathcal{V}$ is a coreflective subvariety of HA, it is closed under arbitrary colimits. Therefore, the coproduct (in HA) of $\mathbf{3}$ with itself belongs to $\mathcal{V}$. As this is an infinite finitely generated Heyting algebra, it follows that $\mathcal{V}$ cannot be locally finite. But all the varieties listed above except fHA are locally finite. Thus, $\mathcal{V} = \mathrm{fHA}$. Consequently, if $\mathcal{V}$ is a nontrivial proper coreflective subvariety of HA, then $\mathcal{V} = \mathrm{BA}$ or $\mathcal{V} = \mathrm{fHA}$. ∎