Random Walk in the Quadrant
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Chapter 0

Introduction and overview

0.0 Introduction

Imagine a chess board, unbounded to the North and to the East. The King is put on some square, say in the South-West corner. A player throws a die and, depending on the outcome, moves the King to one of the neighbouring squares (because the King has eight possible moves, the player needs a die — not necessarily fair — with eight faces). After the move, the player throws the same die again, the King is moved accordingly, and so on. The game is over as soon as the King falls off the chess board.

We are interested primarily in the duration of the game, i.e. the total number of moves of the King until his fall; as this number varies from play to play, our interest concerns the statistical properties of this number. Can we be sure it is finite? What is its expectation (i.e. the average number of moves per play in the long run, if the play is repeated again and again)? Or is its expectation infinite (i.e. the average number of moves per play in the long run grows beyond all bounds, if the play is repeated again and again)?

In technical terms, the King performs a Markov chain of a very special kind, viz. a stopped random walk. The squares of the chess board are called the states of the Markov chain.

We might add to the chess board a column of squares to the West, and a row of squares to the South; these added squares are called boundary states. In our game the boundary states are called absorbing since the King, once moved into such a state, remains in this state forever.

This kind of problem arises in queueing theory. In our case the queueing system consists of two servers, where each server has its own queue of waiting customers. Arriving customers join either one of the queues. Here every integer pair (rather than square) represents a state in the system: the first (resp. second) integer represents the number of waiting customers in the first (resp. second) queue. Every “move” to a neighbouring point can be interpreted in an obvious way; so a move into N.-W. direction can be caused by a customer having been
served by the first server and now having joined the second queue; and a move into N.-E. direction means the simultaneous arrival of two customers, each of them joining a different queue.

Thus the length of each queue separately may change only with one unit at a time; but the two queues may change simultaneously. So moves are possible to neighbouring points only; in other words, the random walk is skipfree (or continuous). In this book we consider only skipfree random walks because of the special position of these walks.

In the queuing model, a busy period begins at once and ends when one of the servers becomes idle; so the first move of the King corresponds with the start of a busy period and a move of the King into the boundary corresponds with the end of it. Thus the duration of the game equals the length of this busy period.

The statistical properties of the duration of the game obviously depend on the statistical properties of the die used. In parts of this book the attention is focused especially on driftless random walk; that means that in the long run the mean displacement per move of the King is zero; because the random walks under consideration are skipfree this amounts to saying that the probability of a move in a westward (i.e. N.-W., W., or S.-W.) direction equals the probability of a move in an eastward direction, and the probability of a move in a northward direction equals the probability of a move in a southward direction.

Closely related to the expectation of the duration of the walk is the probability of the occurrence of a ladder epoch. A ladder epoch is a time when the King is more eastern than ever before and simultaneously more northern than ever before in unrestricted random walk (i.e. on a chess board unbounded to all sides). It is well-known that there is with probability 1 a first ladder epoch for the unrestricted walk (starting at the origin) precisely then if the expected duration of the walk with absorbing boundary (starting at the origin) is infinite, see p. 44, cf. GREENWOOD AND SHAKED [26].

The work on this book started with the question — asked by P. Greenwood, see p. 46 — whether there is almost surely a first ladder epoch for a very special random walk, viz. the unrestricted even (skipfree, all-sided, driftless, see I.1.(4)–(7)) random walk (equivalently, whether the expected duration of the corresponding random walk with absorbing boundary is infinite). For nonnegative correlation coefficient the answer was known to be in the affirmative (cf. GREENWOOD AND SHAKED [25]). For negative correlation coefficient ($\neq -1$) the question is settled in this book. For the case of a positive correlation coefficient we give a proof using stochastic monotonicity. In order to settle the question in the case of a negative correlation coefficient we have computed the generating function $P(x, y, s)$ of the transition probabilities explicitly, using an analytic method due to J. Groeneveld. In this case the answer turns out to be negative. The computation — which involves the uniformization of an algebraic curve, and, subsequently, the solving of a functional equation, and, finally, the determining
of the limit of the solution — is complicated; however, the resulting formula for the probability of the occurrence of a first ladder epoch appears to be simple, and was found originally by approaching the limit numerically on a computer.

These results have been extended to more general random walk, at first to symmetric (skipfree, all-sided, driftless) walk, and, finally, in this book, to general (skipfree, all-sided, driftless) walk (announced in KLEIN HANEVELD [34]).

Afterwards, this simple formula in the case of a negative correlation coefficient has been proved by means of martingale techniques in a simple way in a more general setting, viz. for general left-continuous, driftless random walk, see KLEIN HANEVELD AND PITTINGER [35]. This result has been found independently and proved analytically by Cohen, see COHEN [10, Th. II.2.4.4].

In literature, different methods are used for the analysis of two-dimensional (and higher-dimensional) semi-homogeneous (see p. 15) and left-continuous random walks on the integer pairs of the positive quarter plane. ADAN et al. [2, 1, 3], HOOGHIEMSTRA et al. [30] and BLANC [6] use numerical-iterative methods.

Other authors use generating function methods. These methods all consist of solving some functional equation (where the coefficient function of the unknown generating function is called the kernel, cf. p. 17) by restricting the domain of the functional equation to the zero locus of the kernel. The discrete time and the continuous time cases lead to similar equations. Different analytic techniques have been used to solve the resulting equations.

FAYOLLE AND IASNORODSKI [18], BLANC [5], COHEN AND BOXMA [13], and COHEN [8, 9] transform the equation into a boundary value problem; in the extensive monograph COHEN [10] the author uses the Hitting Point Identity (which is Wald’s exponential equality in a more general setting).

A uniformization technique has been applied only for skipfree random walk. In this technique the zero locus of the kernel is uniformized (parametrized); it is the conformal image of a torus (nondegenerate case, the genus is 1, the uniformizing functions are elliptic), or of a sphere (degenerate case, the genus is 0, the uniformizing functions are rational) (cf. FORD [23]). The solution is obtained, subsequently, via analytic continuation.

Kingman [33] and Flatto and McKean [22], Hofri [29] and Adan et al. [4], and Jaffé [32] study models involving skipfree, one-sided random walks with drift (see Fig. 1 and 2, 3, and 4). Because of one-sidedness, the zero locus of the kernel has genus 0. Cohen [12] determines the solution via analytic continuation without prior uniformization of the zero locus of the kernel.

Flatto and Hahn [21, 20] and Wright [53] study models involving skipfree, all-sided random walks with drift (see Fig. 5 and 6). (Similar random walks are studied in Cohen [11].) Here the zero locus of the kernel has genus 1.
Figure 1: Kingman (1961) (continuous time)

Figure 2: Flatto and McKean (1977) (continuous time)

Figure 3: Hofri (1978) (continuous time)

Figure 4: Jaffe (1992) (discrete time)

Figure 5: Flatto and Hahn (1984–1985) (continuous time)

Figure 6: Wright (1992) (continuous time)
[We note that the random walks studied in Jaffe [32], Flatto and Hahn [21, 20], and Wright [53] are of some more special type; they behave at the boundary in such a way that in the functional equation restricted to the zero locus of the kernel the variables are “separated”.

A basic study is Malyshev [41], cf. also [40]. He develops the above uniformization technique in the nondegenerate case but, however, doesn’t study specific models. Some analytical techniques from [41] are used in this book in the analysis of the zero locus of the kernel.

The above authors are interested in the generating function of the steady state probabilities; so they consider positively recurrent random walk and solve a time-independent (homogeneous) functional equation. Also, they use analytic continuation as an essential step in the process of identifying the solution. Another method to identify the solution via uniformization of the zero locus of the kernel is the application of Fourier analysis (or the use of Laurent series). A pioneer in the field, J. Groeneveld was the first\footnote{Actually, the problem discussed in Kingman [33] was solved by Groeneveld in April, 1958, but it has not been published.} to apply the latter method successfully. He considered continuous time, semi-homogeneous, nondegenerate random walk. He was interested in (the generating function of) the transition probabilities rather than (the generating function of) the steady state probabilities, and therefore considered a time-dependent (inhomogeneous) functional equation (see the appendix, p. 15). In this equation, the time dependence is connected with a variable corresponding in this book with the variable $s$ (see p. 17).

Groeneveld’s [27]\textsuperscript{2} method consists essentially of two elements. One is the uniformization of the kernel; in this book this uniformization takes place in Chapter V (cf. Sec. V.4). The second one is consideration of certain functional equations; in this book the relevant functional equation is considered in Chapter IV (and solved in Sec. IV.5).

In this book we use Groeneveld’s method to determine the generating function $P(x, y, s)$ of the transition probabilities of discrete time random walk with absorbing boundary, with some adaptations. On the one hand, Groeneveld’s problem (concerning a random walk on the quarter plane with reflecting (i.e. not absorbing) boundary) is more general than our problem (concerning a random walk on the quarter plane with absorbing boundary), where in the functional equation (restricted to the zero locus of the kernel) the variables are “separated”. On the other hand, the solution of our problem requires knowledge of the generating

\textsuperscript{1}Actually, the problem discussed in Kingman [33] was solved by Groeneveld in April, 1958, but it has not been published.

\textsuperscript{2}Unfortunately, this has not been published yet. Prof. Runnenburg provided me with a copy of part of Groeneveld’s manuscript [27]. This copy served as an introduction to this analytic method. Consequently, it also strongly influenced us in notation and terminology. However, the full content of Groeneveld’s manuscript is not discussed in this book, because the basic functional equation considered by Groeneveld (cf. the appendix, p. 15) is more general than the relevant functional equation in this book, as is amplified below.
function \( P(x, y, s) \) for \( s = 1 \), rather than for \( s \) close to 0.

Unfortunately, for \( s = 1 \) in the case of skipfree, driftless random walk the functional equation doesn’t determine the generating function. We have circumvented this difficulty by (first step) determining the generating function in the case \( 0 < s < 1 \) for general all-sided random walk (hence, nondegenerate zero locus), and (second step) determining the limit for \( s \uparrow 1 \) of the generating function in the case of driftless all-sided random walk. Although in principle the first step could have been accomplished by applying Groeneveld’s solving method for small values of \( s \), and next continuing the solution for larger values of \( s \), \( 0 < s < 1 \), we have chosen to adapt the solving method.

Here is a sketch of the first step (determining the generating function in the case \( 0 < s < 1 \) for general all-sided random walk). The unknown generating function is the solution of the functional equation. Together with regularity conditions this equation is called in this book the “backward conditions” as they are derived from the Kolmogorov backward equation. Because of the time dependence, the functional equation is not homogeneous. The kernel is a modification of the generating function of the one-step transition probabilities in the interior of the quadrant; it is quadratic in each one of its two variables (here \( x_1 \) and \( x_2 \)) because the random walk is skipfree. We use an analytic method to solve the functional equation.

First, \( x_1 \) and \( x_2 \) are restricted to that part of the zero locus of the kernel where both variables are within the unit circle. So the generating function drops out from the equation, and there remains a linear functional equation in two unknown functions (the “marginal functions”) of one variable, and an unknown constant. In our case the coefficient functions of the unknown marginal functions depend on one variable only, the same variable as the marginal function; so the variables are “separated”. The zero locus of the kernel defines a relationship between the two variables \( x_1 \) and \( x_2 \).

Next, by substituting for these variables suitably chosen second order elliptic functions of \( z_1 \) resp. \( z_2 \) with period 1 (here the zero locus of the kernel is uniformized), this complicated relationship is transformed into a simple linear one of a special type; the unknown functions are even and have period 1; also, the transforms of the unit disc consist of simply connected strips with period 1, so the Fourier coefficients of the unknown functions are well-defined. This enables us to apply Fourier analysis to determine the unknown functions and the unknown constant, because the equations are transformed into linear equations in the unknown Fourier coefficients one can solve.

Here is a sketch of second step (determining the limit for \( s \uparrow 1 \) of the generating function in the case of driftless, all-sided random walk). In the expression for the generating function derived in the first step, integrals occur with integrands involving \( \theta \)-functions. Unfortunately, these \( \theta \)-functions occur in a form only suit-
able for determining the behaviour of the integrands for $s$ close to zero. We use Jacobi’s Imaginary Transformation to transform them into a form suitable for determining the behaviour of the integrands for $s$ close to 1. This enables us to determine the limits of the integrals for $s \uparrow 1$, and subsequently the limit of the generating function, which in the case of $s = 1$, $|x| < (1,1)$, $|y| \leq (1,1)$ is not always known to converge in advance (in advance we know there is convergence if $\rho < 0$, cf. Cor. I.1.12, and divergence if both $\rho \geq 0$ and $y = (1,1)$, cf. Th. I.1.16).

Appendix: The basic functional equation considered by Groeneveld (from the manuscript [27], in our notation). Groeneveld considered continuous time skipfree random walk $(X_t)_{t \geq 0}$ on the nonnegative integer pairs (with independent, exponentially distributed sojourn times). So the walk is a Markov chain with stationary transition probabilities. The chain is semi-homogeneous, i.e. the transition probabilities (and the sojourn time distributions) on the positive $X$-axis, the transition probabilities (and the sojourn time distributions) on the positive $Y$-axis, and the transition probabilities (and the sojourn time distributions) in the interior of the quadrant do not depend on the position; so the associated $Q$-matrix has the form

$$q(i,i+h) = \alpha_0(h) \chi\{i_1 = 0, i_2 = 0\} + \alpha_1(h) \chi\{i_1 \geq 1, i_2 = 0\} + \alpha_2(h) \chi\{i_1 = 0, i_2 \geq 1\} + \alpha_3(h) \chi\{i_1 \geq 1, i_2 \geq 1\}$$

(where $i$ and $h$ are integer pairs with $i = (i_1, i_2) \geq (0,0)$ and $h \in \{-1,0,1\}^2$, and $\chi$ means indicator function), see Fig. 7.

Purpose is to compute the transition probability function

$$p(i,j,n,t) \overset{\text{def}}{=} \Pr\{X_t = j, N_t = n \mid X_0 = i\}$$

where $N_t$ is the number of steps up to time $t$.

Starting point is the forward equation (in differential form; cf. Chung [7, II.17.(14), p.250] for the forward equation in integral form)

$$\begin{cases}
\frac{\partial}{\partial t} p(i,j,n,t) = -q_j p(i,j,n,t) + \sum_{k \neq j} p(i,k,n-1,t) q(k,j) \\
p(i,j,-1,t) \overset{\text{def}}{=} 0 \\
p(i,j,n,0) = \delta(i,j) \delta(n,0)
\end{cases}$$

(Kronecker delta)

where $i, j, k$ are integer pairs $\geq (0,0)$, $n$ is an integer $\geq 0$, and $t$ is a real number $\geq 0$. Define the transform $P(x,y|z,r)$ as follows:

$$p(i,j|z,r) \overset{\text{def}}{=} \sum_{n \geq 0} z^n \int_0^\infty dt \exp(-rt) p(i,j,n,t)$$

$$P(x,y|z,r) \overset{\text{def}}{=} \sum_{i \geq (0,0)} x^i \sum_{j \geq (0,0)} y^j p(i,j|z,r)$$

This transform has the decomposition
Figure 7: Groeneveld (1965–1966, unpublished)
(continuous time)

\[ P(x, y|z, r) = P_0(x|z, r) + \sum_{\ell=1,2} P_\ell(x, y_\ell|z, r) + P_3(x, y|z, r) \]

with

\[ P_0(x|z, r) \overset{\text{def}}{=} \sum_{i \geq (0,0)} x^i p(i, (0, 0)|z, r) \]
\[ P_1(x, y_1|z, r) \overset{\text{def}}{=} \sum_{i \geq (0,0)} x^i \sum_{j_1 \geq 1} y_1^{j_1} p(i, (j_1, 0)|z, r) \]
\[ P_2(x, y_2|z, r) \overset{\text{def}}{=} \sum_{i \geq (0,0)} x^i \sum_{j_2 \geq 1} y_2^{j_2} p(i, (0, j_2)|z, r) \]
\[ P_3(x, y|z, r) \overset{\text{def}}{=} \sum_{i \geq (0,0)} x^i \sum_{j \geq (1,1)} y_j^j p(i, j|z, r) \]

Also define \((\ell = 0, \ldots, 3)\):

\[ \alpha_\ell \overset{\text{def}}{=} \sum_{k \neq (0,0)} \alpha_\ell(k) \]
\[ A_\ell(y) \overset{\text{def}}{=} \sum_{k \neq (0,0)} y^k \alpha_\ell(k) \]

(with 8 values for \(k\) if \(\ell = 3\), with 5 values for \(k\) if \(\ell = 2\) or 1, and with 3 values for \(k\) if \(\ell = 0\)), and

\[ Q_\ell(y|z, r) \overset{\text{def}}{=} r + \alpha_\ell - z A_\ell(y) \].

Then (I) transforms into (II):

\[ Q_0(y|z, r) P_0(x|z, r) + \sum_{\ell=1,2} Q_\ell(y|z, r) P_\ell(x, y_\ell|z, r) + \]
\[ + Q_3(y|z, r) \, P_3(x, y|z, r) = \prod_{k=1,2} (1 - x_k y_k)^{-1}. \]

In this equation the coefficient function \( Q_3(y|z, r) \) is the kernel.

The basic functional equation of Groeneveld results from (II) by restricting \( y = (y_1, y_2) \) to the zero locus of the kernel, and is an equation in the unknown marginal functions \( P_\ell, \ell = 0, 1, 2 \). In this book \( s = \alpha_3 \cdot z/(r + \alpha_3) \).
0.1 Overview

Overview of chapter I  Here we sketch the probabilistic background. In addition, it contains probabilistic amplifications of the main (analytic) arguments in the following chapters, and probabilistic consequences of the analytic results. Finally, the transformation of the probabilistic problem into an analytic one (a functional equation, called here the “backward conditions”) takes place, viz. in the third section.

Overview of chapter II  This chapter logically links up with Chapter VI. Here we construct a function (viz. \( \hat{P}(x, y) \)) which subsequentially is identified with the generating function of the transition probabilities \( P(x, y, s) \) with \( s = 1 \) for skipfree, all-sided, driftless random walk. However, the proof of this identification depends on Basic Theorem II.4.55 which is proved only in Chapter VI.

Also, we uniformize the zero locus of the kernel (see p. 17) in the \( s = 1 \) case, and express all constants involved explicitly in the transition probabilities of the random walk. The functional equation is solved in this case; however, the solution is not unique.

In addition, the way solutions can be continued is demonstrated; however, this technique is not used in the sequel.

Overview of chapter III  This chapter is independent of the other chapters, and can be read separately. It deals with the well-known theory of the construction of second order elliptic functions. This is, partly, for the convenience of the reader, and, partly, because here the fundamental elliptic function in which all other ones are expressed is neither the Weierstrass \( \wp \)-function nor one of the Jacobian functions, but a different one, viz. \( v \mapsto m(v|\tau) \), which we find more convenient for our needs.

Also, the well-known Jacobian \( \theta \)-functions are introduced. These functions are an excellent means to evaluate elliptic functions; moreover, the logarithmic derivative of a \( \theta \)-function is needed in the solution of the functional equation in Chapter IV.

Overview of chapter IV  This chapter is devoted to the solution of a well-defined system consisting of a functional equation with additional conditions, and can be read, therefore, separately. The system is solved with the aid of Fourier analysis. The system is a slight generalization of the transformed backward conditions, and the solution is applied straightforwardly in Sec. V.7 to solve the backward conditions.

Overview of chapter V  Here we assume that the walk is all-sided (see I.1.(5)) and that \( 0 < s < 1 \). In this chapter we solve the functional equation from Chapter I, obtaining an expression for the generating function of the transition
probabilities $P(x, y, s)$. For that purpose we first uniformize the zero locus of the kernel (see p. 17) by means of elliptic functions, and next, in the last section, use the machinery developed in Chapter IV to solve the backward conditions.

In Sec. V.4 the elliptic functions we need are identified by means of the equivalent differential equations. This identification can also be achieved by means of geometrical methods, but the analytical method used is more direct and precise.

**Overview of chapter VI** Here we relate the expression for $P(x, y, s)$ found in Chapter V for all-sided walk in the case $0 < s < 1$ to the function $\hat{P}(x, y)$ constructed in Chapter II for all-sided, driftless random walk in the case $s = 1$. We prove that $\hat{P}(x, y)$ arises as the limit of $P(x, y, s)$ for $s \to 1$, which is the content of Basic Theorem II.4.55. For *non-even* random walk the proof is contained in Secs. VI.1–3, and for *even* random walk in Secs. VI.4–6.

### 0.2 Numbering and Notations

**Numbering** Within a section, the theorems, propositions, lemmas, and properties are numbered consecutively; however, the remarks are numbered separately. Also, the formulæ are numbered consecutively.

Numbered items, and references to numbered items within the same section as well, bear neither the chapter number nor the section number. E.g. see the references to the formulæ (58) and (63), and the reference to Th. 13 on p. 40.

If the numbered item referred to is in a different section, then also the chapter number and the section number is mentioned. E.g. see the reference to Th. II.4.59 on p. 40.

**Notations** Generally, $i$, $j$, and $k$ are indices from $\{1, 2\}$; always, $i$ and $j$ are different, $i + j = 3$. The characters $\kappa$, $\lambda$, $\mu$, and $\nu$ are indices from $\pm$; in formulæ they stand for $\pm 1$.

At some places we use the abbreviation “tg” in stead of “tan” for the tangens-function.

Several names of constants, sets, and functions in Chapter II carry a hat; they are the analogue or the limit (for $s \uparrow 1$) of corresponding constants, sets, or functions from Chapter V.